$x^\mu = (ct, \vec{r})$: four vector

$x^0 = ct$

$x'^\mu \equiv \Lambda^\mu{}_\nu x^\mu$: Covariant vector

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$-c^2t^2 + \vec{r}^2 = \underline{x^\mu} \underline{x_\mu} = x^0 x_0 + \vec{r}$: is invariant

$x_0 = -ct$

$x_\mu \equiv (-ct, \vec{r})$: Contravariant

$-c^2t^2 + \vec{r}^2$	$c^2t^2 - \vec{r}^2$
$x^2 = -c^2t^2 + \vec{r}^2$	$x^2 = c^2t^2 - \vec{r}^2$
$x^\mu = (ct, \vec{r})$	$x^\mu = (ct, \vec{r})$
$x_\mu = (-ct, \vec{r})$	$x_\mu = (ct, -\vec{r})$

$x^0 = ct$

$x_0 = -ct = -x^0$

$d\vec{r}$ dt

$c^2(dt)^2 - (d\vec{r})^2 = (d\tau)^2 c^2$

$d\tau = \frac{1}{c}\sqrt{c^2(dt)^2 - d\vec{r}^2}$

$$\boxed{d\tau = dt\sqrt{1 - \frac{u^2}{c^2}}}$$

τ : proper time

$\vec{u} \equiv \frac{d\vec{r}}{dt}$

$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} > 1$

$$\eta^\mu = \frac{dx^\mu}{d\tau}$$

$$\eta'^\mu = \frac{dx'^\mu}{d\tau} = \frac{d}{d\tau}\left(\Lambda^\mu{}_\nu x^\nu\right) = \sum_\nu \Lambda^\mu{}_\nu \underbrace{\frac{dx^\nu}{d\tau}}_{\eta^\nu}$$

$$\eta'^\mu = \Lambda^\mu{}_\nu \eta^\nu$$

η^μ: four velocity

$$\boxed{\eta^\mu = \frac{dx^\mu}{d\tau}}$$

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{\sqrt{1-\frac{u^2}{c^2}}\,dt} = \boxed{\frac{c}{\sqrt{1-\frac{u^2}{c^2}}} = \eta^0}$$

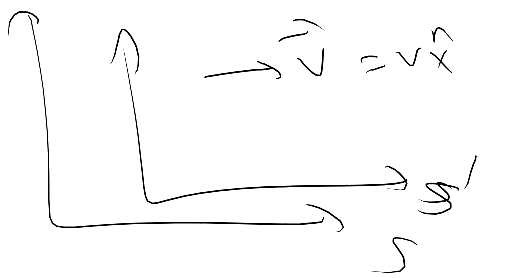
$$\eta^i = \frac{dx^i}{d\tau} = \frac{dx^i}{\sqrt{1-\frac{u^2}{c^2}}\,dt} = \frac{u^i}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\boxed{\vec{\eta} = \frac{\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}}$$

$$(\eta^0)^2 - (\vec{\eta})^2 = \frac{c^2}{\left(\sqrt{1-\frac{u^2}{c^2}}\right)^2} - \frac{u^2}{\left(\sqrt{1-\frac{u^2}{c^2}}\right)^2}$$

$$\boxed{-(\eta^0)^2 + (\vec{\eta})^2 = -c^2}$$

$$\eta'^0 = \gamma(\eta^0 - \beta\eta^1)$$



$$\eta'^1 = \gamma(\eta^1 - \beta\eta^0)$$

$$\eta'^2 = \eta^2$$

$$\eta'^3 = \eta^3$$

ordinary velocity addition

$$\frac{dx'}{dt'} = \frac{\cancel{\gamma}(dx - v\,dt)\,\frac{1}{dt}}{\cancel{\gamma}\left(dt - \frac{v}{c^2}dx\right)\frac{1}{dt}} = \frac{u - v}{1 - \frac{vu}{c^2}}$$

$$\boxed{u'_x = \frac{u - v}{1 - \frac{vu}{c^2}}}$$

$$\frac{dy'}{dt'} = \frac{dy\,\frac{1}{dt}}{\gamma\left(dt - \frac{v}{c^2}dx\right)\frac{1}{dt}} =$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)} \qquad u'_z = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

$$\boxed{p^\mu = m\eta^\mu} \qquad p^2 = m^2\eta^\mu\eta_\mu = -m^2c^2$$

$$p^i = m\eta^i = \frac{m\,u^i}{\sqrt{1 - u^2/c^2}} \Rightarrow \vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}$$

