$$\vec{\nabla}\cdot\vec{D} = 0 \Longrightarrow D_\perp \text{ is cont} \quad \cancel{\vec{E} = \hat{k}\times\vec{B}}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \Longrightarrow E_\parallel \text{ is cont}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Longrightarrow B_\perp \text{ is cont}$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \Longrightarrow H_\parallel \text{ is cont}$$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-wt)}$$

$$\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-wt)}$$

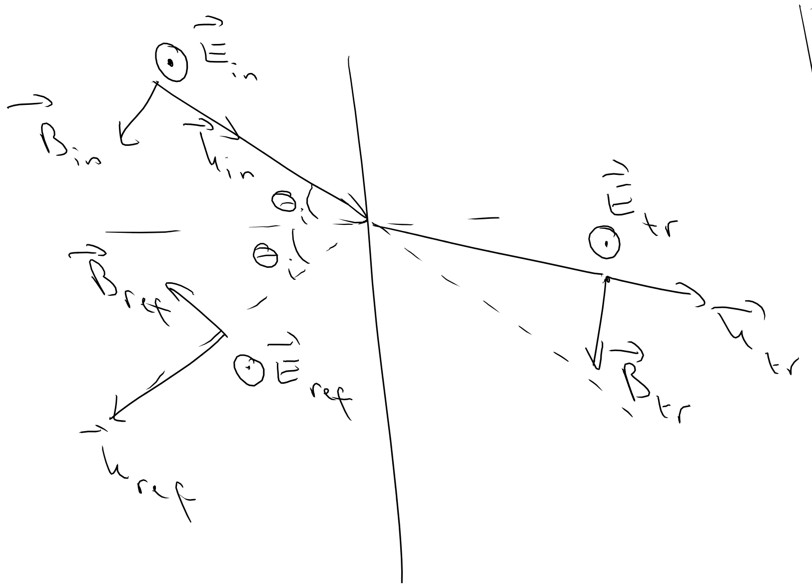
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Longrightarrow \boxed{\vec{k}\times\vec{E}_0 = \frac{w}{c}\vec{B}_0}$$

$$k = \frac{w}{v} = \frac{w}{c}n$$

$$\vec{k} = k\hat{k}$$

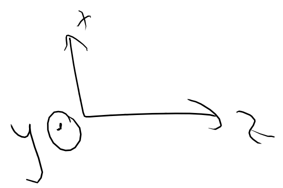
$$\underbrace{k}_{=\cancel{\frac{w}{c}}n}\hat{k}\times\vec{E}_0 = \cancel{\frac{w}{c}}\vec{B}_0$$

$$\boxed{n\hat{k}\times\vec{E} = \vec{B}}$$



$$\boxed{\vec{B} = n\,\hat{k} \times \vec{E}}$$

$$B = nE$$



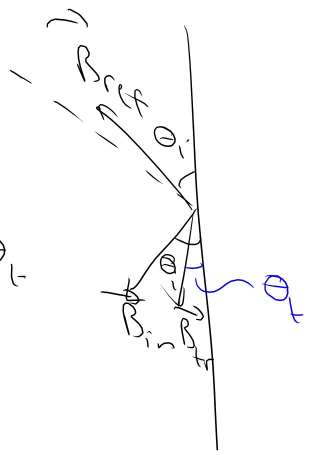
$\vec{D}_\perp$ is continuous

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

$E_\parallel$ is cont

$$\vec{E}^0_{in} + \vec{E}^0_{ref} = \vec{E}^0_{tr}$$

$$\boxed{E^0_{in} + E^0_{ref} = E^0_{tr}}$$



$B_\perp$ is cont.

