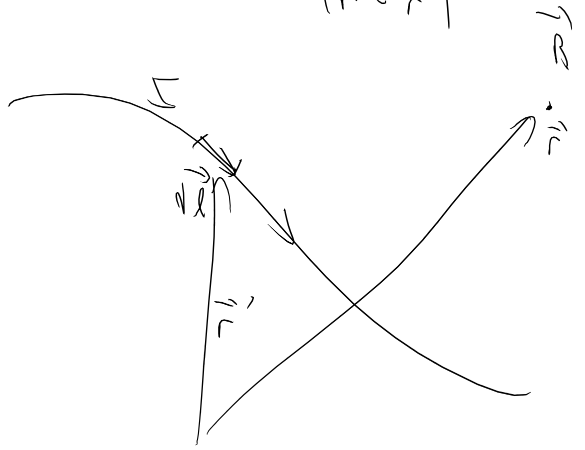
$$\vec B(\vec r) = \frac{1}{c} I \int \frac{d\vec l \times (\vec r - \vec r\,')}{|\vec r - \vec r\,'|^3} = \frac{1}{c}\int dV' \, \frac{\vec J(\vec r\,') \times (\vec r - \vec r\,')}{|\vec r - \vec r\,'|^3}$$



$$\vec B = \vec\nabla \times \underbrace{\left(\frac{1}{c}\int dV' \, \frac{\vec J(\vec r\,')}{|\vec r - \vec r\,'|}\right)}_{\vec A} \Rightarrow \vec B = \vec\nabla \times \vec A$$

$$\boxed{\begin{aligned} \vec\nabla \cdot \vec B &= 0 \\ \vec\nabla \times \vec B &= \frac{4\pi}{c}\vec J \end{aligned}} \longrightarrow \vec B = \vec\nabla \times \vec A$$

gauge transformation

$\Downarrow$

replace $\vec A \to \vec A + \vec\nabla \Lambda = \vec A'$

$$\vec\nabla \times \vec A' = \vec\nabla \times \vec A + \underbrace{\vec\nabla \times \vec\nabla \Lambda}_{=0}$$

$$\vec\nabla \times \vec A' = \vec\nabla \times \vec A = \vec B$$

$$\vec\nabla \times (\vec\nabla \times \vec A') = \vec\nabla(\vec\nabla \cdot \vec A') - \nabla^2 \vec A' = \frac{4\pi}{c}\vec J$$

