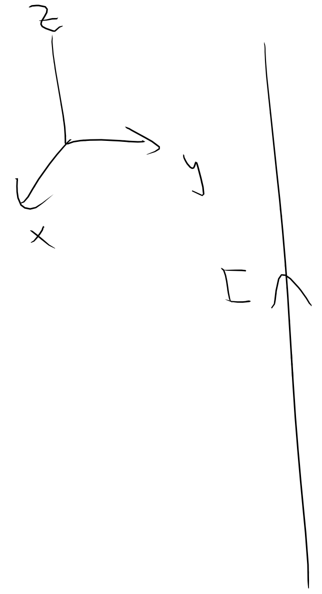
Example $\vec{J}\,dV = I\,d\vec{l}$



$$\vec{A} = \frac{1}{c}\int \frac{\vec{J}(\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|}$$

$\vec{r}$: $\quad \vec{J}(\vec{r}) = \hat{z}\, I\, \delta(x)\,\delta(y)$

$$I \stackrel{?}{=} \int \vec{J}(\vec{r})\cdot \overbrace{(dx\,dy\,\hat{z})}^{d\vec{S}}$$

$$= \int I\,\delta(x)\,\delta(y)\,dx\,dy = I$$

$$A_z(x,y) = \frac{1}{c}\int \frac{I\,dl}{|\vec{r}-\vec{r}'|}$$

$\vec{r} = x\hat{x} + y\hat{y} \qquad |\vec{r}-\vec{r}'| = (x^2+y^2+z'^2)^{1/2}$

$\vec{r}' = z'\hat{z}$

$dl = dz'$

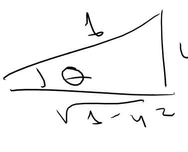
$$A_z = \frac{I}{c}\int_{-L}^{L} \frac{dz'}{(s^2+z'^2)^{1/2}}$$

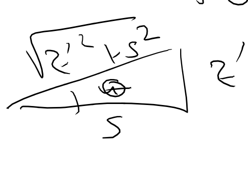
$\rightarrow \boxed{z' = s\tan\theta} \qquad dz' = \frac{s}{\cos^2\theta}\,d\theta$

$$A_z = \frac{I}{c}\int \left(\frac{s}{\cos^2\theta}\right)\frac{1}{s\underbrace{(1+\tan^2\theta)^{1/2}}_{\frac{1}{\cos\theta}}}\,d\theta$$

$$A_z = \frac{I}{c}\int \frac{d\theta}{\cos\theta} \Leftarrow =$$

$\tan\theta = \frac{z'}{s}$

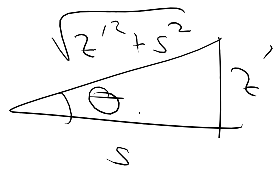




$$u = \frac{z'}{\sqrt{z'^2+s^2}}$$

$$A_z = \frac{I}{c} \int \frac{\cos\theta \, d\theta}{1-\sin^2\theta}$$

$\sin\theta = u$



$$= \frac{I}{c} \int \frac{du}{1-u^2}$$

$\int \frac{1}{1-u} du = -\ln(1-u)$

$u = \sin\theta = \frac{z'}{\sqrt{z'^2+s^2}}$

$$= \frac{I}{c} \int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$$

$$= \frac{I}{c} \frac{1}{2} \ln\left(\frac{1+u}{1-u}\right) \qquad u = \frac{z'}{\sqrt{z'^2+s^2}}$$

$$A_z = \frac{I}{c} \frac{1}{2} \ln \frac{1+\frac{z'}{\sqrt{z'^2+s^2}}}{1-\frac{z'}{\sqrt{z'^2+s^2}}} \Bigg|_{z'=-L}^{L}$$

$$= \frac{I}{2c} \ln\left(\frac{\sqrt{z'^2+s^2}+z'}{\sqrt{z'^2+s^2}-z'}\right)\Bigg|_{z'=-L}^{L}$$

