Covariant Formalism

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$(ct, \vec{r}) \equiv x^\mu$$

$$\left.\begin{aligned} x^0 = ct &\to \gamma(\underset{x^0}{ct} - \beta x^1) \\ x^1 &\to \gamma(x^1 - \beta x^0) \\ x^2 &\to x^2 \\ x^3 &\to x^3 \end{aligned}\right\} \quad x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$V^\mu = (V^0, \vec{V}) \text{ if } V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

$$\Rightarrow V^\mu \text{ is a 4-vector}$$

$$T^{\mu\nu} \equiv V_1^\mu V_2^\nu$$

$$T^{\mu\nu} = V_1^\mu V_2^\nu \longrightarrow V_1'^\mu V_2'^\nu$$

$$= \Lambda^\mu{}_\alpha V_1^\alpha \Lambda^\nu{}_\beta V_2^\beta$$

$$T^{\mu\nu} \longrightarrow \boxed{T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}}$$

$$T'^{00} = \Lambda^{0}{}_{0}\Lambda^{0}{}_{0}T^{00} + \Lambda^{0}{}_{1}\Lambda^{0}{}_{0}T^{10} + \Lambda^{0}{}_{2}\Lambda^{0}{}_{0}T^{20}$$

$$+ \dots$$

$$T^{\mu\nu} = -T^{\nu\mu} \qquad \left(\text{e.g. } T^{\mu\nu} = V_1^{\mu}V_2^{\nu} - V_1^{\nu}V_2^{\mu}\right)$$

$$\begin{array}{c|c} & \xrightarrow{\mu} \\ \hline \nu\downarrow & \begin{pmatrix} 0 & T^{10} & T^{20} & T^{30} \\ T^{01} & 0 & T^{21} & T^{31} \\ T^{02} & -T^{21} & 0 & T^{32} \\ T^{03} & -T^{31} & -T^{32} & 0 \end{pmatrix} \end{array} = T^{\mu\nu}$$

$$T^{00} = -T^{00}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & & 0 \\ -\gamma\beta & \gamma & & \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}$$

$$\Lambda^{0}{}_{1}, \Lambda^{0}{}_{0} \neq 0$$

$$T'^{10} = \Lambda^{1}{}_{\alpha}\Lambda^{0}{}_{\beta}T^{\alpha\beta} =$$

$$= \Lambda^{1}{}_{1}\Lambda^{0}{}_{0}T^{10} + \Lambda^{1}{}_{0}\Lambda^{0}{}_{1}T^{01}$$

$$T'^{10} = \gamma^2 T^{10} + \gamma^2\beta^2\left(-T^{10}\right)$$

$$T'^{10} = \cancel{\gamma^2}\cancel{(1-\beta^2)}\, T^{10} = T^{10}$$

$$\boxed{T'^{10} = T^{10}}$$

$$E'_x = E_x$$

$$B'_x = B_x$$

$$T'^{20} = \Lambda^2{}_\alpha \Lambda^0{}_\beta T^{\alpha\beta} = \Lambda^2{}_2 \Lambda^0{}_0 T^{20} + \Lambda^2{}_2 \Lambda^0{}_1 T^{21}$$

$$T'^{20} = \gamma T^{20} - \beta\gamma T^{21}$$

$$\boxed{T'^{20} = \gamma\left(T^{20} - \beta T^{21}\right)}$$

$$E'_y = \gamma(E_y - \beta c B_z)$$

$$cB'_y = \gamma(cB_y + \beta E_z)$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ & 0 & -E_z/c & E_y/c \\ & & 0 & -E_x/c \\ & & & 0 \end{pmatrix}$$

$\vec{\nabla}\cdot\vec{B} = 0$ $\quad$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$

$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ $\quad$ $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

if $\rho = 0$ $\quad$ $\vec{J} = 0$

$\vec{\nabla}\cdot\vec{B} = 0$ $\quad$ $\vec{\nabla}\times\vec{B} = \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t}$

$\vec{\nabla}\cdot\vec{E} = 0$ $\quad$ $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