$$\vec\nabla \cdot \vec A' = \vec\nabla \cdot \vec A + \nabla^2 \Lambda = 0$$

$$\Rightarrow \nabla^2 \Lambda = -\vec\nabla \cdot \vec A$$

$$\Rightarrow \exists \Lambda \text{ st } \vec{\nabla}\cdot\vec{A}' = 0$$

$$\nabla^2 \vec{A}' = -\frac{4\pi}{c}\vec{J} \Leftarrow$$

$\vec{\nabla}\cdot\vec{A}' = 0$: Coulomb Gauge

$$\nabla^2 A_{x,y,z} = -\frac{4\pi}{c} J_{x,y,z}$$

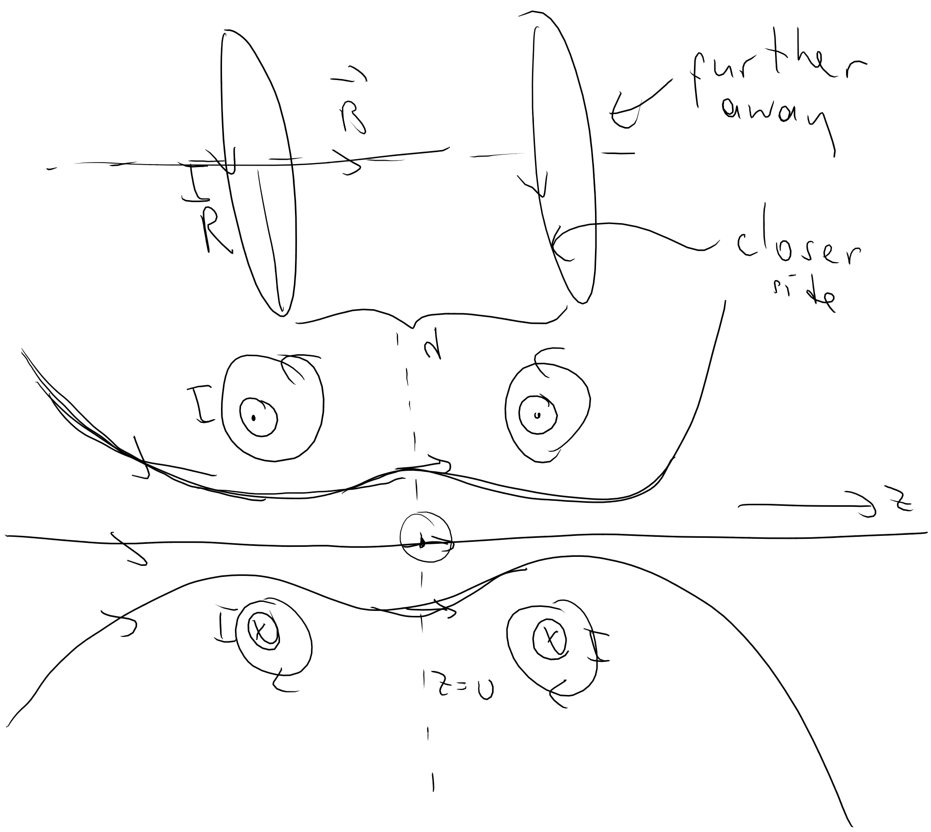
$$\Rightarrow \boxed{A_{x,y,z} = \int dV' \frac{J_{x,y,z}(\vec{r}')/c}{|\vec{r}-\vec{r}'|}}$$

$$A_{r,\phi,\theta} \neq \int dV' \frac{J_{r,\phi,\theta}(\vec{r}')/c}{|\vec{r}-\vec{r}'|}$$

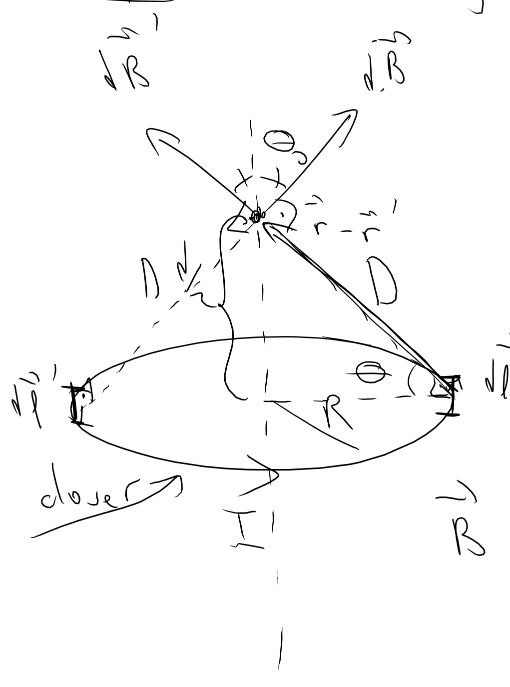
$$A_\phi(\vec{r}) = \vec{A}\cdot\hat{\phi} = \frac{1}{c}\int dV' \frac{\vec{J}(\vec{r}')\cdot\hat{\phi}(\vec{r})}{|\vec{r}-\vec{r}'|}$$

$$\vec{J}(\vec{r}')\cdot\hat{\phi}(\vec{r}) \neq \vec{J}(\vec{r}')\cdot\hat{\phi}(\vec{r}') = J_\phi(\vec{r}')$$