$$A_z = \frac{I}{c} \ln\left(\frac{\sqrt{L^2+s^2}+L}{\sqrt{L^2+s^2}-L}\right)$$

$$\vec{A} = \frac{I}{c} \ln\left(\frac{\sqrt{L^2+s^2}+L}{\sqrt{L^2+s^2}-L}\right)\hat{z}$$

$$\sqrt{L^2+s^2} = L\sqrt{1+\frac{s^2}{L^2}} = L\left(1+\frac{1}{2}\frac{s^2}{L^2}\right) + \dots$$

$$A_z = \frac{I}{c}\ln\left(\frac{2L + \frac{1}{2L}s^2}{\frac{1}{2L}s^2}\right)$$

$$= \frac{I}{c}\ln\left(\frac{4L^2+s^2}{s^2}\right)$$

$$= \frac{I}{c}\ln\left[4L^2\left(1+\frac{s^2}{4L^2}\right)\frac{1}{s^2}\right]$$

$$= \frac{I}{c}\left\{\ln\left[\frac{4L^2}{L_0^2}\right] + \ln\left(1+\frac{s^2}{4L^2}\right) + \ln\left(\frac{L_0^2}{s^2}\right)\right\}$$

$$\boxed{\vec{A} = \frac{I}{c}\ln\left(\frac{L_0^2}{s^2}\right)\hat{z}}$$

Example Rotating spherical surface charge density

$$K = \frac{dI}{dl} \Leftarrow$$

$$\vec{J} = \vec{K}\,\delta(r-R)$$

$\vec{v} = \vec{\omega}\times\vec{r}$

$|\vec{v}|_{equator} = \omega R$

$|\vec{\omega}\times\vec{r}|_{equator} = \omega R$

$$\vec{J}(\vec{r}) = \rho(\vec{r})\,\vec{v}(\vec{r})$$

$$= \sigma\,\delta(r-R)$$

$$J(\vec{r}) = \sigma\,\vec{\omega}\times\vec{r}\,\delta(r-R)$$

$$\Rightarrow \boxed{\vec{K} = \sigma\,\vec{\omega}\times\vec{r}}$$

$$dq = 2\pi R\sin\theta \, dl \, \sigma$$

$$dt = \frac{2\pi}{\omega}$$

$$dI = \frac{dq}{dt} = \omega R \sin\theta \, \sigma \, dl$$

$$K = \frac{dI}{dl} = \omega R \sin\theta \, \sigma$$

$$K = \sigma \left| \vec{\omega} \times \vec{r} \right|$$

$$\vec{K} = \sigma (\vec{\omega} \times \vec{r})$$

$$\vec{A}(\vec{r}) = \frac{1}{c} \int \frac{\vec{K}(\vec{r}')\delta(r'-R) \, dV'}{|\vec{r} - \vec{r}'|}$$

<u>method II</u>

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$$

If $\vec{r}$ is not on the sphere

$$\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} \Phi_M$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \nabla^2 \Phi_M = 0$$

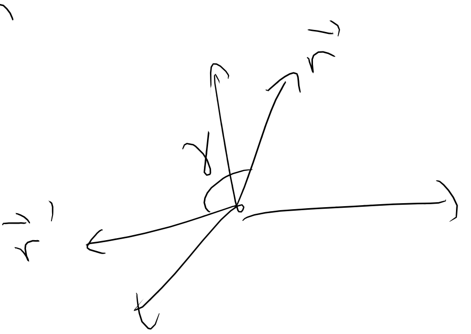
$$\Phi_M(\vec{r}) = \sum_{lm} \left(a_{lm} r^l + b_{lm} \frac{1}{r^{l+1}} \right) Y_{lm}(\theta, \phi)$$

Φ_M should be independent of ϕ coordinate.
$\Rightarrow$ only the $m=0$ terms can exist in the sum.

