Midterm Date 25 December, Sunday

$9^{40} - 13^{00}$

$\vec{\nabla}\cdot\vec{B} = 0 \iff \vec{B} = \vec{\nabla}\times\left(\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}'\right)$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \oint_S \vec{B}\cdot d\vec{S} = 0$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \oint\vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$I_{enc} = \int\vec{J}\cdot d\vec{A}$

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{A}+\vec{\nabla}f)$

↑ vector potential

$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\phi$

$\vec{A}_0 \longrightarrow \vec{A} = \vec{A}_0 + \vec{\nabla}f$: gauge transformation

$\vec{\nabla}\times(\overbrace{\vec{\nabla}\times\vec{A}}^{\vec{B}}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$

$\vec{\nabla}\cdot\vec{A} = 0$ Coulomb Gauge

$\nabla^2\vec{A} = -\mu_0\vec{J}$

$\nabla^2 A_x = -\mu_0 J_x$ $\quad$ $\nabla^2 A_z = -\mu_0 J_z$

$\nabla^2 A_y = -\mu_0 J_y$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}$$
$$\forall\ \vec{A}, \vec{B}, \vec{C}$$

$$\vec{B} \times (\vec{B} \times \vec{C})$$

$$A_{x,y,z}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J_{x,y,z}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \Longleftarrow$$

warning

$$A_\phi(\vec{r}) = \vec{A}(\vec{r}) \cdot \hat{\phi}(\vec{r})$$
$$= \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \cdot \hat{\phi}(\vec{r})}{|\vec{r} - \vec{r}'|} d\tau'$$
$$\neq \frac{\mu_0}{4\pi} \int \frac{J_\phi(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$
$$J_\phi(\vec{r}') = \vec{J}(\vec{r}') \cdot \hat{\phi}(\vec{r}')$$

assume: $\vec{A}_0$ $\quad \vec{\nabla} \times \vec{A}_0 = \vec{B}$ $\quad$ & $\vec{\nabla} \cdot \vec{A}_0 \neq 0$

$$\vec{A} = \vec{A}_0 + \vec{\nabla} f \qquad \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2 f = 0$$

$$\Rightarrow \nabla^2 f = -\vec{\nabla} \cdot \vec{A}_0$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

$$\nabla \cdot \vec{A} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \cdot \left(\nabla \frac{1}{|\vec{r} - \vec{r}'|} \right) d^3r'$$

$$\nabla \frac{1}{|\vec{r} - \vec{r}'|} = -\nabla' \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\nabla \cdot \vec{A} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \cdot \left(-\nabla' \right) \frac{1}{|\vec{r} - \vec{r}'|} d^3r'$$

$$= -\frac{\mu_0}{4\pi} \int \left[\nabla' \cdot \left(\frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) - \frac{1}{|\vec{r} - \vec{r}'|} \nabla' \cdot \vec{J}(\vec{r}') \right] d^3r'$$

$$= 0$$

$$= -\frac{\mu_0}{4\pi} \oint \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \cdot d\vec{S}'$$

$$= 0$$

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}$$

$$\vec{\nabla} \cdot \vec{A} = 0$$

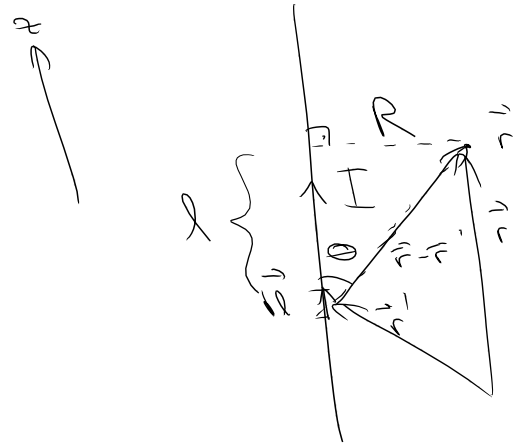
$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla} \phi$$

Example



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}'$$

$$= \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}}{|\vec{r} - \vec{r}'|}$$

$$d\vec{l} = dl\,\hat{z} \qquad \vec{J} = J_z \hat{z}$$

$$\vec{J}\, dV = I\, d\vec{l} = \vec{K}\, dS \qquad A_x = A_y = 0$$

$$A_z = \frac{\mu_0}{4\pi} I \int \frac{dl}{|\vec{r} - \vec{r}'|} \Rightarrow \frac{R}{|\vec{r} - \vec{r}'|} = \sin\theta$$

$$\frac{R}{l} = \tan\theta \Rightarrow l = R\cot\theta \qquad \frac{1}{|\vec{r} - \vec{r}'|} = \frac{\sin\theta}{R}$$

$$dl = -\frac{R}{\sin^2\theta} d\theta$$

$$A_z = \frac{\mu_0}{4\pi} I \int \left(-\frac{R}{\sin^2\theta} d\theta\right) \frac{\sin\theta}{R}$$

