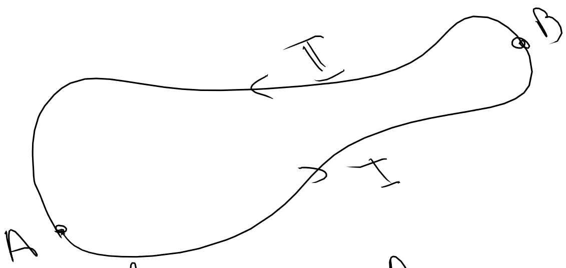
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}\,')\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0} \quad ; \quad \underline{\vec{\nabla}\times\vec{E} = 0} \Longleftarrow$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0} \qquad \oint \vec{E}\cdot d\vec{l} = 0$$

$$\oint \vec{E}\cdot d\vec{l} = 0$$



$$\oint \vec{E}\cdot d\vec{l} = \int_{A,I}^{B} \vec{E}\cdot d\vec{l} + \int_{B,II}^{A} \vec{E}\cdot d\vec{l}$$

$$= \int_{A,I}^{B} \vec{E}\cdot d\vec{l} - \int_{A,II}^{B} \vec{E}\cdot d\vec{l} = 0$$

$$\Rightarrow \int_{I} \vec{E}\cdot d\vec{l} = \int_{II} \vec{E}\cdot d\vec{l}$$

$$\int_{A}^{B} \vec{E}\cdot d\vec{l} \equiv f(A,B)$$

$$V(A) = -\int_{O}^{A} \vec{E}\cdot d\vec{l}$$

$$V'(A) = -\int_{O'}^{A} \vec{E}\cdot d\vec{l} = \underbrace{-\int_{O'}^{O} \vec{E}\cdot d\vec{l}}_{K} - \int_{O}^{A} \vec{E}\cdot d\vec{l}$$

$$K = -\int_{O'}^{O} \vec{E}\cdot d\vec{l}$$

$$\boxed{V'(A) = K + V(A)}$$

$$V(A) - V(B) = -\int_{O}^{A} \vec{E}\cdot d\vec{l} + \int_{O}^{B} \vec{E}\cdot d\vec{l}$$

$$\int_{B}^{A} dV = -\left[\int_{O}^{A} \vec{E}\cdot d\vec{l} + \int_{B}^{O} \vec{E}\cdot d\vec{l}\right]$$

$$\int_{B}^{A} (\vec{\nabla} V)\cdot d\vec{l} = -\int_{B}^{A} \vec{E}\cdot d\vec{l}$$

$$\vec{\nabla} V = -\vec{E} \Rightarrow \boxed{\vec{E} = -\vec{\nabla} V}$$

$$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times(\vec{\nabla} V) = 0$$

$$dV = (\vec{\nabla} V)\cdot d\vec{l}$$

Example

$$\boxed{\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}$$

$$\vec{r}' = 0$$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$$

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}(\vec{r}) \cdot d\vec{l}$$

$\vec{E} = E_r \hat{r}$ $\qquad\qquad$ $\vec{E} \cdot d\vec{l} = E_r \, dr$

$d\vec{l} = dr\,\hat{r} + \ldots$

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \frac{1}{4\pi\varepsilon_0} \frac{q}{r'^2} \, dr'$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{r'} \Bigg|_{r'=r_{\mathcal{O}}}^{r'=r} = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r} - \frac{q}{r_{\mathcal{O}}} \right)$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} - \frac{q}{4\pi\varepsilon_0} \frac{1}{r_{\mathcal{O}}}$$

Consider $r_{\mathcal{O}} \to \infty$

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r}}$$

If $\vec{r}' \neq 0$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{\rho}{\varepsilon_0}$$

$$\Downarrow$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}} \quad \text{Poisson's Eqn}$$

if $\rho = 0$ in a given region

$\nabla^2 V = 0 \Rightarrow$ Laplace eqn.

Superposition of the Electric Potential

$$\vec E(\vec r) = \sum_i \vec E_i(\vec r)$$

$$V(\vec r) = -\int_O^{\vec r} \vec E \cdot d\vec l$$

$$V(\vec r) = -\sum_i \int_O^{\vec r} \vec E_i(\vec r) \cdot d\vec l$$

$$\boxed{V(\vec r) = \sum_i V_i(\vec r)}$$

$$V(\vec r) = \int d^3 r' \, \frac{1}{4\pi\varepsilon_0} \frac{\rho(\vec r')}{|\vec r - \vec r'|}$$

$$\vec E(\vec r) = \int d^3 r' \, \frac{1}{4\pi\varepsilon_0} \frac{\rho(\vec r')}{|\vec r - \vec r'|} \frac{\vec r - \vec r'}{|\vec r - \vec r'|^2}$$

$$\vec\nabla \frac{1}{|\vec r - \vec r'|} = -\frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} = -\frac{1}{|\vec r - \vec r'|} \frac{\vec r - \vec r'}{|\vec r - \vec r'|^2}$$

$$\vec E = -\vec\nabla V$$

For any vector field $\int_{A,I}^{B} \vec V \cdot d\vec l = -\int_{B,I}^{A} \vec V \cdot d\vec l$

