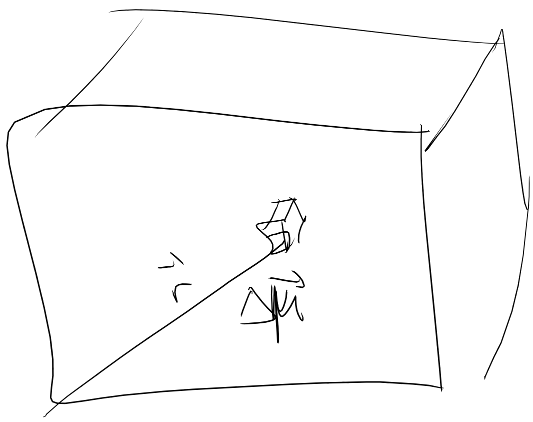


$$\vec{M}(\vec{r}) = \frac{\Delta\vec{\mu}}{\Delta V}$$

$$\vec{A}(\vec{r}) \qquad \vec{J}_b = +c\vec{\nabla}\times\vec{M} \qquad \vec{K}_b = -c\,\hat{n}\times\vec{M}$$

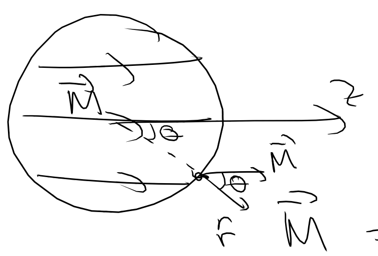
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_{tot} = \frac{4\pi}{c}\left(\vec{J}_f + c\vec{\nabla}\times\vec{M}\right)$$

$$\vec{\nabla}\times(\underbrace{\vec{B} - 4\pi\vec{M}}_{\vec{H}}) = \frac{4\pi}{c}\vec{J}_f$$

$\vec{B}$: magnetic induction intensity

$\vec{H}$: magnetic field intensity

<u>Example</u> uniformly magnetised sphere



$$\vec{J}_b(r<R) = \vec{\nabla}\times\vec{M} = 0$$

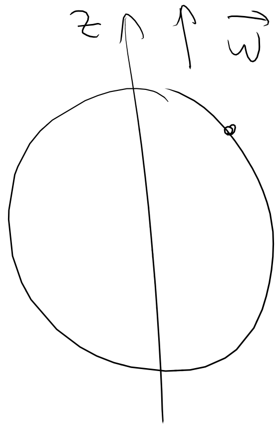
$$\vec{K}_b(r=R) = -\hat{n}\times\vec{M}$$

$$\vec{M} = M_0\hat{z}$$

$$\hat{n} = \hat{r}$$

$$\vec{K}_b = -(\hat{n}\times\vec{M}) = \hat{\theta}(M_0\sin\theta)$$

$$= -M_0\,\hat{r}\times\hat{z} = (M_0\hat{z})\times\hat{r}$$



$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{K} = \sigma \vec{v} = (\sigma \vec{\omega}) \times \vec{r}$$

$$\sigma \vec{\omega} = (\sigma \omega \hat{z}) \times \vec{r}$$

$$\vec{D} = \vec{E} - 4\pi \vec{P} \qquad \vec{P} = \chi_e \vec{E} \qquad \vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \vec{H} + 4\pi \vec{M}$$

$$\vec{M} = \chi_m \vec{H}$$

χ_m: magnetic susceptibility

$$\vec{B} = \underbrace{(1 + 4\pi\chi_m)}_{\mu} \vec{H}$$

$$\vec{B} = \mu \vec{H}$$

$$\mu = 1 + 4\pi\chi_m$$

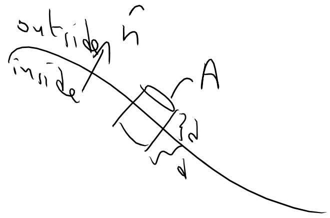
magnetic permitivity

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J}_f$$

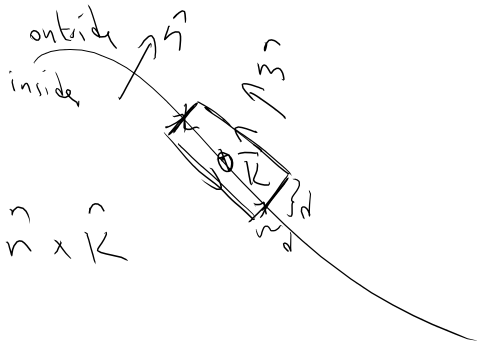
$$\vec{B} = \mu \vec{H}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Longrightarrow \oint \vec{B} \cdot d\vec{S} = 0$$