$$= -\frac{\mu_0}{4\pi} I \int_{\pi}^{0} \frac{d\theta}{\sin\theta}$$

$$= -\frac{\mu_0}{4\pi} I \int_{\pi}^{0} \frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} d\theta$$

$$= -\frac{\mu_0}{4\pi} I \int_{\pi}^{0} \left[\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right] d\theta$$

$$= -\frac{\mu_0}{4\pi} I \left[2\ln\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)\right]_{\theta=\pi}^{0} \quad ?$$

$$R = \sqrt{x^2+y^2}$$

$$A_z^{partial} = -\frac{\mu_0}{8\pi} I \left[2 \ln \left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \right) \right]_{\theta_1}^{\theta_2}$$

$$= -\frac{\mu_0}{4\pi} I \left[\ln \tan\frac{\theta_2}{2} - \ln \tan\frac{\theta_1}{2} \right]$$

$$\frac{\theta_1}{2} + \frac{\theta_2}{2} = \frac{\pi}{2}$$

$$\tan\left(\frac{\theta_1}{2}\right) = \frac{1}{\tan\left(\frac{\theta_2}{2}\right)}$$

$$A_z^{partial} = -\frac{\mu_0}{4\pi} 2I \ln\left(\tan\left(\frac{\theta_2}{2}\right) \right)$$

$$\frac{L/2}{\left[\left(\frac{L}{2}\right)^2 + R^2\right]^{1/2}} = \cos\theta_2 = 2\cos^2\frac{\theta_2}{2} - 1 = 1 - 2\sin^2\frac{\theta_2}{2}$$

$$\cos^2\frac{\theta_2}{2} = \frac{1}{2}\left\{ 1 + \frac{L/2}{\left[\left(\frac{L}{2}\right)^2 + R^2\right]^{1/2}} \right\}$$

$$\sin^2\left(\frac{\theta_2}{2}\right) = \frac{1}{2}\left\{ 1 - \frac{L/2}{\left[\left(\frac{L}{2}\right)^2 + R^2\right]^{1/2}} \right\}$$

$$\tan^2\left(\frac{\theta_2}{2}\right) = \frac{\sqrt{\left(\frac{L}{2}\right)^2 + R^2} - \frac{L}{2}}{\sqrt{\left(\frac{L}{2}\right)^2 + R^2} + \frac{L}{2}}$$

$$A_z^{partial} = -\frac{\mu_0}{4\pi} 2I \ln \tan\left(\frac{\theta_2}{2}\right) = -\frac{\mu_0}{4\pi} I \ln \tan^2\left(\frac{\theta_2}{2}\right)$$

$$A_z^{partial} = -\frac{\mu_0}{4\pi} I \ln\left(\frac{\sqrt{\left(\frac{L}{2}\right)^2 + R^2} - \frac{L}{2}}{\sqrt{\left(\frac{L}{2}\right)^2 + R^2} + \frac{L}{2}} \right)$$

$$= -\frac{\mu_0}{4\pi} I \ln \frac{\frac{L}{2}\left(\sqrt{1 + \frac{4R^2}{L^2}} - 1\right)}{\frac{L}{2}\left(\sqrt{1 + \frac{4R^2}{L^2}} + 1\right)}$$

$$(1+x)^n \simeq 1+nx$$

$$A_z^{\text{partial}} = -\frac{\mu_0}{4\pi} I \ln \frac{\left(1+\frac{2R^2}{L^2}\right)-1}{\left(1+\frac{2R^2}{L^2}\right)+1}$$

$$= -\frac{\mu_0}{4\pi} I \left[\ln\left(2+\frac{2R^2}{L^2}\right) + \ln\left(\frac{2R^2}{L_0^2}\right) + \ln\left(\frac{L_0^2}{L^2}\right) \right]$$

constant, throw it away!

$$A_z = \lim_{L\to\infty} \left[A_z^{\text{partial}} - \text{constant term} \right]$$

$$A_z = \frac{\mu_0}{4\pi} I \left[\ln 2 - \ln\left(\frac{2R^2}{L_0^2}\right) \right]$$

