Hand in your HW

$\nabla^2 \Phi = 0$

$Y(y=0) = Y(y=a) = 0$

$X(x \to \infty) = 0$

$\Phi = X(x)\,Y(y)$

$Y = A_n \sin\left(\frac{n\pi}{a} y\right)$

$X(x) = B_n e^{-\frac{n\pi}{a}x} + C_n e^{\frac{n\pi}{a}x}$

$\Phi = \sum C_n \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a}y}$

In spherical Coordinates

$\Phi(r,\theta) = R(r)\,Z_\ell(\theta)$

$R(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}$

$Z_\ell(\theta) = P_\ell(\cos\theta)$

$P_0(x) = 1$

$P_1(x) = x$

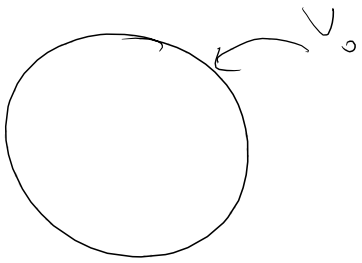
$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\vdots$

$\int_0^\pi d\theta\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \sin\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$

$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

Ex



$$\Phi_{in}(r,\theta) = \sum_{l=0}^{\infty} \left(A_l^{in} r^l + B_l^{in} \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\Phi_{in}(r=0) = \text{finite} \Rightarrow B_l^{in} = 0$$

$$\Phi_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l^{in} r^l P_l(\cos\theta)$$

$$\Phi_{out}(r,\theta) = \sum_{l=0}^{\infty} \left(A_l^{out} r^l + B_l^{out} \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\Phi_{out}(r\to\infty) = 0 \Rightarrow A_l^{out} = 0$$

$$V_0 = \sum_{l=0}^{\infty} A_l^{in} R^l P_l(\cos\theta)$$

$$V_0 = \sum_{l=0}^{\infty} B_l^{out} \frac{1}{R^{l+1}} P_l(\cos\theta)$$

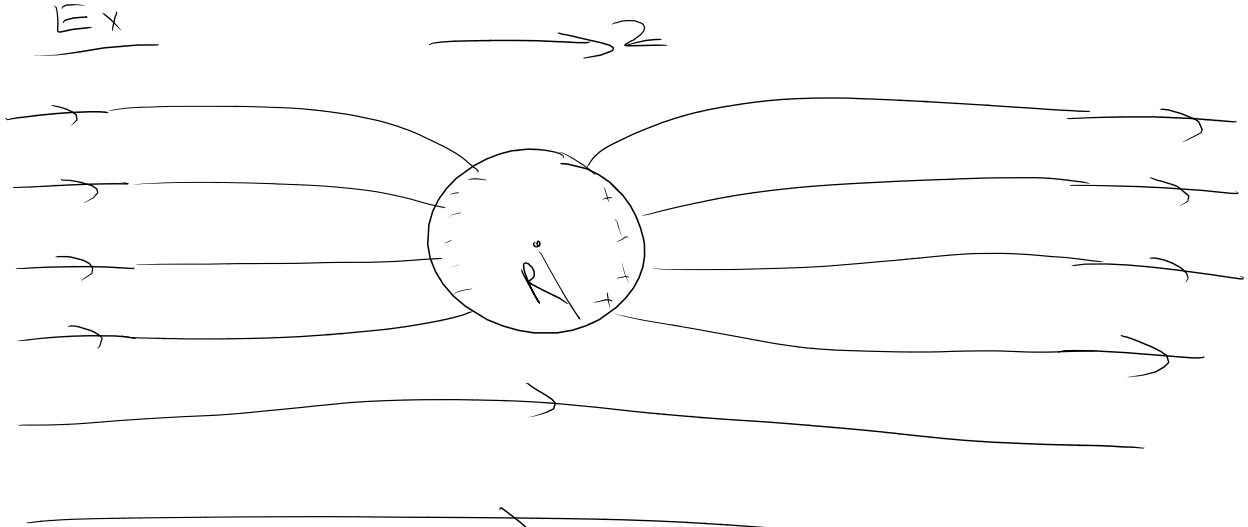
$$\left.\begin{array}{l} A_l^{in} = B_l^{out} = 0 \\ \text{if } l \neq 0 \\ P_0(\cos\theta) = 1 \end{array}\right\}$$

$$V_0 = A_0^{in}$$

$$V_0 = B_0^{out} \frac{1}{R} \Rightarrow B_0^{out} = R V_0$$

$$\Phi(r,\theta) = \begin{cases} V_0 & r < R \\ \dfrac{V_0 R}{r} & r > R \end{cases}$$

