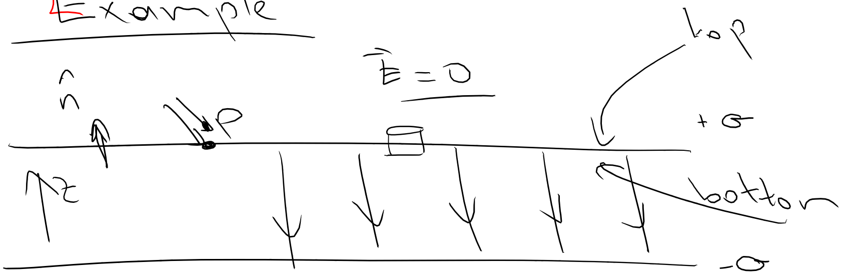
Hand in your HW!

Example

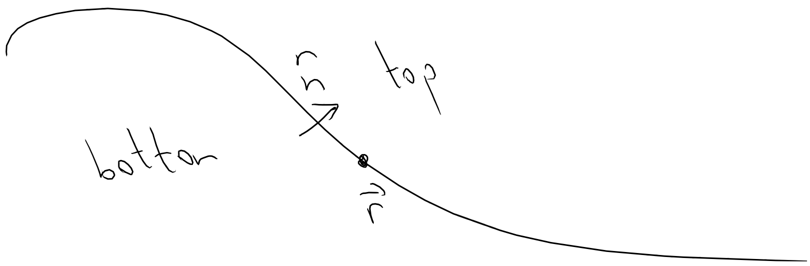


$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z}$$

$$\hat{n} = \hat{z}$$

$$\vec{E}_{top} - \vec{E}_{bottom} = \underbrace{\vec{0}}_{\vec{E}=0} - \left(-\frac{\sigma}{\varepsilon_0}\hat{z}\right) = \frac{\sigma}{\varepsilon_0}\hat{z} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\boxed{\vec{E}_{top}(\vec{r}) - \vec{E}_{bottom}(\vec{r}) = \frac{\sigma(\vec{r})}{\varepsilon_0}\hat{n}}$$

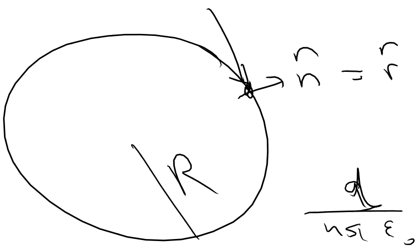


<u>Ex</u> $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\,\Theta(r-R) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} & \text{if } r>R \\ 0 & \text{if } r<R \end{cases}$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \text{if} \quad r \neq R$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right)$$

$$\boxed{\vec{E}_{top}(\vec{r}) - \vec{E}_{bottom}(\vec{r}) = \frac{\sigma(\vec{r})}{\varepsilon_0}\hat{n}}$$



$$\vec{E}_{top} = \frac{q}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r}$$

$$\vec{E}_{bottom} = 0$$

$$\frac{q}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r} = \frac{\sigma(\vec{r})}{\varepsilon_0}\hat{r}$$

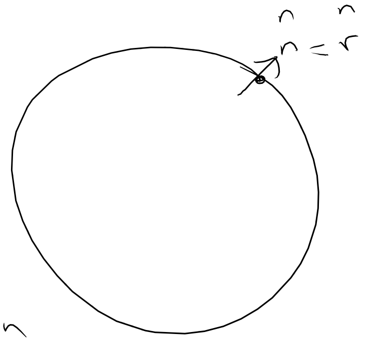
$$\boxed{\sigma(\vec{r}) = \frac{q}{4\pi R^2}}$$

Ex $$\vec{E}(\vec{r}) = \begin{cases} \dfrac{q/2}{4\pi\varepsilon_0}\dfrac{r}{R^3}\hat{r} & r<R \\[2ex] \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{r^2}\hat{r} & r>R \end{cases}$$

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad r>R$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = 3\,\frac{q/2}{4\pi\varepsilon_0 R^3} \equiv \frac{\rho}{\varepsilon_0}$$

$$\boxed{\rho(\vec{r}) = \frac{q/2}{\frac{4\pi}{3}R^3}} \quad \text{if} \quad r<R$$



$$\vec{E}_{top}(\vec{r}) - \vec{E}_{bottom}(\vec{r}) = \frac{\sigma(\vec{r})}{\varepsilon_0}\hat{n}$$

$$\vec{E}_{top} = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\hat{r}$$

$$\vec{E}_{bottom} = \frac{q/2}{4\pi\varepsilon_0}\frac{R}{R^3}\hat{r} = \frac{q/2}{4\pi\varepsilon_0}\frac{1}{R^2}\hat{r}$$

$$\vec{E}_{top} - \vec{E}_{bottom} = \frac{q/2}{4\pi\varepsilon_0}\frac{1}{R^2}\hat{r} \equiv \frac{\sigma(\vec{r})}{\varepsilon_0}\hat{r}$$

$$\boxed{\sigma(\vec{r}) = \frac{(q/2)}{4\pi R^2}}$$

$$\rho(\vec{r}) = \frac{q/2}{\frac{4\pi}{3}R^3}\Theta(R-r) + \frac{q/2}{4\pi R^2}\delta(R-r)$$