$$B^0_{in}\sin\theta_i + B^0_{ref}\sin\theta_i = B^0_{tr}\sin\theta_t$$

$$\cancel{n_1} E^0_{in}\cancel{\sin\theta_i} + \cancel{n_1} E^0_{ref}\cancel{\sin\theta_i} = n_2 E^0_{tr}\cancel{\sin\theta_t}$$

$$n_1\sin\theta_i = n_2\sin\theta_t$$

$$H_\parallel = \frac{B_\parallel}{\mu} \text{ is cont.}$$

$$\frac{B^0_{in}}{\mu_1}\cos\theta_i - \frac{B^0_r}{\mu_1}\cos\theta_i = \frac{B_{tr}}{\mu_2}\cos\theta_t$$

$$\boxed{\frac{n_1}{\mu_1}\left[E^0_{in} - E^0_r\right]\cos\theta_i = \frac{n_2 E^0_{tr}}{\mu_2}\cos\theta_t}$$

$$E^0_{in} + E^0_r = E^0_{tr}$$

$$\frac{n_1}{\mu_1}\left[E^0_{in} - E^0_r\right]\cos\theta_i = \frac{n_2}{\mu_2}\cos\theta_t\left(E^0_{in} + E^0_r\right)$$

$$\left(\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t\right)E^0_{in}$$

$$= \left(\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t\right)E^0_r$$

$$\Rightarrow \boxed{E^0_r = \left[\frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}\right]E^0_{in}}$$

$$E^0_t = E^0_r + E^0_{in} = \frac{2\frac{n_1}{\mu_1}\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}E^0_{in}$$

for $\theta_i = 0 \Rightarrow \theta_t = 0$

$$\boxed{E^0_r = \frac{\frac{n_1}{\mu_1} - \frac{n_2}{\mu_2}}{\frac{n_1}{\mu_1} + \frac{n_2}{\mu_2}}E^0_{in}}$$

$$E_r^0 \overset{?}{=} 0$$

$$\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t = 0$$

$$\frac{n_1}{\mu_1}\cos\theta_i = \frac{n_2}{\mu_2}\cos\theta_t$$

$$\frac{n_1 \sin\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i} = \frac{n_2 \sin\theta_i}{\frac{n_2}{\mu_2}\cos\theta_t}$$

$$\boxed{\mu_1 \tan\theta_i = \mu_2 \tan\theta_t}$$

$$\frac{n_1^2}{\mu_1^2}\cos^2\theta_i = \frac{n_2^2}{\mu_2^2} - \frac{n_2^2 \sin^2\theta_t}{\mu_2^2}$$

$$\frac{n_1^2}{\mu_1^2}\cos^2\theta_i = \frac{n_2^2}{\mu_2^2} - \frac{n_1^2 \sin^2\theta_i}{\mu_2^2}$$

$$n_1^2\left(\frac{\cos^2\theta_i}{\mu_1^2} + \frac{\sin^2\theta_i}{\mu_2^2}\right) = \frac{n_2^2}{\mu_2^2}$$

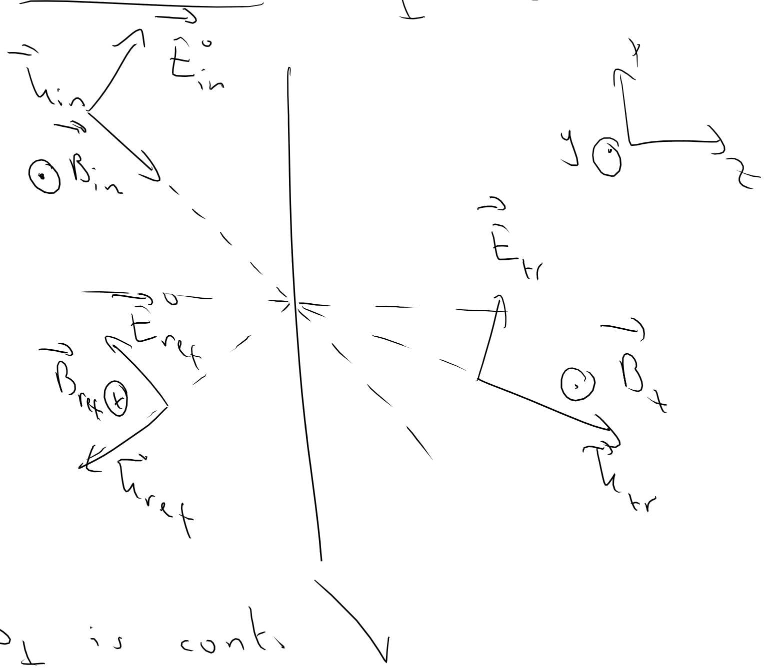
$$n_1^2\left[\sin^2\theta_i\left(\frac{1}{\mu_2^2} - \frac{1}{\mu_1^2}\right) + \frac{1}{\mu_1^2}\right] = \frac{n_2^2}{\mu_2^2}$$

