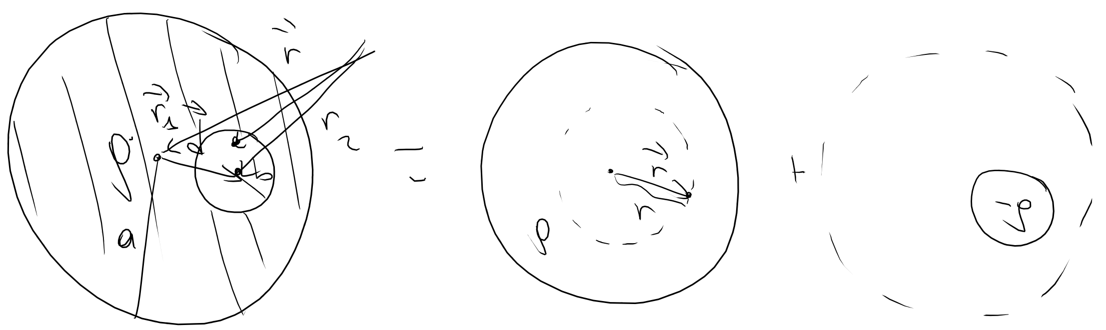
Midterm 2 Date: 24 December 9^{40} - 13^{00}

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}}$$

<u>Example</u> Electrostatics



$\vec{r}_1 + \vec{d} = \vec{r}_2$

$$E = \frac{\left(\frac{4\pi}{3}r^3\right)\rho}{r^2} = \frac{4\pi}{3}\rho r$$

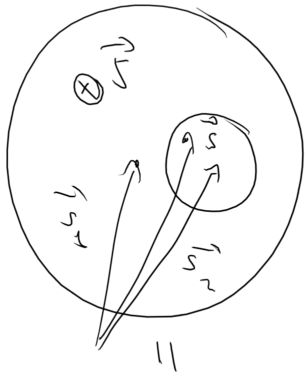
$$\vec{E} = \frac{4\pi}{3}\rho r\hat{r} = \frac{4\pi}{3}\rho\vec{r}$$

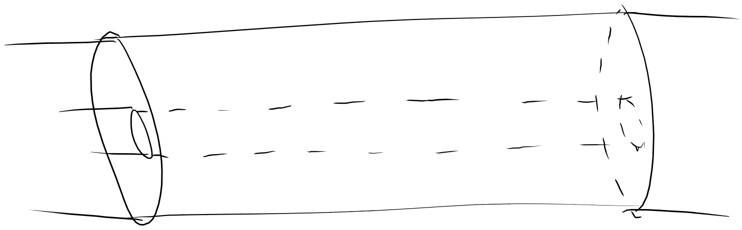
$$\vec{E}(\vec{r}) = \frac{4\pi}{3}\rho\,(\vec{r}-\vec{r}_1) + \frac{4\pi}{3}(-\rho)(\vec{r}-\vec{r}_2)$$

$$= \frac{4\pi}{3}\rho\,(\vec{r}_2-\vec{r}_1)$$

$$\boxed{\vec{E}(\vec{r}) = \frac{4\pi}{3}\rho\,\vec{d}}$$

Example





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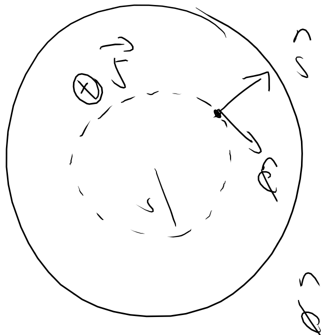