$$\boxed{\Phi_M(\vec r) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$

$$Y_{\ell 0}(\theta,\phi) \propto P_\ell(\cos\theta) \qquad Y_{\ell m}(\theta,\phi) \propto P_\ell^m(\cos\theta) e^{im\phi}$$

$$\sum_m Y^*_{\ell m}(\Omega)\, Y^*_{\ell m}(\Omega') \propto P_\ell(\cos\gamma)$$



inside the sphere

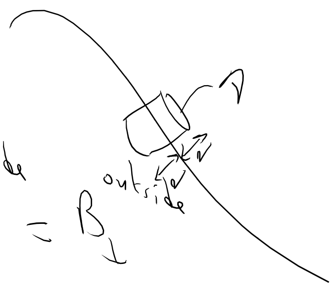
$$\boxed{\Phi_M(\vec r) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)}$$

outside the sphere

$$\boxed{\Phi_M(\vec r) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)}$$

$$\vec\nabla\cdot\vec B = 0 \Rightarrow \oint \vec B\cdot d\vec S = \int(\vec\nabla\cdot\vec B)\,dV = 0$$

$$\vec\nabla\times\vec B = \frac{4\pi}{c}\vec J$$



$$B_\perp^{inside} = B_\perp^{outside}$$

$$\vec B = -\nabla\Phi_M \Rightarrow B_\perp = -\frac{\partial\Phi_M}{\partial r}$$

inside the sphere

$$\boxed{\Phi_M(\vec{r}) = \sum_l A_l r^l P_l(\cos\theta)}$$

outside the sphere

$$\boxed{\Phi_M(\vec{r}) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)}$$

$\Downarrow$

$$-\sum_l A_l\, l\, R^{l-1} P_l(\cos\theta) = \sum_l (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta)$$

$$\Rightarrow \boxed{A_l\, l\, R^{l-1} + B_l\frac{(l+1)}{R^{l+2}} = 0}$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$$

$$\hat{m} = \hat{n}\times\hat{J}$$

$$\oint \vec{B}\cdot d\vec{l} = \int(\vec{\nabla}\times\vec{B})\cdot d\vec{A}$$

$$= \frac{4\pi}{c}\int \vec{J}\cdot d\vec{A}$$

$$= \frac{4\pi}{c} I_{enc}$$

$$\oint \vec{B}\cdot d\vec{l} = \int_b^a \vec{B}\cdot d\vec{l} + \int_d^c \vec{B}\cdot d\vec{l}$$

$$\frac{4\pi}{c} K \cancel{l} = \cancel{l}\, \vec{B}^{inside}\cdot\hat{m} + \cancel{l}\, \vec{B}^{outside}\cdot(-\hat{m})$$

$$\boxed{\left(\vec{B}^{inside} - \vec{B}^{outside}\right)\cdot\hat{m} = \frac{4\pi}{c}K}$$

$$\left(\vec{B}^{\,inside} - \vec{B}^{\,outside}\right) \cdot (-\hat{\theta}) = \frac{4\pi}{c} K$$

$$\hat{m} = \hat{n} \times \hat{K} = -\hat{\theta}$$

$$B_\theta^{outside} - B_\theta^{inside} = \frac{4\pi}{c} K$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\Phi_M$$

$$B_\theta^{inside} = -\sum_l A_l R^{l-1} \frac{d}{d\theta} P_l(\cos\theta)$$

$$B_\theta^{outside} = -\sum_l \frac{B_l}{R^{l+2}} \frac{d}{d\theta} P_l(\cos\theta)$$

$$\sum_l \left(A_l R^{l-1} - \frac{B_l}{R^{l+2}}\right) \frac{d}{d\theta} P_l(\cos\theta) = \frac{4\pi}{c} \sigma \omega R \sin\theta$$

