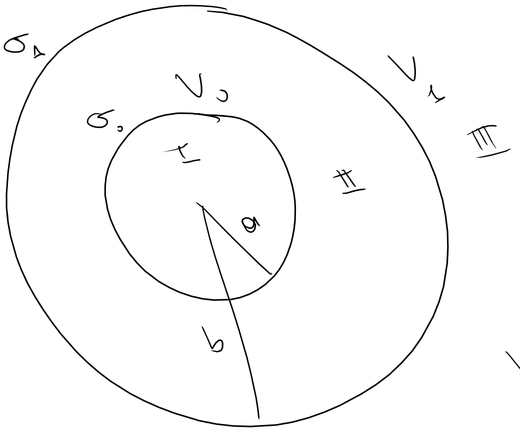


$$V_1 = \frac{1}{4\pi\varepsilon_0}\left(Q_1 + Q_0\right)\frac{1}{b}$$

$$Q_1 = 4\pi b^2 \sigma_1$$

$$Q_0 = 4\pi a^2 \sigma_0$$

$$V_0 = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{a} + V_1 \Rightarrow Q_0$$

$$V = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)P_l(\cos\theta)$$

region I: V is finite at $r=0 \Rightarrow b_l = 0$

$$V^{I}(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$V_0 = V^{I}(r=a,\theta) = \sum_{l=0}^{\infty} a_l a^l P_l(\cos\theta)$$

$$\sum_{l=0}^{\infty} a_l a^l P_l(\cos\theta) = V_0 = V_0 P_0(\cos\theta)$$

multiply both sites by $\sin\theta\, P_{l'}(\cos\theta)$ integrate over θ from 0 to π

$$\sum_{l=0}^{\infty} a_l a^l \underbrace{\int_0^{\pi} \sin\theta\, P_{l'}(\cos\theta) P_l(\cos\theta)\, d\theta}_{\frac{2}{2l+1}\delta_{ll'}} = V_0 \underbrace{\int d\theta \sin\theta\, P_{l'}(\cos\theta) P_0(\cos\theta)}_{\frac{2}{2l'+1}\delta_{l'0}}$$

$$a_{l'} a^{l'} \cancel{\frac{2}{2l'+1}} = \cancel{\frac{2}{2l'+1}}\delta_{l'0} V_0$$

$$a_{l'} = \begin{cases} V_0 & \text{if } l' = 0 \\ 0 & \text{if } l' \neq 0 \end{cases}$$

$$\boxed{V^{I}(r,\theta) = a_0 = V_0}$$

region II

$$V^{II}(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

at $r = a$

$$V_0 = \sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos\theta)$$

$$A_l a^l + \frac{B_l}{a^{l+1}} = \begin{cases} V_0 & \text{if } l=0 \\ 0 & \text{if } l\neq 0 \end{cases}$$

at $r = b$

$$V_1 = \sum_{l=0}^{\infty}\left(A_l b^l + \frac{B_l}{b^{l+1}}\right)P_l(\cos\theta)$$

$$A_l b^l + \frac{B_l}{b^{l+1}} = \begin{cases} V_1 & \text{if } l=0 \\ 0 & \text{if } l\neq 0 \end{cases}$$

$l \neq 0$

$$\left.\begin{aligned} A_l a^l + \frac{B_l}{a^{l+1}} &= 0 \\ A_l b^l + \frac{B_l}{b^{l+1}} &= 0 \end{aligned}\right\} A_l = B_l = 0$$

$l = 0$

$$A_0 + \frac{B_0}{a} = V_0$$

$$-\quad A_0 + \frac{B_0}{b} = V_1$$

$$B_0\left(\frac{1}{a} - \frac{1}{b}\right) = V_0 - V_1$$

$$\boxed{B_0 = \frac{ab}{b-a}(V_0 - V_1)}$$

$$A_0 = V_0 - \frac{B_0}{a} = V_0 - \frac{b}{b-a}(V_0 - V_1) = -\frac{a}{b-a}V_0 + \frac{b}{b-a}V_1$$

$$V = A_0 + \frac{B_0}{r}$$

$$\boxed{V^{II} = -\frac{a}{b-a}V_0 + \frac{b}{b-a}V_1 + \frac{ab}{b-a}(V_0 - V_1)\frac{1}{r}}$$

