$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \quad \vec{A}\to\vec{A}+\vec{\nabla}\Lambda$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \Rightarrow \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \; ; \; \phi\to\phi-\frac{\partial\Lambda}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow$ Coulomb Gauge

$$\phi(\vec{r},t) = \int \frac{1}{4\pi\varepsilon}\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

$\vec{\nabla}\cdot\vec{A} + \mu\varepsilon\frac{\partial\phi}{\partial t} = 0 \Rightarrow$ Lorentz Gauge

$$\underbrace{\left(\mu\varepsilon\frac{\partial^2}{\partial t^2} - \nabla^2\right)}_{\Box^2 \text{: d'Alambertian.}}\begin{pmatrix}\phi\\ \vec{A}\end{pmatrix} = \begin{pmatrix}+\rho/\varepsilon\\ +\mu\vec{J}\end{pmatrix}$$

$\phi = 0$ is it possible?

$$\phi_0(\vec{r},t) - \frac{\partial\Lambda(\vec{r},t)}{\partial t} = 0$$

$$\Lambda(\vec{r},t) = \int_{t_0}^{t}\phi_0(\vec{r},t')\,dt' \Rightarrow \frac{\partial\Lambda}{\partial t} = \phi_0(\vec{r},t)$$

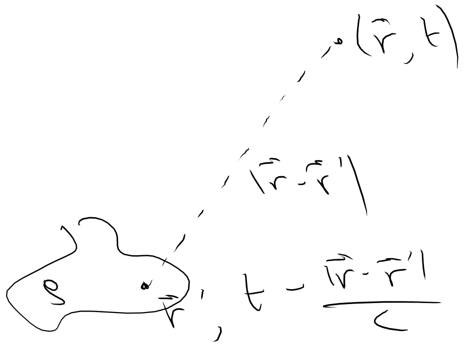
$$\boxed{\phi_0 - \frac{\partial\Lambda}{\partial t} = 0}$$

$$\left(\mu\varepsilon \frac{\partial^2}{\partial t^2} - \nabla^2\right) f(r,t) = g(r,t)$$

$$\underbrace{\left(\mu\varepsilon \frac{\partial^2}{\partial t^2} - \nabla^2\right)}\begin{pmatrix}\phi \\ \vec{A}\end{pmatrix} = \begin{pmatrix}+\rho/\varepsilon \\ +\mu \vec{J}\end{pmatrix}$$

$$f(\vec{r},t) = \int \frac{1}{4\pi} \frac{g\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} d^3r'$$

ϕ



$$c = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$f(\vec{r},t) = \int \frac{1}{4\pi} \frac{g\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} d^3r'$$

$$\frac{\partial^2 f}{\partial t^2} = \int \frac{1}{4\pi} \frac{\ddot{g}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} d^3r'$$

$$\dot{g}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) = \left.\frac{\partial g(\vec{r}',t')}{\partial t'}\right|_{t' = t - \frac{|\vec{r}-\vec{r}'|}{c}}$$

$$\nabla^2 f = \partial_i \partial_i f$$

$$\partial_i |\vec{r}-\vec{r}'| = \partial_i \sqrt{(x_j - x_j')^2} = \frac{1}{\cancel{2}} \frac{\cancel{2}(x_j - x_j')\delta_{ij}}{\sqrt{(\ )}}$$

$$\boxed{\partial_i |\vec r - \vec r'| = \frac{(x_i - x_i')}{|\vec r - \vec r'|}}$$

$$\partial_i^2 f(\vec r, t) = \int \frac{d^3 r'}{4\pi} \partial_i^2 \frac{g(\vec r', \overbrace{t - \frac{|\vec r - \vec r'|}{c}}^{t_r})}{|\vec r - \vec r'|}$$

$$= \int \frac{d^3 r'}{4\pi} \partial_i \left[\dot g(\vec r, t_r) \left(-\frac{1}{c}\right) \frac{(x_i - x_i')}{|\vec r - \vec r'|} \frac{1}{|\vec r - \vec r'|} + g\, \partial_i \frac{1}{|\vec r - \vec r'|} \right]$$

$$= \int \frac{d^3 r'}{4\pi} \left[\ddot g(\vec r', t_r) \left\{ \left(-\frac{1}{c}\right) \frac{\cancel{x_i - x_i'}}{\cancel{|\vec r - \vec r'|}} \right\}^2 \frac{1}{|\vec r - \vec r'|} \right.$$

$$+ \dot g(\vec r', t_r) \left(-\frac{1}{c}\right) \frac{\partial}{\partial x_i} \frac{(x_i - x_i')}{|\vec r - \vec r'|^2}$$

$$+ \dot g(\vec r', t_r) \left(-\frac{1}{c}\right) \frac{(x_i - x_i')}{|\vec r - \vec r'|} \partial_i \frac{1}{|\vec r - \vec r'|}$$

