Magnetostatics

charges are moving

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \vec{j}}{\partial t} = 0, \ \dots \ \frac{\partial}{\partial t} = 0$$

$\Downarrow$

$$\vec{B} = k\,\frac{3\vec{m} - (\vec{m}\cdot\hat{r})\hat{r}}{r^3} \Rightarrow \alpha\vec{B} = (\alpha k)\,\frac{3\vec{m} - (\vec{m}\cdot\hat{r})\hat{r}}{r^2}$$

$$\vec{\tau} = k'(\vec{m}\times\vec{B}) = \left(\frac{k'}{\alpha}\right)\vec{m}\times(\alpha\vec{B})$$

$$mB, \quad \frac{B}{m}$$

i)

$z \otimes$

$$\vec{\tau} = k' m B \sin\theta\, \hat{z}$$

$$(\vec{L})_z = I(-\dot{\theta})$$

$$\vec{L} = I\dot{\theta}(-\hat{z})$$

$$\frac{dL_z}{dt} = \tau_z$$

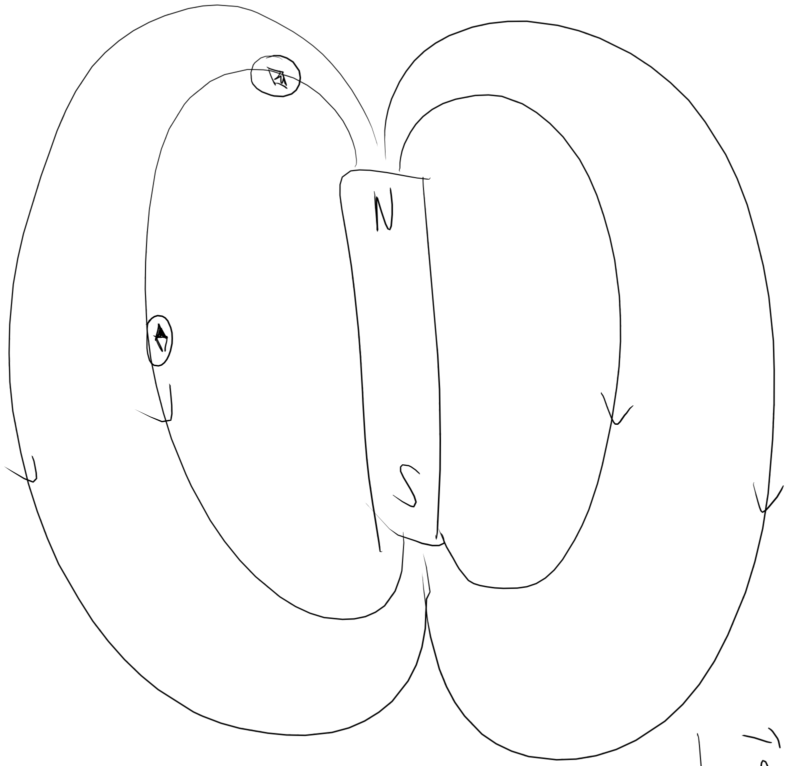
$$\boxed{-I\ddot{\theta} = k' m B \sin\theta} \approx k m B \theta$$

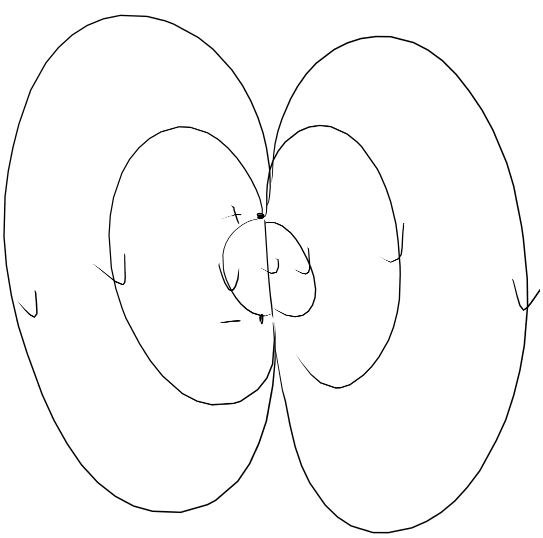
$$\theta = \theta_0 \cos(\omega t + \delta)$$

