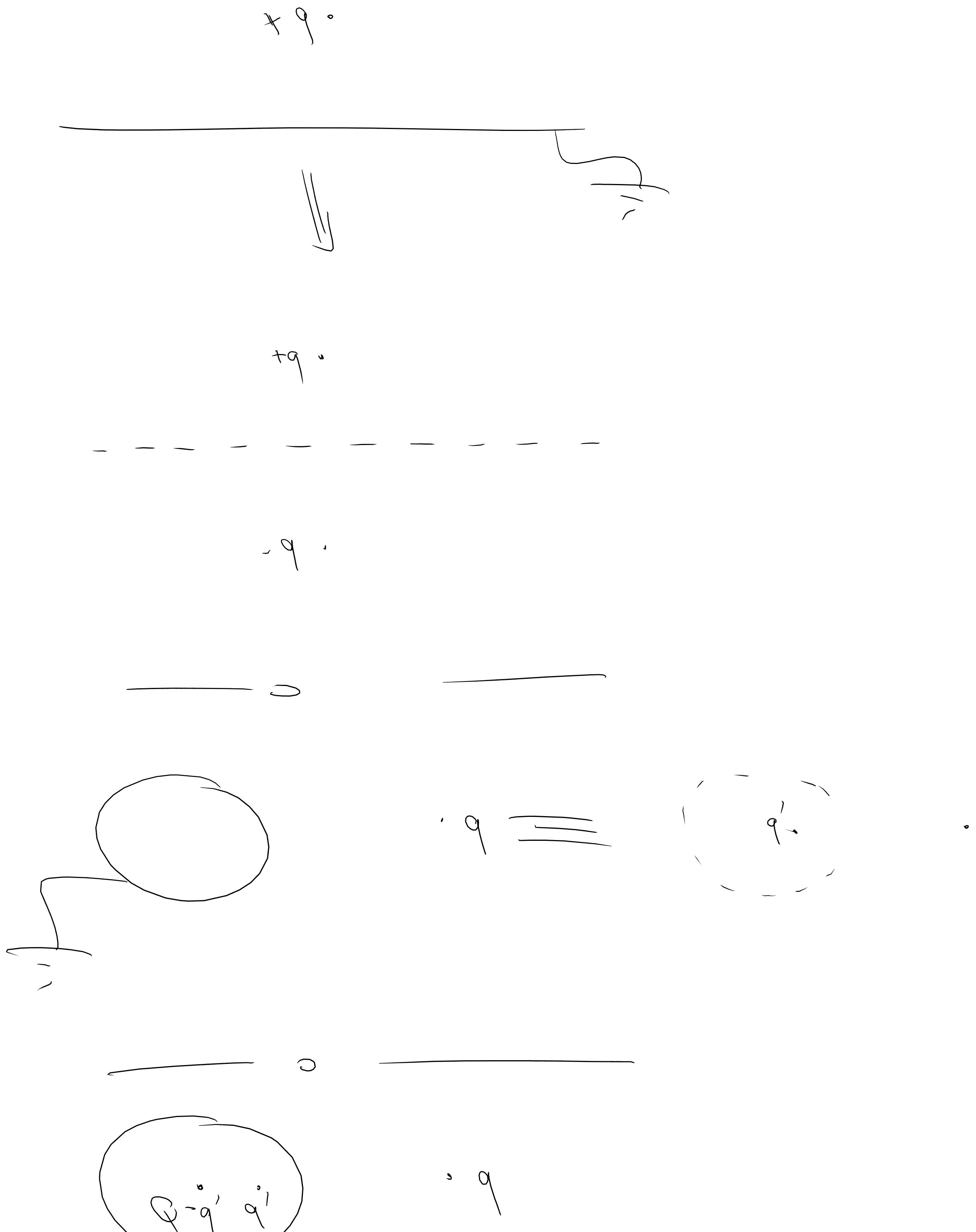
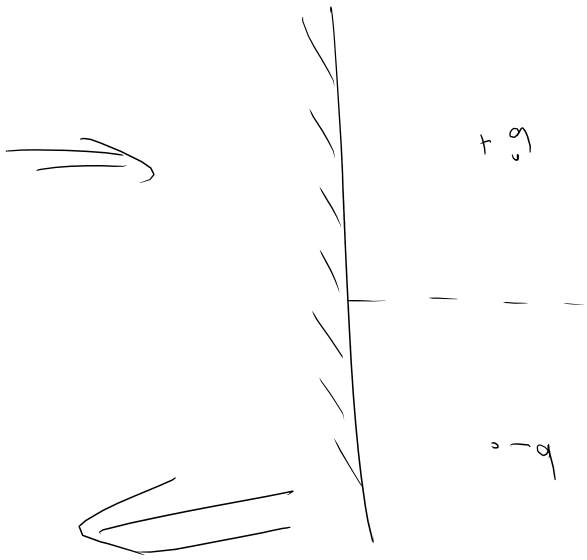