Ex



$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\Phi(r=R,\theta) = 0$$

$$\vec{E}_\infty = E_0 \hat{z} \Rightarrow \quad -E_0 z = \Phi^{\vec{E}_0}(\vec{r})$$

$$\Delta\Phi = -\int \vec{E} \cdot d\vec{l} = -\vec{E}_0 \cdot \Delta\vec{l}$$

$$\Phi(r\to\infty,\theta) = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

$$\lim_{r\to\infty} \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta) = -E_0 r P_1(\cos\theta)$$

$$A_l = 0 \quad \text{if} \quad l \geq 2$$

$$A_1 = -E_0$$

$$\Phi = A_0 - E_0 r P_1(\cos\theta) + \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi(r=R,\theta) = 0$$

$$A_0 - E_0 R P_1(\cos\theta) + \sum_{l=0}^{\infty} B_l \frac{1}{R^{l+1}} P_l(\cos\theta) = 0$$

$$A_0 + \frac{B_0}{R^1} = 0 \qquad (l=0)$$

$$-E_0 R + B_1 \frac{1}{R^2} = 0 \qquad (l=1)$$

$$\frac{B_l}{R^{l+1}} = 0 \qquad \text{for } l \geq 2$$

$$B_0 = -R A_0$$

$$B_1 = E_0 R^3$$

$$B_l = 0 \qquad \text{for } l > 1$$

$$\Phi = A_0 - E_0 r P_1(\cos\theta) + \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

$$= A_0 - E_0 r P_1(\cos\theta) - R A_0 \frac{1}{r} + E_0 R^3 \frac{1}{r^2} P_1(\cos\theta)$$

$$\boxed{\Phi = A_0\left(1 - \frac{R}{r}\right) - E_0 P_1(\cos\theta)\, r\left(1 - \frac{R^3}{r^3}\right)}$$

$$\Phi(r \to \infty) = A_0 - E_0 r P_1(\cos\theta) \simeq -E_0 r P_1(\cos\theta)$$

$$A_0 = 0$$

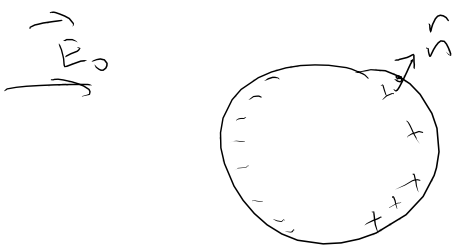
$$\boxed{\Phi(r,\theta) = -E_0 r \cos\theta \left(1 - \frac{R^3}{r^3}\right)}$$

$$\vec{E} = -\vec{\nabla}\Phi = -\frac{\partial \Phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\theta}$$

$$\vec{E} = +E_0 \cos\theta \left(1 + \frac{2R^3}{r^3}\right)\hat{r} - E_0 \sin\theta \left(1 - \frac{R^3}{r^3}\right)\hat{\theta}$$

$$\vec{E}(r \to \infty) = E_0 (\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}) = E_0 \hat{z}$$

$$\vec{E}^{\text{outside}} - \vec{E}^{\text{inside}} = \frac{\sigma}{\varepsilon_0} \hat{n}$$