$$\Rightarrow \left(\vec{B}^{\,outside} - \vec{B}^{\,inside}\right) \cdot \hat{n} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}_f \iff \oint \vec{H}\cdot d\vec{l} = \frac{4\pi}{c} I_{enc}$$



$$\hat{m} = \hat{n}\times\hat{K}$$

$$\left(\vec{H}^{inside} - \vec{H}^{outside}\right)\cdot\hat{K} = 0$$

$$\left(\vec{H}^{inside} + \vec{H}^{outside}\right)\cdot\hat{m} = -K\frac{4\pi}{c}$$

$$\left(\vec{H}^{inside} - \vec{H}^{outside}\right)\cdot\hat{m} = \frac{4\pi}{c}K$$

$$\boxed{\left(\vec{H}^{inside} - \vec{H}^{outside}\right)\cdot\left(\hat{n}\times\hat{K}\right) = \frac{4\pi}{c}K}$$

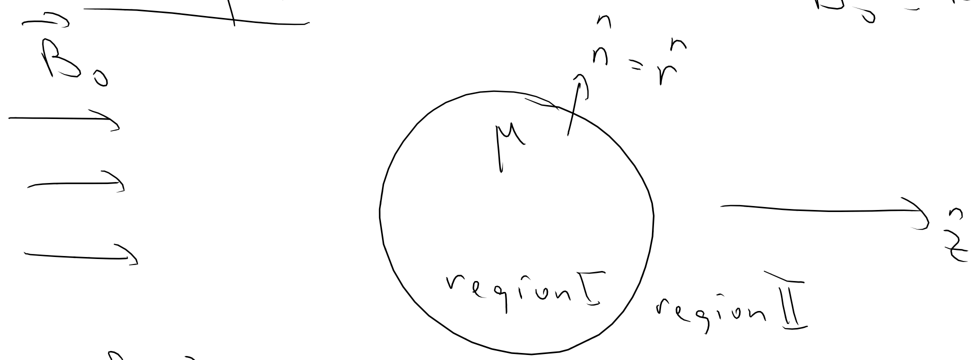
$$\left(\vec{H}^{inside} - \vec{H}^{outside}\right)\cdot\hat{K} = 0$$

$$\left(\vec{H}^{inside} - \vec{H}^{outside}\right)_{\parallel} = 0\,\hat{K} + \left(\hat{n}\times\hat{K}\right)\frac{4\pi}{c}K$$

$$= \frac{4\pi}{c}\,\hat{n}\times\vec{K}$$

$$\boxed{\left(\vec{H}^{outside} - \vec{H}^{inside}\right)_{\parallel} = \frac{4\pi}{c}\vec{K}\times\hat{n}}$$

Example



$\vec{B}_0 = B_0 \hat{z}$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{B} = \mu \vec{H} \Rightarrow \vec{H} = \frac{1}{\mu} \vec{B}$$

$$\vec{\nabla} \times \vec{H} = 0$$

$$\vec{\nabla} \times \left(\frac{1}{\mu} \vec{B} \right) = 0$$

in region I & II, $\mu = \text{const}$

$$0 = \vec{\nabla} \times \left(\frac{1}{\mu} \vec{B} \right) = \frac{1}{\mu} \vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{\nabla} \times \vec{B} = 0$$

$$\left. \begin{array}{l} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = 0 \end{array} \right\} \text{ in region I \& II } \underline{\text{separately!}}$$

$$\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} \phi_M$$

$$\nabla^2 \phi_M = 0$$

$$\phi_M = \sum_{lm} \left(a_{lm} r^l + \frac{b_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \phi)$$

$$Y_{l0} \propto P_l(\cos\theta)$$

$$\phi_M = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

inside the sphere, ϕ_M is finite at $r=0$

$$\boxed{\phi_M^{I} = \sum_l A_l r^l P_l(\cos\theta)}$$

$\cos\theta = P_1(\cos\theta)$

in region II, $\phi_M \to -B_0 r\cos\theta$ as $r\to\infty$

$$\boxed{\phi_M^{II} = -B_0 r\cos\theta + \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)}$$

