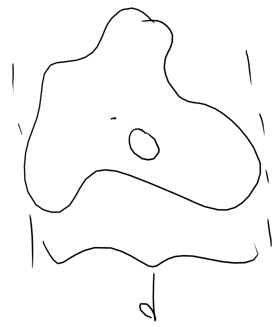
$$\vec{A} = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV'$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \dots$$



$r \gg d$

$$J_i = \frac{\partial}{\partial x_j}(x_i J_j) \qquad \vec{\nabla}\cdot\vec{J} = 0$$

$$\frac{\partial}{\partial x_j}(x_i J_j) = \underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}} J_j + x_i \underbrace{\frac{\partial J_j}{\partial x_j}}_{}$$

$$= J_i$$

$$\int J_i'\,dV' = \int \frac{\partial}{\partial x_j'}(x_i' J_j')\,dV' \overset{\Downarrow}{=} \oint x_i' J_j'\,dS_j = 0$$

$$\int (\vec{\nabla}\cdot\vec{V})\,dV = \int \vec{V}\cdot d\vec{S}$$

$$\int \left(\frac{\partial}{\partial x_j} V_j\right) dV = \int V_j\,dS_j$$

$$\int (\vec{r}\cdot\vec{r}\,')\vec{J}\,'\,dV' = \vec{r}\times\vec{\mu}$$

$$\left[\int (\vec{r}\cdot\vec{r}\,')\,\vec{J}\,'dV'\right]_i = \int \underbrace{x_j x_j'}_{\vec{r}\cdot\vec{r}\,'} J_i'\,dV' = x_j \int x_j' J_i'\,dV'$$

$$\int x_j' J_i' \, dV' = \int x_j' \frac{\partial}{\partial x_k}(x_i' J_k') \, dV'$$

$$= \int \left[\frac{\partial}{\partial x_k'}(x_j' x_i' J_k') - \left(\frac{\partial x_j'}{\partial x_k'} \right) x_i' J_k' \right] dV'$$

$$= - \int \delta_{jk} x_i' J_k' \, dV'$$

$$\boxed{\int x_j' J_i' \, dV' = - \int x_i' J_j' \, dV'}$$

$$\vec{r} \times (\vec{r}' \times \vec{J}') = \vec{r}' (\vec{r} \cdot \vec{J}') - \vec{J}' (\vec{r} \cdot \vec{r}')$$

$$(\vec{r} \cdot \vec{r}') \vec{J}' = \vec{r}' (\vec{r} \cdot \vec{J}') - \vec{r} \times (\vec{r}' \times \vec{J}')$$

$$\int (\vec{r} \cdot \vec{r}') \vec{J}' \, dV' = \int \vec{r}' (\vec{r} \cdot \vec{J}') dV' - \vec{r} \times \int (\vec{r}' \times \vec{J}') dV'$$

$$\int x_j' x_i J_i' \, dV' = x_i \int x_j' J_i' \, dV'$$

$$= - x_i \int x_i' J_j' \, dV'$$

$$= - \left(\int (\vec{r} \cdot \vec{r}') \vec{J} \, dV' \right)_j$$

$$\int (\vec{r} \cdot \vec{r}') \vec{J}' \, dV' = - \int (\vec{r} \cdot \vec{r}') \vec{J} \, dV' - \vec{r} \times (\vec{r}' \times \vec{J}') dV$$

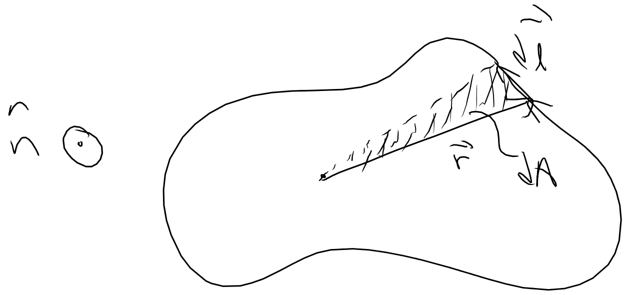
$$\Rightarrow \int (\vec{r} \cdot \vec{r}') \vec{J}' \, dV' = - \frac{1}{2} \vec{r} \times \int (\vec{r}' \times \vec{J}') \, dV'$$

$$\vec{A}(\vec{r}) = \frac{1}{cr^3}\int (\vec{r}\cdot\vec{r}')\,\vec{J}'\,dV'$$

$$= -\frac{1}{2cr^3}\,\vec{r}\times\int(\vec{r}'\times\vec{J}')\,dV' = \frac{\vec{\mu}\times\vec{r}}{r^3}$$