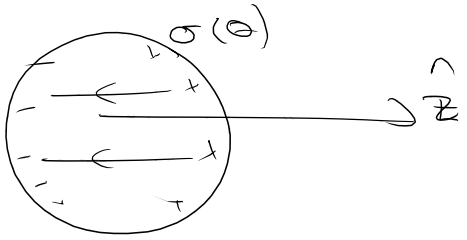
$$\vec{E}^{\text{inside}} = 0$$

$$\vec{E}^{\text{outside}} = 3E_0 \cos\theta\, \hat{r}$$

$$3E_0 \cos\theta = \frac{\sigma}{\varepsilon_0}$$

$$\boxed{\sigma = 3\varepsilon_0 E_0 \cos\theta}$$

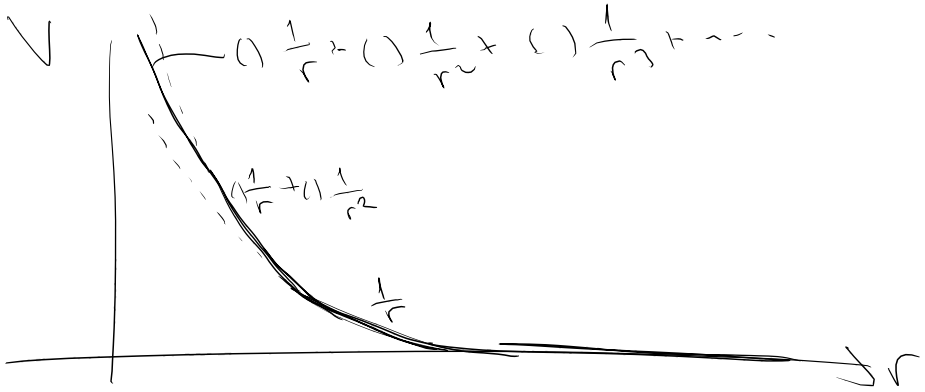
Ex



$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta$$

$$\vec{E}_{\text{inside}} = -E_0 \hat{z}$$

Multipole Expansion



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r'$$

$r' < d$ $\qquad \frac{r'}{r} << 1$

$r >> d$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$|\vec{r}-\vec{r}'|^{-1} = \left[(\vec{r}-\vec{r}')^2\right]^{-1/2}$$

$$= \left[r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'\right]^{-1/2}$$

$$= \frac{1}{r}\left[1 + x^2 - 2x\cos\theta\right]^{-1/2} \qquad x = \frac{r'}{r}$$

$$\hat{r}\cdot\hat{r}' = \cos\theta$$

$$= \frac{1}{r}\sum_l x^l P_l(\cos\theta)$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\sum_l \frac{r'^l}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_l \frac{1}{r^{l+1}}\int d^3r'\,\rho(\vec{r}')\, r'^l\, P_l(\cos\theta)$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\underbrace{\int d^3r'\,\rho(\vec{r}')}_{\text{monopole } (l=0)} + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\underbrace{\int d^3r'\,\rho(\vec{r}')\,r'\cos\theta}_{\text{dipole}} + \dots$$

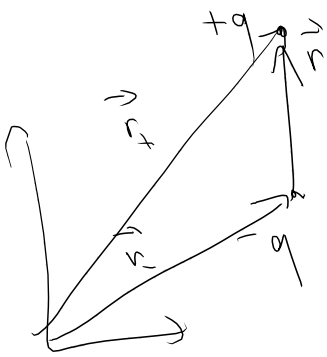
$$\vec{p} = q\,d\,\hat{r}$$

$$\int d^3r'\,\rho(\vec{r})\,r'\cos\theta = \int d^3r'\,\rho(\vec{r})\,\vec{r}'\cdot\hat{r}$$

$$\boxed{\vec{p} = \int d^3r'\,\rho(\vec{r}')\,\vec{r}'}$$

$$V(\vec{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \dots$$

Ex



$$\rho(\vec{r}) = (-q)\delta^{(3)}(\vec{r}-\vec{r}_-) + (+q)\delta^{(3)}(\vec{r}-\vec{r}_+)$$

$$\vec{p} = \int d^3r'\,\rho(\vec{r}')\,\vec{r}'$$

$$= \int d^3r'\,q\left[\delta^{(3)}(\vec{r}'-\vec{r}_+) - \delta^{(3)}(\vec{r}'-\vec{r}_-)\right]\vec{r}'$$