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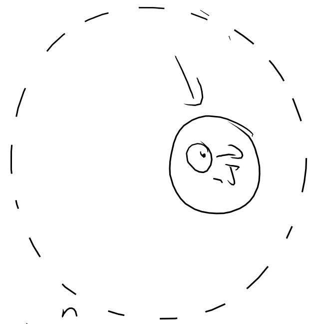


$$\hat\phi = \hat J \times \hat s$$

$$\oint \vec B \cdot d\vec l = \frac{1}{c} I_{enc}$$

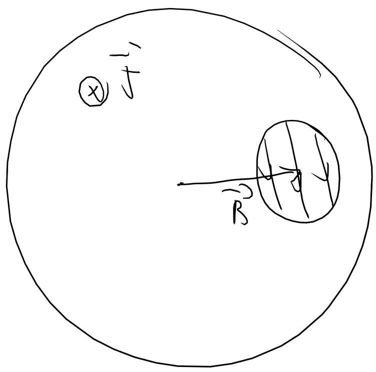
$$2\pi s\, B(s) = \frac{1}{c} J (\pi s^2)$$

$$\boxed{B(s) = \frac{1}{2c} J s}$$

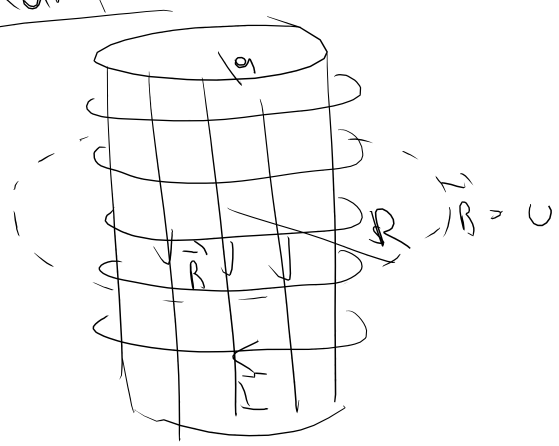
$$\vec B = \frac{1}{2c} J s \hat\phi$$
$$= \frac{1}{2c} J s (\hat J \times \hat s)$$
$$= \frac{1}{2c} (J \hat J) \times (s \hat s)$$

$$\boxed{\vec B = \frac{1}{2c} \vec J \times \vec s}$$

$$\vec B(\vec s) = \frac{1}{2c} \vec J \times (\vec s - \vec s_1)$$
$$+ \frac{1}{2c} (-\vec J) \times (\vec s - \vec s_2)$$
$$= \frac{1}{2c} \vec J \times (\vec s_2 - \vec s_1)$$



Example



$$|\vec{B}| = \frac{1}{c} I n$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$= -\frac{d}{dt}(\pi a^2)\mu_0 I n$$

$$E(R)\, 2\pi R = -\pi a^2 \mu_0 n \frac{dI}{dt}$$

$$E(R) = -\frac{\pi a^2 \mu_0 n}{2\pi R} \frac{dI}{dt} \neq 0$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

displacement current

$$\Downarrow$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\frac{\partial}{\partial t}\overbrace{\left(\frac{1}{4\pi}\vec{\nabla} \cdot \vec{D}\right)}^{\rho} + \vec{\nabla} \cdot \underbrace{\left(\frac{c}{4\pi}\vec{\nabla} \times \vec{H} - \frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t}\right)}_{\vec{J}} = 0$$

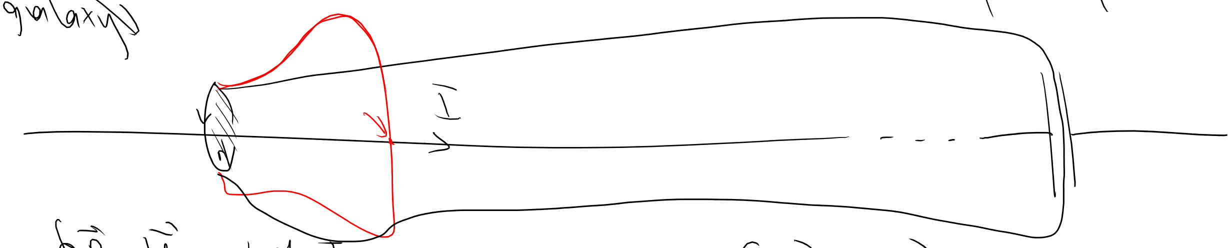
$$\vec{J} = \frac{c}{4\pi}\vec{\nabla} \times \vec{H} - \frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

$$\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B}$$

our galaxy

galaxy far far away



$$\oint_{\partial S} \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enc}$$

