

Green's Thm

$$\underset{B,C}{\int} \nabla^2 \phi = -4 \sup_{\vec{r}} (\vec{r})$$

$$4 \nabla^2 \phi = \vec{\nabla} \cdot (2(\vec{\nabla} \phi)) - (\vec{\nabla} \psi) \cdot (\vec{\nabla} \phi)$$

$$\underset{V}{\int} d^3 r (4 \nabla^2 \phi) = \underset{V}{\int} \vec{\nabla} \cdot (2(\vec{\nabla} \phi)) dr - \underset{V}{\int} (\vec{\nabla} \psi) \cdot (\vec{\nabla} \phi) dr$$

$$\begin{aligned} & \underset{V}{\int} d^3 r (4 \nabla^2 \phi) + \underset{V}{\int} d^3 r (\vec{\nabla} \psi) \cdot (\vec{\nabla} \phi) \\ &= \oint_{\partial V} (4 \vec{\nabla} \phi) \cdot d\vec{s} \\ &= \oint_{\partial V} 4 \frac{\partial \phi}{\partial n} ds \end{aligned}$$

$$\frac{\partial \phi}{\partial n} = \hat{n} \cdot \vec{\nabla} \phi$$

$$\underset{V}{\int} d^3 r (4 \nabla^2 \phi) + \underset{V}{\int} d^3 r (\vec{\nabla} \psi) \cdot (\vec{\nabla} \phi) = \oint_{\partial V} 4 \frac{\partial \phi}{\partial n} ds$$

$$\underset{V}{\int} d^3 r (\phi \nabla^2 \psi) + \underset{V}{\int} d^3 r (\vec{\nabla} \phi) \cdot (\vec{\nabla} \psi) = \oint_{\partial V} \phi \frac{\partial \psi}{\partial n} ds$$

$$\Rightarrow \boxed{\underset{V}{\int} d^3 r (4 \nabla^2 \phi - \phi \nabla^2 \psi) = \oint_{\partial V} \left(4 \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) ds}$$

$$4\nabla^2\phi - \phi\nabla^2u = \vec{\nabla} \cdot (u\vec{\nabla}\phi - \phi\vec{\nabla}u)$$

$$\int d\vec{r}' (4\nabla^2\phi - \phi\nabla^2u) = \oint_{\partial V} \left(u \frac{\partial \phi}{\partial n} - \phi \frac{\partial u}{\partial n} \right) ds'$$

$$\nabla^2\phi = -4\pi g(\vec{r}) + \text{B.C.} \Rightarrow \phi = ?$$

$$\nabla^2 G(\vec{r}, \vec{r}') = -4\pi g \delta^{(3)}(\vec{r} - \vec{r}') \Leftarrow$$

$$\text{Let } u(\vec{r}') = G(\vec{r}, \vec{r}')$$

$$\int d\vec{r}' \left[G(-4\pi g) - \phi(-4\pi g \delta^{(3)}(\vec{r} - \vec{r}')) \right] = \oint_{\partial V} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) ds'$$

$$\phi(\vec{r})_{\text{HSI}} = 4\pi g \int d\vec{r}' G(\vec{r}, \vec{r}') g(\vec{r}') + \oint_{\partial V} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) ds'$$

$$\boxed{\phi(\vec{r}) = \int d\vec{r}' G(\vec{r}, \vec{r}') f(\vec{r}') + \frac{1}{4\pi g} \oint_{\partial V} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) ds'}$$

If ϕ on ∂V is known, set $G = 0$ on the boundary

If $\frac{\partial \phi}{\partial n}$ on ∂V is known, set $\frac{\partial G}{\partial n} = \frac{4\pi g}{V}$ on the boundary