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-\frac{u^2}{c^2}}} \equiv \frac{E}{c}$$

$$E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \Rightarrow E\Big|_{u=0} = mc^2$$

$$\boxed{KE = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2}$$

$$= mc^2\left(1-\frac{u^2}{c^2}\right)^{-1/2} - mc^2$$

$$\frac{u}{c} \ll 1 \quad \simeq mc^2\left(\cancel{1}+\frac{u^2}{2c^2}\right) - \cancel{mc^2} = \frac{1}{2}mu^2$$

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \; ; \; \vec{E} = \frac{m}{\sqrt{1-u^2/c^2}}c^2$$

$$m_{\substack{eff \\ rel}} = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}$$

the mass of an object $\equiv m$ = constant, doesnot depend on the speed.

Energy = mass

Energy & momentum is conserved!

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right)$$

$$\vec{F} = \underbrace{\frac{m}{\sqrt{1-u^2/c^2}}}_{m_{rel}} \frac{d\vec{u}}{dt} + m\vec{u}\,\frac{d}{dt}\frac{1}{\sqrt{1-u^2/c^2}}$$

Ex

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

$$p_1^0 = \frac{mc}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{5}{4}mc \qquad p_2^0 = \frac{5}{4}mc$$

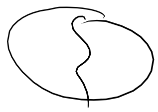
$$p_1^x = \frac{m\frac{3}{5}c}{\sqrt{1-u^2/c^2}} = \frac{3}{4}mc \qquad p_2^x = -\frac{3}{4}mc$$

$$p_1^y = 0 \qquad p_2^y = 0$$

$$p_1^z = 0 \qquad p_2^z = 0$$

$$\left.\begin{aligned} p_1 &= \left(\frac{5}{4}mc, \frac{3}{4}mc, 0, 0\right) \\ p_2 &= \left(\frac{5}{4}mc, -\frac{3}{4}mc, 0, 0\right) \end{aligned}\right\} p_1 + p_2 = \left(\frac{5}{2}mc, \vec{0}\right)$$

after collision



$$P_f = (Mc, \vec{0}) = \left(\frac{5}{2}mc, \vec{0}\right)$$

$$M \to \frac{5}{2}m = 2m + \frac{m}{2}$$

$$P = \left(\frac{mc}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \right) = \left(\frac{E}{c}, \vec{p}\right)$$

Example Constant Force $(\vec{u}(t=0) = \vec{0})$

$$\frac{d\vec{p}}{dt} = \vec{F} \Rightarrow \vec{p} = \vec{F}t + \vec{p}_0 \qquad \vec{p}_0 = \vec{0}$$

$$\frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} = \vec{F}t$$

$$\vec{F} = F\hat{x} \qquad \frac{mu}{\sqrt{1-u^2/c^2}} = (Ft)$$

$$\vec{u} = u\hat{x}$$

$$m^2u^2 = \left(1 - \frac{u^2}{c^2}\right)(Ft)^2$$

$$u^2\left[1 + \left(\frac{Ft}{mc}\right)^2\right] = \left(\frac{Ft}{m}\right)^2$$

