$\nabla^2 \Phi = 0$ $\qquad$ $\Phi = 0$ on the ∂V

$\Rightarrow \Phi = 0$ everywhere in V

$\nabla^2 \Phi = -\frac{1}{\varepsilon_0}\rho$ with $\Phi = \Phi_{B.C}$ on the boundary

if Φ_1 & Φ_2 are solns, then

$\Phi_3 = \Phi_1 - \Phi_2$

$\nabla^2 \Phi_3 = 0$ $\qquad$ on the boundary $\Phi_3 = 0$

$\Rightarrow \Phi_3 = 0$ everywhere $\Rightarrow \Phi_1 = \Phi_2$ everywhere

everywhere : inside the boundary

Second uniqueness thm

Let Φ_1 & Φ_2 be two solns.

$\Phi_3 = \Phi_1 - \Phi_2$

$\nabla^2 \Phi_3 = 0$

$$\Rightarrow \vec{\nabla}\cdot(\Phi_3 \vec{E}_3) = (\vec{\nabla}\Phi_3)\cdot\vec{E}_3 + \Phi_3(\vec{\nabla}\cdot\vec{E}_3) \qquad \vec{E}_3 = -\vec{\nabla}\Phi_3$$

$\vec{\nabla}\cdot\vec{E}_3 = 0$ in the region between conductors

$$\vec{\nabla}\Phi_3 = -\vec{E}_3$$

$$\vec{\nabla}\cdot(\Phi_3\vec{E}_3) = -\vec{E}_3^{\,2}$$

integrate over the volume, V, between the conductors

$$\int_V \vec{\nabla}\cdot(\Phi_3\vec{E}_3)\,dV = \int_{\partial V} \Phi_3\vec{E}_3\cdot d\vec{S}$$

$$= \sum_{\text{surfaces } S_i} \oint \Phi_3 \vec{E}_3\cdot d\vec{S}$$

$$\Phi_3 = \Phi_1 - \Phi_2 \qquad = \sum \Phi_{3i} \oint_{S_i} \vec{E}_3\cdot d\vec{S}$$

$$\oint_{S_i} \vec{E}_3\cdot d\vec{S} = \oint_{S_i} \vec{E}_1\cdot d\vec{S} - \oint_{S_i} \vec{E}_2\cdot d\vec{S}$$

$$= \left(\frac{Q_i}{\varepsilon_0}\right) - \left(\frac{Q_i}{\varepsilon_0}\right) = 0$$

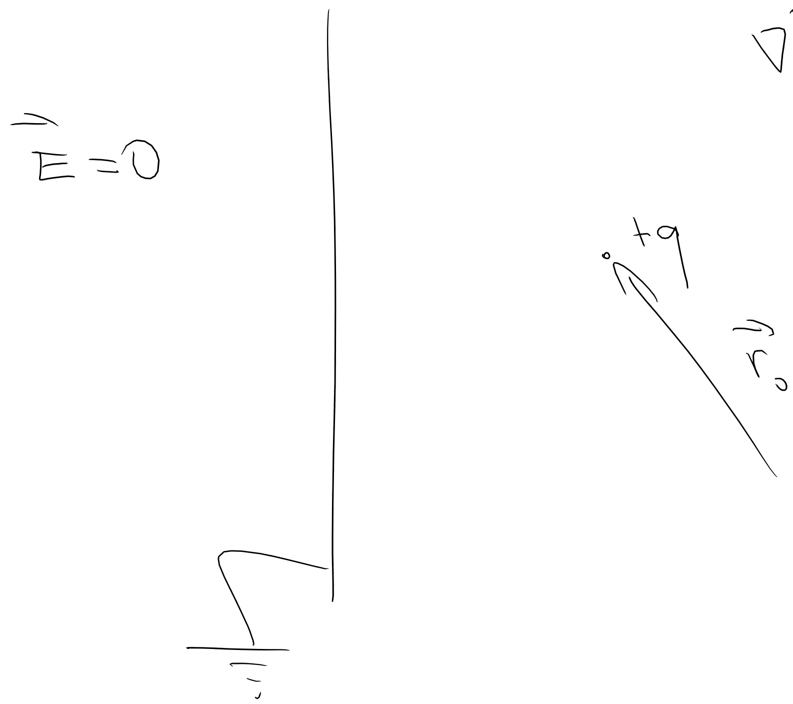
$$\int_V \underbrace{\vec{\nabla}\cdot(\Phi_3\vec{E}_3)}_{=} \,dV = 0$$

