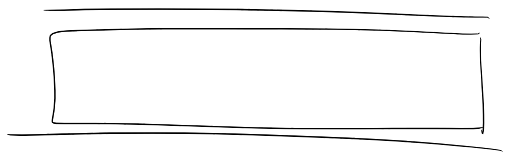
$$\left.\begin{aligned}\vec{P} &= \varepsilon_0 \chi_e \vec{E} \\ \vec{D} &= \varepsilon_0 (1+\chi_e) \vec{E}\end{aligned}\right\}$$

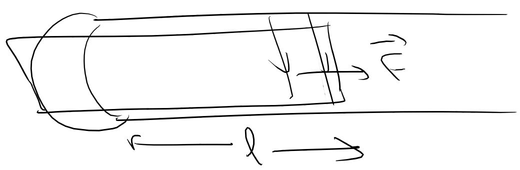
$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$$

$$W = \int \frac{1}{2} \vec{E} \cdot \vec{D} \, d^3r$$



$$C = \varepsilon_r C^{(0)}$$

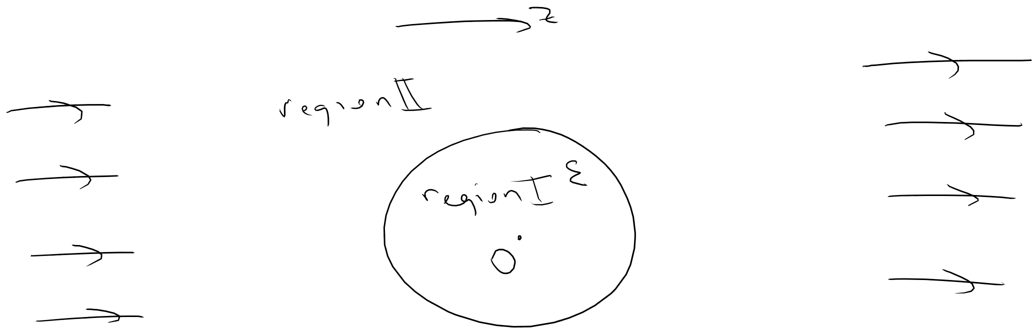


$$U$$

$$F = \frac{dU}{dl}$$

Boundary Value Problems with Dielectrics

$$\vec{E}^{(0)} = E^{(0)} \hat{z}$$



$$V^{(0)} = -E^{(0)} z = -E^{(0)} r \cos\theta$$

$$V^{I} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V^{II} = -E^{(0)} r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla} V$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{\nabla}\cdot\vec{D} = 0 \Rightarrow \vec{\nabla}\cdot\left(\varepsilon(-\vec{\nabla} V)\right) = 0$$

$$\vec{\nabla}\cdot(\varepsilon\vec{\nabla} V) = 0$$

region I & II ε is constant

$$\varepsilon \nabla^2 V = 0 \Rightarrow \boxed{\nabla^2 V = 0}$$

$$V^{I} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V^{II} = -E^{(0)} r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V^{I}(r=R) = V^{II}(r=R)$$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E^{(0)} R P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

$$\boxed{\begin{aligned} A_1 R &= -E^{(0)} R + \frac{B_1}{R^2} \\ A_l R^l &= \frac{B_l}{R^{l+1}} \qquad l \neq 1 \end{aligned}}$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$\nabla \times \vec{E} = 0 \Rightarrow E_{\parallel}$ is continuous

$\Rightarrow V$ is continuous accross the boundary.