Work and Energy

$$W = \int_A^B \vec{F} \cdot d\vec{l}$$

if $\vec{F}$ is constant $\quad W = \vec{F} \cdot (\vec{r}_B - \vec{r}_A)$

if $\oint \vec{F} \cdot d\vec{l} = 0 \Rightarrow \vec{F}$ is a conservative force.

$$\vec{F} = q\vec{E}$$

$$\oint \vec{F} \cdot d\vec{l} = q \oint \vec{E} \cdot d\vec{l} = q \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = 0$$

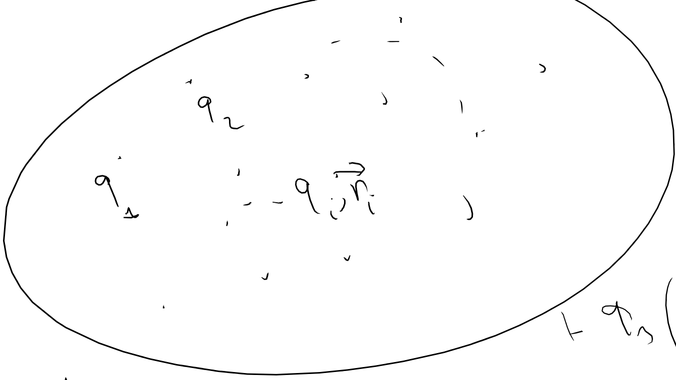
$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{l} = -\int_{O}^{\vec{r}} q\vec{E} \cdot d\vec{l}$$

$$= q \underbrace{\left(-\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l} \right)}_{V(\vec{r})}$$

$$\boxed{U(\vec{r}) = qV(\vec{r})}$$

→ potential ~~energy~~

→ potential ENERGY



$U = ?$

$$U = q_1 \, 0 \, \frac{1}{4\pi\varepsilon_0} + \frac{q_2 q_1}{|\vec{r}_1 - \vec{r}_2|} \frac{1}{4\pi\varepsilon_0} + q_3 \left(\frac{q_1}{|\vec{r}_3 - \vec{r}_1|} \frac{1}{4\pi\varepsilon_0} + \frac{q_2}{|\vec{r}_2 - \vec{r}_1|} \frac{1}{4\pi\varepsilon_0} \right) - - - -$$

$$q_i \sum_{j=1}^{i-1} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \frac{1}{4\pi\varepsilon_0} + \ldots$$

$$U = \sum_{i=1}^{N} q_i \underline{V_i(\vec{r}_i)}$$

V_i: potential created by the charges $q_1, q_2, \dots, q_{i-1}$

$$\boxed{U = \sum_{i=1}^{N} q_i \sum_{j=1}^{i-1} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \frac{1}{4\pi\varepsilon_0}}$$

$$q_1 \cdot 0 + q_2 \left(\frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_1 - \vec{r}_2|} \right) + q_3 \left(\frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_1 - \vec{r}_3|} + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{|\vec{r}_2 - \vec{r}_3|} \right)$$

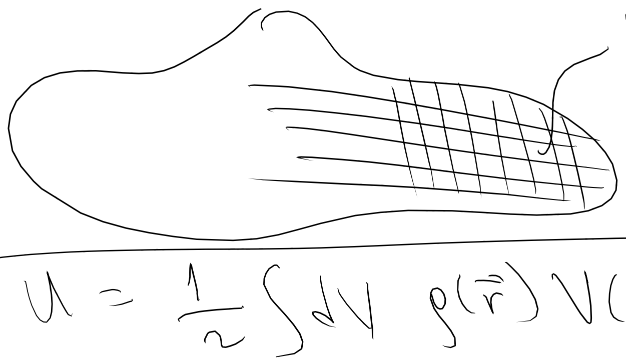
$$= \frac{1}{2} q_1 \left(\frac{1}{4\pi\varepsilon_0} \frac{q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{1}{4\pi\varepsilon_0} \frac{q_3}{|\vec{r}_1 - \vec{r}_3|} \right)$$

$$+ \frac{1}{2} q_2 \left(\frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_2 - \vec{r}_1|} + \frac{1}{4\pi\varepsilon_0} \frac{q_3}{|\vec{r}_2 - \vec{r}_3|} \right)$$

$$+ \frac{1}{2} q_3 \left(\frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{1}{4\pi\varepsilon} \frac{q_2}{|\vec{r}_3 - \vec{r}_2|} \right)$$

$$= \sum_i \frac{1}{2} q_i V_i'(\vec{r}_i)$$

$V_i'(\vec{r}_i)$ is the potential created by all other charges in the system.