Example Helmholtz Coils



Example single ring



$$\vec{B} = \frac{1}{c} I \int d\vec{l} \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$|d\vec{B}| = \frac{1}{c} I \frac{dl}{|\vec{r}-\vec{r}'|^2} = \frac{I}{c} \frac{dl}{D^2}$$

$$|d\vec{B}'| = |d\vec{B}|$$

$$\vec{B} = B\hat{z}$$

$$= \hat{z} \frac{1}{c} I \int \frac{dl}{D^2} \cos\theta_0$$

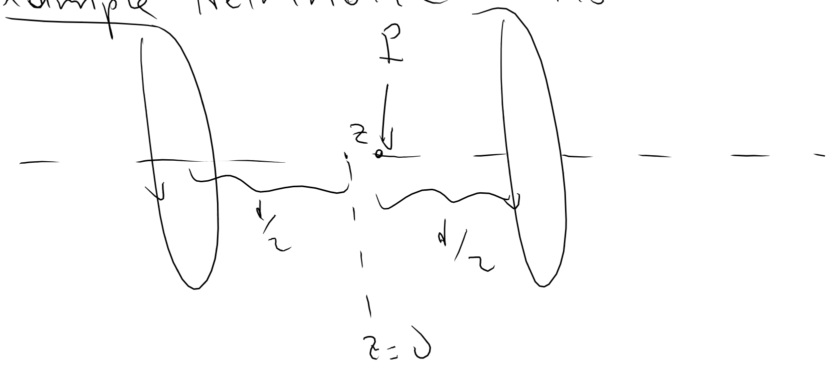
$$\vec{B} = \frac{I}{c} \frac{R}{D^3} \underbrace{2\pi R}\, \hat{z}$$

$$\boxed{\vec{B} = 2\frac{I(\pi R^2)}{c}\frac{\hat{z}}{D^3}} \qquad \boxed{(\pi R^2) I \hat{z} \equiv \vec{\mu}}$$

$$\vec{B} = \frac{2}{c}\frac{\vec{\mu}}{D^3} \qquad \text{compare} \qquad \vec{E} = \frac{2\vec{d}}{D^3}$$

$$D = R^2 + \left(\frac{d}{2}\right)^2 \qquad \vec{B} = \frac{2}{c}\frac{\mu}{\left[R^2+h^2\right]^{3/2}}$$

Example Helmholtz Coils



$$|\vec{B}| = \frac{2}{c}\frac{\mu}{\left[R^2 + \left(\frac{d}{2}+z\right)^2\right]^{3/2}} + \frac{2}{c}\frac{\mu}{\left[R^2 + \left(\frac{d}{2}-z\right)^2\right]^{3/2}} \equiv f(z)$$

$$f(z) = f(0) + f''(z=0)\frac{z^2}{2!} + \dots$$

since $f(-z) = f(z)$

$$f'(z) = \frac{2}{c}\mu\left(-\frac{3}{\cancel{2}}\right)\frac{1}{\left[\quad\right]^{5/2}}\cancel{2}\left(\frac{d}{2}+z\right) + \dots$$

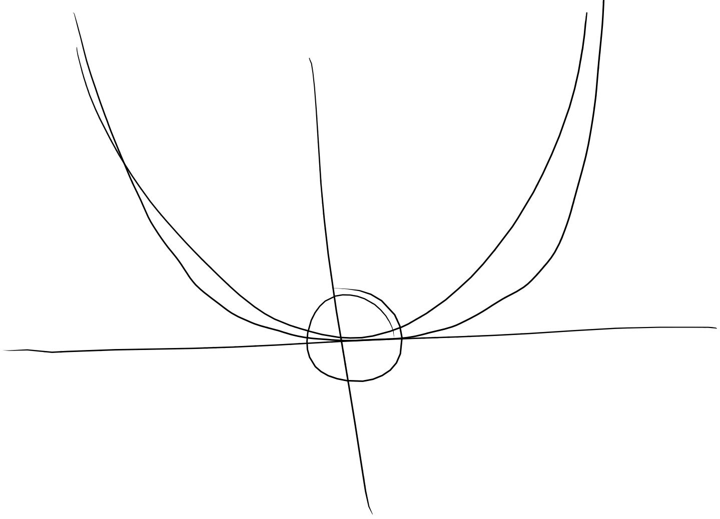
$$f''(z)\Big|_{z=0} = -6\frac{\mu}{c}\left\{\frac{1}{\left[R^2+\left(\frac{d}{2}\right)^2\right]^{5/2}} + \left(\frac{d}{2}\right)\left(-\frac{5}{\cancel{2}}\right)\frac{\cancel{2}\left(\frac{d}{2}\right)}{\left[R^2+\left(\frac{d}{2}\right)^2\right]^{7/2}}\right\}$$

$$f''(z=0) = -6\frac{M}{c}\frac{1}{\left[R^2+\left(\frac{d}{2}\right)^2\right]^{7/2}}\left\{R^2+\left(\frac{d}{2}\right)^2-5\left(\frac{d}{2}\right)^2\right\}$$