$$2\pi r B = \frac{4\pi I}{c}$$

$$\boxed{B = \frac{1}{2\pi c}\frac{I}{r}}$$

$$I_{enc} = \int_S \vec{J}\cdot d\vec{S}$$

$$E_{\text{in the capacitor}} = 4\pi \frac{Q}{A}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$Q = \left(E_{\text{in the capacitor}} A\right)\frac{1}{4\pi}$$

$$Q = \int_S \vec{E}\cdot d\vec{S}\ \frac{1}{4\pi}$$

$$\frac{dQ}{dt} = I$$

$$I_{enc} \Rightarrow \int \vec{J}\cdot d\vec{S} + \underline{\int \frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}\cdot d\vec{S}}$$

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{S} + \frac{1}{c}\int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{S}$$

$$\int (\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{S} + \frac{1}{c}\int \frac{\partial \vec{E}}{\partial t}$$

Conservation Laws

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \implies \frac{d}{dt}\left(\int \rho \, dV\right) = -\int \vec{j}\cdot d\vec{S}$$

Conservation of Energy

$$P = \vec{F}\cdot\vec{v}$$

$$\frac{dE_{mech}}{dt} = P = \int(\vec{f}\cdot\vec{v})\,dV$$

$$= \int\left(\rho\vec{E} + \frac{\vec{j}}{c}\times\vec{B}\right)\cdot\vec{v}\,dV$$

$$= \int \rho\vec{E}\cdot\vec{v}\,dV$$

$$= \int \vec{j}\cdot\vec{E}\,dV$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{j} = \frac{c}{4\pi}\left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\frac{dE_{mech}}{dt} = \frac{c}{4\pi}\int(\vec{\nabla}\times\vec{B})\cdot\vec{E}\,dV - \frac{1}{4\pi}\int\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}\,dV$$

$$\frac{d}{dt}\left(E_{mech} + \frac{1}{8\pi}\int\vec{E}^2\,dV\right) = \frac{c}{4\pi}\int\underbrace{(\vec{\nabla}\times\vec{B})\cdot\vec{E}}\,dV$$

$$(\vec{\nabla}\times\vec{B})\cdot\vec{E} = E_i\,\varepsilon_{ijk}\frac{\partial}{\partial x_j}B_k$$

$$= \frac{\partial}{\partial x_j}\left(\varepsilon_{ijk}E_iB_k\right) - \varepsilon_{ijk}B_k\left(\frac{\partial}{\partial x_j}E_i\right)$$

$$= -\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) - B_k\left(-\vec{\nabla}\times\vec{E}\right)_k$$

$$\left(\vec{\nabla}\times\vec{B}\right)\cdot\vec{E} = -\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) + \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)$$

$$= -\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) + \vec{B}\cdot\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right)$$

$$\frac{d}{dt}\left(E_{mech} + \frac{1}{8\pi}\int\vec{E}^2\,dV\right) = \frac{c}{4\pi}\int dV\left[-\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)\right] - \frac{1}{4\pi}\int dV\,\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(E_{mech} + \frac{1}{8\pi}\int\vec{E}^2\,dV + \frac{1}{8\pi}\int\vec{B}^2\,dV\right) + \frac{c}{4\pi}\int d\vec{A}\cdot\left(\vec{E}\times\vec{B}\right) = 0$$

$\frac{c}{4\pi}\vec{E}\times\vec{B} = \vec{S}$: Poynting vector

$$\frac{d\vec{P}_{mech}}{dt} = \int \vec{f}\, dV$$

$$= \int \left(\rho \vec{E} + \frac{\vec{j}}{c} \times \vec{B}\right) dV$$

$$= \int dV \left\{ \frac{1}{4\pi}(\vec{\nabla}\cdot\vec{E})\,\vec{E} + \frac{\cancel{c}}{4\pi}\frac{1}{\cancel{c}}\left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} \right\}$$

$$= \frac{1}{4\pi}\int dV \left\{ \vec{E}(\vec{\nabla}\cdot\vec{E}) + (\vec{\nabla}\times\vec{B})\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\times\vec{B} \right\}$$

