$l=1$ $\quad A_1 - \dfrac{B_1}{R^3} = -\mu_0 \omega R \sigma$

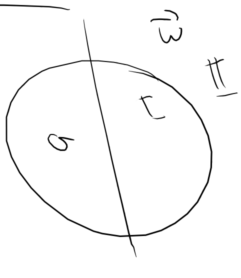
$l \neq 1$ $\quad A_l R^{l-1} - \dfrac{B_l}{R^{l+2}} = 0$

$$-A_l\, l\, R^{l-1} = \frac{B_l}{R^{l+2}}(l+1)$$

$$A_l = 0 \quad B_l = 0 \quad \text{if } l \neq 1$$

Midterm II: December 30, 2016 (if ∃ an empty classroom)

Example



$$\Phi_M^{I} = \sum A_l r^l P_l(\cos\theta)$$

$$\Phi_M^{II} = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\vec{B}^{\text{inside}} - \vec{B}^{\text{outside}} = (\pm)\mu_0\, \hat{n} \times \vec{K}$$

$$\vec{B} = -\nabla \Phi_M$$

$$\nabla^2 \Phi = 0 \iff \Phi = \sum \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

if Φ is indep. of coordinate ϕ

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \implies \vec{B}_{\parallel}$ is discontinuous

$\vec{\nabla} \cdot \vec{B} = 0 \implies \vec{B}_{\perp}$ is continuous.

$$A_1 - \frac{B_1}{R^3} = -\mu_0 \omega R \sigma$$

$$-A_1 = -2\frac{B_1}{R^3} \Rightarrow A_1 = -2\frac{B_1}{R^3}$$

$$-\frac{3B_1}{R^3} = -\mu_0 \omega R \sigma \Rightarrow \boxed{B_1 = \frac{\mu_0}{3}\omega R^4 \sigma}$$

$$A_1 = -2\frac{B_1}{R^3} = -\frac{2}{3}\mu_0 \omega R \sigma$$

$$\Phi^{I} = A_1 r P_1(\cos\theta) = -\frac{2}{3}\mu_0 \omega R \sigma \underbrace{r\cos\theta}$$

$$\vec{B}^{I} = \left(\frac{2}{3}\mu_0 \omega R \sigma\right)\hat{z} = \frac{2}{3}\mu_0 R \sigma \vec{\omega}$$

$$\Phi^{II} = \frac{B_1}{r^2} P_1(\cos\theta) = \frac{\mu_0 \omega}{3} R^4 \sigma \frac{\cos\theta}{r^2}$$

$$= \frac{\mu_0 R^4}{3} \sigma \frac{\vec{\omega}\cdot\vec{r}}{r^3}$$

$$\vec{\omega}\cdot\vec{r} = \omega r \cos\theta$$

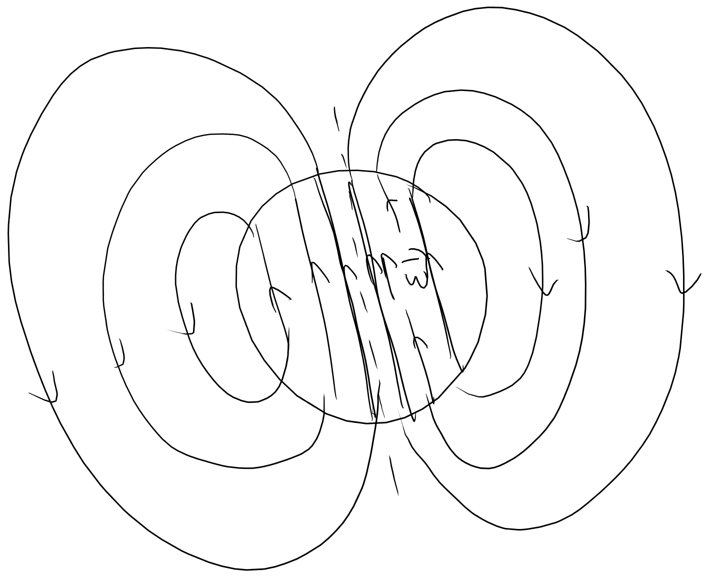
$$\vec{m} = \frac{4\pi R^4}{3} \sigma \vec{\omega}$$