$$\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow D_\perp^{\text{outside}} - D_\perp^{\text{inside}} = \sigma_f = 0$$

$$D_\perp = \varepsilon E_r = -\varepsilon \frac{\partial V}{\partial r}$$

$$\Rightarrow \varepsilon \left.\frac{\partial V^{I}}{\partial r}\right|_{r=R} = \left.\frac{\partial V^{II}}{\partial r}\right|_{r=R}$$

$$V^{I} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V^{II} = -E^{(0)} r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\varepsilon \sum_{l=0}^{\infty} A_l l R^{l-1} P_l(\cos\theta) = -E^{(0)} \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1) B_l}{R^{l+2}} P_l(\cos\theta)$$

$$P_1(\cos\theta)$$

$$\boxed{\varepsilon A_1 = -E^{(0)} - \frac{2B_1}{R^3}} \qquad (1)$$

$$\varepsilon l A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} \qquad l \neq 1 \qquad (2)$$

$$A_1 R = -E^{(0)} R + \frac{B_1}{R^2} \qquad (3)$$

$$A_l R^l = \frac{B_l}{R^{l+1}} \qquad l \neq 1 \qquad (4)$$

$$(2) \;\&\; (4) \Rightarrow A_l = 0 \quad B_l = 0 \qquad l \neq 1$$

$$(3) \Rightarrow A_1 = -E^{(0)} + \frac{B_1}{R^3}$$

$$(1) \Rightarrow \varepsilon\left(-E^{(0)} + \frac{B_1}{R^3}\right) = -E^{(0)} - \frac{2B_1}{R^3}$$

$$(\varepsilon + 2)\frac{1}{R^3} B_1 = (\varepsilon - 1) E^{(0)}$$

$$\boxed{B_1 = R^3 \frac{\varepsilon - 1}{\varepsilon + 2} E^{(0)}}$$

$$A_1 = -E^{(0)} + \frac{B_1}{R^3}$$

$$= -E^{(0)} + \frac{\varepsilon - 1}{\varepsilon + 2} E^{(0)}$$

$$= \frac{-\varepsilon - 2 + \varepsilon - 1}{\varepsilon + 2} E^{(0)}$$

$$\boxed{A_1 = -\frac{3}{\varepsilon + 2} E^{(0)}}$$

$$V^{I} = A_1 r^1 P_1(\cos\theta)$$

$$= -\frac{3 E^{0}}{\varepsilon + 2} r \cos\theta$$

$$\boxed{V^{I} = -\frac{3 E^{(0)}}{\varepsilon + 2} z} \Rightarrow \vec{E}^{I} = \frac{3}{\varepsilon + 2} \vec{E}^{(0)} = \vec{E}^{(0)} + \vec{E}_p$$

$$\vec{E}_p = \frac{1 - \varepsilon}{\varepsilon + 2} \vec{E}^{(0)}$$

$$V^{II} = -E^{(0)} r \cos\theta + R^3 \frac{\varepsilon - 1}{\varepsilon + 2} E^{(0)} \frac{1}{r^2} P_1(\cos\theta)$$

$$\boxed{V^{II} = -E^{(0)} r \cos\theta + \frac{\varepsilon - 1}{\varepsilon + 2} E^{(0)} \frac{R^3}{r^2} P_1(\cos\theta)}$$

external electric field

potential of a pure dipole.

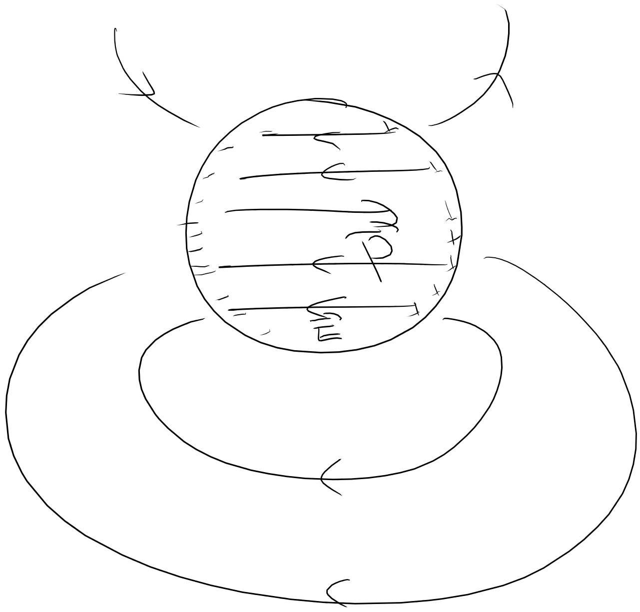
$$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