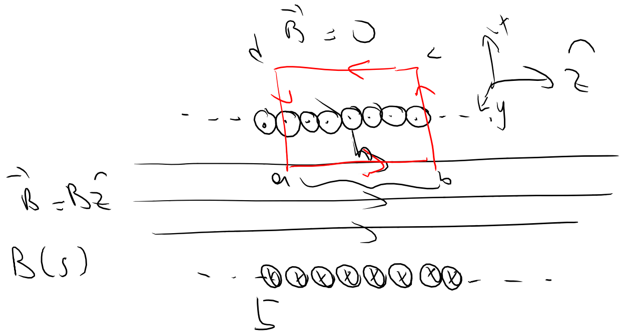
$$= -6\frac{M}{c}\frac{1}{\left(R^2+\left(\frac{d}{2}\right)^2\right)^{7/2}}\left\{R^2-\cancel{4}\,\frac{d^2}{\cancel{4}}\right\}$$

$$f''(z=0) = 0 \quad \text{if} \quad \boxed{R = d}$$

$$|\vec{B}| = |\vec{B}|\Big|_{z=0} + \mathcal{O}(z^4)$$



Solenoid



$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$$

$$\int_A (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} = \int_A \frac{4\pi}{c}\vec{J} \cdot d\vec{A}$$

$$\Rightarrow \oint_{\partial A} \vec{B} \cdot d\vec{l} = \frac{4\pi}{c}\int_A \vec{J} \cdot d\vec{A}$$

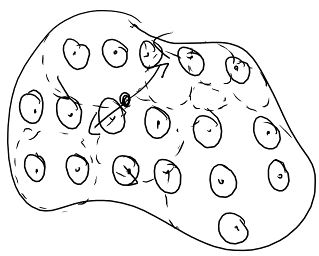
$$\int_{\partial A} \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{enc}$$

$$\oint \vec{B} \cdot d\vec{l} = \int_a^b \vec{B} \cdot d\vec{l} + \overset{0}{\cancel{\int_b^c \vec{B} \cdot d\vec{l}}} + \overset{0}{\cancel{\int_c^d \vec{B} \cdot d\vec{l}}} + \overset{0}{\cancel{\int_d^a \vec{B} \cdot d\vec{l}}}$$

$$= Bh = \frac{4\pi}{c} I N$$

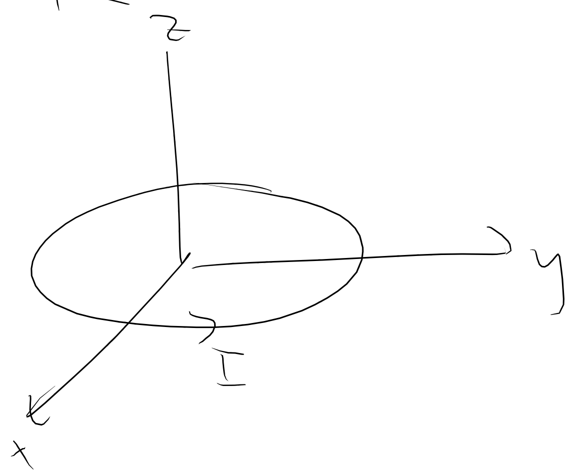
$$B = \frac{4\pi}{c} I \underbrace{\left(\frac{N}{h}\right)}_{= n}$$

$$\boxed{B(s) = \frac{4\pi}{c} I n}$$



Vector Potential

Example



$$\vec{A}(\vec{r}) = \frac{1}{c}\int dV' \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\Uparrow$$

$$\vec{J}(\vec{r}) = \oint \delta(r-R)\,\delta(z)\,I\,D$$

$D = ?$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{c}\int dV'\ \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)$$

$$= \frac{1}{c}\int dV'\ \vec{J}(\vec{r}\,')\cdot\vec{\nabla}\ \frac{1}{|\vec{r}-\vec{r}\,'|}$$