$$= -\frac{4\pi}{c} \sigma \omega R \frac{d}{d\theta}(\cos\theta)$$

$$= -\frac{4\pi}{c} \sigma \omega R \frac{d}{d\theta} P_1(\cos\theta)$$

$$A_l R^{l-1} - \frac{B_l}{R^{l+2}} = 0 \quad \text{if } l \neq 1$$

$$\boxed{A_1 - \frac{B_1}{R^3} = -\frac{4\pi}{c} \sigma \omega R}$$

$$A_l l R^{l-1} + \frac{B_l (l+1)}{R^{l+2}} = 0$$

$$A_l = 0 \; ; \; B_l = 0 \quad \forall \, l \neq 1 \quad \cancel{\boxed{A_l = B_l = 0}}$$

$$A_1 - \frac{B_1}{R^3} = -\frac{4\pi}{c} \sigma \omega R$$

$$A_1 + \frac{2B_1}{R^3} = 0 \Rightarrow A_1 = -\frac{2B_1}{R^3}$$

$$-\frac{3B_1}{R^3} = -\frac{4\pi}{c} \sigma \omega R$$

$$B_1 = \frac{4\pi}{3c} \sigma \omega R^4 = \left(\frac{4\pi}{3} R^3\right)\left(\frac{\sigma \omega R}{c}\right)$$

$$A_1 = -\frac{8\pi}{3c} \sigma \omega R = -\frac{8\pi}{3}\left(\frac{\sigma \omega R}{c}\right)$$

inside $\Phi_M = A_1 r P_1(\cos\theta)$

$$\Phi_M = -\frac{8\pi}{3}\left(\frac{\sigma \omega R}{c}\right) r \cos\theta$$

$$\Phi_M = -\frac{8\pi}{3}\left(\frac{\sigma \omega R}{c}\right) z$$

$$\boxed{\vec{B}(r<R) = \frac{8\pi}{3}\left(\frac{\sigma \omega R}{c}\right)\hat{z}}$$

uniform magnetic field inside

<u>outside</u>

$$\Phi_M = \frac{B_1}{r^2} P_1(\cos\theta)$$

$$\Phi_M = \left(\frac{4\pi}{3} R^3\right)\left(\frac{\sigma \omega R}{c}\right)\frac{1}{r^2}\cos\theta$$

$$\Phi_M = \frac{\vec{\mu}\cdot\vec{r}}{r^3}$$

$$\Rightarrow \vec{\mu} = \left(\frac{4\pi}{3} R^3\right)\left(\frac{\sigma \omega R}{c}\right)\hat{z}$$

$$\vec{B}(r>R) = \frac{3\vec{\mu} - (\vec{\mu}\cdot\hat{r})\hat{r}}{r^3}$$

$$\boxed{\vec{A} = \frac{\vec{\mu}\times\hat{r}}{r^2}}$$ HW

$$\Phi_M(\vec{r}) = \begin{cases} -\frac{8\pi}{3}\left(\frac{\sigma\omega R}{c}\right) r\cos\theta & \text{if } r<R \\ \left(\frac{4\pi}{3}R^3\right)\left(\frac{\sigma\omega R}{c}\right)\frac{1}{r^2}\cos\theta & \text{if } r>R \end{cases}$$

Magnetic Dipole

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV' \qquad |\vec{r}| \gg d$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{lm} \frac{r'^{l}}{r^{l+1}} Y_{lm}(\Omega) Y^{*}_{lm}(\Omega')$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right)$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'}^{\,-1} = \frac{1}{r}\left(1 + \frac{r'^2}{r^2} - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right)^{-1/2}$$

$$\simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right)$$

$$\vec{A} = \frac{1}{c}\int \vec{J}(\vec{r}\,')\left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3}\right] dV'$$

$$\vec{A} = \frac{1}{rc}\underbrace{\int \vec{J}(\vec{r}\,')\, dV'}_{=0} + \frac{1}{r^3 c}\int \vec{J}(\vec{r}\,')(\vec{r}\cdot\vec{r}\,')\, dV'$$

$$\int \vec{J}(\vec{r}\,')\, dV' = 0$$

$$\int J_i'\, dV' = \int \delta_{ij} J_j'\, dV'$$

$$= \int \left[\left(\frac{\partial x_i'}{\partial x_j'}\right) J_i'\right] dV'$$