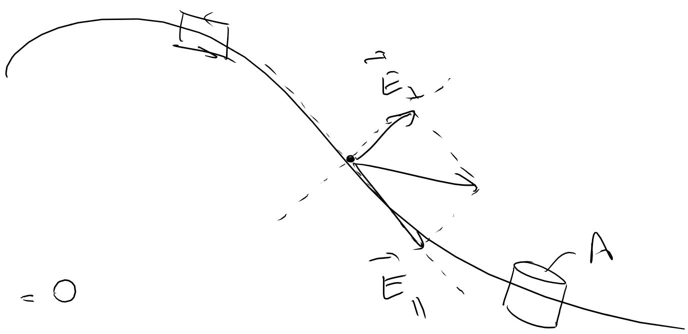
$$\vec{\nabla} \cdot \vec{P} = -\rho_b$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$$\sigma_b = \vec{P} \cdot \hat{n} \ , \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$





$$\oint \vec{D} \cdot d\vec{S} = Q^f_{enc}$$

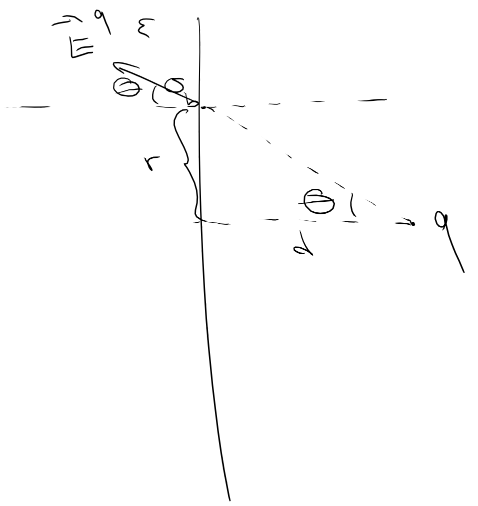
$$Q^f_{enc} = \sigma_f$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \iff \Delta E_\parallel = 0$$

$$\vec{\nabla} \cdot \vec{D} = \rho_f \iff \Delta D_\perp = \sigma_f$$

Example



$$E_z^q = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2+d^2}\frac{d}{\sqrt{r^2+d^2}}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{q\,d}{(r^2+d^2)^{3/2}}$$

$\longrightarrow z$

method i

$$\sigma_b = \vec{P}\cdot\hat{n} = P_z = \varepsilon_0\chi_e E_z$$

$$E_z = E_z^q + \frac{\sigma_b}{2\varepsilon_0}$$

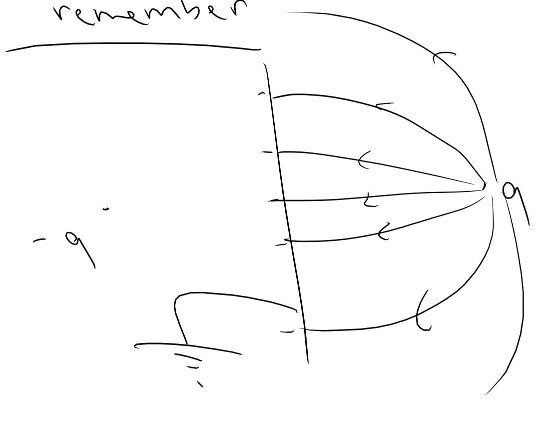
$$\sigma_b = \varepsilon_0\chi_e\left(E_z^q + \frac{\sigma_b}{2\varepsilon_0}\right)$$