$[\vec E] = N/C = V/m$

$[V] = N/C \, m = V = \text{Volt}$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} + \vec{\nabla}\times\vec{E} = 0 + \text{B.C.}}$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}V \qquad V(\vec{r}) = -\int^{\vec{r}} \vec{E}\cdot d\vec{l}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

Examples

1)

Q

$$dq = \frac{Q}{L}\,dl$$

$$r = \sqrt{R^2+d^2}$$

$$V = \int \left(\frac{Q}{L}\,dl\right) \frac{1}{\sqrt{R^2+d^2}} \frac{1}{4\pi\varepsilon_0}$$

$$V = \frac{Q}{\cancel{L}} \frac{1}{\sqrt{R^2+d^2}} \frac{1}{4\pi\varepsilon_0} \cancel{\int dl}$$

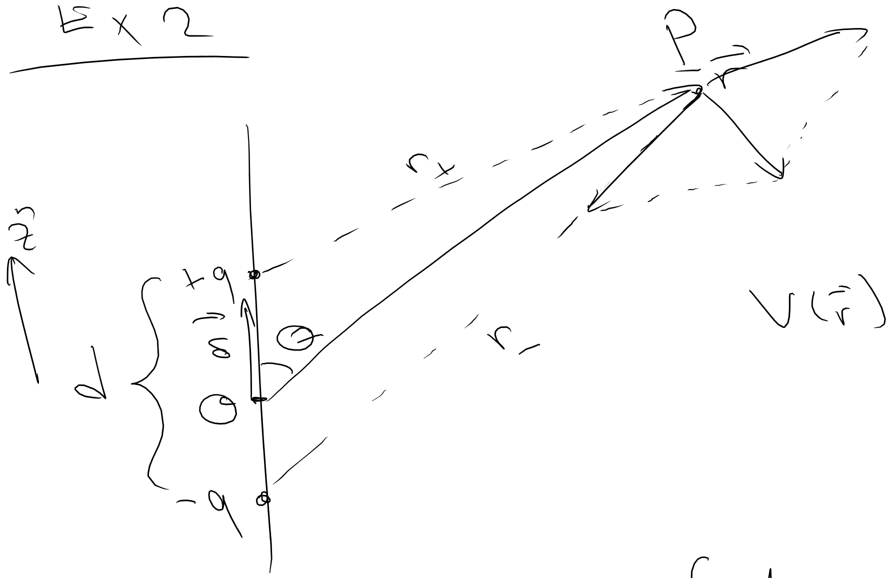
$$\boxed{V = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{R^2+d^2}}}$$

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

$$E_z = -\frac{\partial V}{\partial d}\hat{z} = +\frac{Q}{4\pi\varepsilon_0}\left(+\frac{1}{\cancel{2}}\right)\frac{\cancel{2}d}{(R^2+d^2)^{3/2}}$$

$$\boxed{E_z = \frac{Q}{4\pi\varepsilon_0} \frac{d}{(R^2+d^2)^{3/2}}}$$

Ex 2



$$\vec{r}_+ = \vec{r} - \vec{\delta}$$
$$\vec{r}_- = \vec{r} + \vec{\delta}$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r_+} + \frac{(-q)}{4\pi\varepsilon_0}\frac{1}{r_-}$$

$$\vec{\delta} = \frac{d}{2}\hat{z} \qquad V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{r_- - r_+}{r_+ r_-}$$

$$r_\mp \equiv |\vec{r}_\mp| = \sqrt{\vec{r}_\mp^{\,2}} = \sqrt{(\vec{r} \mp \vec{\delta})^2}$$
$$= \sqrt{r^2 + \delta^2 \pm 2r\delta\cos\theta}$$
$$= r\left(1 + \left(\frac{\delta}{r}\right)^2 \pm 2\frac{\delta}{r}\cos\theta\right)^{1/2}$$

$$\frac{d}{r} \ll 1 \qquad r_\mp \simeq r\left[1 + \frac{1}{2}\left(\left(\frac{\delta}{r}\right)^2 \pm 2\frac{\delta}{r}\cos\theta\right)\right]$$
$$(1+x)^n \simeq 1 + nx$$

$$r_\mp \simeq r\left[1 \pm \frac{\delta}{r}\cos\theta\right]$$

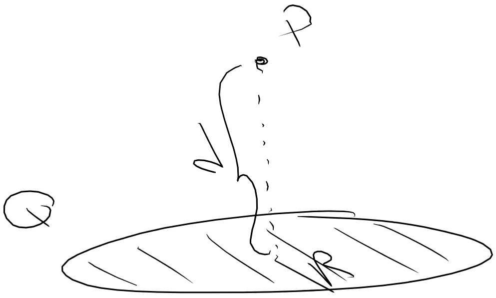
$$r_\mp \simeq r \pm \delta\cos\theta$$

$$\boxed{r_\mp \simeq r \pm \frac{\vec{r}\cdot\vec{\delta}}{r}}$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{r_- - r_+}{r_+ r_-} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}\cdot(2\vec{\delta})}{r}\frac{1}{r^2} + O\left(\left(\frac{d}{r}\right)^2\right)$$