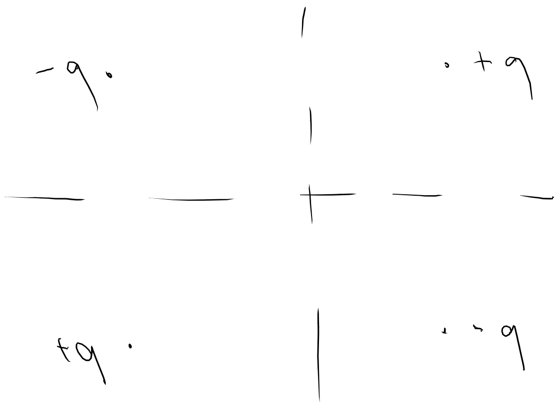
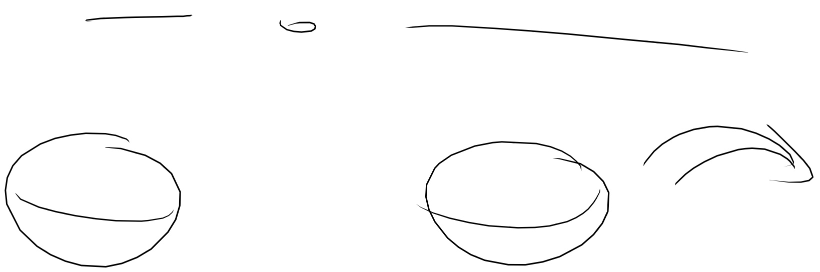