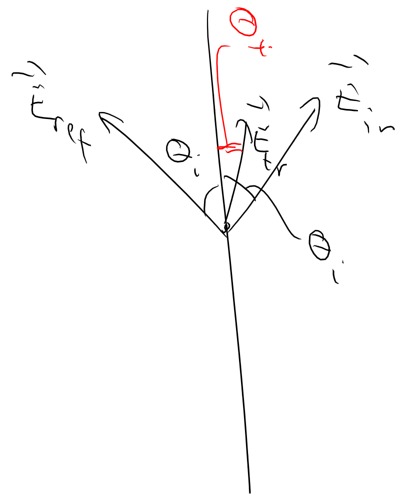
$$\boxed{\sin^2\theta_i = \frac{\frac{n_2^2}{\mu_2^2} - \frac{n_1^2}{\mu_1^2}}{\frac{1}{\mu_2^2} - \frac{1}{\mu_1^2}}}$$

$$n^2 = \varepsilon\mu$$

$$\sin^2\theta_i = \frac{\varepsilon_2^2 - \varepsilon_1^2}{\frac{1}{\mu_2^2} - \frac{1}{\mu_1^2}}$$

<u>case II</u> $\quad B_\perp = 0 \qquad n\hat{k}\times\vec{E} = \vec{B}$





$B_\perp$ is cont.

$H_\parallel$ is cont

$$\frac{B^0_{in}}{\mu_1} - \frac{B^0_r}{\mu_1} = \frac{B^0_t}{\mu_2} \Rightarrow \boxed{\frac{n_1}{\mu_1}\left(E^0_{in} - E^0_r\right) = \frac{n_2}{\mu_2}E^0_t}$$

$D_\perp$ is cont

$$\left(E^0_{in} - E^0_r\right)\cancel{\sin\theta_i}^{\frac{n_2}{n_1}}\varepsilon_1 = E_{tr}\,\varepsilon_2\cancel{\sin\theta_t}$$

$$\sin\theta_i = \frac{n_2}{n_1}\sin\theta_t$$

$$\left(E^0_{in} - E^0_r\right)\frac{\varepsilon_1}{n_1} = E_{tr}\frac{\varepsilon_2}{n_2}$$

$$\frac{\varepsilon_i}{n_i} = \frac{n_i}{\mu_i} \quad \text{since } n_i^2 = \varepsilon_i\mu_i$$

$$\left(E_{in}^{0} - E_{r}^{0}\right)\frac{n_1}{\mu_1} = E_{tr}\frac{n_2}{\mu_2}$$