$$\int_V (-\vec{E}_3^{\,2})\,dV = 0 \Rightarrow \vec{E}_3^{\,2} = 0 \text{ everywhere}$$

$$\Rightarrow \vec{E}_1 = \vec{E}_2$$

Method of Images

$$\nabla^2 \Phi = -4\pi\rho$$



Pot. on the LHS of the plate:

$$\nabla^2 \Phi = 0$$

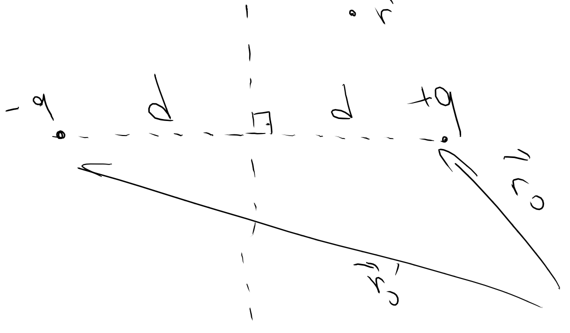
$\Phi = 0$ on the boundary

$$\boxed{\Phi = 0}$$

Pot. on the RHS of the plate

$$\nabla^2 \Phi = -\frac{q}{\varepsilon_0}\delta^{(3)}(\vec{r}-\vec{r}_0)$$

$\Phi = 0$ on the boundary



if $\vec{r}$ is on the middle plane $|\vec{r}-\vec{r}_0| = |\vec{r}-\vec{r}_0'|$

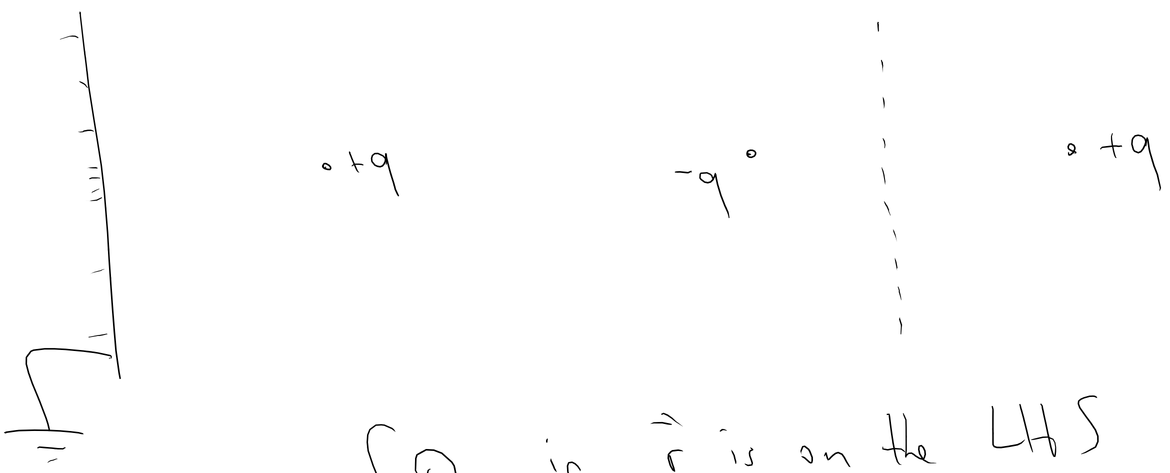
$$\Phi(\vec{r}) = \frac{+q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|} + \frac{(-q)}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_0'|}$$

$\Phi(\vec{r}) = 0$ on the middle plane.

$$\nabla^2\Phi(\vec{r}) = -\frac{q}{\varepsilon_0}\delta^{(3)}(\vec{r}-\vec{r}_0) - \frac{(-q)}{\varepsilon_0}\delta^{(3)}(\vec{r}-\vec{r}_0')$$