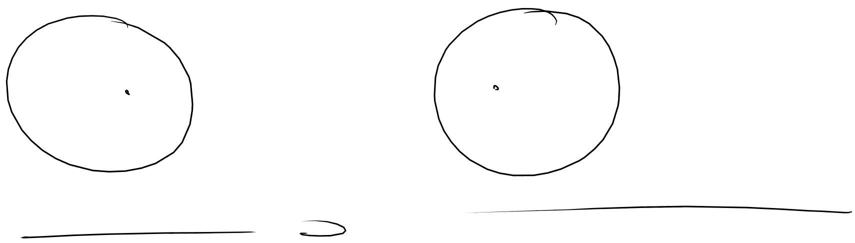
+q
+q
-q
q
q'
q - q'
q'
q

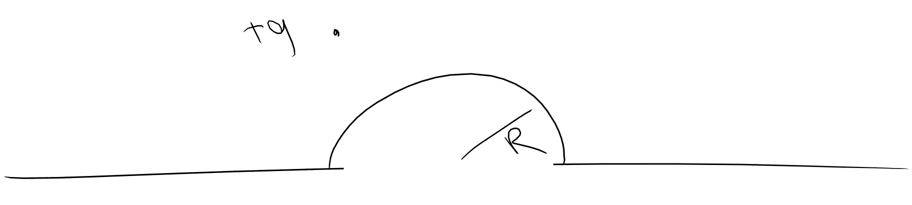

+q
θ = π/2

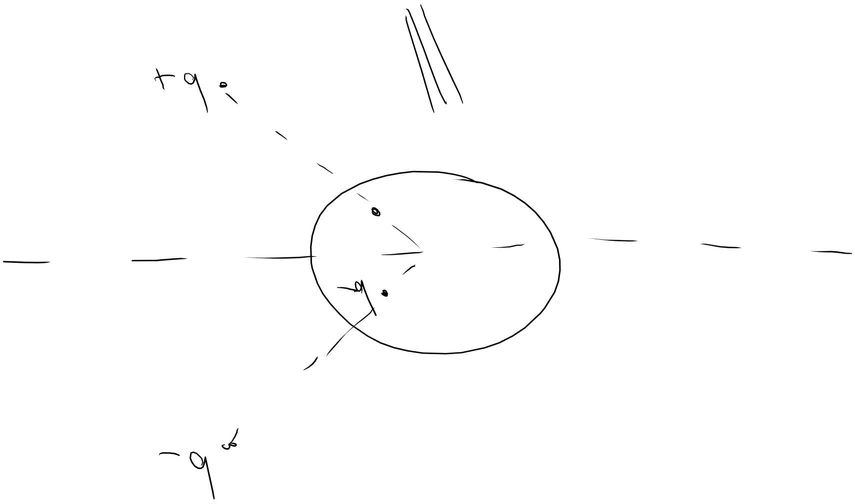

+q
-q


-q
+q
+q
-q

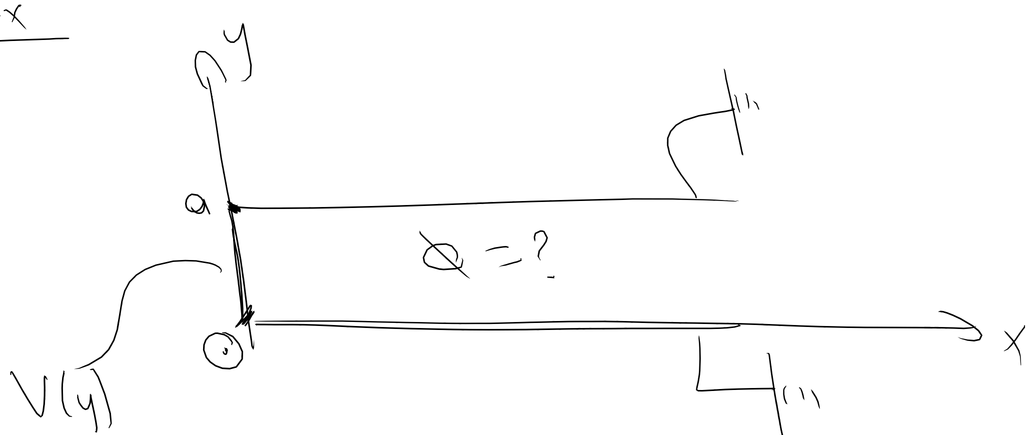





+q
R



Ex



$$\phi(x,y,z) = \phi(x,y) = \underline{X(x)Y(y)}$$

$Y(0) = 0$
$Y(a) = 0$
$X(x \to \infty) = 0$

$$0 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\frac{0}{XY} = \left(Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2}\right)\frac{1}{XY}$$

$$0 = \underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{x} + \underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{y}$$

$$\boxed{\frac{1}{Y}\frac{d^2 Y}{dy^2} = C_1 \qquad \frac{1}{X}\frac{d^2 X}{dx^2} = C_2}$$