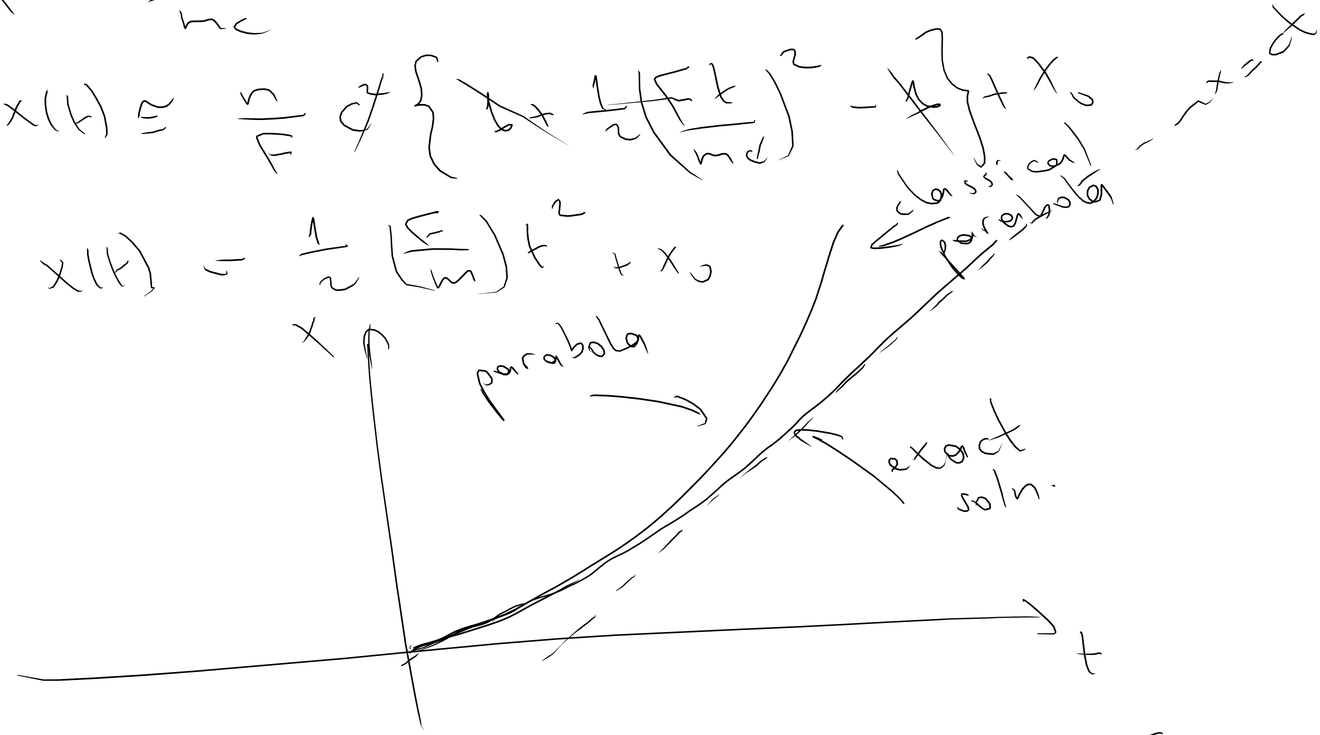
$$\frac{dx}{dt} = u = \frac{Ft/m}{\left[1 + \left(\frac{Ft}{mc}\right)^2\right]^{1/2}} = \frac{mc^2}{F}\frac{d}{dt}\left[1 + \left(\frac{Ft}{mc}\right)^2\right]^{1/2}$$

$$x(t) = \frac{m}{F}c^2\left\{\left[1+\left(\frac{Ft}{mc}\right)^2\right]^{1/2} - 1\right\} + x(t=0)$$

If $\frac{Ft}{mc} \ll 1$

$$x(t) \simeq \frac{m}{F}c^{2}\left\{\cancel{1} + \frac{1}{2}\left(\frac{Ft}{mc}\right)^2 - \cancel{1}\right\} + x_0$$

$$x(t) = \frac{1}{2}\left(\frac{F}{m}\right)t^2 + x_0$$



$$x(t) = \frac{m}{F}c^2\left\{\left[1+\left(\frac{Ft}{mc}\right)^2\right]^{1/2} - 1\right\} + \underbrace{x(t=0)}_{=0}$$

$$= \frac{m}{F}c^2\left\{\left(\frac{Ft}{mc}\right)\left[1+\left(\frac{mc}{Ft}\right)^2\right]^{1/2} - 1\right\} + \overset{0}{\overset{\shortparallel}{x_0}}!$$

$$= \frac{m}{F}c^2\left\{\frac{Ft}{mc}\left[1+\frac{1}{2}\left(\frac{mc}{Ft}\right)^2\right] - 1\right\}$$

$$x(t) \underset{t\to\infty}{=} \frac{m}{F}c^2\left\{\frac{Ft}{mc} - 1\right]$$

$$\boxed{x(t) \underset{t\to\infty}{=} ct - \frac{mc^2}{F}}$$

$$\vec{F} = \frac{d\vec{p}}{dt} \Leftarrow \text{not a 4-vector}$$

$$K^\mu = \frac{dp^\mu}{d\tau} \Leftarrow \text{4-vector}$$

$$\boxed{d\tau = \sqrt{1-\frac{u^2}{c^2}}\ dt}$$ proper time.

$$\frac{u}{\sqrt{1-u^2/c^2}} = \frac{dx}{d\tau} = \eta = \text{const}$$

$$\frac{L}{\eta} = \left(\frac{L}{u}\right)\sqrt{1-\frac{u^2}{c^2}}$$