check

$$V(r=a) = -\frac{a}{b-a}V_0 + \cancel{\frac{b}{b-a}V_1} + \frac{b}{b-a}(V_0 - \cancel{V_1})$$

$$= \frac{b-a}{b-a}V_0 = V_0$$

$$V(r=b) = \cancel{-\frac{a}{b-a}}V_0 + \frac{b}{b-a}V_1 + \frac{a\cancel{b}}{b-a}(\cancel{V_0} - V_1)\frac{1}{\cancel{b}}$$

$$= \frac{b-a}{b-a}V_1 = V_1 \quad \checkmark$$

$$\boxed{\frac{1}{4\pi\varepsilon_0}Q_0 = \frac{ab}{b-a}(V_0 - V_1)}$$

region III

$$V^{III}(r) = \left(C_0 + \frac{D_0}{r}\right)\underbrace{P_0(\cos\theta)}_{1}$$

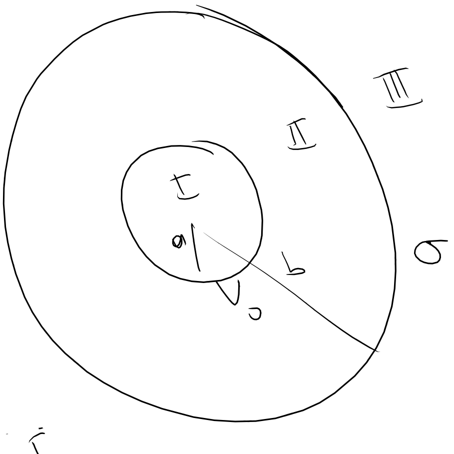
$$V^{III}(r) = C_0 + \frac{D_0}{r}$$

$$V^{III}(r\to\infty) = C_0 \stackrel{!}{=} 0$$

$$V^{III}(r=b) = \frac{D_0}{b} \stackrel{!}{=} V_1 \Rightarrow D_0 = bV_1$$

$$\boxed{V^{III}(r) = \frac{b}{r}V_1}$$

$$\frac{1}{4\pi\varepsilon_0}(Q_0 + Q_1) = bV_1$$



$$\sigma = k\cos\theta$$

$$V^{I} = V_0$$

$$V^{II} = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$V^{II}(r=a,\theta) = V_0 = \sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos\theta)$$

$$A_l a^l + \frac{B_l}{a^{l+1}} = \begin{cases} 0 & \text{if } l \neq 0 \\ V_0 & \text{if } l = 0 \end{cases} \qquad (1)$$

$$\boxed{V^{III}(r,\theta) = \sum_{l=0}^{\infty}\frac{C_l}{r^{l+1}}P_l(\cos\theta)}$$

$$V^{II}(r=b,\theta) = V^{III}(r=b,\theta)$$

$$\sum_l\left(A_l b^l + \frac{B_l}{b^{l+1}}\right)P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{C_l}{b^{l+1}}P_l(\cos\theta)$$

$$\boxed{A_l b^l + \frac{B_l}{b^{l+1}} = \frac{C_l}{b^{l+1}}} \qquad (2)$$

$$\vec{E}^{\,outside} - \vec{E}^{\,inside} = \frac{\sigma}{\varepsilon_0}\hat{n} = \frac{\sigma}{\varepsilon_0}\hat{r}$$

$$\vec{E}^{\,outside} = -\vec{\nabla}V^{III}\Big|_{r=b}$$

$$\vec{E}^{\,inside} = -\vec{\nabla}V^{II}\Big|_{r=a}$$

$$\left(-\frac{\partial V^{III}}{\partial r}\right)\Bigg|_{r=b} - \left(-\frac{\partial V^{II}}{\partial r}\right)\Bigg|_{r=b} = \frac{\sigma}{\varepsilon_0} = \frac{k P_1(\cos\theta)}{\varepsilon_0}$$

$$\sum_{l=0}^{\infty} C_l \frac{l+1}{b^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty}\left(-A_l l b^{l-1} + B_l \frac{(l+1)}{b^{l+2}}\right) P_l(\cos\theta)$$