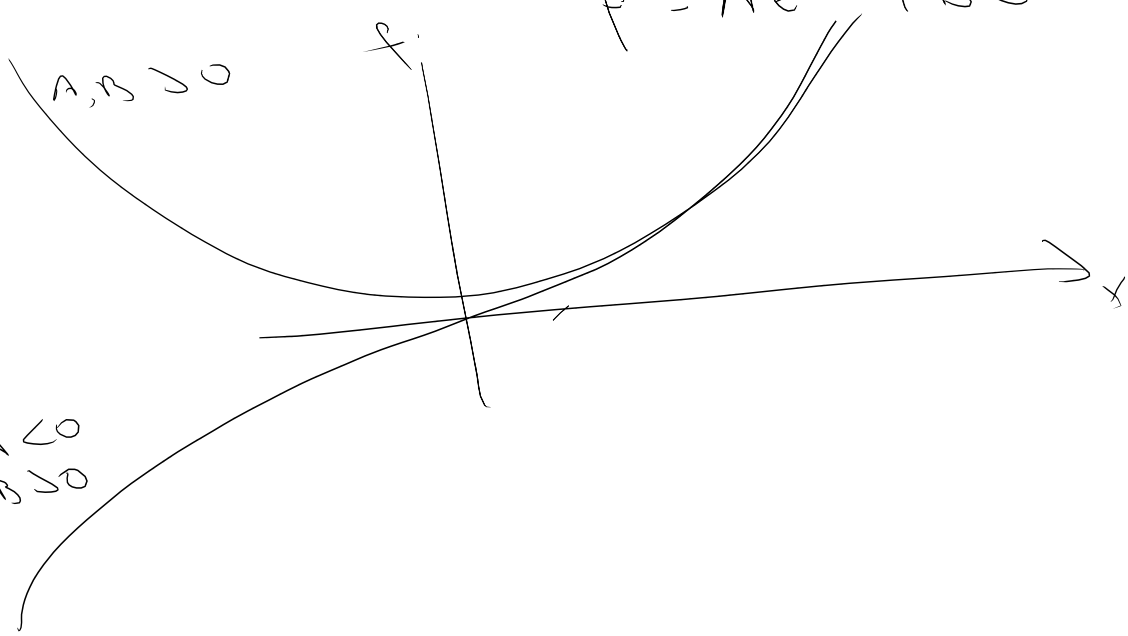
$$C_1 + C_2 = 0$$

$$\frac{d^2 f}{dx^2}\frac{1}{f} = \text{const}$$

case i const > 0 $k > 0$

$$\frac{d^2 f}{dx^2}\frac{1}{f} = k^2 \Rightarrow \frac{d^2 f}{dx^2} = k^2 f$$

$$f = Ae^{kx} + Be^{-kx}$$



case ii   const < 0

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -k^2 \Longrightarrow f = A\cos(kx + \delta)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = C_1 = -k^2 \qquad \frac{1}{X}\frac{d^2X}{dx^2} = C_2 = k^2$$

$$Y = A_k \sin(ky + \delta_k) \qquad k > 0$$

$$X = B_k e^{kx} + C_k e^{-kx}$$

$$\Downarrow$$

$$\Phi(x,y,z) = \Phi(x,y) = X(x)Y(y) \qquad Y(0) = 0$$

$$Y(a) = 0$$

$$\Phi(x,y) = A_k \sin(ky + \delta_k)(B_k e^{kx} + C_k e^{-kx})$$

$$\Phi(x, y=0) = A_k \sin(\delta_k)(B_k e^{kx} + C_k e^{-kx}) = 0$$

$$\Rightarrow \delta_k = 0 \quad \forall k$$

$$\Phi(x,y) = A_k \sin(ky)(B_k e^{kx} + C_k e^{-kx}) = 0$$

$$\Phi(x, y=a) = A_k \sin(ka)(B_k e^{kx} + C_k e^{-kx}) = 0$$

$$k = n\frac{\pi}{a} \quad ; \; n = 1, 2, \dots$$

$$\Phi(x \to \infty, y) = 0 \Rightarrow B_k = 0 \quad \forall k$$

$$\boxed{\Phi_n(x,y) = D_n \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a} x}} \qquad n = 1, 2, 3, \dots$$

$$\Phi(x,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a}y\right) e^{-\frac{n\pi}{a}x}$$

If it possiple to choose D_n s.t.

$$V(y) = \Phi(x=0,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a}y\right)$$

$$\int_0^a dy \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right) = \int_0^a dy \frac{1}{2}\left[\cos\left(\frac{n-n'}{a}y\right) - \cos\left(\frac{n+n'}{a}y\right)\right]$$

if $n' \neq n$

$$= \frac{1}{2}\left[\left.\frac{+\sin\left(\frac{(n-n')}{a}\pi y\right)}{\frac{n-n'}{a}\pi}\right|_{y=0}^{a} - \left.\frac{+\sin\left(\frac{n+n'}{a}\pi y\right)}{\frac{n+n'}{a}\pi}\right|_{y=0}^{a}\right]$$