$F^{\mu\nu}$: Field strength tensor

$G^{\mu\nu}$: Dual field strength tensor

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu}$$

$\underline{\nu = 0}$ $\quad$ $F^{00} = 0$ $\quad$ $F^{10} = -E_x/c$

$F^{20} = -E_y/c$

$F^{30} = -E_z/c$

$$\frac{\partial F^{\mu 0}}{\partial x^\mu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = -\frac{1}{c}\vec{\nabla}\cdot\vec{E}$$

$$A^\mu A_\mu = -(A^0)^2 + (\vec{A})^2$$

$$\boxed{\frac{\partial F^{\mu 0}}{\partial x^\mu} = -\frac{\rho}{c}\frac{1}{\varepsilon_0}} = -\frac{\rho c}{c^2 \varepsilon_0} = -\mu_0 \rho c$$

$\underline{\nu = 1}$

$$F^{01} = E_x / c$$
$$F^{11} = 0$$
$$F^{21} = -B_z$$
$$F^{31} = B_y$$

$$\frac{\partial F^{\mu 1}}{\partial x^\mu} = \frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3}$$

$$= \frac{1}{c^2}\frac{\partial E_x}{\partial t} + 0 - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}$$

$$= +\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} - \left(\vec{\nabla} \times \vec{B}\right)_x$$

$$= -\mu_0 J_x$$

$$\boxed{\frac{\partial F^{\mu(1,2,3)}}{\partial x^\mu} = -\mu_0 J_{x,y,z}}$$

$$\left(\vec{\nabla} \times \vec{B}\right)_x = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix}_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}$$

$$A^\mu B_\mu = -A^0B^0 + \vec{A}\cdot\vec{B}$$
$$= A^0 B_0 + \vec{A}\cdot\vec{B}$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = \frac{\partial F^{0\nu}}{\partial x^0} + \frac{\partial F^{i\nu}}{\partial x^i}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ & 0 & -E_z/c & E_y/c \\ & & 0 & -E_x/c \\ & & & 0 \end{pmatrix}$$

$$\frac{\partial G^{\mu 0}}{\partial x^\mu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{10}}{\partial x^1} + \frac{\partial G^{20}}{\partial x^2} + \frac{\partial G^{30}}{\partial x^3}$$
$$= -\vec{\nabla}\cdot\vec{B} = 0$$

$$G^{00} = 0$$
$$G^{10} = -B_x$$
$$G^{20} = -B_y$$
$$G^{30} = -B_z$$

$$\frac{\partial G^{\mu 1}}{\partial x^\mu} = \frac{\partial G^{01}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{21}}{\partial x^2} + \frac{\partial G^{31}}{\partial x^3}$$

$$G^{01} = B^x$$
$$G^{11} = 0$$
$$G^{21} = E_z/c$$
$$G^{31} = -E_y/c$$

$$= +\frac{\partial}{c\,\partial t} B_x + 0 + \frac{1}{c}\frac{\partial E_z}{\partial y} - \frac{1}{c}\frac{\partial E_y}{\partial z}$$

$$\frac{\partial G^{\mu 1}}{\partial x^\mu} = \frac{1}{c}\left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right)_x = 0$$

$$\frac{\partial G^{\mu 1,2,3}}{\partial x^\mu} = \frac{1}{c}\left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right)_{x,y,z} = 0$$

$$\frac{\partial G^{\mu\nu}}{\partial x^\mu} = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = -\mu_0 \left(\rho c, \vec{J}\right)_\nu$$

$V, \rho \xrightarrow{\vec{v}} q \Longrightarrow V_0, \rho_0$ q at rest

$$\rho = \frac{q}{V} = \frac{q}{V/\gamma} \qquad \rho_0 = \frac{q}{V_0}$$

$$\boxed{\rho = \gamma \rho_0 = \rho_0 \frac{1}{\sqrt{1 - v^2/c^2}}} \qquad \vec{J} = \rho \vec{v} = \rho_0 \frac{\vec{v}}{\sqrt{1 - v^2/c^2}}$$

$$\eta^\mu = \frac{\partial x^\mu}{\partial \tau} = \frac{\partial x^\mu}{\sqrt{1 - v^2/c^2}\, dt}$$

$$\eta^0 = \frac{c}{\sqrt{1 - v^2/c^2}}, \quad \vec{\eta} = \frac{\partial \vec{r}}{\sqrt{1 - v^2/c^2}\, dt} = \frac{\vec{v}}{\sqrt{1 - V^2/c^2}}$$