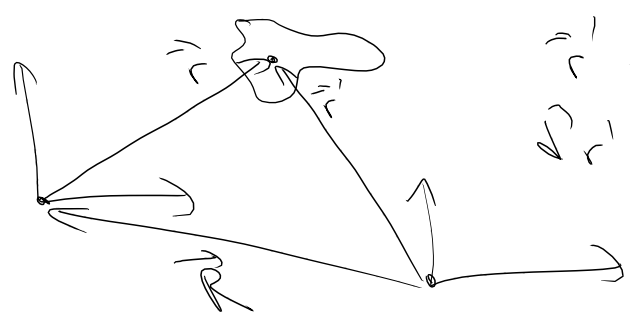
$$= q\vec{r}_+ - q\vec{r}_-$$

$$\boxed{\vec{p} = q(\vec{r}_+ - \vec{r}_-)}$$

Ex

$$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}-\vec{r}_0)$$

$$\vec{p} = \int d^3r'\,\vec{r}'\,q\,\delta^{(3)}(\vec{r}-\vec{r}_0) = q\vec{r}_0$$



$$\vec{r}' = \vec{r} + \vec{R}$$
$$d^3r' = d^3r$$

$$\rho(\vec{r}) = \rho'(\vec{r}')$$
$$\vec{p} = \int \rho(\vec{r})\vec{r}\, d^3r = \int d^3r' (\vec{r}' - \vec{R})\rho'(\vec{r}')$$
$$\vec{p} = \int d^3r' \vec{r}' \rho'(\vec{r}') - \vec{R}\int d^3r' \rho'(\vec{r}')$$
$$\boxed{\vec{p} = \vec{p}' - \vec{R}Q}$$

Ex Point charge at the origin

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum \frac{1}{r^{l+1}}\int d^3r' \rho(\vec{r}')r'^l P_l(\cos\theta)$$

$$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r})$$
$$\int d^3r'\, q\,\delta^{(3)}(\vec{r})r^l P_l(\cos\theta) = q\,\delta_{l,0}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

ideal dipole

$$\lim_{\substack{d\to 0\\ q\to\infty\\ qd = p = \text{const}}} \left(\begin{matrix} +q \\ -q \end{matrix}\right\} d$$

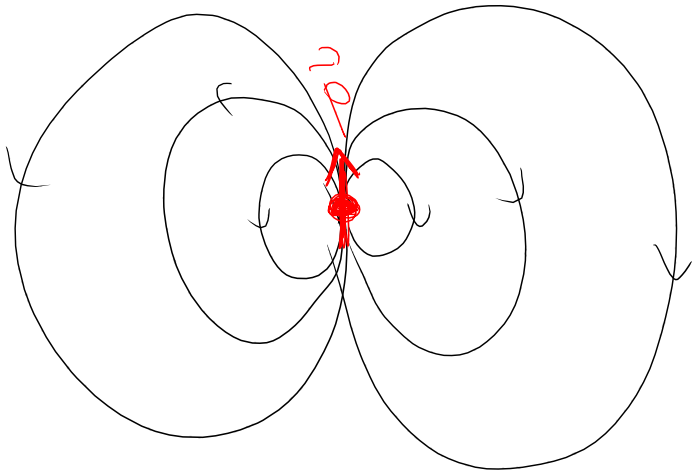
$$V_{\text{ideal dipole}}(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$V = \frac{p\cos\theta}{r^2}\,\frac{1}{4\pi\varepsilon_0}$$

$$\vec{E} = -\nabla V = -\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{r^2}\right)\hat{r}\,\frac{1}{4\pi\varepsilon_0} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{p\cos\theta}{r^2}\right)\hat{\theta}\,\frac{1}{4\pi\varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}\hat{r} + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}p\sin\theta\,\hat{\theta}$$

$$\stackrel{?}{=} \frac{1}{4\pi\varepsilon_0}\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}$$



ideal dipole