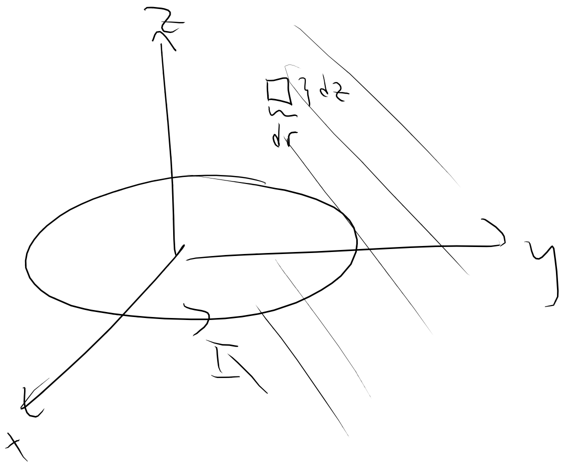
$$= \frac{1}{c}\int dV'\ \vec{J}(\vec{r}\,')\cdot\left(-\vec{\nabla}_{\vec{r}\,'}\right)\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\int dV'\left[\vec{\nabla}'\cdot\left(\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) - \underbrace{\frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}_{\overset{!}{=}0}\right]$$

$$= -\frac{1}{c}\int d\vec{A}'\cdot\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = 0$$

$$\Downarrow$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \qquad \text{in statics } \frac{\partial\rho}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$$



$$\vec{J}(\vec{r}) = \hat{\phi}\, \delta(r-R)\, \delta(z)\, I D$$

$$D = ?$$

integrate $\vec{J}$ over the $x = 0, y > 0$ half plane

$$\int \vec{J}(\vec{r}) \cdot \underbrace{(\hat{x}\, dS)}_{d\vec{S}} = -I \qquad \hat{\phi} = -\hat{x}$$

$$-\cancel{I} = -\int \cancel{I} D\, \delta(r-R)\, dS \qquad \boxed{\vec{J} = \delta(r-R)\, \delta(z)\, I\, \hat{\phi}}$$

$$1 = D \int_0^\infty dr \int_{-\infty}^{\infty} dz\, \delta(r-R) = D$$

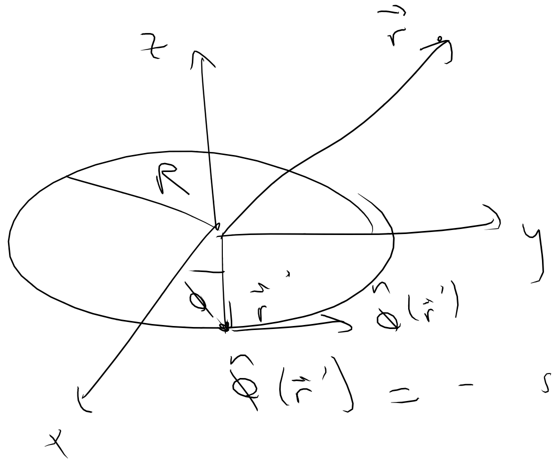
$$\vec{A}(\vec{r}) = \frac{1}{c} \int \delta(r'-R)\, \delta(z')\, I\, \hat{\phi}(\vec{r}')\, \frac{1}{|\vec{r}-\vec{r}'|}\, dV'$$