$$\boxed{\ddot{\theta} = -\frac{k'}{I} m B \theta}$$

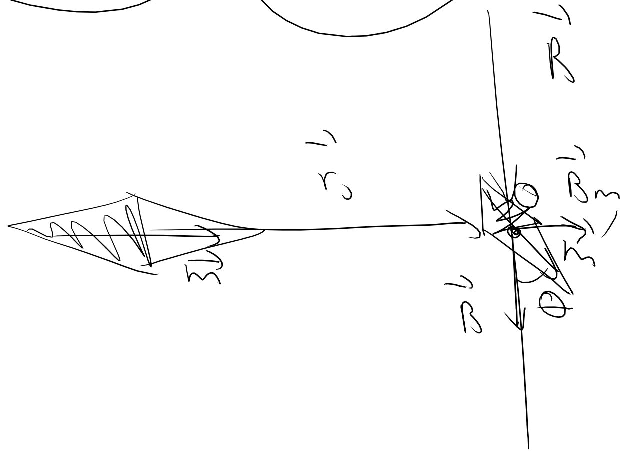
$$\omega^2 = \frac{k'}{I} m B \Rightarrow \boxed{mB = \frac{I}{k'}\omega^2}$$

$$\omega \neq \dot{\theta}$$





ii)



$$\vec B_m = k\,\frac{3\vec m - (\vec m \cdot \hat r_0)\,\hat r_0}{r_0^3} = k\,\frac{3\vec m - m\hat m}{r_0^2}$$

$$\hat r_0 = \hat m \qquad\qquad = 2\vec m\,\frac{k}{r_0^2}$$

$$\vec m = m\hat m$$

$$\tan\theta = \frac{2\frac{k}{r_0^2}m}{B} \Rightarrow \begin{cases} \boxed{\dfrac{m}{B} = \dfrac{r_0^3}{2k}\tan\theta} \\ \boxed{mB = \dfrac{I}{k'}\omega^2} \end{cases}$$

$$\boxed{\begin{aligned} m^2 &= \frac{r_0^2 I\omega^2}{2kk'}\tan\theta \\ B^2 &= \frac{I\omega^2}{r_0^3\tan\theta}\,\frac{2k}{k'} \end{aligned}}$$

$$[\vec{B}] = [\vec{E}]$$

$$u_E = \frac{1}{8\pi}\vec{E}^2$$

$$[u_E] = \frac{J}{m^3} \Rightarrow [\vec{E}^2] = \frac{J}{m^3}$$

$$\left[\frac{I\omega^2}{r_0^3\tan\theta}\right] = \frac{kg\, m^2 \frac{1}{s^2}}{m^3} = \frac{kg\left(\frac{m}{s}\right)^2}{m^2} = \frac{J}{m^3}$$

$$[\vec{B}^2] = [\vec{E}^2]\left[\frac{k}{k'}\right]$$

Let $[k] = [k']$

choose $\boxed{k = k'}$

$$\vec{B} = k\,\frac{3\vec{m} - (\vec{m}\cdot\hat{r})\hat{r}}{r^3} \qquad [\vec{B}^2] = \frac{J}{m^3}$$

$$\vec{\tau} = k(\vec{m}\times\vec{B})$$

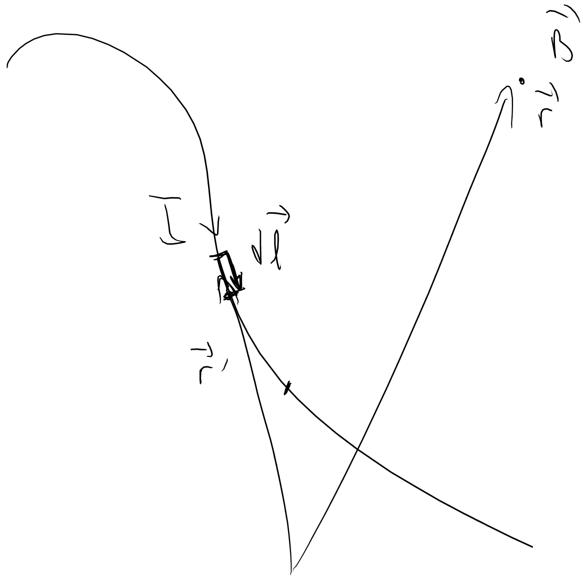
$$[\vec{B}] = [k]\,\frac{[m]}{m^3} = \sqrt{\frac{J}{m^3}}$$

$$\boxed{[k]^2[m]^2 = J\, m^3}$$

$$J = [\vec{\tau}] = [k]\,[m]\sqrt{\frac{J}{m^3}}$$

$$\boxed{J\, m^3 = [k]^2[m]^2}$$

$$\boxed{k = 1}$$



$$\boxed{d\vec{B} = kI \frac{d\vec{l} \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}$$