<u>$D_{\parallel}$ is cont</u>

$$D_{\parallel} = \varepsilon E_{\parallel}$$

$$\boxed{\varepsilon_1\left(E_{in}^{0} + E_{r}^{0}\right)\cos\theta_i = \varepsilon_2 E_t \cos\theta_t}$$

$$\frac{n_1}{\mu_1}\left(E_{in}^{0} - E_{r}^{0}\right) = \frac{n_2}{\mu_2}E_t^{0}$$

$$\varepsilon_1\left(E_{in}^{0} + E_{r}^{0}\right)\cos\theta_i = \varepsilon_2\frac{\mu_2}{n_2}\frac{n_1}{\mu_1}\cos\theta_t\left(E_{in}^{0} - E_{r}^{0}\right)$$

$$E_r^{0} = \frac{\left[\varepsilon_2\frac{\mu_2}{\mu_1}\frac{n_1}{n_2}\cos\theta_t - \varepsilon_1\cos\theta_i\right]}{\varepsilon_2\frac{\mu_2}{\mu_1}\frac{n_1}{n_2}\cos\theta_t + \varepsilon_1\cos\theta_i}E_{in}^{0}$$

$$E_t^{0} = \frac{\mu_2}{n_2}\frac{n_1}{\mu_1}\left(E_{in}^{0} - E_{r}^{0}\right)$$

$$\boxed{E_t^{0} = \frac{\mu_2}{n_2}\frac{n_1}{\mu_1}\left(\frac{2\varepsilon_1\cos\theta_i}{\varepsilon_2\frac{\mu_2}{\mu_1}\frac{n_1}{n_2}\cos\theta_i + \varepsilon_1\cos\theta_i}\right)E_{in}^{0}}$$

$E_r^{0} = 0$ if

$$\varepsilon_2\frac{\mu_2}{\mu_1}\frac{n_1}{n_2}\cos\theta_t = \varepsilon_1\cos\theta_i$$

$$\varepsilon_1^2 \cos^2\theta_i = \left(\varepsilon_2 \frac{\mu_2}{\mu_1}\frac{n_1}{n_2}\right)^2 \left(1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i\right)$$

$$\varepsilon_1^2 \cos^2\theta_i = \left(\quad \right)^2 \left(1 - \frac{n_1^2}{n_2^2} + \frac{n_1^2}{n_2^2}\cos^2\theta_i\right)$$

$$\cos^2\theta_i = \frac{\left(\cancel{\varepsilon_2}^{\,n_2} \frac{\mu_2}{\mu_1}\frac{n_1}{\cancel{n_2}}\right)^2 \left(1 - \frac{n_1^2}{n_2^2}\right)}{\varepsilon_1^2 - \left(\cancel{\varepsilon_2}^{\,n_2} \frac{\mu_2}{\mu_1}\frac{n_1}{\cancel{n_2}}\right)^2 \left(\frac{n_1}{n_2}\right)^2} = \frac{\frac{n_2^2 n_1^2}{\mu_1}\left(1 - \frac{n_1^2}{n_2^2}\right)}{\varepsilon_1^2 - \frac{n_1^4}{\mu_1}}$$

$$\mu \sim 1$$
$$n \sim \sqrt{\varepsilon}$$

$$\cos^2\theta_i \simeq \frac{\cancel{\varepsilon_2^2}\,\frac{\varepsilon_1^2}{\cancel{\varepsilon_2^2}}\left(1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}\right)}{\varepsilon_1^2 - \cancel{\varepsilon_2^2}\,\frac{\varepsilon_1^2}{\cancel{\varepsilon_2^2}}\,\frac{\varepsilon_1^2}{\varepsilon_2^2}}$$

$$\cos^2\theta_i = \frac{\varepsilon_1^2\left(1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}\right)}{\varepsilon_1^2\left(1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}\right)} \simeq 1$$

when $B_\perp = 0$

$$E_r^0 = \left[\frac{\varepsilon_2 \frac{\mu_2}{\mu_1}\frac{n_1}{n_2}\cos\theta_t - \varepsilon_1 \cos\theta_i}{\varepsilon_2 \frac{\mu_2}{\mu_1}\frac{n_1}{n_2}\cos\theta_t + \varepsilon_1 \cos\theta_i}\right] E_{in}^0$$

when $E_\perp = 0$

$$E_r^0 = \left[\frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}\right] E_{in}^0$$