$$= 0$$

if $n = n'$ $\quad = \frac{a}{2}$

$$\int_0^a dy \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right) = \frac{a}{2}\delta_{n,n'} \Leftarrow$$

$$\Downarrow$$

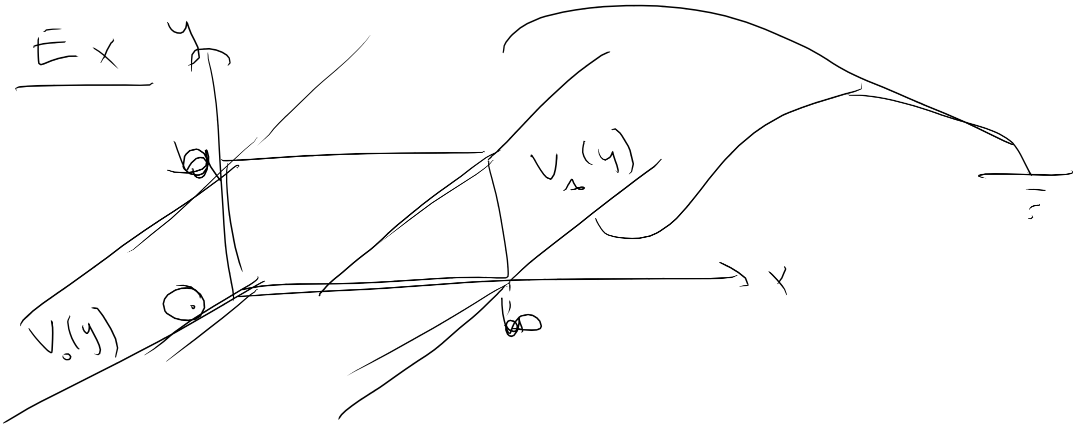
$$\int_0^a \left(V(y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a}y\right)\right] \sin\left(\frac{n'\pi}{a}y\right) dy$$

$$\int_0^a V(y)\sin\left(\frac{n'\pi}{a}y\right)dy = \sum_{n=1}^{\infty} D_n \underbrace{\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy}_{\frac{a}{2}\delta_{nn'}}$$

$$\boxed{\frac{a}{2}D_{n'} = \int_0^a V(y)\sin\left(\frac{n'\pi}{a}y\right)dy}$$

$$\Phi(x,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a}y\right) e^{-\frac{n\pi}{a}x} \Leftarrow$$

Ex



$$\phi(x,y) = A_k \sin(ky + \delta_k)\left(B_k e^{kx} + C_k e^{-kx}\right)$$

$$\phi(x, y=0) = \phi(x, y=a) = 0$$

$$\Rightarrow \delta_k = 0\,,\quad k = \frac{n\pi}{a} \qquad n = 1, 2, \dots$$

$$\phi_n(x,y) = A_n \sin\left(\frac{n\pi}{a} y\right)\left(B_n e^{\frac{n\pi}{a}x} + C_n e^{-\frac{n\pi}{a}x}\right)$$

$$= \sin\left(\frac{n\pi}{a} y\right)\left(\underbrace{(B_n A_n)}_{D_n} e^{\frac{n\pi}{a}x} + \underbrace{(C_n A_n)}_{E_n} e^{-\frac{n\pi}{a}x}\right)$$

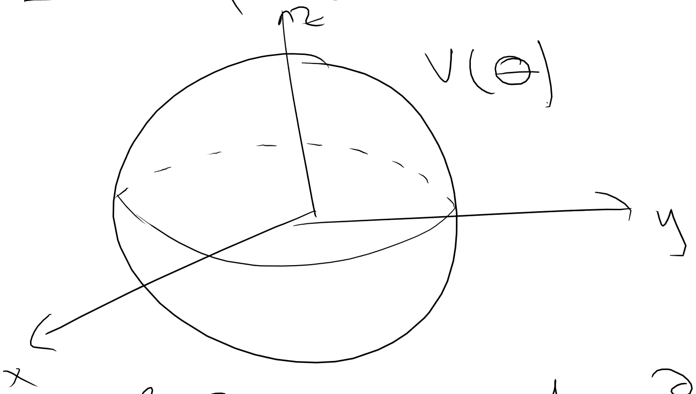
$$\phi_n(x,y) = \sin\left(\frac{n\pi}{a} y\right)\left(D_n e^{\frac{n\pi}{a}x} + E_n e^{-\frac{n\pi}{a}x}\right)$$