$$\left. + g(\vec r', t_r) \partial_i^2 \frac{1}{|\vec r - \vec r'|} \right]$$

$$\boxed{t_r = t - \frac{|\vec r - \vec r'|}{c}}$$

$$\partial_i^2 \frac{1}{|\vec r - \vec r'|} = \nabla^2 \frac{1}{|\vec r - \vec r'|} = 4\pi \delta^{(3)}(\vec r - \vec r')$$

$$\nabla^2 f = g(\vec r, t) + \mu \varepsilon \frac{\partial^2 f}{\partial t^2}$$

$$+ \frac{1}{4\pi} \int d^3 r' \; \dot g(\vec r', t_r) \left(-\frac{1}{c}\right) \left[\frac{\partial}{\partial x_i} \frac{x_i - x_i'}{|\vec r - \vec r'|^2} + \frac{(x_i - x_i')}{|\vec r - \vec r'|} \frac{\partial}{\partial x_i} \frac{1}{|\vec r - \vec r'|} \right]$$

$$\left(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right) f = g +$$

$$\frac{1}{4\pi} \int d^3 r' \; \dot g(\vec r', t_r) \left(-\frac{1}{c}\right) \left[\cancel{\frac{1/3}{|\vec r - \vec r'|^2}} + \frac{\cancel{(x_i - x_i')}(-\cancel{2})}{|\vec r - \cancel{\vec r'}|^{2}} \cancel{\frac{(x_i - x_i')}{|\vec r - \vec r'|}} \right.$$

$$+\cancel{\frac{(x_i-x_i')}{|\vec r-\vec r'|}}\ \cancel{\frac{-1}{|\vec r-\vec r'|^2}}\ \cancel{\frac{(x_i-x_i')}{|\vec r-\vec r'|}}\Bigg]$$

$$\boxed{\left(\nabla^2-\mu\varepsilon\frac{\partial^2}{\partial t^2}\right)f=g}\qquad f(\vec r,t)=\int\frac{d^3r'}{4\pi}\frac{g(\vec r',t_r)}{|\vec r-\vec r'|}$$

$$\frac{\partial(x_i-x_i')}{\partial x_i}=\delta_{ii}=3$$

$$\phi(\vec r,t)=\int d^3r'\frac{1}{4\pi\varepsilon}\frac{\rho(\vec r',t_r)}{|\vec r-\vec r'|}\qquad t_r=t-\frac{|\vec r-\vec r'|}{c}$$

$$\vec A(\vec r,t)=\int d^3r'\frac{\mu}{4\pi}\frac{\vec J(\vec r',t_r)}{|\vec r-\vec r'|}$$

$$\vec E=-\nabla\phi-\frac{\partial\vec A}{\partial t}$$

$$\vec E(\vec r,t)=\int d^3r'\frac{1}{4\pi\varepsilon}\Bigg[\rho(\vec r',t_r)\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^3}+\dot\rho(\vec r',t_r)\frac{1}{c}\left(\vec\nabla|\vec r-\vec r'|\right)\frac{1}{|\vec r-\vec r'|}$$
$$-\frac{1}{c^2}\frac{\dot{\vec J}(\vec r',t_r)}{|\vec r-\vec r'|}\Bigg]$$

$$\vec B(\vec r,t)=\vec\nabla\times\vec A=\int d^3r'\frac{\mu}{4\pi}\Bigg[\frac{\vec J(\vec r',t_r)\times(\vec r-\vec r')}{|\vec r-\vec r'|^3}+\frac{\dot{\vec J}(\vec r',t_r)\times\left(\frac{1}{c}\right)\vec\nabla|\vec r-\vec r'|}{|\vec r-\vec r'|}\Bigg]$$

Jefimenko's eqns

<u>Exercise</u> A point charge, q, following the trajectory $\vec r(t)=\vec w(t)$

$$\rho(\vec r,t)=q\,\delta^{(3)}(\vec r-\vec w(t))$$
$$\vec J=q\,\vec v(t)\,\delta^{(3)}(\vec r-\vec w(t))\qquad\boxed{\vec v(t)=\frac{d\vec w(t)}{dt}=\dot{\vec w}(t)}$$

$$\phi(\vec r,t) = \int \frac{1}{4\pi\varepsilon} \frac{q\,\delta^{(3)}(\vec r' - \vec w(t_r))}{|\vec r - \vec r'|}\, d^3r'$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec r - \vec w(t_r)|} \frac{1}{|\vec\nabla_{r'}(\vec r' - \vec w(t_r))|}$$

$$\int dx\, \delta(f(x))\, g(x) = \sum_{x_0} \frac{g(x_0)}{|f'(x_0)|} \Leftarrow \qquad \vec r' = \vec w\left(t - \frac{|\vec r - \vec r'|}{c}\right)$$