$$\Phi^{II}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m}\cdot\vec{r}}{r^3}$$

compare with potential of an electric dipole

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3}$$

$$\vec{B}^{II}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}$$



$$\Phi^{II}(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec m\cdot\vec r}{r^3}$$

$$\vec B = -\frac{\mu_0}{4\pi}\vec\nabla\left(\frac{\vec m\cdot\vec r}{r^3}\right)$$

$$\left[\vec\nabla\times(\vec a\times\vec b)\right]_i = \varepsilon_{ijk}\,\partial_j\,(\vec a\times\vec b)_k$$

$$= \varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,a_l\,b_m$$

assume $\partial_j A_l = 0$

$$= \varepsilon_{ijk}\,\varepsilon_{klm}\,a_l\,(\partial_j b_m)$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)a_l\,(\partial_j b_m)$$

$$= a_i(\vec\nabla\cdot\vec b) - (\vec a\cdot\vec\nabla)\,b_i$$

$$\left[\vec\nabla\times(\vec a\times\vec b)\right] = \vec a\,(\vec\nabla\cdot\vec b) - (\vec a\cdot\vec\nabla)\,\vec b$$

$$\vec B = -\frac{\mu_0}{4\pi}\vec\nabla\left(\frac{\vec m\cdot\vec r}{r^3}\right) = \vec\nabla\times\left(\frac{\mu_0}{4\pi}\,\vec m\times\frac{\vec r}{r^3}\right)$$

$\vec\nabla\cdot\left(\frac{\vec r}{r^3}\right) = 4\pi\delta^{(3)}(\vec r) = 0$ in region II

$$\left[(\vec a\cdot\vec\nabla)\vec b\right]_i = a_j\,\partial_j\, b_i$$

$$\left[\vec\nabla\,(\vec a\cdot\vec b)\right]_i = \partial_i\, a_j b_j = a_j\,\partial_i\, b_j$$

$$\vec\nabla\times\left(\frac{\vec r}{r^3}\right) = 0 \Longrightarrow \partial_i\left(\frac{\vec r}{r^3}\right)_j = \partial_j\left(\frac{\vec r}{r^3}\right)_i$$

$$(\vec a\cdot\vec\nabla)\,\vec b = \vec\nabla\,(\vec a\cdot\vec b) \quad \text{if } \vec\nabla\times\vec b = 0$$

$$\vec b \equiv \frac{\vec r}{r^3} \;;\; \vec a = \frac{\mu_0}{4\pi}\vec m$$

$$\vec B = -\frac{\mu_0}{4\pi}\vec\nabla\left(\frac{\vec m\cdot\vec r}{r^3}\right) = \frac{\mu_0}{4\pi}\vec\nabla\times\left(\vec m\times\frac{\vec r}{r^3}\right)$$

$$\boxed{\vec A = \frac{\mu_0}{4\pi}\,\vec m\times\frac{\vec r}{r^3}}$$

$$\vec B^{I} = \frac{2}{3}\mu_0 R\sigma\vec\omega \qquad \vec m = \frac{4\pi R^4}{3}\sigma\vec\omega$$

$$\Downarrow$$

$$\sigma\vec\omega R = \frac{\vec m}{\frac{4\pi}{3}R^3}$$

$$\boxed{\vec B^{I} = \frac{2}{3}\mu_0\left(\frac{\vec m}{\frac{4\pi}{3}R^3}\right)}$$

$$\vec\nabla\times(\vec m\times\vec r) = \vec m\,(\vec\nabla\cdot\vec r) - (\vec m\cdot\vec\nabla)\,\vec r$$

$$\vec\nabla\cdot\vec r = \frac{\partial}{\partial x_i}x_i = \delta_{ii} = 3$$

$$\left[(\vec m\cdot\vec\nabla)\,\vec r\right]_i = m_i\frac{\partial}{\partial x_i}x_j = m_i\,\delta_{ij} = m_j$$

$$\vec\nabla\times(\vec m\times\vec r) = \vec m\,3 - \vec m = 2\vec m \Longrightarrow \vec m = \frac{1}{2}\vec\nabla\times(\vec m\times\vec r)$$