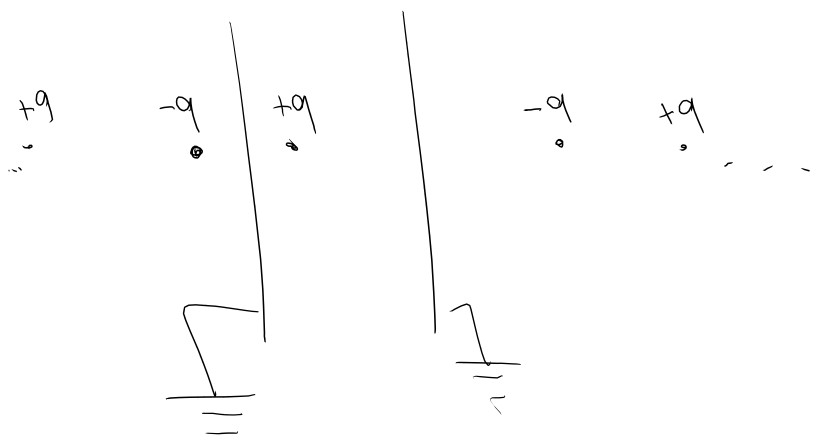
on the RHS $\vec{r} \neq \vec{r}_0'$

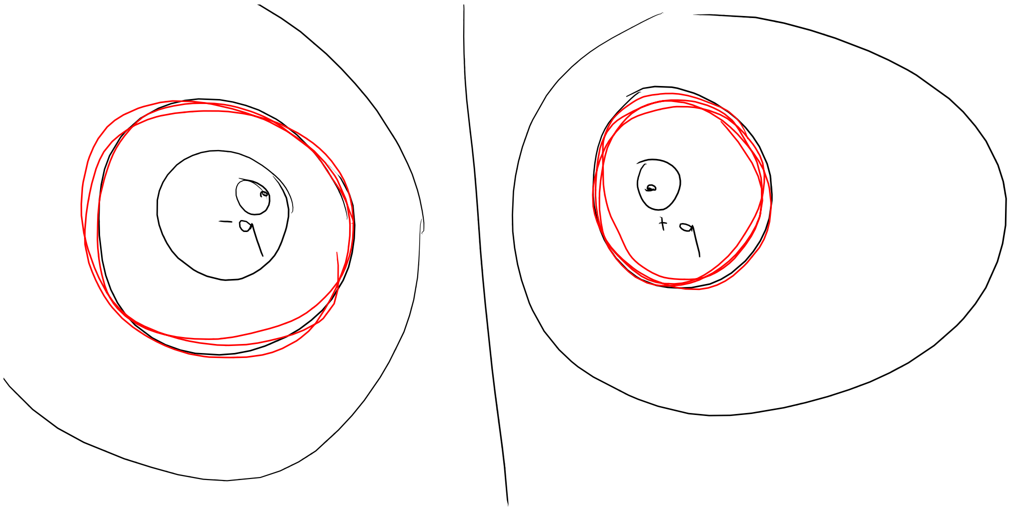
on the RHS, $\nabla^2\Phi(\vec{r}) = -\frac{q}{\varepsilon_0}\delta^{(3)}(\vec{r}-\vec{r}_0)$



$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\begin{cases} 0 & \text{if } \vec{r} \text{ is on the LHS} \\ \frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{|\vec{r}-\vec{r}_0'|} & \text{if } \vec{r} \text{ is on the RHS} \end{cases}$$

Ex

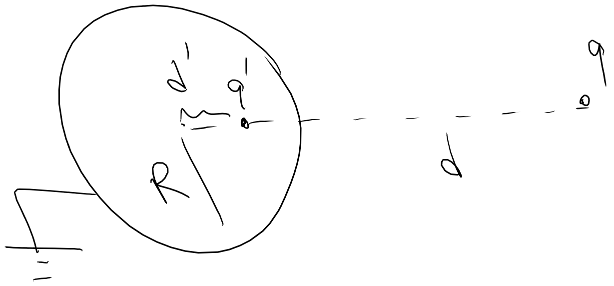


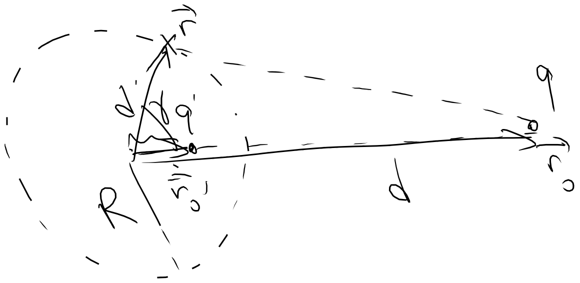


$\phi(\vec{r}) = ?$

$$\phi(\vec{r}) = \frac{-q}{4\pi\varepsilon_0 r_-} + \frac{q}{4\pi\varepsilon_0}\frac{1}{r_+}$$