$B_r = -\frac{\partial \phi_M}{\partial r}$ is continuous on the boundary $(r=R)$

$$\sum_l A_l l R^{l-1} P_l(\cos\theta) = -B_0 P_1(\cos\theta) - \sum_l \frac{B_l(l+1)}{R^{l+2}} P_l(\cos\theta)$$

$$\boxed{\begin{array}{ll} l=1 & A_1 = -B_0 - \frac{2B_1}{R^3} \\ l\neq 1 & A_l l R^{l-1} = -\frac{B_l(l+1)}{R^{l+2}} \end{array}}$$

$\vec{H}_{\parallel}$ is continuous

$$\vec{H}_{\parallel} = H_\theta\hat{\theta} + \underbrace{H_\phi}_{=0}\hat{\phi} = H_\theta\hat{\theta}$$

$$H_\theta = \frac{1}{\mu}B_\theta = -\frac{1}{\mu}\frac{1}{r}\frac{\partial}{\partial\theta}\phi_M$$

$$\phi_M^{I} = \sum_l A_l r^l P_l(\cos\theta)$$

$$\phi_M^{II} = -B_0 r\cos\theta + \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\frac{1}{\mu}\sum_l A_l R^l \frac{\partial}{\partial\theta} P_l(\cos\theta)$$

$$= -B_0 R \frac{\partial}{\partial\theta} P_1(\cos\theta) + \sum_l \frac{B_l}{R^{l+1}} \frac{\partial}{\partial\theta} P_l(\cos\theta)$$

$$l=1 \quad \frac{1}{\mu} A_1 R = -B_0 R + \frac{B_1}{R^2}$$

$$\boxed{\frac{1}{\mu} A_1 = -B_0 + \frac{B_1}{R^3}}$$

$$l \neq 1 \quad \boxed{\frac{1}{\mu} A_l R^l = \frac{B_l}{R^{l+1}}} \quad \Leftarrow$$

$$l=1 \quad A_1 = -B_0 - \frac{2B_1}{R^3}$$

$$l \neq 1 \quad A_l \, l \, R^{l-1} = -\frac{B_l (l+1)}{R^{l+2}} \quad \Leftarrow$$

$$A_l = 0, \; B_l = 0 \quad \text{for } l \neq 1$$

$$\frac{1}{\mu} A_1 = -B_0 + \frac{B_1}{R^3}$$

$$A_1 = -B_0 - \frac{2B_1}{R^3}$$

$$\frac{1}{\mu}\left(-B_0 - 2\frac{B_1}{R^3}\right) = -B_0 + \frac{B_1}{R^3}$$

$$\left(1 - \frac{1}{\mu}\right) B_0 = \frac{B_1}{R^3}\left(\frac{2}{\mu} + 1\right)$$

$$\boxed{B_1 = \frac{\mu - 1}{\mu + 2} B_0 R^3}$$

$$A_1 = -B_0 - 2\frac{B_1}{R^3} = -B_0 - 2\frac{\mu-1}{\mu+2} B_0$$

$$= -B_0\left(1 + 2\frac{\mu-1}{\mu+2}\right)$$

$$= -B_0\left(\frac{\mu+2+2\mu-2}{\mu+2}\right)$$

$$\boxed{A_1 = -B_0 \frac{3\mu}{\mu+2}}$$

$$\phi_M^{I} = \sum A_\ell r^\ell P_\ell(\cos\theta)$$

$$= -B_0 \frac{3\mu}{\mu+2} r\cos\theta$$

$$= -B_0 r\cos\theta - B_0\left(\frac{3\mu}{\mu+2} - 1\right) r\cos\theta$$

$$= -B_0 r\cos\theta - B_0\left(\frac{2\mu-2}{\mu+2}\right) r\cos\theta$$

$$\phi_M^{I} = -B_0 r\cos\theta - 2B_0\left(\frac{\mu-1}{\mu+2}\right) r\cos\theta$$