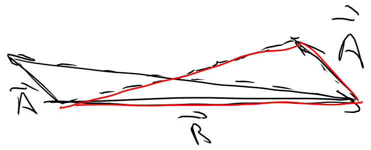
$$\boxed{\vec{\mu} = \frac{1}{2c}\int(\vec{r}'\times\vec{J}')\,dV'}$$

$$\vec{J}'\,dV' = I\,d\vec{l} = \vec{K}\,dS$$



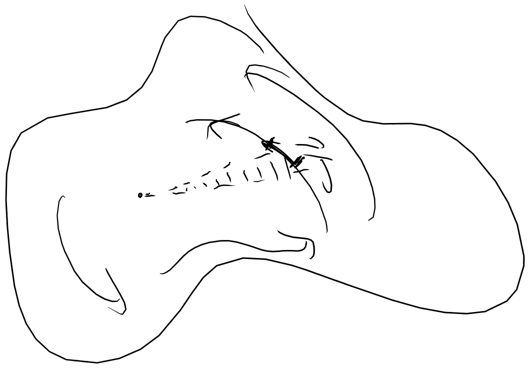
$$\vec{\mu} = \frac{1}{2c}\int \vec{r}'\times d\vec{l}'$$

$$\frac{\vec{r}'\times d\vec{l}'}{2} = (dA)\,\hat{n}$$



area $= \frac{1}{2}|\vec{A}\times\vec{B}|$

$$\boxed{\vec{\mu} = \frac{1}{c}\,a\,\hat{n}}$$



Origin dependence

El. dipole moment:

$$\vec d = \int \vec r \, \rho(\vec r)\, dV \longrightarrow \int (\vec r + \vec a)\, \rho(\vec r + \vec a)\, dV$$

$$= \int \vec r \, \rho \, dV + \vec a \int \rho \, dV$$

$$= \vec d' + \vec a\, Q$$

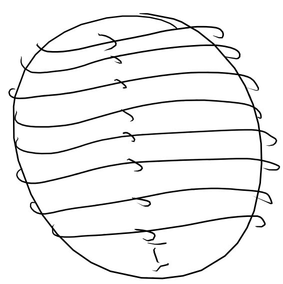
mag. dipole moment

$$\int \vec r \times \vec J \, dV \longrightarrow \int (\vec r + \vec a) \times \vec J \, dV$$

$$= \int \vec r \times \vec J \, dV + \vec a \times \underbrace{\int \vec J \, dV}_{= 0}$$

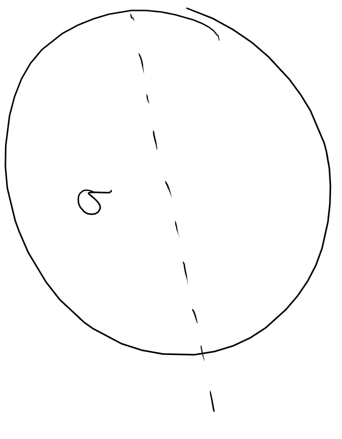
$$= \int (\vec r \times \vec J)\, dV$$

Example



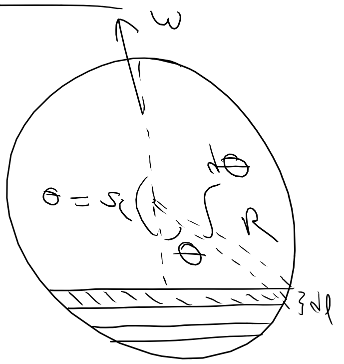
$$\vec \mu \neq 0$$

$\vec{\omega} = \text{const}$ $\qquad \sigma$ uniform.



Method I

$dl = R\,d\theta$



$$d\vec{\mu} = (dI)\, a\, \hat{\omega}$$

$$a = (R\sin\theta)^2 \pi$$

$$dI = \frac{\cancel{2\pi}(R\sin\theta)\, dl\, \sigma}{\frac{\cancel{2\pi}}{\omega}}$$

$$d\vec{\mu} = \pi(R\sin\theta)^2 (R\sin\theta)\,\sigma\,\omega\, dl\, \hat{\omega}$$

$$d\vec{\mu} = \pi\sigma\omega R^3 \sin^3\theta\, dl\, \hat{\omega}$$

$$\Rightarrow d\vec{\mu} = \pi\sigma R^3 \sin^3\theta\, dl\, \vec{\omega}$$

$$\Rightarrow \vec{\mu} = \int \pi\sigma R^3 \sin^3\theta\, dl\, \vec{\omega}$$

$$= \int \pi\sigma R^4 \sin^3\theta\, d\theta\, \vec{\omega}$$

$$= \pi\sigma R^4 \vec{\omega} \int \sin\theta(1-\cos^2\theta)\, d\theta$$

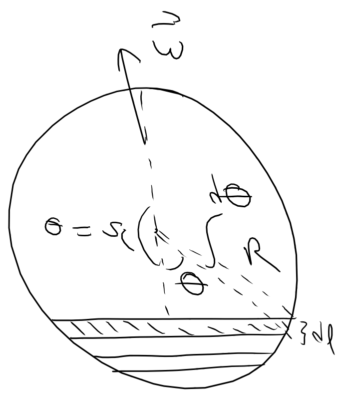
$$= \pi\sigma R^4 \vec{\omega} \left(\cos\theta - \frac{\cos^3\theta}{3}\right)(-1)\Big|_{\theta=0}^{\pi}$$