$$= \frac{k}{\varepsilon_0} P_1(\cos\theta)$$

$$\Rightarrow \boxed{\begin{array}{l} C_1 \frac{2}{b^3} - \left(-A_1 + B_1 \frac{2}{b^3}\right) = \frac{k}{\varepsilon_0} \\ (C_l - B_l)\frac{l+1}{b^{l+2}} + A_l l b^{l-1} = 0 \quad \text{if } l \neq 1 \end{array}} \qquad (3)$$

$$A_l b^l + \frac{B_l}{b^{l+1}} = \frac{C_l}{b^{l+1}} \qquad (2)$$

$$A_l a^l + \frac{B_l}{a^{l+1}} = \begin{cases} 0 & \text{if } l \neq 0 \\ V_0 & \text{if } l = 0 \end{cases} \qquad (1)$$

$$A_l = 0 \ ; \ B_l = 0 \ ; \ C_l = 0 \quad \text{if } l \geq 2$$

$\underline{l = 0}$

$C_0 = B_0$ from eq (3)

$A_0 = 0$ from eq (2)

$\frac{B_0}{a} = V_0 \Rightarrow B_0 = a V_0$ from eq (1)

$\underline{l = 1 \text{ case}}$

$$C_1 \frac{2}{b^3} - \left(-A_1 + B_1 \frac{2}{b^3}\right) = \frac{k}{\varepsilon_0}$$

$$\boxed{(C_1 - B_1)\frac{2}{b^3} + A_1 = \frac{k}{\varepsilon_0}} \qquad (4)$$

$$\boxed{A_1 a + \frac{B_1}{a^2} = 0} \qquad (5)$$

$$A_1 b + \frac{B_1}{b^2} = \frac{C_1}{b^2} \Rightarrow \boxed{(C_1 - B_1)\frac{1}{b^2} = A_1 b} \qquad (6)$$

$$\text{Eq } 6 \Rightarrow C_1 - B_1 = A_1 b^3$$

$$\text{Eq } 4 \Rightarrow A_1 b^3 \frac{2}{b^3} + A_1 = \frac{k}{\varepsilon_0}$$

$$\Rightarrow \boxed{A_1 = \frac{k}{3\varepsilon_0}}$$

$$C_1 - B_1 = A_1 b^3 = \boxed{\frac{k}{3\varepsilon_0} b^3 = C_1 - B_1}$$

$$\text{Eq } 5 \Rightarrow B_1 = -A_1 a^3$$

$$\boxed{B_1 = -\frac{k}{3\varepsilon_0} a^3}$$

$$C_1 = \frac{k}{3\varepsilon_0} b^3 + B_1$$

$$\boxed{C_1 = \frac{k}{3\varepsilon_0}(b^3 - a^3)}$$

$$V^{II} = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$= \frac{aV_0}{r} + \left(\frac{k}{3\varepsilon_0} r - \frac{k}{3\varepsilon_0}\frac{a^3}{r^2}\right) P_1(\cos\theta)$$

$$V^{II} = \frac{aV_0}{r} + \frac{k}{3\varepsilon_0} r\cos\theta \left(1 - \frac{a^3}{r^3}\right)$$

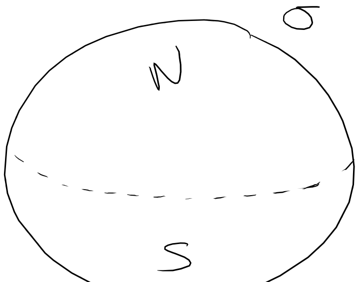
$$\boxed{V^{II} = \frac{aV_0}{r} + \frac{k}{3\varepsilon_0} z \left(1 - \frac{a^3}{r^3}\right)}$$

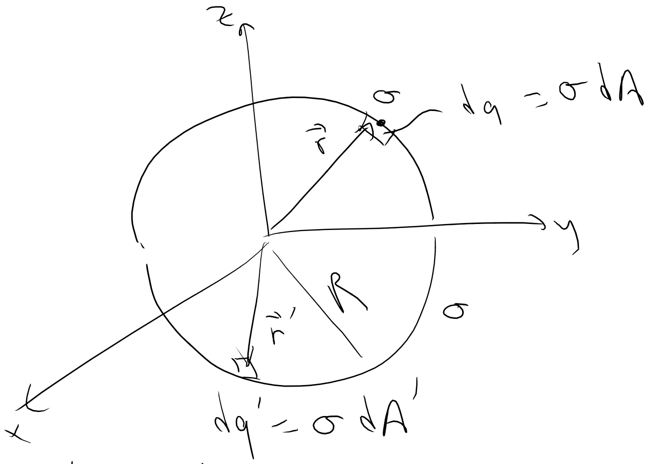
$$V^{III} = \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{C_0}{r} P_0(\cos\theta) + \frac{C_1}{r^2} P_1(\cos\theta)$$