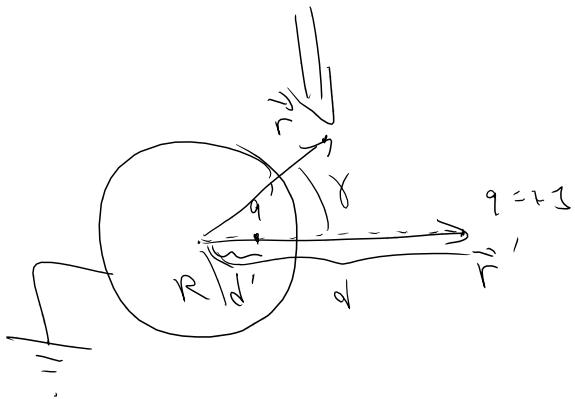
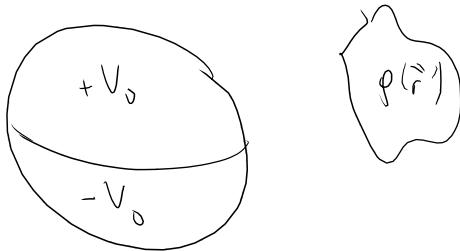
$$\nabla^2 G(\vec{r}, \vec{r}') = -4\pi \delta^{(3)}(\vec{r} - \vec{r}')$$

$$G = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}')$$

$$\nabla^2 G = -4\pi \delta^{(3)}(\vec{r} - \vec{r}') + \nabla^2 F$$

$$\Rightarrow \boxed{\nabla^2 F = 0}$$

Ex



$$\begin{aligned} q' &= -q \frac{R}{2} = -\frac{R}{2} \\ d' &= \frac{R^2}{2} \\ d &= |\vec{r}'| \end{aligned}$$

$$V(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}'|} + \frac{q'}{|\vec{r} - \frac{d' \vec{r}'}{2}|}$$

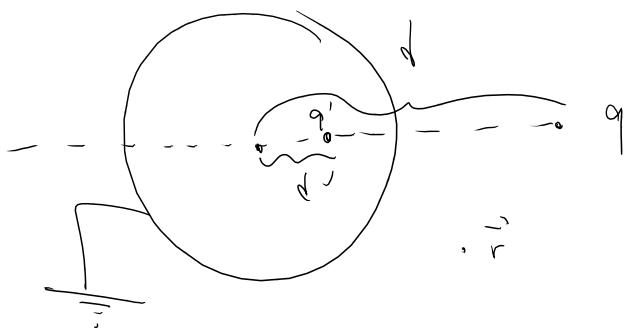
$$|\vec{r} - \vec{r}'| = \sqrt{(\vec{r} - \vec{r}')^2} = \sqrt{r^2 + r'^2 - 2rr' \cos\gamma}$$

$$\begin{aligned} \left| \vec{r} - \frac{d' \vec{r}'}{2} \right| &= \sqrt{\left(\vec{r} - \vec{r}' \frac{d'}{2} \right)^2} = \sqrt{\left(\vec{r} - \vec{r}' \frac{\frac{R^2}{2}}{2} \right)^2} \\ &\approx \left(r^2 + r'^2 \frac{R^2}{4} - 2rr' \frac{R^2}{4} \cos\gamma \right)^{1/2} \end{aligned}$$

$$\left| \vec{r} - d' \frac{\vec{r}'}{r'} \right| = \left(r^2 + \frac{R^4}{r'^2} - 2r \frac{R^2}{r'} \cos\gamma \right)^{1/2}$$

$$\begin{aligned} d &= r' \\ \cos\gamma &\stackrel{?}{=} \cos\theta \cos\phi' - \sin\theta \sin\phi' \cos(\psi - \psi') \end{aligned}$$

Example



$$V(\vec{r}) = \frac{q}{|\vec{r} - \vec{d}|} + \frac{q'}{|\vec{r} - \vec{d}'|}$$

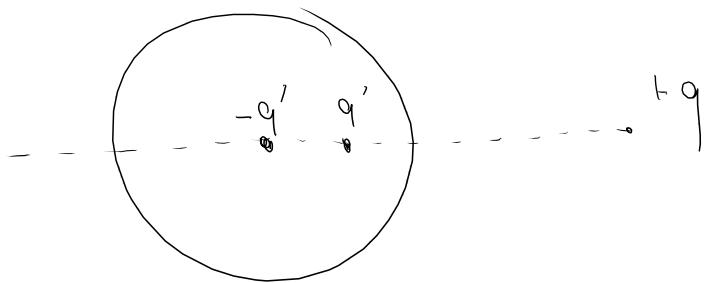
$$\Rightarrow V(\vec{r}) = \frac{q}{\sqrt{r^2 + d^2 - 2rd \cos\gamma}} + \frac{q'}{\sqrt{r^2 + d'^2 - 2rd' \cos\gamma}}$$