$dq = \rho(\vec{r})\, dV$

$$\boxed{U = \frac{1}{2} \int_V dV\, \rho(\vec{r})\, V(\vec{r})}$$

$$V(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\rho = -\varepsilon_0 (\nabla^2 V) = \varepsilon_0 (\vec{\nabla} \cdot \vec{E})$$

$$U = \frac{1}{2} \int d^3 r \; \varepsilon_0 (\vec{\nabla} \cdot \vec{E}) V$$

$$(\vec{\nabla} \cdot \vec{E}) V = \vec{\nabla} \cdot (V \vec{E}) - \vec{E} \cdot \vec{\nabla} V$$

$$\vec{\nabla} \cdot (V \vec{E}) = (\vec{\nabla} V) \cdot \vec{E} + V (\vec{\nabla} \cdot \vec{E})$$

$$U = \frac{1}{2} \int_V d^3 r \, \varepsilon_0 \vec{\nabla} \cdot (V \vec{E}) + \frac{1}{2} \int_V d^3 r \, \varepsilon_0 \vec{E}^2$$

$$\boxed{U = \frac{1}{2} \varepsilon_0 \int_{\partial V} d\vec{S} \cdot V \vec{E} + \int_V d^3 r \left(\frac{1}{2} \varepsilon_0 \vec{E}^2 \right)}$$

if $V \to \infty$

$$U = \int d^3 r \; u_E(\vec{r})$$

$$\boxed{u_E(\vec{r}) = \frac{1}{2} \varepsilon_0 \vec{E}(\vec{r})^2}$$

$$= \frac{1}{2} \int d^3 r \, \rho(\vec{r}) V(\vec{r})$$

Example

$$U = \frac{q_1 q_2}{4\pi \varepsilon_0} \frac{1}{|\vec{r}_1 - \vec{r}_2|}$$

$q_1, \vec{r}_1$ $\quad$ $q_2, \vec{r}_2$

$$\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$$

$$\vec{E}_i = \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i)$$

$$\vec{E}^2 = \vec{E}_1^2(\vec{r}) + \vec{E}_2^2(\vec{r}) + 2 \vec{E}_1 \cdot \vec{E}_2 \qquad i = 1, 2$$

$$U = \frac{1}{2} \varepsilon_0 \int d^3 r \left(\vec{E}_1^2 + \vec{E}_2^2 + 2 \vec{E}_1 \cdot \vec{E}_2 \right)$$

$$= U_1 + U_2 + U_{12}$$

$$U_i = \frac{1}{2}\varepsilon_0 \int d^3r \left(\frac{1}{4\pi\varepsilon_0}\frac{q_i}{|\vec{r}-\vec{r}_i|^2}\right)^2$$

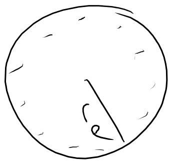
$$U_i = q_i^2 \left(\frac{1}{2}\varepsilon_0\right)\int d^3r \left(\frac{1}{4\pi\varepsilon_0 r^2}\right)^2$$

$$\int d^3r \, \frac{1}{r^4} = 4\pi\int_0^\infty dr\, r^2 \frac{1}{r^4} = 4\pi \left. -\frac{1}{r}\right|_{r=0}^{\infty} = \frac{4\pi}{0}!$$

$$U_{12} = \int d^3r \, \frac{1}{2}\varepsilon_0 \vec{E}_1(\vec{r})\cdot\vec{E}_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_1-\vec{r}_2|}$$

$$U_i = \int d^3r \left(\frac{1}{2}\varepsilon_0 \vec{E}_i^2\right)$$: self energy!

<u>Example</u> Classical Electron Radius.



$$E_{self} = m_e c^2 \Rightarrow r_e = ?$$

$$\vec{E} = \begin{cases} 0 & \text{if } r < r_e \\ \frac{q_e}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} & r \geq r_e \end{cases}$$

$$E_{self} = \frac{1}{2}\varepsilon_0 \int d^3r \, \vec{E}^2 = \frac{1}{2}\varepsilon_0 \int_{r_e}^{\infty} dr\, r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \left(\frac{q_e}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4}$$

$$= \frac{1}{3\cancel{2\pi^2}\varepsilon_0} q_e^2 \cancel{4\pi} \int_{r_e}^{\infty} \frac{dr}{r^2}$$

$$= \frac{2}{3}\frac{q_e^2}{\pi\varepsilon_0} \left. -\frac{1}{r}\right|_{r=r_e}^{\infty} = \frac{2q_e^2}{3\pi\varepsilon_0 r_e} = m_e c^2$$

$$\boxed{r_e = \frac{2 q_e^2}{3\pi \varepsilon_0 m_e c^2}}$$