$$\vec{B}^{I} = -\vec{\nabla}\phi_M^{I} = +\boxed{B_0\hat{z} + 2B_0\left(\frac{\mu-1}{\mu+2}\right)\hat{z} = \vec{B}^{I}}$$

$$\phi_M^{II} = -B_0 r\cos\theta + \sum \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\phi_M^{II} = -B_0 r\cos\theta + \frac{\mu-1}{\mu+2} B_0 R^3 \frac{1}{r^2}\cos\theta$$

$$= -B_0 r\cos\theta + \frac{\mu-1}{\mu+2} B_0 R^3 \frac{r\cos\theta}{r^3}$$

$$= -B_0 r\cos\theta + \frac{\mu-1}{\mu+2} R^3 \frac{\vec{B}\cdot\vec{r}}{r^3}$$

$$\vec{M} \equiv \frac{\mu-1}{\mu+2} R^3 \vec{B}$$

$$\phi_M^{II} = -B_0 r \cos\theta + \frac{\vec{M}\cdot\vec{r}}{r^3}$$

$$\boxed{\vec{B}^{II} = \vec{B}_0 + \frac{3(\vec{M}\cdot\hat{r})\hat{r} - \vec{M}}{r^3}}$$

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

$$\vec{B} = \vec{H} + 4\pi\vec{M}$$

outside $\vec{M} = 0$

$$\vec{B}^{II} = \vec{H}^{II}$$

inside

$$\vec{B} = \frac{1}{\mu}\vec{H} = \vec{H} + 4\pi\vec{M}$$

$$\vec{M} = \frac{1}{4\pi}\left[\frac{1}{\mu} - 1\right]\vec{H} = \frac{1}{4\pi}\left[\frac{1}{\mu} - 1\right]\mu\vec{B}$$

$$\vec{M} = \frac{1}{4\pi}(1-\mu)\vec{B}$$

$$\vec{M} = \frac{1}{4\pi}(1-\mu)\left[B_0\hat{z} + 2B_0\left(\frac{\mu-1}{\mu+2}\right)\hat{z}\right]$$

Electrodynamics

$$\rightarrow \vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\wedge\vec{H} = \frac{4\pi}{c}\vec{J}$$

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{4\pi}\vec{\nabla}\cdot\vec{E}\right) + \vec{\nabla}\cdot\Big(\underbrace{\frac{c}{4\pi}\vec{\nabla}\times\vec{B}}_{-\frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}}\Big) = 0$$

$$\Uparrow$$

$$0$$

$$\frac{1}{4\pi}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = 0$$

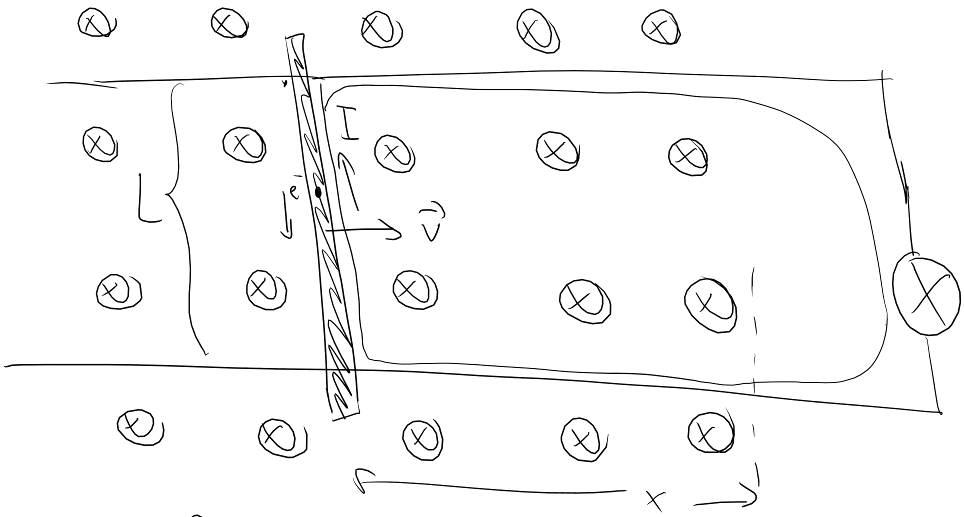
$$-\frac{1}{4\pi}\vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} - \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}} \quad ?$$