$$V(|\vec{r}| = R) = 0 \quad \text{for values of } \gamma$$

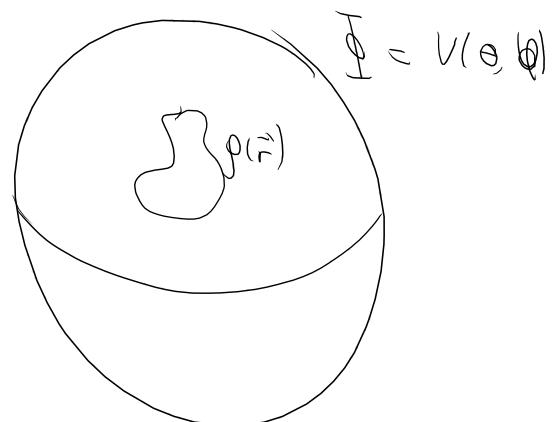
$$1. \quad \frac{q}{d \sqrt{1 + \left(\frac{R}{d}\right)^2 - 2 \frac{R}{d} \cos\gamma}} + \frac{q'}{R \sqrt{1 + \frac{d'^2}{R^2} - 2 \frac{d'}{R} \cos\gamma}}$$

$$2. \quad \frac{R}{d^2} = \frac{d'}{R^2} ; \quad \frac{q}{d} + \frac{q'}{R} = 0 \Rightarrow q' = -q \frac{R}{d}$$

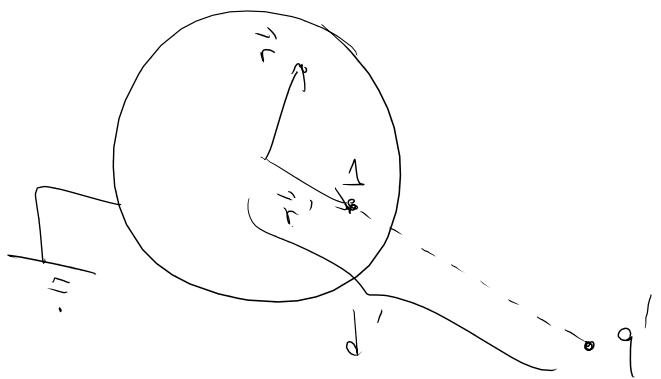
Ex



Ex



What is the Green's Function inside



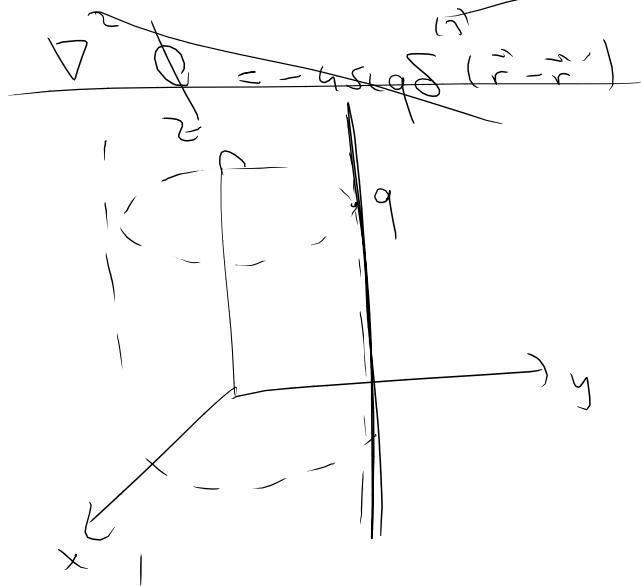
$$d' = \frac{R}{r'} \quad q' = f(\zeta) \frac{R}{r'}$$

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{R}{r'} \frac{1}{|\vec{r} - d' \hat{\vec{r}}'|}$$

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{R}{r'} \frac{1}{\left| \vec{r} - \frac{R^2}{r'} \frac{\vec{r}'}{r'} \right|} = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{R}{\left| \vec{r}' \vec{r} - R^2 \vec{r}' \vec{r}' \right|}$$