$$\phi(x,y) = \sum_n \sin\left(\frac{n\pi}{a} y\right)\left(D_n e^{\frac{n\pi}{a}x} + E_n e^{-\frac{n\pi}{a}x}\right)$$

$$\phi(x=0, y) \equiv V_0(y) = \sum_n \sin\left(\frac{n\pi}{a} y\right)\left(D_n + E_n\right)$$

$$\phi(x=b, y) \equiv V_1(y) = \sum_n \sin\left(\frac{n\pi}{a} y\right)\left(D_n e^{\frac{n\pi}{a}b} + E_n e^{-\frac{n\pi}{a}b}\right)$$

Example



(r, θ, ϕ)

$\Phi(r, \theta)$

$$\nabla^2 \Phi(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right)$$

$$\Phi(r,\theta) = R(r)\, Z_\ell(\theta)$$

$$0 = \nabla^2 \Phi = Z_\ell(\theta)\, \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + R\, \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dZ_\ell}{d\theta}\right)$$

$$0 = \frac{\nabla^2 \Phi}{\Phi} = \frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{r^2}\left[\frac{1}{Z_\ell}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dZ_\ell}{d\theta}\right)\right]$$

$$\frac{1}{Z_\ell}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dZ_\ell}{d\theta}\right) = -\ell(\ell+1)$$

$$\left[\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2} = 0\right] r^2 R$$

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \ell(\ell+1) R = 0$$

$$R = r^n$$

$$n(n-1)\cancel{r^n} + 2n\cancel{r^n} - \ell(\ell+1)\cancel{r^n} = 0$$

$$n(n+1) - \ell(\ell+1) = 0$$

$$n = \ell \quad \text{or} \quad n = -(\ell+1)$$

$$\boxed{R_\ell(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}}$$

$$\frac{1}{Z_\ell}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dZ_\ell}{d\theta}\right) = -\ell(\ell+1)$$

$$\left.\begin{array}{l} Z_\ell(\theta=0) \\ Z_\ell(\theta=\pi) \end{array}\right\} \text{ should be finite.}$$

$\Downarrow$

ℓ is integer

$$Z_\ell(\theta) = P_\ell(\cos\theta)$$

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = \delta_{\ell\ell'}\frac{2}{2\ell+1}$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$\vdots$

$$\frac{2\delta_{\ell\ell'}}{2\ell+1} = \int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta$$

$$\boxed{\begin{array}{l} \text{if } \Phi = R(r)\Theta(\theta)\varphi(\phi) \\ \Theta(\theta) = P_\ell^m(\cos\theta) \end{array}}$$

$$\Phi_\ell = \left(A_\ell r^\ell + B_\ell\frac{1}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$\boxed{\Phi = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell\frac{1}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

Ex

$$\Phi = \sum_{l=0}^{\infty}\left(A_l r^l + B_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta)$$

at $r=0$, Φ should be finite $\Rightarrow B_l = 0$

$$\Phi = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi(r=R,\theta) \equiv V(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

multiply by $\sin\theta\, P_{l'}(\cos\theta)$, integrate over $0 \leq \theta \leq \pi$

$$\int_0^{\pi} V(\theta)\sin\theta\, P_{l'}(\cos\theta)\, d\theta$$

$$= \sum_{l=0}^{\infty} A_l R^l \underbrace{\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta}_{\frac{2}{2l+1}\delta_{ll'}}$$

$$= A_{l'} R^{l'} \frac{2}{2l'+1}$$

Ex $V(\theta) = V_0$

$$\int_0^{\infty} V(\theta)\sin\theta\, P_{l'}(\cos\theta)\, d\theta = V_0 \int_0^{\pi} \sin\theta\, P_{l'}(\cos\theta) P_0(\cos\theta)\, d\theta$$

$$= V_0 \frac{2}{2l'+1}\delta_{l',0} = 2V_0\,\delta_{l',0}$$