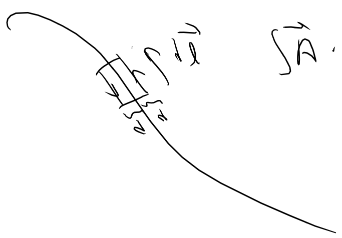
$$\vec{B}^{I} = \frac{2}{3}\mu_0 \frac{\vec{m}}{V} = \frac{2}{3}\mu_0 \frac{1}{V}\frac{1}{2}\vec{\nabla}\times(\vec{m}\times\vec{r})$$

$$= \vec{\nabla}\times\left(\frac{\mu_0}{3V}\vec{m}\times\vec{r}\right)$$

$$\boxed{\vec{A}^{I}(\vec{r}) = \frac{\mu_0}{3V}\vec{m}\times\vec{r}}$$

$$\vec{A}^{II}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{m}\times\frac{\vec{r}}{r^3} = \frac{\mu_0}{3\left(\frac{4\pi}{3}r^3\right)}\vec{m}\times\vec{r}$$

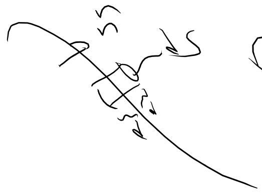
$$\vec{B} = \vec{\nabla}\times\vec{A} \Longrightarrow \oint\vec{A}\cdot d\vec{l} = \int\vec{B}\cdot d\vec{S} = \Phi_B : \text{flux}$$



$$\int\vec{A}\cdot d\vec{l} \xrightarrow[l\to 0]{} \left(\vec{A}^{\text{outside}} - \vec{A}^{\text{inside}}\right)\cdot d\vec{l} = 0$$

$$\vec{\nabla}\cdot\vec{A} = 0$$

$$\oint\vec{A}\cdot d\vec{S} = 0$$



$$0 = \int\vec{A}\cdot d\vec{S} = \left(\vec{A}^{\text{outside}} - \vec{A}^{\text{inside}}\right)\cdot\hat{n}S$$

$$\vec{A}(\vec{r}) = \int\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV' = \int\frac{I\,d\vec{l}}{|\vec{r}-\vec{r}\,'|}$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \xrightarrow[\text{statics}]{} \vec{\nabla}\cdot\vec{J} = 0$$

$$\partial_i J_i = 0$$

$$\vec{\nabla}\cdot(x_i\vec{J}) = \frac{\partial}{\partial x_j}(x_i J_j) = \underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}} J_j + x_i\underbrace{\frac{\partial}{\partial x_j}J_j}_{\vec{\nabla}\cdot\vec{J}=0}$$

$$\boxed{\vec{\nabla}\cdot(x_i\vec{J}) = J_i}$$

$$\int x_i J_j \, dV = \int x_i \vec{\nabla}\cdot(x_j \vec{J})\, dV$$

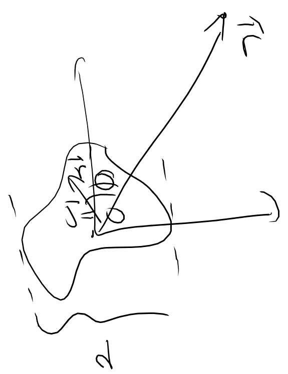
$$= \int x_i \frac{\partial}{\partial x_k}(x_j J_k)\, dV$$

$$= \int \left[\frac{\partial}{\partial x_k}(x_i x_j J_k) - (x_j J_k)\frac{\partial x_i}{\partial x_k}\right] dV$$

$$= \int \vec{\nabla}\cdot(x_i x_j \vec{J})\, dV - \int x_j J_k \delta_{ik}\, dV$$

$$\boxed{\int x_i J_j \, dV = -\int x_j J_i \, dV}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV'$$