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{R}{(r'^2 r^2 + R^4 - 2r' R^2 r \cos\gamma)^{1/2}}$$

Example $\frac{1}{|\vec{r} - \vec{r}'|}$ in cylindrical coordinates



$$\sum_l (l) Y_{lm}(sr) Y_{lm}(sr') = P_l(sr)$$

inside & outside the cylinder, $\nabla^2 \phi = 0$

$$\nabla^2 \phi = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial \phi}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\Phi(s, \varphi, z) = S(s) P(\varphi) Z(z)$$

$$0 = \frac{1}{s} \nabla^2 \phi = \frac{1}{S} + \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial S}{\partial s} \right) + \frac{1}{P} \frac{1}{s^2} \frac{\partial^2 P}{\partial \varphi^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = C$$

doesn't depend on z depends only on z

$$\Rightarrow \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = C_1$$

$$\frac{1}{S} - \frac{1}{s} \frac{\partial^2}{\partial s^2} \left(s \frac{\partial S}{\partial s} \right) + \frac{1}{s^2} \left(\frac{1}{P} \frac{\partial^2 P}{\partial p^2} \right) + C_1 = 0$$

only part
 that might
 depend on ψ

$$s^2 \frac{\partial^2}{\partial s^2} \left(\frac{1}{P} \frac{\partial^2 P}{\partial p^2} \right) = -m^2$$

$$s^2 \left[\frac{1}{S} - \frac{1}{s} \frac{\partial^2}{\partial s^2} \left(s \frac{\partial S}{\partial s} \right) - \frac{1}{s^2} m^2 + C_1 \right] = 0$$

$$s^2 \frac{\partial^2}{\partial s^2} \left(\frac{1}{S} \right) + s \frac{\partial^2}{\partial s^2} \left(s \frac{\partial S}{\partial s} \right) - m^2 S + C_1 s^2 S = 0$$

P is independent of $\psi \Rightarrow C_1 = 0$

$$s^2 \frac{\partial^2}{\partial s^2} \left(\frac{1}{S} \right) + s \frac{\partial^2}{\partial s^2} \left(s \frac{\partial S}{\partial s} \right) - m^2 S = 0$$

$$S = s^n$$

$$n(n-1) s^{n-2} + n s^{n-1} - m^2 s^n = 0$$

$$n^2 - m^2 = 0 \Rightarrow n = \pm m$$

$$S = a_m s^m + \frac{b_m}{s^m} \quad \text{if } m \neq 0$$

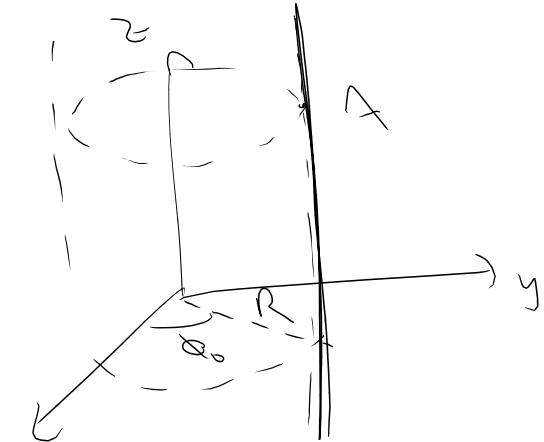
$$0 = s \frac{\partial^2}{\partial s^2} \left(\frac{1}{S} \right) + \frac{\partial}{\partial s} \left(\frac{1}{S} \right) = \frac{1}{s} \left(s \frac{\partial^2}{\partial s^2} \left(\frac{1}{S} \right) \right)$$

$$s \rightarrow \frac{C}{s} = a_0$$

$$\frac{d \ln s}{ds} = \frac{a_0}{s} \Rightarrow \ln s = a_0 \ln \left(\frac{s}{s_0} \right)$$

$$U = a_0 \ln s - \underbrace{a_0 \ln s_0}_{b_0}$$

$$\boxed{\Phi(s, \phi) = b_0 \ln \left(\frac{s}{s_0} \right) + \sum_{m=1}^{\infty} \left(a_m s^m + b_m \frac{1}{s^m} \right) \cos(n\phi + \delta_m)}$$