$E_\perp = 0$ & $B_\perp = 0$ when $\theta_i = \theta_t = 0$

$$\frac{\varepsilon_2 \frac{\mu_2}{\mu_1}\frac{n_1}{n_2} - \varepsilon_1}{\varepsilon_2 \frac{\mu_2}{\mu_1}\frac{n_1}{n_2} + \varepsilon_1} \stackrel{?}{=} \frac{\frac{n_1}{\mu_1} - \frac{n_2}{\mu_2}}{\frac{n_1}{\mu_1} + \frac{n_2}{\mu_2}}$$

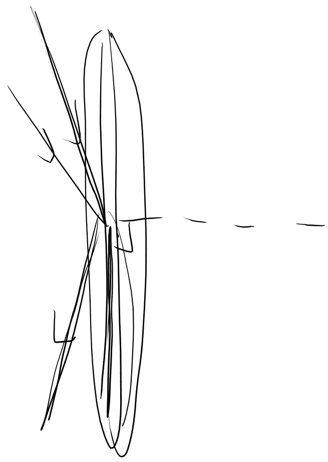
$$\frac{\overset{n_2}{\frac{\varepsilon_2 \cancel{\mu_2}}{n_2}}\frac{\cancel{\varepsilon_1}}{n_1} - \cancel{\varepsilon_1}}{\frac{\varepsilon_2 \mu_2}{n_2}\frac{n_2 \cancel{\varepsilon_1}}{n_1} + \cancel{\varepsilon_1}} = \left(\frac{\frac{\varepsilon_2\mu_2}{n_2 n_1} - 1}{\frac{\varepsilon_2\mu_2}{n_2 n_1} + 1}\right)\frac{\frac{n_1}{\mu_2}}{\frac{n_1}{\mu_2}}$$

$$= \frac{\frac{\varepsilon_2}{n_2} - \frac{n_1}{\mu_2}}{\frac{\varepsilon_2}{n_2} + \frac{n_1}{\mu_2}} = \boxed{\frac{\frac{n_2}{\mu_2} - \frac{n_1}{\mu_2}}{\frac{n_2}{\mu_2} + \frac{n_1}{\mu_2}}}$$

$$= \frac{n_2 - n_1}{n_2 + n_1}$$

$$n_1 \sin\Theta_1 = n_2 \sin\Theta_2$$

$$\Theta_2 = \frac{\pi}{2} \Rightarrow \sin\Theta_1^0 = \frac{n_2}{n_1}$$

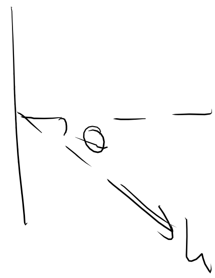


$$\Theta_1 > \Theta_1^0$$

$$\sin\Theta_2 = \frac{n_1}{n_2} \sin\Theta_1 > 1$$

$$\Rightarrow \Theta_2 \in \mathbb{C}$$

$$\vec{k}_t = k_\parallel \hat{n}_\parallel + k_\perp \hat{n}_\perp$$



$$k_\perp$$

$$k_\parallel = k \sin\Theta \in \mathbb{R}$$

$$k_\perp = k \cos\Theta = k\sqrt{1 - \sin^2\Theta} \in \mathbb{C}$$

$$e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$i\vec{k}\cdot\vec{r} = i\vec{k}_\parallel \cdot \vec{r}_\parallel + ik_\perp r_\perp$$

$$e^{i\vec{k}\cdot\vec{r}} = e^{i\vec{k}_\parallel\cdot\vec{r}_\parallel} e^{-\kappa_\perp r_\perp}$$

EM Waves in Conductors

vacuum | conductor ε, μ, σ

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \longrightarrow \boxed{\vec{k} \times \vec{E} = \frac{\omega}{c}\vec{B}}$$