$$= -\pi\sigma R^4 \vec{\omega}\left(-1+\frac{1}{3}\right)2$$

$$\boxed{\vec{\mu} = \frac{4}{3}\pi\sigma R^4 \vec{\omega}}$$

$$\vec{\mu} = (4\pi R^2 \sigma)\,\frac{1}{3} R^2 \vec{\omega}$$

$$\boxed{\vec{\mu} = Q\,\frac{R^2}{3}\vec{\omega}}$$



$$d\vec{L} = (dI)\,\vec{\omega} = (dm)(R\sin\theta)^2\vec{\omega}$$

$$\vec{L} = \vec{r}\times\vec{p}$$

$$dm = 2\pi(R\sin\theta)\,dl\,\sigma_m$$

σ_m : surface mass density.

$$d\vec{L} = 2\pi(R\sin\theta)\,dl\,\sigma_m\,(R\sin\theta)^2\vec{\omega}$$

$$d\vec{\mu} = \pi(R\sin\theta)^2(R\sin\theta)\,\sigma\,dl\,\vec{\omega}$$

$$d\vec{\mu} = d\vec{L}\,\frac{1}{2}\frac{\sigma}{\sigma_m} = d\vec{L}\,\frac{Q}{2M}$$

$$\boxed{\vec{\mu} = \frac{Q}{2M}\vec{L}}$$

$$L_z = n\hbar$$

$\frac{\hbar}{2M} \equiv \mu_B$: Bohr magneton

$$\vec{\mu}_s = g\,\frac{Q}{2M}\,\vec{S}$$ g: gyromagnetic ratio

for electrons $g=2$

$g-2$

$$\boxed{\vec{\mu} = \frac{Q}{2M}\left(\vec{L} + 2\vec{S}\right)}$$ for an electron

$\vec{\nabla}\cdot\vec{B} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{E} = \vec{\nabla}\times\vec{A}_E$

Force acting on magnetic dipoles

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

$$= q\vec{E} + \frac{(q\vec{v})}{c}\times\vec{B}$$

$$\boxed{\vec{F}_B = \frac{1}{c}\int \vec{J}(\vec{r}\,')\times\vec{B}(\vec{r}\,')\,dV'} = \frac{1}{c}I\int d\vec{l}\times\vec{B}$$

$$\vec{B}(\vec{r}\,') = \vec{B}\left(\vec{r} + (\vec{r}\,' - \vec{r})\right)$$

$$= \vec{B}(\vec{r}) + \left[(\vec{r}\,' - \vec{r})\cdot\vec{\nabla}\right]\vec{B}(\vec{r}) + \dots$$

For simplicity of notation $\vec{r} = 0$

$$\vec{B}(\vec{r}') = \vec{B}(0) + (\vec{r}' \cdot \vec{\nabla})\vec{B}(0) + \ldots$$

$$B_i = B_i^0 + x_j' \left(\frac{\partial B_i}{\partial x_j}\right|_{x_j = 0}$$

$$\frac{\partial B_i}{\partial x_j} = 0$$

$$\frac{\partial B_i}{\partial x_j} = \frac{\partial B_j}{\partial x_i} \quad \text{since } \vec{\nabla} \times \vec{B} = 0$$

$$(\vec{\nabla} \times \vec{B})_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \Rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

$$\vec{F} = \frac{1}{c} \int \vec{J}(\vec{r}') \times \vec{B}(\vec{r}')\, dV'$$

$$F_i = \frac{1}{c} \int \varepsilon_{ijk} J_j' \left[B_k(0) + x_l' \frac{\partial B_k(0)}{\partial x_l} + \ldots \right] dV'$$

$$F_i = \frac{1}{c} \cancel{\varepsilon_{ijk} B_k(0)} \int J_j' \, dV' \quad 0$$

$$+ \frac{1}{c} \varepsilon_{ijk} \frac{\partial}{\partial x_l} B_k(0) \int x_l' J_j' \, dV'$$

$$\boxed{\vec{\mu} = \frac{1}{2c} \int \vec{r}' \times \vec{J}' \, dV'} \qquad \int \vec{J}\, dV = 0$$

$$F_i = \frac{1}{c} \varepsilon_{ijk} \frac{\partial B_k}{\partial x_l} \int x_l' J_j' \, dV'$$

$$\left[M_i = \frac{1}{2c} \varepsilon_{ijk} \int x_j' J_k' \, dV' \right] \varepsilon_{ilm}$$