$$[B] = [k] \frac{[q]}{[t]} \frac{[dl]}{[l^2]}$$

$$\cancel{[B]} = [k] \frac{[l]}{[t]} \cancel{[E]}$$

$$[k] = \frac{1}{[l/t]} = \frac{1}{[v]}$$

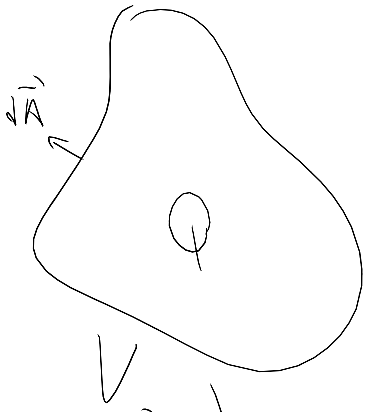
$$\boxed{k = \frac{1}{c}}$$

$$I = \lim_{t \to 0} \frac{dq}{dt}$$

$$J = \frac{dI}{dA} = \frac{dq}{dt\, dA}$$

$$\vec{J} = J\, d\hat{A}$$

$$\boxed{I = \int \vec{J} \cdot d\vec{A}}$$



$$Q = \int_V \rho(\vec{r}, t)\, d^3\vec{r}$$

$$\frac{dQ}{dt} = -\int_{\partial V} \vec{J} \cdot d\vec{A}$$

$$\frac{d}{dt}\left(\int \rho(\vec{r}, t)\, d^3 r\right) = \int_V \frac{\partial \rho(\vec{r}, t)}{\partial t}\, d^3 r$$

$$= -\int_{\partial V} \vec{J} \cdot d\vec{A} = -\int_V (\vec{\nabla} \cdot \vec{J})\, d^3 r$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) d^3 r = 0$$

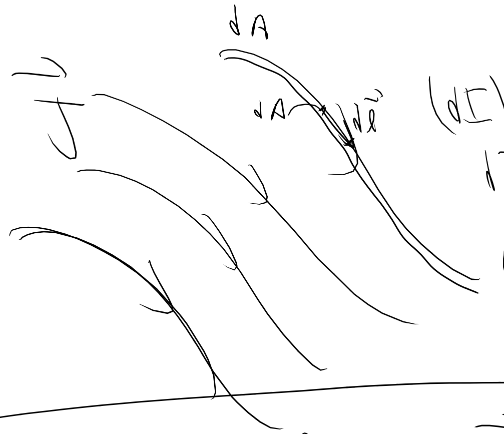
$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

continuity eqn

in statics ; $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = 0}$

$$d\vec{B} = \frac{1}{c} I \frac{d\vec{l} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

$$\vec{B} = \frac{1}{c} \int I \, d\vec{l} \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$



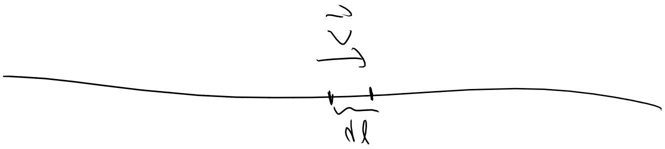
$(dI) = J \, dA$

$d\vec{l} \parallel \vec{J}$

$(dI)\, d\vec{l} = J \, dA \, d\vec{l} = \vec{J} \, dA \, dl$

$$\boxed{\vec{B} = \frac{1}{c} \int dV \, \vec{J} \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}}$$

$$\boxed{\vec{J} = q \vec{v}_0 \, \delta^{(3)}(\vec{r} - \vec{r}_0)}$$



dt

$dl = v\, dt$

$dq = v\, dt\, \rho$

$I = \frac{dq}{dt} = v\rho \frac{dt}{dt} = \rho v \Leftarrow \rho$ linear charge density of moving charges

$\boxed{\vec{J} = \rho \vec{v}} \Longleftarrow \rho$ volume charge density.

$$\vec{B} = \frac{1}{c} \int dV \; \vec{J}(\vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

for a point charge

$$\vec{B} = \frac{1}{c} \int dV' \; q\vec{v}_0 \, \delta^{(3)}(\vec{r}' - \vec{r}_0) \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$= \frac{1}{c} \, q\vec{v}_0 \times \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3}$$

approximate

$$= \left(\frac{\vec{v}_0}{c}\right) \times \underbrace{\left(q \, \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3}\right)}_{\vec{E}(\vec{r})}$$

$$\boxed{\vec{B} = \frac{\vec{v}_0}{c} \times \vec{E}(\vec{r})}$$

correct if $\vec{v}_0$ is const.