$$\int dV\, \frac{\partial \vec{E}}{\partial t}\times\vec{B} = \int dV \left[\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial \vec{B}}{\partial t} \right]$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}} \qquad = \frac{d}{dt}\left(\int dV\, (\vec{E}\times\vec{B})\right) + \int dV\, c\,\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\frac{d\vec{P}_{mech}}{dt} = \frac{1}{4\pi}\int dV \left\{ \vec{E}(\vec{\nabla}\cdot\vec{E}) + (\vec{\nabla}\times\vec{B})\times\vec{B} - \vec{E}\times(\vec{\nabla}\times\vec{E}) \right\}$$

$$- \frac{1}{4\pi c}\frac{d}{dt}\int dV\, (\vec{E}\times\vec{B})$$

$$\frac{d}{dt}\left[\vec{P}_{mech} + \frac{1}{c^2}\int dV\, \vec{S}\right]$$

$$= \frac{1}{4\pi}\int dV \left\{ \vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - \vec{B}\times(\vec{\nabla}\times\vec{B}) \right\}$$

$$\left[\vec{E}\,(\vec{\nabla}\cdot\vec{E}) + \vec{B}\,(\vec{\nabla}\cdot\vec{B}) + (\vec{\nabla}\times\vec{E})\times\vec{E} + (\vec{\nabla}\times\vec{B})\times\vec{B}\right]_i$$

$$\left[\vec{E}\,(\vec{\nabla}\cdot\vec{E}) + (\vec{\nabla}\times\vec{E})\times\vec{E}\right]_i \qquad \partial_i \equiv \frac{\partial}{\partial x_i}$$

$$E_i\,\partial_j E_j + \varepsilon_{ijk}(\vec{\nabla}\times\vec{E})_j\,E_k$$

$$= E_i\,\partial_j E_j + \varepsilon_{ijk}\,\varepsilon_{jlm}(\partial_l E_m)E_k$$

$$= E_i\,\partial_j E_j - \underbrace{\varepsilon_{jik}\,\varepsilon_{jlm}}_{(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})}(\partial_l E_m)E_k$$

$$= E_i\,\partial_j E_j - (\partial_i E_k)E_k + (\partial_k E_i)E_k$$

$$= E_i(\partial_k E_k) - \frac{1}{2}\partial_i(E_k)^2 + E_k(\partial_k E_i)$$

$$= \partial_k(E_i E_k) - \frac{1}{2}\partial_i(E_k)^2$$

$$= \partial_j(E_i E_j) - \frac{1}{2}\partial_i(E_k)^2$$

$$= \partial_j\Big(\underbrace{E_i E_j - \frac{1}{2}\delta_{ij}(E_k)^2}_{T^E_{ij}}\Big)$$

$$\left[\vec{E}\,(\vec{\nabla}\cdot\vec{E}) + \vec{B}\,(\vec{\nabla}\cdot\vec{B}) + (\vec{\nabla}\times\vec{E})\times\vec{E} + (\vec{\nabla}\times\vec{B})\times\vec{B}\right]_i$$

$$= \partial_j \left(T_{ij}^{E} + T_{ij}^{B}\right)$$

$$T_{ij}^{E} = E_i E_j - \frac{1}{2}\delta_{ij} E_k^2$$

$$T_{ij}^{B} = E_i E_j - \frac{1}{2}\delta_{ij} B_k^2$$

$$T_{ij} = -\left(T_{ij}^{E} + T_{ij}^{B}\right)\frac{1}{4\pi}$$ energy-momentum tensor.