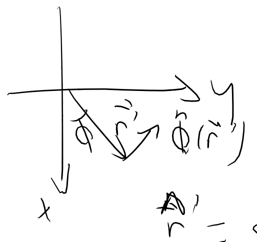
$$\vec{A}(\vec{r}) = \frac{I}{c} \int_{-\infty}^{\infty} dz' \int_0^\infty r'\, dr' \int_0^{2\pi} d\phi'\, \delta(r'-R)\, \delta(z')\, \hat{\phi}(\vec{r}')\, \frac{1}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r}) = \frac{I}{c} R \int_0^{2\pi} d\phi'\, \hat{\phi}(\vec{r}')\, \frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{c} \int \underbrace{(R\, d\phi')\, \hat{\phi}(\vec{r}')}_{d\vec{l}}\, I\, \frac{1}{|\vec{r}-\vec{r}'|} \qquad \vec{J}\, dV \to I\, d\vec{l}$$

$$\vec{A}(\vec{r}) = \frac{I}{c} R \int_0^{2\pi} d\phi' \, \hat{\phi}(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|}$$





$\hat{r}' = \cos\phi \hat{x} + \sin\phi \hat{y}$

$$\hat{\phi}(\vec{r}') = -\sin\phi' \hat{x} + \cos\phi' \hat{y}$$

choose $\vec{r}$ to be on the yz plane $(y>0)$

$$|\vec{r}-\vec{r}'| = |(r\hat{y} + z\hat{z}) - (r'\cos\phi' \hat{x} + r'\sin\phi' \hat{y})|$$

$$= \left[r'^2 \cos^2\phi' + (r - r'\sin\phi')^2 + z^2\right]^{1/2}$$

$$= \left[r'^2 \cos^2\phi' + r^2 - 2rr'\sin\phi' + r'^2 \sin^2\phi' + z^2\right]^{1/2}$$

$$|\vec{r}-\vec{r}'| = \left[r'^2 + r^2 - 2rr'\sin\phi' + z^2\right]^{1/2}$$

$$\vec{A}(\vec{r}) = \frac{I}{c} R \int_0^{2\pi} d\phi' \; \hat{\phi}(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|}$$

$$\hat{\phi}(\vec{r}') = -\sin\phi' \, \hat{x} + \cos\phi' \, \hat{y}$$

$$|\vec{r}-\vec{r}'| = \left[r'^2 + r^2 - 2rr' \sin\phi' + z^2\right]^{1/2}$$

$$\vec{A}(\vec{r}) = \frac{I}{c} R \int_{-\pi}^{\pi} d\phi' \frac{\left(-\sin\phi' \, \hat{x} + \cos\phi' \, \hat{y}\right)}{\left[r'^2 + r^2 - 2rr' \sin\phi' + z^2\right]^{1/2}}$$

$$\vec{A}(\vec{r}) = \frac{I}{c} R \int_{-\pi}^{\pi} d\phi' \frac{(-\sin\phi')}{\left[r'^2 + r^2 - 2rr' \sin\phi' + z^2\right]^{1/2}} \hat{x}$$

if $\vec{r}$ is in the yz plane $(y > 0)$

For an arbitrary point

$$\boxed{\vec{A}(\vec{r}) = \hat{\phi} \frac{I}{c} R \int_{-\pi}^{\pi} d\phi' \frac{\sin\phi'}{\left[R^2 + r^2 - 2rR \sin\phi' + z^2\right]^{1/2}}}$$

$$\vec{A}(\vec{r}) = \hat{\phi} \, A(r, z)$$

$$\vec{\nabla} \times \vec{A} = -\frac{\partial A}{\partial z} \hat{r} + \frac{1}{r} \frac{\partial}{\partial r}(rA) \hat{z}$$

$$\boxed{\vec{B} = -\frac{\partial A}{\partial z} \hat{r} + \hat{z}\left[\frac{\partial A}{\partial r} + \frac{1}{r} A\right]}$$

$$\vec{A}(\vec{r}) = \hat{r}\, f(r)$$

$$\vec{\nabla} \times \vec{A} = 0$$

$$\int_{-\pi}^{\pi} d\phi' \frac{\cos\phi'}{\left[r'^2 + r^2 - 2rr'\sin\phi' + z^2\right]^{1/2}}$$

$$= \frac{2}{-2rr'}\left[r'^2 + r^2 - 2rr'\sin\phi' + z^2\right]^{1/2} \Bigg|_{\phi' = -\pi}^{\phi' = +\pi} = 0$$