$\vec{k} = k\hat{z}$

$\vec{E} = E\hat{x}$

$\vec{k} \times \vec{E} = kE\hat{y} = \frac{\omega}{c}B\hat{y}$

$$\boxed{B = k\frac{\omega}{c}E}$$

$k\frac{\omega}{c} \equiv n = n_1 + in_2$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{j} = \sigma\vec{E}$$

$$\frac{1}{\mu}\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\sigma\vec{E} + \frac{1}{c}\epsilon\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{E})}_{0} - \nabla^2\vec{E} = -\frac{1}{c}\mu\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\sigma\vec{E} + \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t}\right)$$

$$\boxed{\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = -4\pi\frac{\mu\sigma}{c^2}\frac{\partial \vec{E}}{\partial t}} \qquad v = \frac{c}{\sqrt{\varepsilon\mu}}$$

Look for a PW solution of the form:

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)} \Rightarrow \boxed{\left(-\frac{\omega^2}{v^2} + k^2\right) = -\frac{4\pi\mu\sigma}{c^2}(-i\omega)}$$

$$k^2 = \frac{\omega^2 n^2}{c^2} + i\frac{4\pi\mu\sigma}{c^2}\omega$$

$$k = k_1 + ik_2 \qquad k_i \in \mathbb{R}$$

$$\boxed{\begin{cases} k_1^2 - k_2^2 = \dfrac{\omega^2 n^2}{c^2} \\ k_1 k_2 = \dfrac{2\pi\mu\sigma}{c^2} \end{cases}} \qquad \boxed{\lambda = \frac{1}{k_2} \text{ : skin depth}}$$

$$\vec{E} = \vec{E}_0 e^{i(k_1 z - \omega t)} e^{-k_2 z}$$

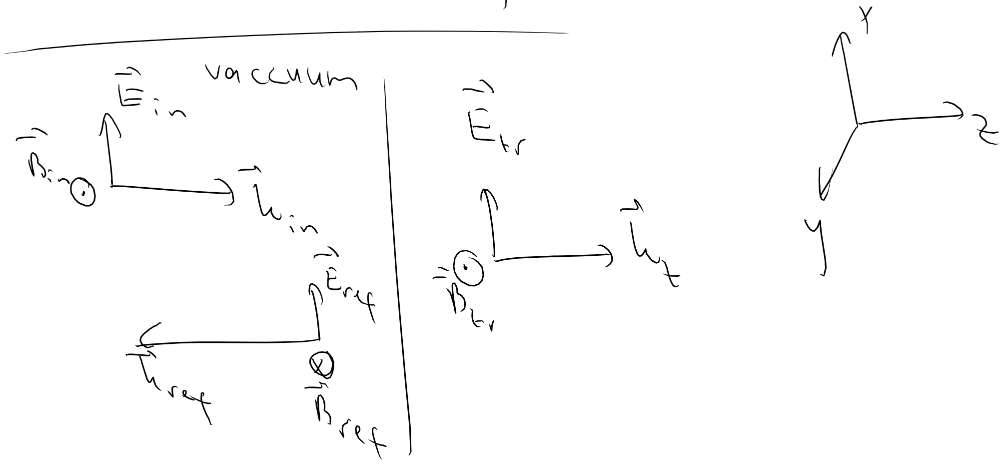
$$= \vec{E}_0 e^{i(k_1 z - \omega t)} \exp\left\{-\frac{2\pi\mu\sigma}{c^2 k_1} z\right\}$$

Small sigma limit: $k_1 \approx \frac{\omega n}{c}$

$$k_2 \approx \frac{2\pi\mu\sigma}{\cancel{c}\frac{\omega n}{\cancel{c}}} = \boxed{\frac{2\pi\mu}{n}\frac{\sigma}{\omega} = k_2}$$

$$\boxed{\lambda \approx \frac{n}{2\pi\mu}\frac{\omega}{\sigma}}$$

Reflection on a Conductive Surface (Normal Incidence)