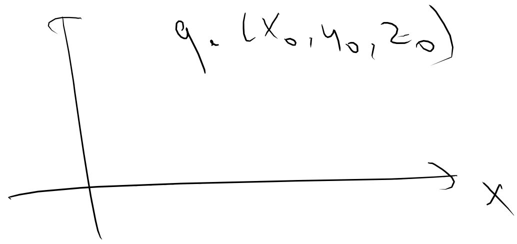
$$\rho = \frac{\rho_0}{c}\eta^0 \quad , \quad \vec{J} = \rho_0 \vec{\eta}$$

$$(\rho c, \vec{J}) = \rho_0 (\eta^0, \vec{\eta})$$

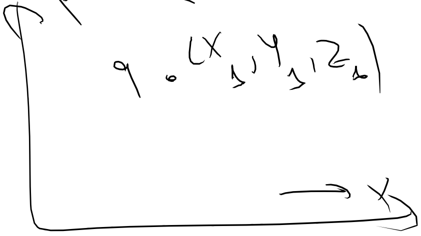
$\Rightarrow$ $\underline{(\rho c, \vec{J})}$ is a four vector.

__Example__ Point charge $\quad x_1 = \sqrt{1 - v^2/c^2}\, x_0$

rest frame
of the point charge



moving
frame



$$\rho = q\,\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)$$

$$\rho' = q\,\delta(x' - x_1)\,\delta(y' - y_1)\,\delta(z' - z_1)$$

$y' = y \qquad y_1 = y_0$

$z' = z \qquad z_1 = z_0$

$x' = \frac{1}{\gamma} x \qquad x_1 = \frac{1}{\gamma} x_0$

$$\rho' = q\,\delta\left(\frac{x - x_0}{\gamma}\right)\delta(y - y_0)\,\delta(z - z_0)$$

$$= \gamma q\,\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = \gamma \rho$$

$$\delta\left(\frac{x - x_0}{\gamma}\right) = \frac{\delta(x - x_0)}{\left|\frac{1}{\gamma}\right|} = \gamma\,\delta(x - x_0)$$

$$\delta(f(x)) = \sum_{x_0} \frac{\delta(x-x_0)}{\left| \frac{\partial f}{\partial x}\Big|_{x=x_0} \right|} \qquad \Theta\left(\frac{x-x_0}{r}\right) = \Theta(x-x_0)$$

$f(x_0) = 0$

Lorentz transformation of charge & current densities:

$$\rho' c = \gamma\left(\rho c - \frac{v}{c} J_x\right)$$

$$J'_x = \gamma\left(J_x - \frac{v}{c} \rho' c\right)$$

$$J'_y = J_y$$

$$J'_z = J_z$$

$s = \sqrt{x^2+z^2}$

$\vec{J} = J(s)\,\hat{y}$

$\rho = 0$

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial y} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0$$

$$c\rho' = \gamma\left(c\rho - \frac{v}{c} J_x\right) = 0$$

$$J'_x = \gamma\left(J_x - v\rho\right) = 0$$

$$J'_y = J_y = J(s)$$

$$J'_z = J_z = 0$$

$$J'(s') = J(s)$$

$$J^\mu = (\rho c, \vec{J})$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$= \frac{\partial(\rho c)}{\partial(ct)} + \frac{\partial J^i}{\partial x^i} = 0$$

$$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^i}{\partial x^i} = \boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$$

$$E^x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$E^x = -\frac{\partial \phi}{\partial x^1} - \frac{\partial (c A_x)}{\partial x^0}$$

$$F^{01} = \frac{E_x}{c} = -\frac{\partial}{\partial x^1}(\phi/c) - \frac{\partial A_x}{\partial x^0}$$

$$= +\frac{\partial}{\partial x^1}\left(-\frac{\phi}{c}\right) - \frac{\partial}{\partial x^0}(A^1)$$

$$= -\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\phi}{c}\right)}_{\equiv A^0} + \frac{\partial}{\partial x_0}(A^1)$$

$$F^{01} = \frac{\partial}{\partial x_0} A^1 - \frac{\partial}{\partial x_1} A^0$$

$$A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$$

$$\boxed{F^{\mu\nu} = \frac{\partial}{\partial x_\mu} A^\nu - \frac{\partial}{\partial x_\nu} A^\mu}$$

$$G^{\mu\nu} = (\,)\, \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$