$$\sigma_b = \frac{\chi_e}{2}\sigma_b + \varepsilon_0\chi_e E_z^q$$

$$\boxed{\sigma_b = \frac{2\varepsilon_0\chi_e}{2+\chi_e}E_z^q}$$

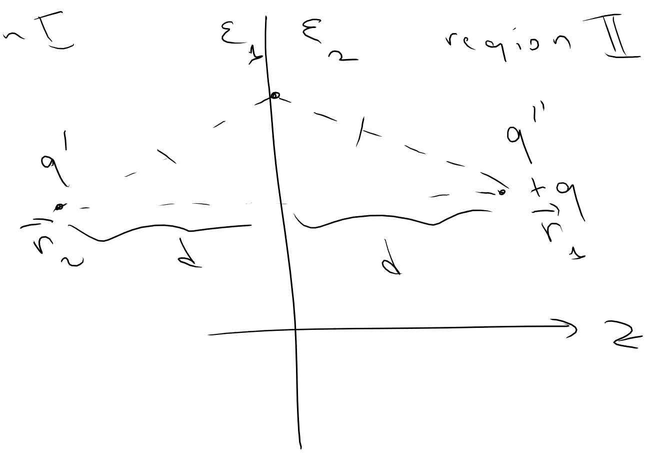
$$\sigma_b = \frac{2\cancel{\varepsilon_0}\chi_e}{2+\chi_e}\left(\frac{-1}{4\pi\cancel{\varepsilon_0}}\right)\left(\frac{q\,d}{(r^2+d^2)^{3/2}}\right) = -\frac{2\chi_e}{2+\chi_e}\frac{1}{4\pi}\frac{q\,d}{(r^2+d^2)^{3/2}}$$

remember



$$\sigma_{induced} \propto \frac{1}{(r^2+d^2)^{3/2}}$$

method II — method of images



$q'' \stackrel{?}{=} q+q'$

$$V^{II}(\vec r) = \frac{1}{4\pi\varepsilon_2}\frac{q}{|\vec r - \vec r_1|} + \frac{1}{4\pi\varepsilon_2}\frac{q'}{|\vec r - \vec r_2|}$$

$$V^{I}(\vec r) = \frac{1}{4\pi\varepsilon_1}\frac{q''}{|\vec r - \vec r_1|}$$

$V^I = V^{II}$ on the interface: $|\vec r - \vec r_1| = |\vec r - \vec r_2|$

$$\boxed{\frac{q+q'}{\varepsilon_2} = \frac{q''}{\varepsilon_1}}$$

$$\varepsilon_1 \frac{\partial V^I}{\partial z}\bigg|_{o.b.} = \varepsilon_2 \frac{\partial V^{II}}{\partial z}\bigg|_{\text{on the boundary}}$$

$$\frac{\partial}{\partial z}\frac{1}{|\vec r - \vec r_1|}\bigg|_{z=0} = \frac{\partial}{\partial z}\frac{1}{|(x\hat x + y\hat y + z\hat z) - d\hat z|}\bigg|_{z=0} = \frac{\partial}{\partial z}\frac{1}{[x^2+y^2+(z-d)^2]^{1/2}}\bigg|_{z=0}$$

$$= \frac{-(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}}\bigg|_{z=0} = \frac{d}{[x^2+y^2+d^2]^{3/2}} = \frac{d}{|\vec r - \vec r_1|^3}\bigg|_{o.b.}$$

$$\frac{\partial}{\partial z}\frac{1}{|\vec r - \vec r_2|} = \frac{-d}{|\vec r - \vec r_2|^3}\bigg|_{o.b.}$$

$$V^{II}(\vec{r}) = \frac{1}{4\pi\varepsilon_2}\frac{q}{|\vec{r}-\vec{r}_1|} + \frac{1}{4\pi\varepsilon_2}\frac{q'}{|\vec{r}-\vec{r}_2|}$$