$$\begin{aligned} \Phi_{\text{inside}} &= b_0 + \sum_{m=1}^{\infty} a_m s^m \cos(n\phi + \delta_m) \\ \Phi_{\text{outside}} &= b_0 \ln \left(\frac{s}{s_0} \right) + \sum_{m=1}^{\infty} \left(a_m s^m + b_m \frac{1}{s^m} \right) \cos(n\phi + \delta_m) \end{aligned}$$

$$\Rightarrow \tilde{\Phi}_{\text{outside}} - \tilde{\Phi}_{\text{inside}} = h \ln \Omega n$$

$$\sigma(\phi, z) = \lambda \tilde{\Phi}(\phi - \phi_0)$$

$$\tilde{b}_0 + \sum_{n=1}^{\infty} a_n R^n \cos(n\phi + \delta_n) = b_0 \ln \left(\frac{R}{s_0} \right) + \sum_{n=1}^{\infty} \left(\tilde{a}_n R^n + \tilde{b}_n \frac{1}{R^n} \right) \cos(n\phi + \tilde{\delta}_n)$$

$$\boxed{\begin{aligned} \tilde{b}_0 &= b_0 \ln \left(\frac{R}{s_0} \right) \\ \tilde{a}_n R^n &= \tilde{a}_n R^n + \tilde{b}_n \frac{1}{R^n} \\ \tilde{\delta}_n &= \delta_n \end{aligned}}$$

$$\left(-\frac{\partial \phi}{\partial s} \right)_{s \rightarrow R^+} - \left(-\frac{\partial \phi}{\partial s} \right)_{s \rightarrow R^-} = 4\pi \lambda \delta(\phi - \phi_0)$$

$$\left. -\frac{\partial \phi}{\partial s} \right|_{s \rightarrow R^-} = -\frac{\partial}{\partial s} \left[b_0 + \sum_{m=1}^{\infty} a_m s \cos(m\phi + \delta_m) \right]$$

$$\boxed{\left. -\frac{\partial \phi}{\partial s} \right|_{s=R^-} = -\sum_{m=1}^{\infty} a_m R^{n-1} m \cos(n\phi - \delta_m)}$$

$$\left. -\frac{\partial \phi}{\partial s} \right|_{s=R^+} = -\frac{\partial}{\partial s} \left[b_0 \ln\left(\frac{s}{s_0}\right) + \sum_{n=1}^{\infty} \left(\tilde{a}_n R^{n-1} - n \tilde{b}_n \frac{1}{R^{n+1}} \right) \cos(n\phi + \delta_n) \right]_{s=R}$$

$$\left(-\frac{\partial \phi}{\partial s} \right)_{s=R^+} - \left(-\frac{\partial \phi}{\partial s} \right)_{s=R^-}$$

$$= -\frac{b_0}{R} - \sum_{n=1}^{\infty} n \left(\tilde{a}_n R^{n-1} - \frac{\tilde{b}_n}{R^{n+1}} + a_n R^{n-1} \right) \cos(n\phi + \delta_n)$$

$$= 4\pi \lambda \delta(\phi - \phi_0)$$

$$\boxed{\delta_n = -n\phi_0}$$

$$\int_0^{\pi} d\phi \cos(n(\phi - \phi_0)) \cos(n(\phi - \phi_0)) = \frac{\pi}{2} \delta_{nn}$$

$$\int_0^{\pi} d\phi \cos(n(\phi - \phi_0)) \delta(\phi - \phi_0) = \cos(0) = 1$$

$$\int dx \delta(x - x_0) f(x) = f(x_0)$$

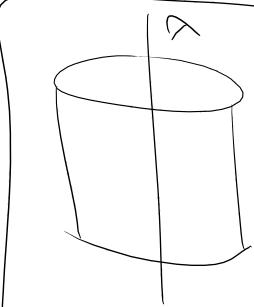
$$-\frac{b_0}{R} = 4\pi\lambda \Rightarrow b_0 = -4\pi R \lambda$$

$$-\ln \left(\tilde{a}_n R^{n-1} - \frac{\tilde{b}_n}{R^{n+1}} + \tilde{a}_m R^{m-1} \right) \frac{s}{2} = 4\pi\lambda$$

$$\tilde{b}_0 = b_0 \ln \left(\frac{R}{s_0} \right)$$

$$\tilde{a}_n R^n = \tilde{a}_n R^n + \tilde{b}_n \frac{1}{R^m} \quad \Downarrow$$