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

$$\varepsilon_{ilm} M_i = \frac{1}{2c} \left(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \right) \int x_j' J_k' \, dV'$$

$$= \frac{1}{2c} \int x_l' J_m' \, dV' - \frac{1}{2c} \int x_m' J_l' \, dV'$$

$$\int x_i' J_j' \, dV' = -\int x_j' J_i' \, dV'$$

$$\varepsilon_{ilm} M_i = \frac{1}{c} \int x_l' J_m' \, dV'$$

$$\boxed{\int x_l' J_m' \, dV' = c \, \varepsilon_{ilm} M_i}$$

$$F_i = \frac{1}{c} \varepsilon_{ijk} \frac{\partial B_k}{\partial x_l} \int x_l' J_j' \, dV'$$

$$= \frac{1}{c} \varepsilon_{ijk} \frac{\partial B_k}{\partial x_l} \, c \, \varepsilon_{nlj} M_n$$

$$= - \left(\varepsilon_{ikj} \varepsilon_{nlj} \right) M_n \frac{\partial B_k}{\partial x_l}$$

$$F_i = - \left(\delta_{in} \delta_{kl} - \delta_{il} \delta_{kn} \right) M_n \frac{\partial B_k}{\partial x_l}$$

$$F_i = -\left(\mu_i \frac{\partial B_k}{\partial x_k} - \mu_k \frac{\partial B_k}{\partial x_i}\right)$$

$$= -\left[\mu_i \underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0} - \frac{\partial}{\partial x_i}(\mu_k B_k)\right]$$

$$F_i = \left[\frac{1}{c}\vec{\nabla}(\vec{\mu}\cdot\vec{B})\right]_i$$

$$\boxed{\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})} \Leftarrow$$

$$F_i = \mu_k \frac{\partial}{\partial x_i} B_k = \mu_k \frac{\partial}{\partial x_k} B_i$$

$$F_i = (\vec{\mu}\cdot\vec{\nabla}) B_i$$

$$\boxed{\vec{F} = (\vec{\mu}\cdot\vec{\nabla})\vec{B}} \Leftarrow$$

Torque acting on a magnetic dipole

$$\vec{\tau} = \vec{r}\times\vec{F}$$

$$\vec{\tau} = \int \vec{r}\,' \times d\vec{F}\,'$$

$$\vec{\tau} = \int \vec{r}\,' \times \underbrace{\left(\frac{1}{c}\vec{J}\,'\times\vec{B}\,'\right) dV'}_{d\vec{F}}$$

$$\vec{\tau} = \frac{1}{c}\int \vec{r}\,' \times (\vec{J}\,' \times \vec{B}\,')\, dV'$$

$$= \frac{1}{c}\int \vec{r}\,' \times \left[\vec{J}\,' \times \left(\vec{B}(0) + (\vec{r}\,' \cdot \vec{\nabla})\vec{B}\,'(0)\right)\right] dV'$$

$$= \frac{1}{c}\int \vec{r}\,' \times \left[\vec{J}\,' \times \vec{B}(0)\right] dV'$$

$$+ \frac{1}{c}\int \vec{r}\,' \times \left[\vec{J}\,' \times \left\{(\vec{r}\,' \cdot \vec{\nabla})\vec{B}\,'(0)\right\}\right] dV'$$

$$+ \cdots$$

$$\Downarrow$$

$$\vec{r}\,' \times \left[\vec{J}\,' \times \vec{B}(0)\right] = \vec{J}\,' (\vec{r}\,' \cdot \vec{B}(0)) - \vec{B}(0)\left[\vec{r}\,' \cdot \vec{J}\,'\right]$$

$$\left[\int x_k' J_l' \, dV' = -\int x_l' J_k' \, dV'\right] \delta_{kl}$$

$$\int (\vec{r}\,' \cdot \vec{J}\,')\, dV' = -\int (\vec{r}\,' \cdot \vec{J}\,')\, dV' = 0$$

$$\vec{\tau} = \frac{1}{c}\int \vec{J}\,' (\vec{r}\,' \cdot \vec{B}(0))\, dV' + \cdots$$

$$\tau_i = \frac{1}{c}\int J_i' x_k' B_k \, dV'$$

$$\tau_i = \frac{1}{c} B_k \int x_k' J_i' \, dV'$$

$$\boxed{\int x_l' J_m' \, dV' = c\, \varepsilon_{ilm} M_i}$$

$$\tau_i = \cancel{\tfrac{1}{2}} B_k \cancel{c}\, \varepsilon_{nki} \mu_n$$

$$= \varepsilon_{ink} \mu_n \times B_k$$

$$\tau_i = (\vec{\mu} \times \vec{B})_i \implies \boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

$$\boxed{\vec{F} = \vec{\nabla} (\vec{\mu} \cdot \vec{B})}$$