$$V^{I}(\vec{r}) = \frac{1}{4\pi\varepsilon_1}\frac{q''}{|\vec{r}-\vec{r}_1|}$$

$$\cancel{\varepsilon_1}\,\cancel{\frac{1}{4\pi\varepsilon_1}}\,\frac{q''\,d}{\cancel{|\vec{r}-\vec{r}_1|^3}\Big|_{a.b}} = \cancel{\varepsilon_2}\,\cancel{\frac{1}{4\pi\varepsilon_2}}\left[\frac{q\,d}{\cancel{|\vec{r}-\vec{r}_1|^3}\Big|_{a.b.}} + \frac{q'(-d)}{\cancel{|\vec{r}-\vec{r}_2|^3}\Big|_{a.b.}}\right]$$

$$q''\,d = q\,d - q'\,d$$

$$\boxed{q'' = q - q'}$$

$$\frac{q+q'}{\varepsilon_2} = \frac{q''}{\varepsilon_1} = \frac{q-q'}{\varepsilon_1}$$

$$q\,\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1\varepsilon_2} = q\left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right) = -q'\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right) = \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}(-q')$$

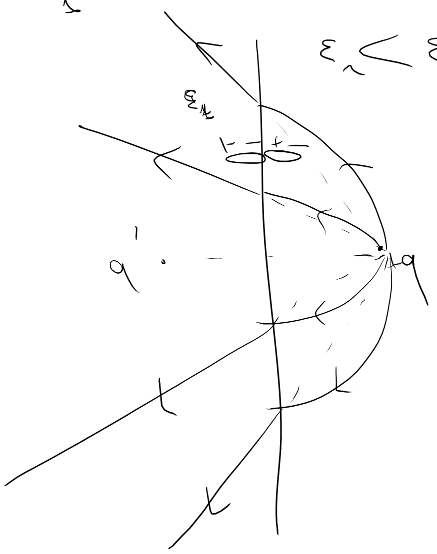
$$\boxed{q' = \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\,q}$$

$$q'' = q - q' = q\left[1 - \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right]$$

$$\boxed{q'' = \frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\,q}$$

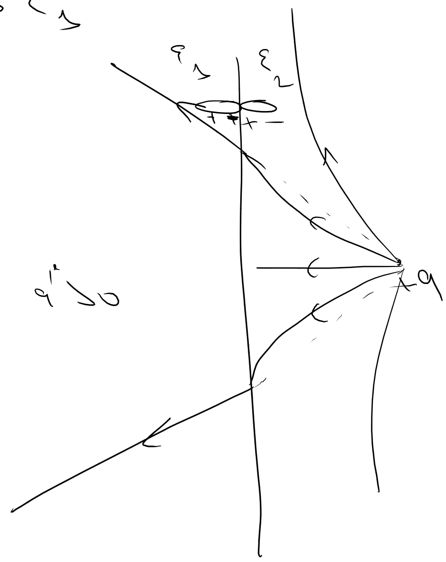
$$q' = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} q \qquad q'' = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} q$$

$\varepsilon_2 < \varepsilon_1$



$$q' = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} q \qquad q'' = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} q$$

$\varepsilon_2 > \varepsilon_1$



$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\int dS\,\sigma_b(\vec r\,')\,\frac{1}{|\vec r-\vec r\,'|} + \frac{1}{4\pi\varepsilon_2}\,q\,\frac{1}{|\vec r-\vec r_1|}$$

$$\vec\nabla\cdot\vec D = -\vec\nabla\cdot(\varepsilon\vec\nabla V) = \rho_f$$

if ε is const.

$$\nabla^2 V = -\frac{\rho_f}{\varepsilon} \iff \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

in region II if $z>0$

$$\int dS\,\frac{\sigma_b(\vec r\,')}{|\vec r-\vec r\,'|} \equiv \frac{q'}{|\vec r-\vec r_2|}$$

how do you relate it to $z<0$?

$$\cancel{\frac{1}{z-i\varepsilon} = \frac{1}{z+i\varepsilon} - 2i\pi\delta(z)}$$