displacement current

$$\boxed{\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}}$$



$$F = \frac{q}{c} v B \qquad\qquad v = -\frac{dx}{dt}$$

$$W = \frac{q}{c} v B L$$

$$\mathcal{E} = \boxed{\frac{W}{q} = \frac{1}{c} v B L} \Leftarrow \qquad \mathcal{E} = -\frac{1}{c}\frac{d}{dt}(xBL)$$

$$= -\frac{1}{c}\frac{d}{dt}(BA)$$

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \qquad = -\frac{1}{c}\frac{d}{dt}\phi_B$$

$$\Rightarrow \phi_B = \int \vec{B}\cdot d\vec{A} \qquad \boxed{\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\phi_B}$$

$$\Uparrow$$

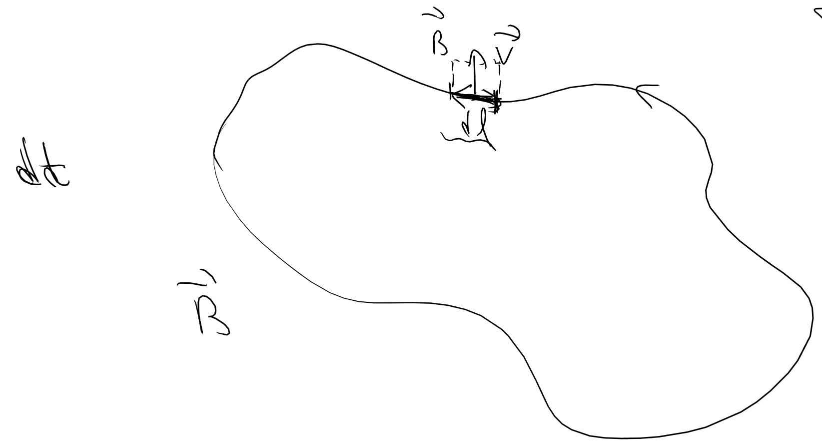
$$\Rightarrow \mathcal{E} = \oint \vec{E}\cdot d\vec{l}$$

$$\oint_{\partial A} \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_A \vec{B}\cdot d\vec{A}$$

$$- \int_A (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\frac{1}{c}\int_A \left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{A}$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}$$

$\mathcal{E} = \oint \vec{E} \cdot d\vec{l}$



$\vec{F} = q\,\vec{v} \times \vec{B}$

$dW = \frac{1}{c}(q\vec{v} \times \vec{B}) \cdot d\vec{l}$

$\frac{dW}{q} \equiv d\mathcal{E} = \left(\frac{\vec{v}}{c} \times \vec{B}\right) \cdot d\vec{l}$

$$\frac{dW}{q} = \left(\frac{\vec{v}}{c} \times \vec{B}\right) \cdot d\vec{l}$$

$$d(d\phi_B) = \vec{B} \cdot d\vec{A} = -\vec{B} \cdot \left[d\vec{l} \times (\vec{v}\, dt)\right]$$

$$\frac{d(d\phi)}{dt} = -\vec{B} \cdot (d\vec{l} \times \vec{v})$$

$$= -d\vec{l} \cdot (\vec{v} \times \vec{B}) = -c\, d\mathcal{E}$$

$$\frac{d}{dt}(d\phi_B) = -c\, d\mathcal{E}$$

$$\boxed{\mathcal{E} = -\frac{1}{c}\frac{d\phi_B}{dt}} \Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}$$