Example





$\vec{r}$ is on the sphere

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|} + \frac{q'}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_0'|}$$

$$|\vec{r}-\vec{r}_0| = \sqrt{(\vec{r}-\vec{r}_0)^2} = \sqrt{r^2+r_0^2-2\vec{r}\cdot\vec{r}_0}$$

$$|\vec{r}-\vec{r}_0| = \sqrt{R^2+r_0^2-2Rr_0\cos\gamma}$$

$$|\vec{r}-\vec{r}_0'| = \sqrt{R^2+r_0'^2-2Rr_0'\cos\gamma_0}$$

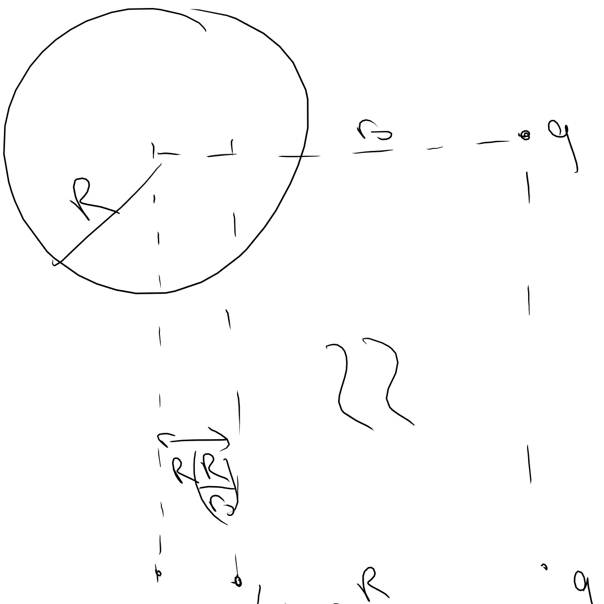
$$0 = \Phi(\vec{r}\in \text{surpace}) = \frac{q}{4\pi\varepsilon_0\sqrt{R^2+r_0^2-2Rr_0\cos\gamma}} + \frac{q'}{4\pi\varepsilon_0\sqrt{R^2+r_0'^2-2Rr_0'\cos\gamma}}$$

$$= \frac{q}{4\pi\varepsilon_0 r_0\sqrt{1+\left(\frac{R}{r_0}\right)^2-2\left(\frac{R}{r_0}\right)\cos\gamma}} + \frac{q'}{4\pi\varepsilon_0 R\sqrt{1+\left(\frac{r_0'}{R}\right)^2-2\left(\frac{r_0'}{R}\right)\cos\gamma}}$$

$$\boxed{\frac{R}{r_0} = \frac{r_0'}{R}} \qquad \boxed{\frac{q}{r_0}+\frac{q'}{R} = 0}$$

$$\boxed{r_0' = \frac{R^2}{r_0}} \qquad \boxed{q' = -q\left(\frac{R}{r_0}\right)}$$

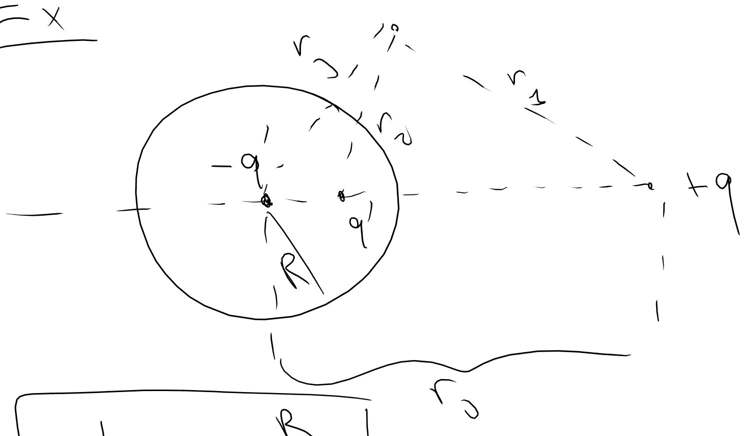
$$\boxed{r_0' = R\left(\frac{R}{r_0}\right)}$$