$$\boxed{V^{III} = \underbrace{\frac{aV_0}{r}}_{\text{monopole}} + \underbrace{\frac{k}{3\varepsilon_0}(b^3-a^3)\frac{\cos\theta}{r^2}}_{\text{dipole}}}$$

HW Q1





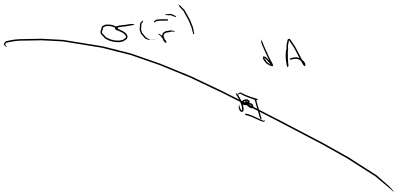
$$d\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} dq' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{E}(\vec{r}) = \int \frac{1}{4\pi\varepsilon_0} \sigma \, dA' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{E}(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_{\pi/2}^{\pi} R^2 d\theta' \sin\theta' \int_0^{2\pi} d\phi' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$d\vec{F} = \vec{E}(\vec{r}) \, \sigma \, dA$$

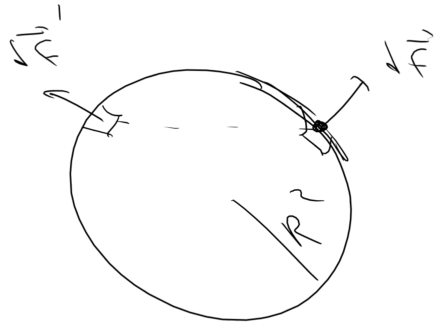
$$\vec{F} = \frac{\sigma^2}{4\pi\varepsilon_0} \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} R^2 d\theta' \sin\theta' \int_0^{2\pi} d\phi' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$



$$dq = \sigma\, dA$$

$$d\vec{F} = \vec{E}^{\text{other}}\, dq = \sigma \vec{E}^{\text{other}}\, dA$$

$$\vec{E}^{\text{other}} = \frac{\vec{E}^{\text{outside}} + \vec{E}^{\text{inside}}}{2}$$



$$\vec{E}^{\text{inside}} = 0$$

$$\vec{E}^{\text{outside}} = \frac{Q_{tot}}{R^2}\frac{1}{4\pi\varepsilon_0}\hat{r}$$

$$d\vec{F} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{Q_{tot}}{R^2}\hat{r}\,\sigma\, dA$$

$$dF_z = \frac{1}{8\pi\varepsilon_0}\frac{Q_{tot}}{R^2}(\hat{r}\cdot\hat{z})\,\sigma\, dA$$

$$dF_z = \frac{1}{8\pi\varepsilon_0}\frac{Q_{tot}}{\cancel{R^2}}\cos\theta\,\sigma\,\underbrace{\cancel{R^2}\sin\theta\, d\theta\, d\phi}_{dA}$$

$$F_z = \frac{\sigma Q_{tot}}{8\pi\varepsilon_0}\int_0^{\pi/2} d\theta\,\sin\theta\cos\theta\int_0^{2\pi} d\phi$$

$$= \frac{\sigma Q_{tot}}{8\pi\varepsilon_0}\,2\pi\,\frac{\sin^2\theta}{2}\Bigg|_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{\sigma Q_{tot}}{4\varepsilon_0}\frac{1}{2} = \boxed{\frac{\sigma Q_{tot}}{8\varepsilon_0} = F_z}$$

$$F_z = \frac{4\pi R^2\sigma\,\sigma}{8\varepsilon_0} = \boxed{\frac{\pi R^2\sigma^2}{2\varepsilon_0} = F_z}$$

$$F_z = F_z^{\text{northern hemisphere}} + F_z^{\text{southern hemisphere}}$$

$$F_z^{\text{northern hemisphere}} = 0$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$ $\Leftarrow$ only in linear dielectrics (for small $|\vec{E}|$)