BONUS

$$\int d^3r'\, \delta^{(3)}(\vec r' - \vec w(t_r))\, g(\vec r') \overset{?}{=} \frac{g(\vec w(t_r))}{|\vec\nabla_{r'}(\vec r' - \vec w(t_r))|} \Leftarrow \text{check}$$

$$\phi(\vec r,t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec r - \vec w(t_r)|} \frac{1}{|\vec\nabla_{r'}\cdot(\vec r' - \vec w(t_r))|}$$

$$\vec\nabla_{r'}\cdot\left(\vec r' - \vec w\left(t - \frac{|\vec r - \vec r'|}{c}\right)\right) = \frac{\partial}{\partial x_i'}\left(x_i' - w_i(t_r)\right)$$

$$= 3 - \dot w_i(t_r) \frac{\partial}{\partial x_i'}\left(t - \frac{|\vec r - \vec r'|}{c}\right)$$

$$= 3 - \dot w_i(t_r)\left(-\frac{1}{c}\right) \frac{x_i - x_i'}{|\vec r - \vec r'|}$$

$$= 3 + \frac{1}{c} \frac{\vec v(t_r)\cdot(\vec r - \vec r')}{|\vec r - \vec r'|}$$

$$\phi(\vec r,t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec r - \underbrace{\vec w(t_r)}_{\vec r'}|} \frac{1}{\left[3 + \frac{1}{c}\vec v(t_r) \frac{(\vec r - \vec r')}{|\vec r - \vec r'|}\right]}$$

$$\vec r' = \vec w(t_r)$$

$$\phi(\vec r,t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\left\{\cancel{3}\,|\vec r - \vec w(t_r)| + \frac{1}{c}\vec v(t_r)\cdot(\vec r - \vec w(t_r))\right\}}$$

special case $\vec w(t) = \vec r_0$ $\qquad \phi(\vec r,t) = \dfrac{1}{4\pi\varepsilon} \dfrac{q}{|\vec r - \vec r_0|}$

$$\phi(\vec r,t)=\frac{1}{4\pi\varepsilon_0}\frac{q}{\left\{\cancel{3}|\vec r-\vec w(t_r)|+\frac{1}{c}\vec v(t_r)\cdot(\vec r-\vec w(t_r))\right\}}$$

$$\vec A(\vec r,t)=\frac{\mu_0}{4\pi}\frac{q\vec v(t_r)}{\left[|\vec r-\vec w(t_r)|+\frac{\vec v(t_r)}{c}\cdot(\vec r-\vec w(t_r))\right]}$$

$$\boxed{\vec A(\vec r,t)=\mu_0\varepsilon_0\vec v(t_r)\phi(\vec r,t)}$$

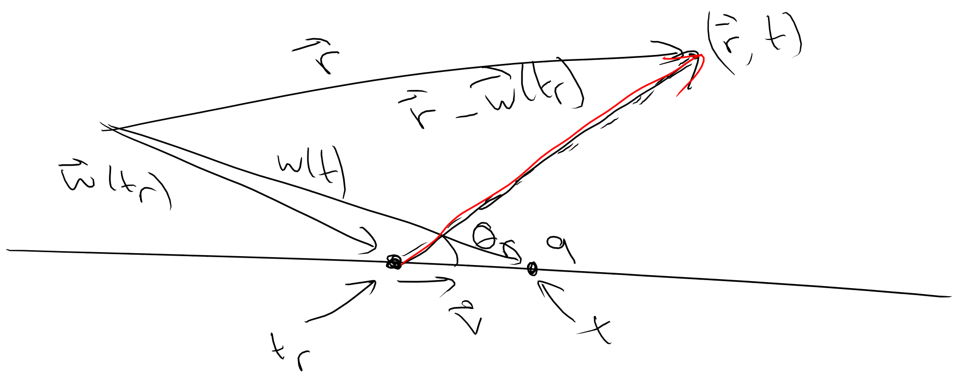
Point charge moving at constant velocity, $\vec v$

$$\phi(\vec r,t)=\frac{1}{4\pi\varepsilon_0}\frac{q}{\left[|\vec r-\vec w(t_r)|+\frac{\vec v}{c}\cdot(\vec r-\vec w(t_r))\right]}$$

$$\vec A(\vec r,t)=\frac{1}{c^2}\vec v\,\phi(\vec r,t)$$

$$\boxed{\vec w(t_r)=\vec r_0+\vec v t}$$

$$\vec r'=\vec r_0+\vec v\left(t-\frac{|\vec r-\vec r'|}{c}\right)$$



$$\boxed{\vec r=\vec w(t)}$$

$$\vec r(t)=\vec r_0+\vec v t$$

$$|\vec r-\vec w(t_r)|=?$$

$$(\vec r-\vec w(t_r))\cdot\vec v$$
$$=|\vec r-\vec w(t_r)|\,v\cos\theta_r$$

$$\vec r'=\vec r_0+\vec v\left(t-\frac{|\vec r-\vec r'|}{c}\right)$$

$$\vec r-\vec w(t_r)=\vec r-\vec r'=?$$

Midterm on April 1st, 13^{30} in P422