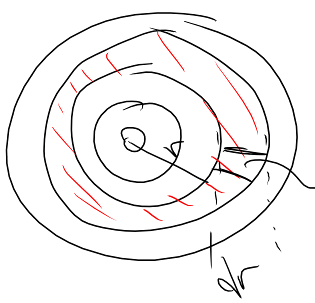
$$\vec{p} \equiv q(2\vec{\delta}) \Rightarrow \boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}}$$

Ex



$$V(P) = ?$$

$$dQ = \frac{Q}{\pi R^2} 2\pi r \, dr$$



$(dr \; r \, d\theta)$

$$dV = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\pi R^2} 2\pi r \, dr\right)\frac{1}{\sqrt{r^2+d^2}}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{Q}{\pi R^2}\frac{2\pi r \, dr}{\sqrt{r^2+d^2}}$$

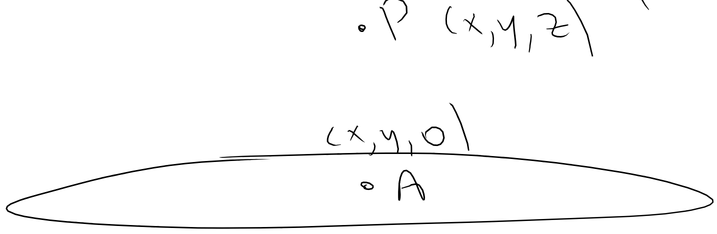
$$= \frac{1}{2\cancel{4}\pi\varepsilon_0}\frac{Q}{\cancel{\pi}R^2}\cancel{2\pi}\sqrt{r^2+d^2}\Big|_{r=0}^{R}$$

$$\boxed{V = \frac{1}{2\pi\varepsilon_0}\frac{Q}{R^2}\left(\sqrt{R^2+d^2} - d\right)}$$

$$\boxed{V = \frac{1}{2\varepsilon_0}\sigma\left(\sqrt{R^2+d^2} - d\right)}$$

$$\sigma = \frac{Q}{\pi R^2}$$

Ex Potential of infinite plane



$$V(P) - V(A) = \frac{1}{2\varepsilon_0}\sigma\left(\sqrt{R^2+d^2} - d\right) - \frac{1}{2\varepsilon_0}\sigma R$$

choose A as the reference point at which $V(A) = 0$

$$\boxed{V(P) = \frac{1}{2\varepsilon_0}\sigma\left[\sqrt{R^2+d^2} - R - d\right]}$$

$$\sqrt{R^2+d^2} = R\sqrt{1+\left(\frac{d}{R}\right)^2} \simeq R\left(1+\frac{1}{2}\left(\frac{d}{R}\right)^2\right) \simeq R + \frac{d^2}{2R}$$

$$V(P) = \frac{1}{2\varepsilon_0}\sigma\left[R + \frac{d^2}{2R} - R - d\right]$$

$$\boxed{V(P) = -\frac{\sigma}{2\varepsilon_0} d} \Rightarrow \boxed{V(P) = -\frac{\sigma}{2\varepsilon_0}|z|}$$

$$V(P) = \frac{\sigma}{2\varepsilon_0}\begin{cases} -z & \text{if } z > 0 \\ z & \text{if } z < 0 \end{cases}$$

$$\vec{E} = -\frac{\partial V}{\partial z}\hat{z} = \frac{\sigma}{2\varepsilon_0}\begin{cases} \hat{z} & \text{if } z > 0 \\ -\hat{z} & \text{if } z < 0 \end{cases}$$

$$\vec{E}_{top} - \vec{E}_{bottom} = \frac{\sigma}{\varepsilon_0} \hat{n}_{above}$$

$$V_{top} - V_{bottom} = -\int_{bottom} \vec{E} \cdot \vec{dl} = 0$$

$$\boxed{\begin{gathered} \Delta V \Big|_{\text{accross the surface}} = 0 \\ \frac{\partial V}{\partial n}\Big|_{top} - \frac{\partial V}{\partial n}\Big|_{bottom} = -\frac{\sigma}{\varepsilon_0} \end{gathered}}$$