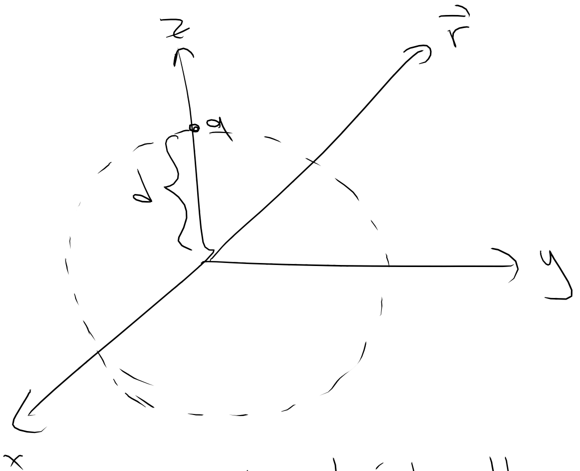
Example



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r} - d\hat{z}|}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{r^2 + d^2 - 2rd\cos\theta}}$$

inside and outside the sphere $\rho = 0$

$$\nabla^2 V = 0$$

$$V(r, \theta)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial V}{\partial \theta} = 0$$

$$V(r, \theta) = R(r)\,\Theta(\theta)$$

$$\frac{1}{R\Theta}\left[\Theta \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{r^2 \sin\theta} R \frac{d}{d\theta}\sin\theta \frac{d\Theta}{d\theta} = 0\right]$$

$$\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{r^2}\left[\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta \frac{d\Theta}{d\theta}\right] = 0$$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = C_1$$

$$\left[\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{C_1}{r^2} = 0\right] r^2 R$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + C_1 R = 0$$

$$\boxed{r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + C_1 R = 0} \Leftarrow$$

$$\sum a_n r^n \frac{d^nR}{dr^n} = 0 \Rightarrow R = r^\nu$$

$$\nu(\nu-1)\cancel{r^2}\,\cancel{r^{\nu-2}} + 2\cancel{r}\,\nu\,\cancel{r^{\nu-1}} + C_1 \cancel{r^\nu} = 0$$

$$\nu(\nu+1) + C_1 = 0$$

$$C_1 \equiv -l(l+1)$$

$$\nu = l \quad \text{or} \quad \nu = -(l+1)$$

$$R(r) = A_l r^l + \frac{B_l}{r^{l+1}}$$

$$\frac{1}{\Theta}\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -l(l+1)$$

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0}$$

$\Theta(\theta)$ should be finite $\forall\ 0 \leq \theta \leq \pi$

$\Theta(\theta) = P_l(\cos\theta)$ $\quad$ l integer

$$\Rightarrow \int_0^\pi d\theta \sin\theta\, P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}$$

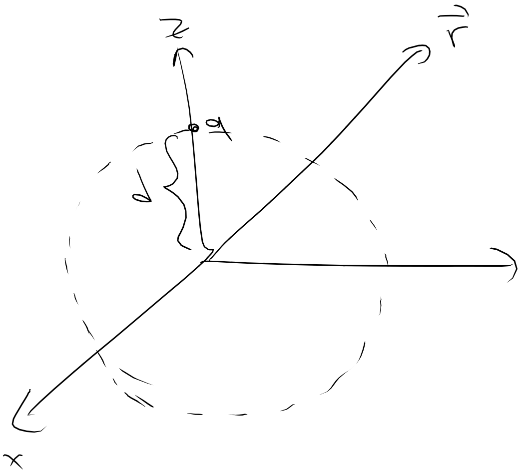
$P_l(1) = 1$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)}$$



$r < d \quad \nabla^2 V^{in} = 0$

$r > d \quad \nabla^2 V^{out} = 0$

$$V^{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V^{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V^{in}(r=d,\theta) = V^{out}(r=d,\theta)$$

$$\boxed{A_l d^l = \frac{B_l}{d^{l+1}}}$$

$$\vec{E}^{outside} - \vec{E}^{inside} = \frac{\sigma}{\varepsilon_0}\hat{n} \Longleftarrow$$

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$V^{in} = \sum_l A_l r^l P_l(\cos\theta)$$

$$\vec{E}^{in} = -\sum_l A_l l r^{l-1} P_l(\cos\theta)\hat{r}$$

$$- \sum_l A_l r^{l-1} \frac{d}{d\theta} P_l(\cos\theta)\hat{\theta}$$

$$V^{out} = \sum_{l=0} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\vec{E}^{out} = +\sum_{l=0}^{\infty} B_l \frac{(l+1)}{r^{l+2}} P_l(\cos\theta)\,\hat{r} \leftarrow$$

$$- \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+2}} \frac{dP_l}{d\theta}(\cos\theta)\,\hat{\theta} =$$

$$\vec{E}^{in} = -\sum_l A_l\, l\, r^{l-1} P_l(\cos\theta)\,\hat{r}$$

$$- \sum_l A_l r^{l-1} \frac{d}{d\theta} P_l(\cos\theta)\,\hat{\theta}$$

$$E_\theta^{inside} = -\sum_l A_l d^{l-1} \frac{d}{d\theta} P_l(\cos\theta)$$

$$E_\theta^{outside} = -\sum_l B_l \frac{1}{d^{l+2}} \frac{d}{d\theta} P_l(\cos\theta)$$