$$= \int \left[\frac{\partial (x_i' J_j')}{\partial x_j'} - x_i' \underbrace{\frac{\partial J_j'}{\partial x_j'}}_{\vec{\nabla} \cdot \vec{J} = 0} \right] dV'$$

$$= \int \vec{\nabla}' \cdot (x_i' \vec{J}') \, dV'$$

$$= \int x_i' \vec{J}' \cdot d\vec{S}' = 0$$

$$\vec{A} = \frac{1}{cr^3} \int \vec{J}' (\vec{r} \cdot \vec{r}') \, dV'$$

$$\vec{r} \times (\vec{J}' \times \vec{r}') = \vec{J}' (\vec{r} \cdot \vec{r}') - \vec{r}' (\vec{r} \cdot \vec{J}')$$

$$\vec{A} = \frac{1}{cr^3} \int \vec{r} \times (\vec{J}' \times \vec{r}') \, dV'$$

$$+ \frac{1}{cr^3} \int \vec{r}' (\vec{r} \cdot \vec{J}') \, dV'$$

$$\vec{A} = \frac{\vec{r}}{cr^3} \times \int (\vec{J}' \times \vec{r}') \, dV'$$

$$+ \frac{1}{cr^3} \int \vec{r}' (\vec{r} \cdot \vec{J}') \, dV'$$

$$\boxed{\vec{M} = \frac{1}{c} \int (\vec{r}' \times \vec{J}') \, dV'} \Leftarrow$$

$$\boxed{\vec{A} = \frac{\vec{\mu}\times\vec{r}}{r^3}} + \frac{1}{cr^3}\int \vec{r}\,'\,(\vec{r}\cdot\vec{J}\,')\,dV'$$

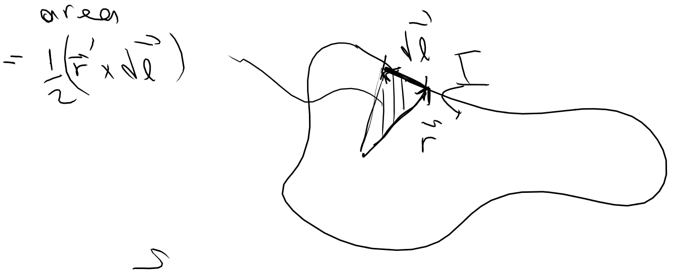
HW 2: calculate $\vec{\mu}$ for the rotating sphere example.

$$\int \vec{r}\,'\,(\vec{r}\cdot\vec{J}\,') = r_i \int \vec{r}\,' J_i\, dV'$$

$$\boxed{\int x_j' J_i'\, dV' \overset{?}{=} 0}$$

Example Assume a current carrying loop

$$\vec{\mu} = \frac{1}{c}\int \vec{r}\,'\times\vec{J}\,'\,dV' = \frac{1}{c}\int I\,\vec{r}\,'\times d\vec{\ell}$$



$$\vec{J}\,' dV' \Leftrightarrow I\, d\vec{\ell}$$

$$|\vec{A}\times\vec{B}| = AB\sin\theta$$
$$= 2\left(\frac{1}{2}(A\sin\theta)B\right)$$
$$= 2(\text{area of triangle})$$

$$\vec{\mu} = \frac{1}{c}\int I\,(\vec{r}\,'\times d\vec{\ell}) = 2\,\frac{I}{c}\int d(\text{area})\,\hat{n}$$

$$\boxed{\vec{\mu} = \frac{2}{c}\, I\,(\text{area})\,\hat{n}}$$

$$\int x'_j J'_i \, dV' \overset{?}{=} 0$$

$$J'_i = \vec{\nabla}' \cdot (x'_i \vec{J}') = \frac{\partial}{\partial x'_j}(x'_i J'_j) = \underbrace{\frac{\partial x'_i}{\partial x'_j}}_{\delta_{ij}} J'_j + x'_i \underbrace{\frac{\partial J'_j}{\partial x'_j}}_{0}$$