$$q' = -q\frac{R}{r_0}$$

$$\boxed{\nabla^2 \Phi = -\frac{1}{\varepsilon_0}\rho}$$

Ex

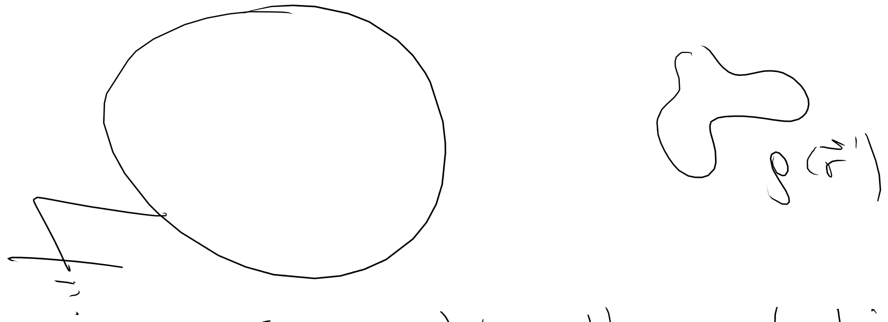


$$q' = -q\frac{R}{r_0}$$

$$\Phi = \frac{q}{4\pi\varepsilon_1 r_1} + \frac{q'}{4\pi\varepsilon_0 r_2} - \frac{q'}{4\pi\varepsilon_0 r_3}$$

$$\boxed{q' = -q\frac{R}{r_0}}$$

Ex



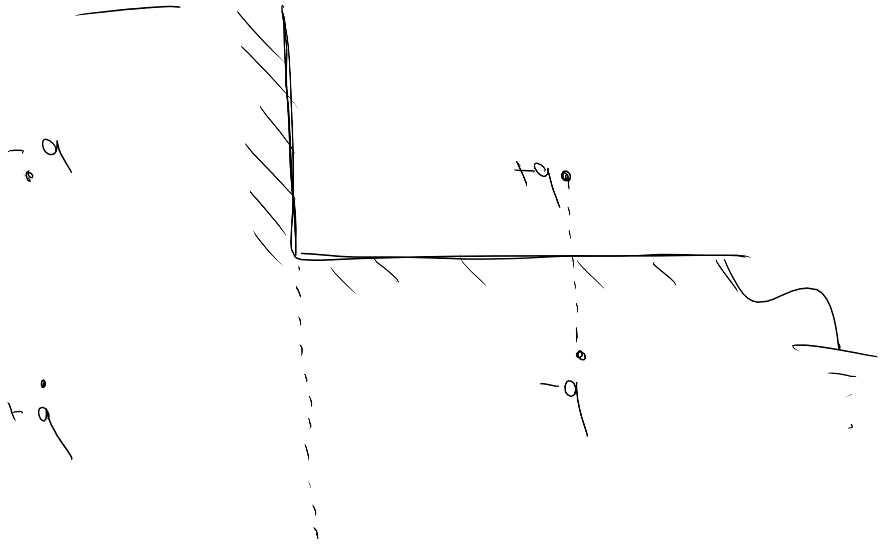
Let $\Phi_1(\vec{r};\vec{r}_0)$ be the potential created by a unit point charge at $\vec{r}_0$

$$\Phi(\vec{r}) = \int d^3r' \, \rho(\vec{r}') \, \Phi_1(\vec{r};\vec{r}')$$

$$\nabla^2 \Phi(\vec{r}) = \int d^3r' \, \rho(\vec{r}') \underbrace{\nabla_r^2 \Phi_1(\vec{r};\vec{r}')}_{-\frac{1}{\varepsilon_0}\delta^{(3)}(\vec{r}-\vec{r}')}$$

$$\boxed{\nabla^2 \Phi(\vec{r}) = -\frac{1}{\varepsilon_0}\rho(\vec{r})}$$

Ex



Ex