$$\Rightarrow A_l d^l = \frac{B_l}{d^{l+1}} \Rightarrow A_l d^{l-1} = \frac{B_l}{d^{l+2}}$$

$$\Rightarrow E_\theta^{inside} = E_\theta^{outside}$$

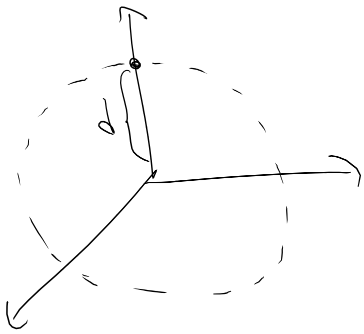
$$\boxed{\vec{E}^{outside} - \vec{E}^{inside} = \frac{\sigma}{\varepsilon_0}\hat{n}} = \frac{\sigma}{\varepsilon_0}\hat{r} =$$

$$E_r^{outside} - E_r^{inside} = \frac{\sigma(\theta)}{\varepsilon_0}$$

$$E_r^{out} = + \sum_{l=0}^{\infty} B_l \frac{(l+1)}{r^{l+2}} P_l(\cos\theta)$$

$$E_r^{in} = -\sum_l A_l\, l\, r^{l-1} P_l(\cos\theta)$$

$$\frac{\sigma}{\varepsilon_0} = E_r^{outside} - E_r^{inside} = \sum_{l=0}^{\infty} B_l \frac{(l+1)}{d^{l+2}} P_l(\cos\theta) + \sum_{l=0}^{\infty} A_l\, l\, d^{l-1} P_l(\cos\theta)$$



$$\sigma(\theta) = A\,\delta(\cos\theta - 1)$$

$$q = \int \sigma(\theta)\, dS = \int A\,\delta(\cos\theta - 1)$$

$$dS \; \{ \; d^2\, d\phi\, d(\cos\theta)$$

$$q = A R^2 2\pi$$

$$\Rightarrow \boxed{A = \frac{q}{2\pi d^2}}$$

$$\boxed{\sigma(\theta) = \frac{q}{2\pi d^2}\,\delta(\cos\theta - 1)}$$

$$\frac{\sigma}{\varepsilon_0} = \sum_{l=0}^{\infty} B_l \frac{(l+1)}{d^{l+2}} P_l(\cos\theta) + \sum_{l=0}^{\infty} A_l\, l\, d^{l-1} P_l(\cos\theta)$$

$$\boxed{\frac{q}{\varepsilon_0 2\pi d^2}\,\delta(\cos\theta - 1) = \sum_{l=0}^{\infty}\left(\frac{B_l (l+1)}{d^{l+2}} + A_l\, l\, d^{l-1}\right) P_l(\cos\theta)}$$

multiply both sites by $\sin\theta\, P_{l'}(\cos\theta)$
integrate over from $\theta = 0$ to $\theta = \pi$

$$\int_0^\pi d\theta \sin\theta\, P_{l'}(\cos\theta) P_l(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}$$

$$\frac{q}{2\pi\varepsilon_0 d^2}\int_0^\pi d\theta \sin\theta\, P_{l'}(\cos\theta)\,\delta(\cos\theta - 1) \Leftarrow$$

$$= \sum_l \left(\frac{B_l(l+1)}{d^{l+2}} + A_l l d^{l-1}\right)\frac{2}{2l+1}\delta_{ll'}$$

$$= \frac{2}{2l'+1}\left[\frac{B_{l'}(l'+1)}{d^{l'+2}} + A_{l'} l' d^{l'-1}\right]$$

Let $u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta$

$$\int_0^\pi d\theta \sin\theta\, P_{l'}(\cos\theta)\,\delta(\cos\theta - 1)$$

$$= -\int_1^{-1} du\, P_{l'}(u)\,\delta(u-1)$$

$$= \int_{-1}^{1} du\, P_{l'}(u)\,\delta(u-1) = P_{l'}(1) = 1$$

$$\frac{q}{2\pi\varepsilon_0 d^2} = \frac{2}{2l+1}\left[\frac{B_l(l+1)}{d^{l+2}} + A_l l d^{l-1}\right]$$

$$A_l d^l = \frac{B_l}{d^{l+1}} \Rightarrow A_l = \frac{B_l}{d^{2l+1}}$$

$$\frac{q}{4\pi\varepsilon_0 \cancel{d^2}} \cancel{(2l+1)} = B_l \frac{(l+1)}{d^{l+2}} + \frac{B_l\, l}{d^{l+2}} = \frac{B_l \cancel{(2l+1)}}{\cancel{d^{l+2}}}$$

$$\boxed{B_l = \frac{q}{4\pi\varepsilon_0} d^l} \qquad A_l = \frac{q}{4\pi\varepsilon_0} \frac{1}{d^{l+1}}$$

$$V_{in} = \sum A_l r^l P_l(\cos\theta) = \frac{q}{4\pi\varepsilon_0} \sum_l \frac{r^l}{d^{l+1}} P_l(\cos\theta)$$

$$V_{out} = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta) = \frac{q}{4\pi\varepsilon_0} \sum_l \frac{d^l}{r^{l+1}} P_l(\cos\theta)$$

$$\left\{ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} \right.$$

$r_> = \max(r, d)$

$r_< = \min(r, d)$

$$V_{in} = \frac{q}{4\pi\varepsilon_0} \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

$$V_{out} = \frac{q}{4\pi\varepsilon_0} \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

$$\boxed{V = \frac{q}{4\pi\varepsilon_0} \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)}$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{(r^2 + d^2 - 2rd\cos\theta)^{+1/2}}$$