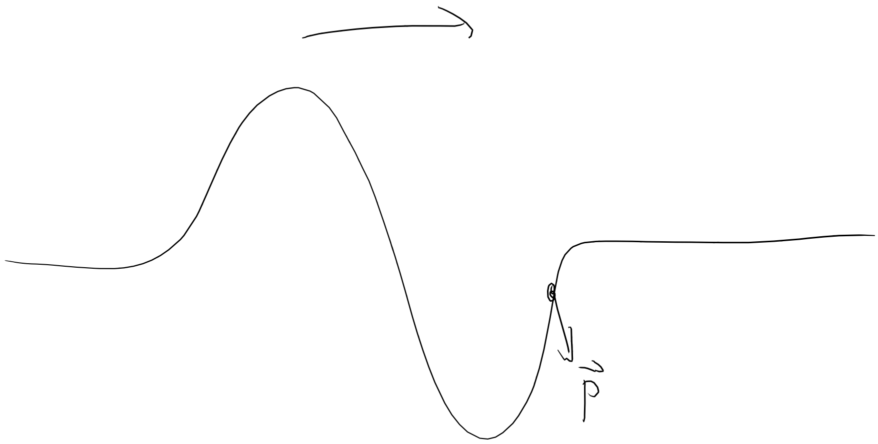
$$\frac{d}{dt}\left[\vec{P}_{mech} + \frac{1}{c^2}\int dV\,\vec{S}\right]_i = -\int \partial_j T_{ij}\, dV$$

$$= -\int T_{ij}\, n_j\, dS$$

$$\boxed{\frac{d}{dt}\left(P^i_{mech} + \frac{1}{c^2}\int dV\, S_i\right) = -\int T_{ij}\, n_j\, dS}$$

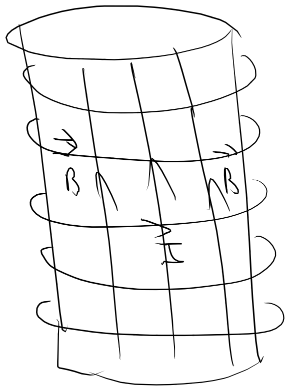
$$\boxed{\vec{L}_{em} = \int \vec{r}\times\frac{\vec{S}}{c^2}\, dV}$$

$$\vec{L} = \vec{r}\times\vec{p}$$

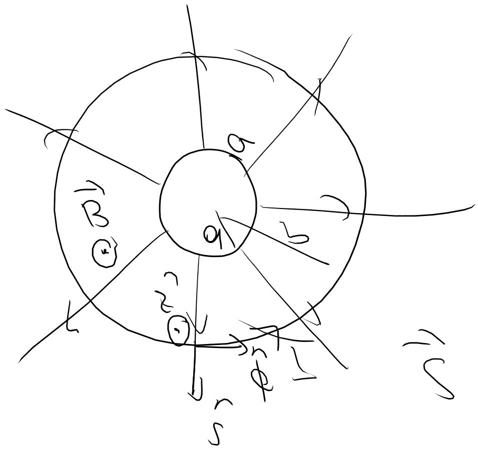


$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

Example



$$\boxed{B = \frac{1}{c} I n}$$



$$\oint \vec{E}\cdot d\vec{s} = 4\pi Q_{enc}$$

$$E\,(2\pi s)\,h = \sigma(2\pi a h)\,4\pi$$

$$\boxed{E = 4\pi\frac{\sigma a}{s}}$$

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$$

$$= \frac{c}{4\pi}\,\frac{4\pi\sigma a}{s}\,\frac{1}{c} I n\,(-\hat{\phi})$$

$$\boxed{\vec{S} = \frac{\sigma a I n}{s}\,\hat{\phi}}$$

$$\boxed{\vec{S} = -\frac{\sigma a I n}{s}\hat{\phi}}$$

$$\frac{\vec{L}_{em}}{\text{length}} = \frac{1}{\ell}\int (\vec{r}\times\vec{S})\, dV$$

$$= \frac{1}{\ell}\int (s\hat{s}+z\hat{z})\times\left(-\frac{\sigma a I n}{s}\hat{\phi}\right) dV$$

$dV = s\,ds\,d\phi\,dz$

$$= \frac{1}{\ell}\iint\left[(-\sigma a I n)\hat{z} + \left(-\frac{\sigma a I n z}{s}\right)(-\hat{s})\right] dV$$

$\int \hat{s}\, d\phi = 0$

$$= (-\sigma a I n)\hat{z}\,\pi(b^2-a^2) + 0$$

$$\boxed{\frac{\vec{L}_{em}}{\text{length}} = \sigma a I n \pi (b^2-a^2)(-\hat{z})}$$

Assume I is decreasing. $\frac{dI}{dt} < 0$