$$\boxed{A_z = \frac{\mu_0}{4\pi} I \ln\left(\frac{L_0^2}{R^2}\right)}$$

$$\vec{r} = R\hat{s} + z\hat{z}$$

$$R^2 = x^2 + y^2$$

$$A_z(\vec{r}) = \frac{\mu_0}{4\pi} I \ln\left(\frac{L_0^2}{x^2+y^2}\right)$$

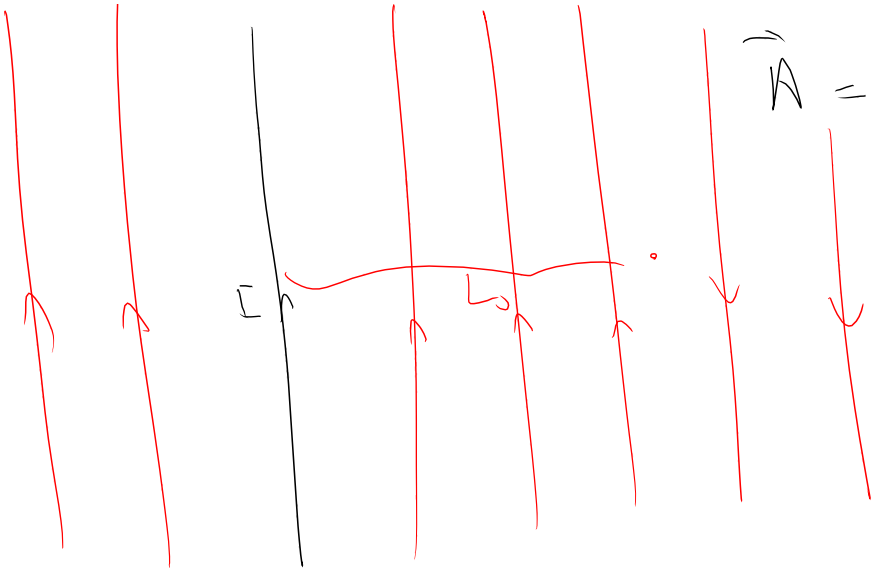
In cylindrical coordinates, $A_s = 0$; $A_\phi = 0$; $A_z \neq 0$

$$\vec{\nabla} \times \vec{A} = \hat{s}\left[\frac{1}{s}\frac{\partial A_z}{\partial \phi}\right] + \hat{\phi}\left[-\frac{\partial A_z}{\partial s}\right] + \hat{z}[0]$$

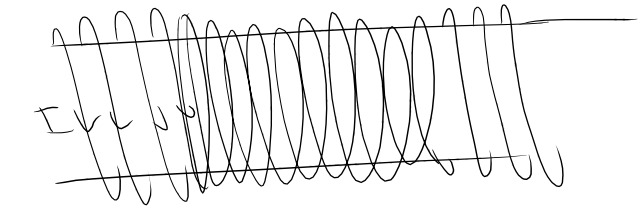
$$= -\frac{\partial}{\partial s}\left[\frac{\mu_0}{4\pi} I \ln\left(\frac{L_0^2}{s^2}\right)\right]\hat{\phi}$$

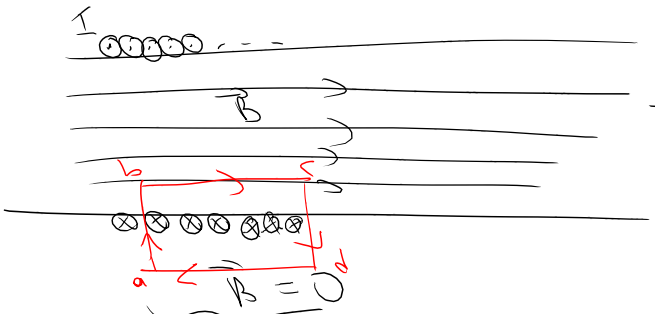
$$= -\frac{\mu_0}{4\pi} I \frac{\partial}{\partial s}\left[-2\ln\left(\frac{s}{L_0}\right)\right]\hat{\phi}$$

$$\vec{\nabla} \times \vec{A} = \frac{\mu_0}{2\pi} I \frac{1}{s}\hat{\phi} = \frac{\mu_0}{2\pi s} I \hat{\phi} = \vec{B}$$



Example Solenoid





$\vec{B}(s, \cancel{\phi}, \cancel{z})$

$\Rightarrow \vec{B} = B(s)\hat{z}$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\oint \vec{B} \cdot d\vec{l} = \int_a^b \vec{B} \cdot d\vec{l} + \int_b^c \vec{B} \cdot d\vec{l} + \int_c^d \vec{B} \cdot d\vec{l} + \int_d^a \vec{B} \cdot d\vec{l}$$

$d\vec{l} = dl\hat{z}$

$$= \int_b^c B(s)\, dl = B(s) L = \mu_0 N I$$

$$B(s) = \mu_0 \left(\frac{N}{L}\right) I$$

$n = \frac{N}{L}$: # of turns per unit length

$B(s) = \mu_0 n I$

$$\vec{B}(\vec{r}) = \begin{cases} \mu_0 n I \hat{z} & \text{if } s < R \\ 0 & \text{if } s > R \end{cases}$$

Assume $\vec{A} = \hat{\phi} A(s)$

$$\vec{\nabla} \times \vec{A} = \left(-\cancel{\frac{\partial A}{\partial z}}^{0} \right) \hat{s} + 0\,\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA)\right]\hat{z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}(sA)\hat{z} = \mu_0 n I \hat{z} \begin{cases} 1 & \text{if } s < R \\ 0 & \text{if } s > R \end{cases}$$