$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum \frac{r'^{\,l}}{r^{l}} P_l(\cos\theta)$$

$$= \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV'$$

$$= \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{r}\,')\, dV' + \frac{\mu_0}{4\pi}\frac{1}{r^3}\int \vec{J}(\vec{r}\,')(\vec{r}\cdot\vec{r}\,')\, dV' + \cdots$$

$$\int J_i(\vec{r}\,')\, dV' = \int \vec{\nabla}'\cdot(x_i' \vec{J}(\vec{r}\,'))\, dV' = \int x_i' \vec{J}(\vec{r}\,')\cdot d\vec{S}\,' = 0$$

$$J_i(\vec{r}\,') = \vec{\nabla}'\cdot(x_i' \vec{J}(\vec{r}\,'))$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \int \vec{J}(\vec{r}\,') (\vec{r} \cdot \vec{r}\,') \, dV'$$

$$\int dV' \vec{r} \times (\vec{J}(\vec{r}\,') \times \vec{r}\,') = \int dV' \vec{J}(\vec{r}\,') (\vec{r} \cdot \vec{r}\,') - \int \vec{r}\,' (\vec{J}(\vec{r}\,') \cdot \vec{r}) \, dV'$$

$$\left[\int \vec{r}\,' (\vec{J}(\vec{r}\,') \cdot \vec{r}) \, dV' \right]_i = \int dV' \, x_i' \, x_j \, J_j(\vec{r}\,')$$

$$= x_j \int dV' \, x_i' \, J_j(\vec{r}\,')$$

$$= - x_j \int dV' \, x_j' \, J_i(\vec{r}\,')$$

$$= \left[- \int dV' (\vec{r} \cdot \vec{r}\,') \vec{J}(\vec{r}\,') \right]_i$$

$$\int dV' \vec{r} \times (\vec{J}(\vec{r}\,') \times \vec{r}\,') = 2 \int \vec{J}(\vec{r}\,') (\vec{r} \cdot \vec{r}\,') \, dV'$$

$$\int \vec{J}(\vec{r}\,') (\vec{r} \cdot \vec{r}\,') \, dV' = \vec{r} \times \left(\frac{1}{2} \int dV' \vec{J}(\vec{r}\,') \times \vec{r}\,' \right)$$

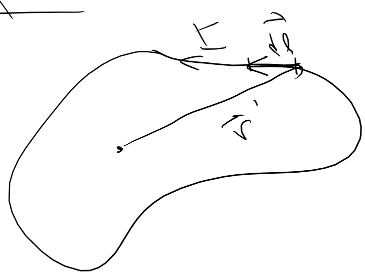
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \int \vec{J}(\vec{r}\,') (\vec{r} \cdot \vec{r}\,') \, dV'$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \vec{r} \times \left(\frac{1}{2} \int dV' \vec{J}(\vec{r}\,') \times \vec{r}\,' \right)$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \vec{m} \times \vec{r}$$

$$\boxed{\vec{m} = \frac{1}{2} \int dV' \vec{r}\,' \times \vec{J}(\vec{r}\,')}$$

Example



$$\vec{m} = \frac{1}{2}\int dV'\, \vec{r}' \times \vec{J}(\vec{r}')$$

$$= \frac{1}{2}\int \vec{r}' \times (I\, \vec{dl})$$

$$= \frac{I}{2}\int \vec{r}' \times \vec{dl}$$

$$dA = \frac{1}{2} r'\, dl \sin\theta$$

$$dA = \frac{1}{2} |\vec{r}' \times \vec{dl}|$$

$$|\vec{r}' \times \vec{dl}| = r'\, dl \sin(\pi - \theta)$$

$$= r'\, dl \sin\theta$$

$$\vec{m} = I \int \vec{dA} = I \int dA\, \hat{n}$$

$$= I A \hat{n}$$

A → area bounded by the loop.