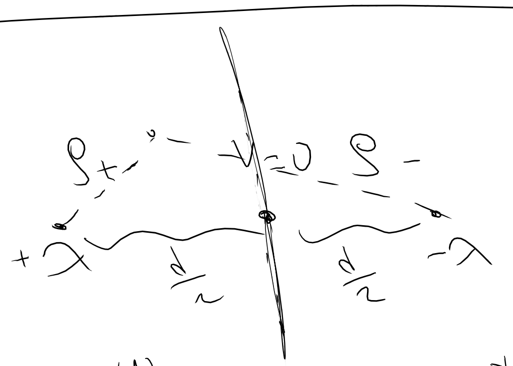
$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{(r_>^2 + r_<^2 - 2r_< r_> \cos\theta)^{1/2}}$$

$$x = \frac{r_<}{r_>} \qquad = \frac{q}{4\pi\varepsilon_0 r_>} (1 + x^2 - 2x\cos\theta)^{-1/2}$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r_>} \sum_l x^l P_l(\cos\theta)$$

$$\boxed{\left(1+x^2-2x\cos\theta\right)^{-1/2} = \sum_l x^l P_l(\cos\theta)}$$

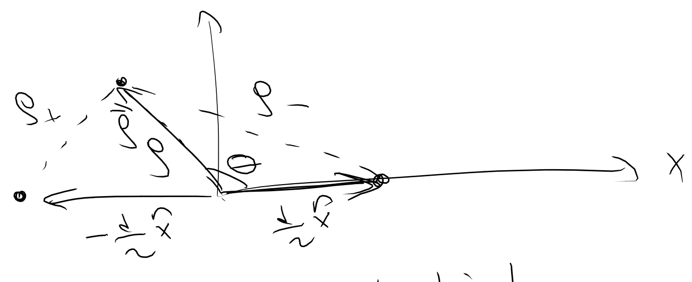
<u>Ex</u>



$$V^{(1)}(\rho) - V^{(1)}(\infty) = -\int \vec{E}^{(1)} \cdot d\vec{l} = -\int_\infty^r \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\,dr$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r}{\infty}\right)!$$

$$V(\rho) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\rho_+}{\rho_-}\right)$$



$$\rho_- = \left|\vec{\rho} - \frac{d}{2}\hat{x}\right|$$

$$= \left[\left(\vec{\rho} - \frac{d}{2}\hat{x}\right)^2\right]^{1/2}$$

$$= \left[\rho^2 + \frac{d^2}{4} - \frac{d}{2}\vec{\rho}\cdot\hat{x}\right]^{1/2}$$

points on an equipotential surface

$$\rho_- = \left[\rho^2 + \frac{d^2}{4} - \frac{d}{2}\rho\cos\theta\right]^{1/2}$$

$$\rho_+ = \left[\rho^2 + \frac{d^2}{4} + \frac{d}{2}\rho\cos\theta\right]^{1/2}$$

$$\frac{\rho_+}{\rho_-} = C$$

$$\frac{\rho_+^2}{\rho_-^2} = C^2 \Rightarrow \left(\rho^2 + \frac{d^2}{4} + \frac{d}{2}\rho\cos\theta\right)$$

$$= C\left(\rho^2 + \frac{d^2}{4} - \frac{d}{2}\rho\cos\theta\right)$$

$$0 = \rho^2(1-c) + \frac{d^2}{4}(1-c) + \frac{d}{2}\rho\cos\theta(1+c)$$

$$0 = \rho^2 + \frac{d^2}{4} + \frac{d}{2}\rho\cos\theta\,\frac{(1+c)}{1-c} \Leftarrow$$

$$\Rightarrow \left(\rho + \frac{d}{4}\cos\theta\,\frac{1+c}{1-c}\right)^2 - \left(\frac{d}{4}\cos\theta\,\frac{1+c}{1-c}\right)^2 + \frac{d^2}{4} = 0$$

$$\overset{?}{\Rightarrow} \boxed{\rho - \rho_0 = A\cos\theta} \Rightarrow \text{eqn of a circle around } \rho_0$$

$$\left.\begin{array}{l} x = \rho\cos\theta \\ y = \rho\sin\theta \end{array}\right\} \; x^2 + y^2 + \frac{d^2}{4} + \frac{d}{2}\,\frac{1+c}{1-c}\,x = 0$$

$$\Downarrow$$

$$(x - x_0)^2 + y^2 = R^2$$