$$\rho = q \, \delta^{(3)}(\vec{r} - \vec{r}_0)$$

$$\frac{\partial \rho}{\partial t} \neq 0$$

$$\vec{B} = \frac{1}{c} \int dV' \, \vec{J}(\vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = -\nabla_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} = + \vec{\nabla}_{\vec{r}'} \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\vec{B}(\vec{r}) = \frac{1}{c} \int dV' \left[\vec{J}(\vec{r}') \times \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \right]$$

$$B_i(\vec{r}) = \frac{1}{c} \varepsilon_{ijk} \int dV' \, J_j(\vec{r}') \frac{\partial}{\partial x_k'} \frac{1}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{c} \varepsilon_{ijk} \int dV' \left[\frac{\partial}{\partial x_k'} \left(J_j(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \right) \right.$$

$$\left. - \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial}{\partial x_k'} J_j(\vec{r}') \right]$$

$$\vec{B} = \frac{1}{c} \int dV' \left[-\vec{\nabla} \times \left(\frac{\vec{J}}{|\vec{r} - \vec{r}'|} \right) + \frac{\vec{\nabla}_{r'} \times \vec{J}}{|\vec{r} - \vec{r}'|} \right]$$

$$\vec{B} = \frac{1}{c}\int dV' \, \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$= -\frac{1}{c}\int dV' \, \vec{J}(\vec{r}') \times \vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$= +\frac{1}{c}\vec{\nabla} \times \int dV' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{A} = \frac{1}{c}\int dV' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

$\vec{A}$: vector potential

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

$$\vec{B} = \frac{1}{c}\int dV' \, \vec{J}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\int dV' \, \vec{\nabla}_r \times \left(\frac{\vec{J}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right)$$

$$(\vec{\nabla} \times \vec{B})_i = \left[\frac{1}{c}\int dV' \, \vec{\nabla} \times \left(\vec{J}' \times f(\vec{r}-\vec{r}')\right)\right]_i$$

$$\vec f(\vec r) = \frac{\vec r}{r^3} \qquad \vec\nabla\cdot\vec f = 4\pi\,\delta^{(3)}(\vec r)$$

$$\left[\vec\nabla\times\left(\vec J'\times\vec f\right)\right]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\vec J'\times\vec f\right)_k$$

$$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}J'_l f_m$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_j}J'_l f_m$$

$$= J'_i\left(\vec\nabla\cdot\vec f\right) - \left(\vec J'\cdot\vec\nabla\right)f_i$$

$$= J'_i\,4\pi\,\delta^{(3)}(\vec r-\vec r') - \left(\vec J'\cdot\vec\nabla\right)f_i$$

$$\vec\nabla\times\vec B = \frac{1}{c}\int dV'\,\vec J(\vec r')\,4\pi\,\delta^{(3)}(\vec r-\vec r')$$

$$- \frac{1}{c}\int dV'\left(\vec J(\vec r')\cdot\vec\nabla_{\vec r}\right)\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}$$

$$\vec\nabla\times\vec B(\vec r) = \frac{4\pi}{c}\vec J(\vec r) + \frac{1}{c}\int dV'\,\vec J(\vec r')\cdot\vec\nabla_{\vec r'}\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}$$

$$\left[\int dV'\left(\vec J(\vec r')\cdot\vec\nabla_{\vec r'}\right)\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}\right]_i$$

$$= \int dV'\left(\vec J'\cdot\vec\nabla_{\vec r'}\right)f_i = \int dV'\Big[\overbrace{\vec\nabla_{\vec r'}\cdot\left(\vec J' f_i\right)}^{=0} - \underbrace{\left(\vec\nabla_{\vec r'}\cdot\vec J'\right)}_{=0}f_i\Big]$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0} \iff \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}}$$

$$\boxed{\text{in SI units } \sqrt{\varepsilon_0\mu_0} = \frac{1}{c}}$$

$$\vec{B} = -\frac{1}{c}\int dV' \, \overbrace{\vec{J}(\vec{r}\,') \times \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|}}^{0}$$

$$B_i = -\frac{1}{c}\varepsilon_{ijk}\int dV' J'_j \frac{\partial}{\partial x_k}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= -\frac{1}{c}\varepsilon_{ijk}\frac{\partial}{\partial x_k}\underbrace{\int dV' \frac{J'_j}{|\vec{r}-\vec{r}\,'|}}_{cA_j}$$

$$= +\,\underbrace{\varepsilon_{ikj}\frac{\partial}{\partial x_k}A_j}$$

$$= (\vec{\nabla}\times\vec{A})$$