$\vec{E} = E^0 \hat{x}$ $\quad \vec{B}_{ref} = -B_r \hat{y}$ $\quad D_\perp = 0$

$\vec{B} = B^0 \hat{y}$ $\quad B_\perp = 0$

$\vec{k}_{in} = k\hat{z}$ $\quad \vec{k}_t = k_t \hat{z}$

$\vec{k}_{ref} = -k\hat{z}$ $\quad k_t = k_1 + ik_2$

$\left.\begin{array}{l} E_\parallel \text{ is cont.} \\ H_\parallel \text{ is cont} \end{array}\right\}$

$E_\parallel$ is cont

$$\boxed{E_{in} + E_r = E_t}$$

$$H_\parallel = \frac{B_\parallel}{\mu} \text{ is cont}$$

$$\Rightarrow B_{in} - B_{ref} = \frac{B_{tr}}{\mu}$$

$$\begin{cases} B_{in} = E_{in} \\ B_{ref} = E_{ref} \\ B_{tr} = n E_{tr} \end{cases}$$

$n = n_1 + in_2$

$$E_{in} - E_r = \frac{n}{\mu} E_{tr} = \frac{n}{\mu}\left(E_{in} + E_r\right)$$

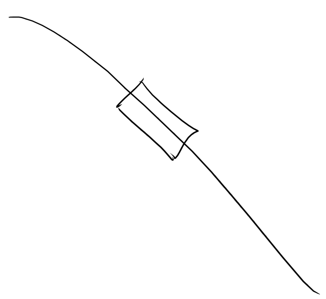
$$\boxed{E_r^0 = \frac{1-\frac{n}{\mu}}{1+\frac{n}{\mu}}\, E_{in}^0} \qquad \boxed{E_{tr} = \frac{2}{1+\frac{n}{\mu}}\, E_{in}^0}$$

<u>Note</u> $\boxed{E_r^0 \ \& \ E_{tr} \text{ are complex}}$

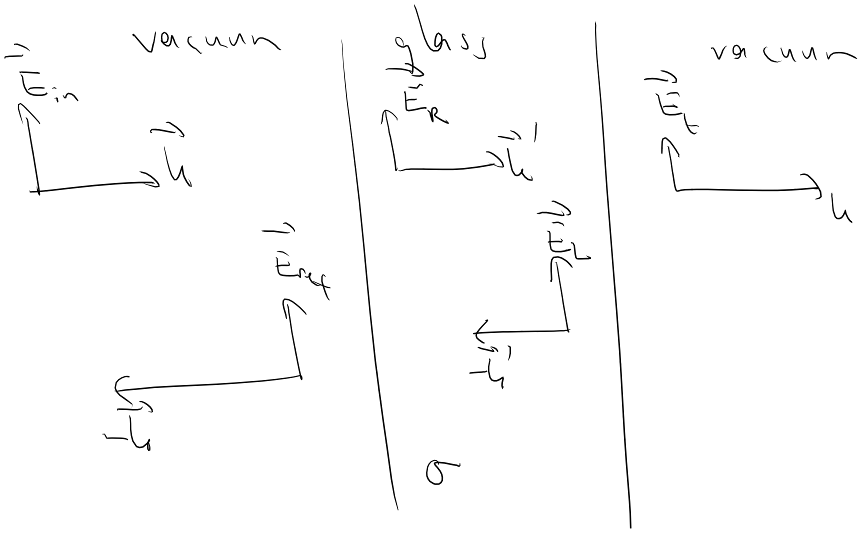
$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\sigma \vec{E} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

$\Rightarrow H_\parallel$ is continuous

$$\int \left(\vec{\nabla} \times \vec{H}\right) \cdot d\vec{S} = \frac{4\pi}{c}\sigma \int \vec{E} \cdot d\vec{S} + \int \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$



Example



Example

perfect metal $\sigma \to \infty$