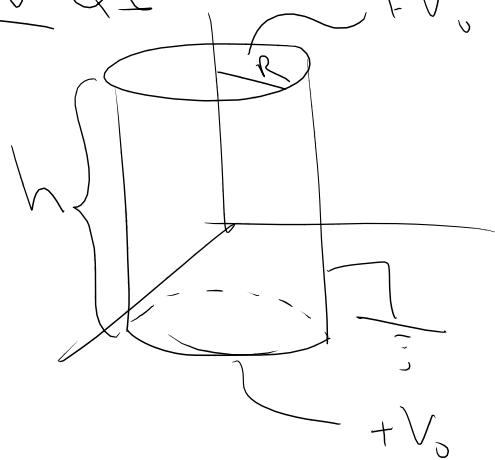
$$\Phi(s, \theta) \xrightarrow[s \rightarrow \infty]{} 2\pi \ln \left(\frac{s}{s_0} \right) \Rightarrow \begin{cases} \tilde{a}_n = 0 \\ b_0 = 2\pi \ln \frac{s_0}{s_0} \end{cases}$$



$$\vec{\Phi} \cdot \vec{r} = 2\pi s h \vec{E} = 4\pi h \lambda$$

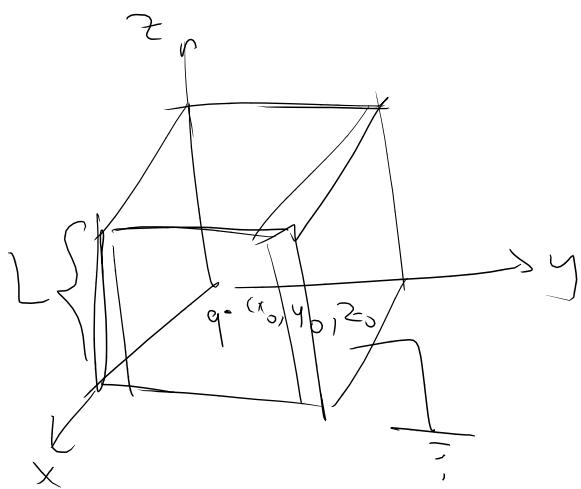
$$\vec{E} = \frac{2}{s} \vec{\lambda}$$

HW Q1



$\vec{\Phi}$ inside the cylinder = ?

HW Q2



$$\nabla^2 \Phi = 0 \quad z < z_0 \\ z > z_0$$

$$\Phi = X(x)Y(y)Z(z)$$

$$\frac{\partial^2 X}{\partial x^2} = -n^2 X ; \quad \frac{\partial^2 Y}{\partial y^2} = -n^2 Y$$

$$\frac{\partial^2 Z}{\partial z^2} = (n^2 + n^2) Z$$

$$\Phi_{z < z_0} = \sum_{nm} a_{nm} \sin \frac{n\pi l}{L} x \sin \frac{m\pi l}{L} y \left(e^{\frac{\sqrt{n^2 + m^2}}{L} z} - e^{-\frac{\sqrt{n^2 + m^2}}{L} z} \right)$$

$$\Phi_{z > z_0} = \sum_{nm} b_{nm} \sin \frac{n\pi l}{L} x \sin \left(\frac{m\pi l}{L} y \right) \left(e^{\frac{\sqrt{n^2 + m^2}}{L} (z-L)} - e^{-\frac{\sqrt{n^2 + m^2}}{L} (z-L)} \right)$$

$$\lim_{z \rightarrow z_0^-} \Phi = \lim_{z \rightarrow z_0^+} \Phi$$

$$\vec{E}_{\text{outside}} - \vec{E}_{\text{inside}} = 4\pi q \delta(x-x_0) \delta(y-y_0) \hat{z}$$