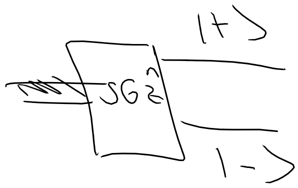
$\{|\alpha\rangle\}$



$|\alpha\rangle \longrightarrow a|\alpha\rangle \qquad a \in \mathbb{C}$

$\checkmark \boxed{a|\alpha\rangle + b|\beta\rangle}$

$|\alpha\rangle \longleftrightarrow \langle\alpha|$

ket $\qquad$ bra

$\langle\beta|\left(|\alpha\rangle\right) \equiv \langle\beta|\alpha\rangle$

$\left(a|\alpha\rangle + b|\beta\rangle\right)^{*} = a^{*}\langle\alpha| + b^{*}\langle\beta|$

Ex

$$\left|\, a|\alpha\rangle + b|\beta\rangle \,\right| = \sqrt{\left(a|\alpha\rangle + b|\beta\rangle\right)^{*}\left(a|\alpha\rangle + b|\beta\rangle\right)}$$

$$= \sqrt{\left(a^{*}\langle\alpha| + b^{*}\langle\beta|\right)\left(a|\alpha\rangle + b|\beta\rangle\right)}$$

$$= \sqrt{|a|^{2}\langle\alpha|\alpha\rangle + |b|^{2}\langle\beta|\beta\rangle + ab^{*}\underbrace{\langle\beta|\alpha\rangle} + ba^{*}\underbrace{\langle\alpha|\beta\rangle}}$$

$$\left(\langle\beta|\alpha\rangle\right)^{*} = \left(\left(\langle\beta|\right)\left(|\alpha\rangle\right)\right)^{*}$$

$$= \left(|\alpha\rangle\right)^{*}\left(\langle\beta|\right)^{*}$$

$$\left(\langle\beta|\alpha\rangle\right)^{*} = \langle\alpha|\beta\rangle$$

Note

$$(\langle\alpha|\beta\rangle)^* \neq (\langle\alpha|)^* (|\beta\rangle)^*$$

$$= |\alpha\rangle\langle\beta| \neq \langle\beta|\alpha\rangle$$

operator   number

$$\Rightarrow \underbrace{(|\alpha\rangle\langle\beta|)}(|\gamma\rangle) = \underbrace{|\alpha\rangle\overbrace{\langle\beta|\gamma\rangle}}$$

$$\cancel{(|\alpha\rangle\langle\beta|)\langle\gamma| = |\alpha\rangle(\langle\beta|\langle\gamma|)}$$

$$(\langle\gamma|)(|\alpha\rangle\langle\beta|) = \langle\gamma|\alpha\rangle\langle\beta|$$

Let $X|\alpha\rangle = |\beta\rangle$

$\langle\beta| = ?$

$\langle\beta| = \langle\alpha|(X^\dagger)$

$X^\dagger$: hermitian conjugate of X

<u>Ex</u> Let $X = |\alpha\rangle\langle\beta|$

$$(X|\gamma\rangle)^* = \langle\gamma| X^\dagger$$

$$
\begin{aligned}
(|\alpha\rangle\langle\beta|\gamma\rangle)^* &= (\langle\beta|\gamma\rangle)^* \langle\alpha| \\
&= \langle\gamma|\beta\rangle\langle\alpha| \\
&= \langle\gamma|(|\beta\rangle\langle\alpha|) \\
&= \langle\gamma| X^\dagger
\end{aligned}
$$

$X^\dagger = |\beta\rangle\langle\alpha| \qquad X^\dagger \neq X$

$X = |\alpha\rangle\langle\beta|$

$$
\begin{aligned}
(|\alpha\rangle\langle\beta|)^* &= (\langle\beta|)^*(|\alpha\rangle)^* \\
&= |\beta\rangle\langle\alpha|
\end{aligned}
$$

$\cancel{|\alpha\rangle X}$

$|\alpha\rangle\langle\beta|$ ✓

$\langle\alpha|\beta\rangle$ ✓

$X|\alpha\rangle$ ✓

$\langle\beta| X$ ✓

$\cancel{|\beta\rangle X}$ ✗

$\cancel{|\alpha\rangle|\beta\rangle}$ ✗

$\cancel{|\alpha\rangle|\beta\rangle}$

$\cancel{\langle\alpha|\langle\beta|}$

$$\underbrace{|\alpha\rangle}_{\substack{|+\rangle \\ |-\rangle}} \otimes \underbrace{|\beta\rangle}_{|\text{position}\rangle} \longrightarrow |\alpha\rangle|\beta\rangle$$

$$|+\rangle_1 \otimes |-\rangle_2 \equiv |+\rangle_1|-\rangle \equiv |+-\rangle$$

If $X^{+} = X$: hermitian operator.

$|\alpha\rangle\langle\alpha|$ is an hermitian operator.

<u>Eigenvalues and Eigenvectors:</u>

If $X|a\rangle = \alpha|\alpha\rangle$

α : eigenvalue

$|\alpha\rangle$: eigenvector

<u>Example</u>

$$X = |\alpha\rangle\langle\beta|$$

$$X|\gamma\rangle = |\alpha\rangle\langle\beta|\gamma\rangle$$

$$X|\alpha\rangle = |\alpha\rangle\underbrace{\langle\beta|\alpha\rangle}$$

Ex $X = |\alpha\rangle\langle\alpha|$

$$X|\alpha\rangle = \big(|\alpha\rangle\langle\alpha|\big)|\alpha\rangle$$
$$= |\alpha\rangle \big(\langle\alpha|\alpha\rangle\big)$$
$$\in \mathbb{R}$$

Let X be a hermition operator.

Let $\{|a_1\rangle, |a_2\rangle, |a_3\rangle, \dots\}$ be the eigenvectors

$$X|a_i\rangle = a_i|a_i\rangle$$

i) Any ket $|\alpha\rangle$ can be written as

$$|\alpha\rangle = \sum \lambda_i |a_i\rangle \quad : \text{completenes}$$

ii) a_i are real, $\langle a_i|a_j\rangle = 0$ if $a_i \neq a_j$

$$\langle a_j| \big(X|a_i\rangle = a_i|a_i\rangle\big)$$
$$X|a_j\rangle = a_j|a_j\rangle$$
$$\langle a_j|X^\dagger = \langle a_j|a_j^*$$
$$X^\dagger = X$$
$$\langle a_j|X|a_i\rangle = a_j^*\langle a_j|a_i\rangle$$
$$\langle a_j|X|a_i\rangle = a_i\langle a_j|a_i\rangle$$

$$\boxed{(a_j^* - a_i)\langle a_j|a_i\rangle = 0} \Leftarrow$$

case i) $|a_i\rangle = |a_j\rangle$

$$(a_i^* - a_i)\underbrace{\langle a_i | a_i\rangle}_{\neq 0} = 0$$

$$a_i^* = a_i \Rightarrow a_i \in \mathbb{R}$$

case ii) $a_i \neq a_j$

$$\underbrace{(a_j - a_i)}_{\neq 0}\langle a_j | a_i\rangle = 0$$

$$\Downarrow$$

$$\langle a_j | a_i\rangle = 0$$

if $|a_i\rangle$ is an eigenvector, so is $\lambda |a_i\rangle$

$\lambda \in \mathbb{C}, \lambda \neq 0$

X is linear

$$X(\lambda |a_i\rangle) = \lambda \underbrace{(X|a_i\rangle)}_{a_i|a_i\rangle}$$

$$= a_i (\lambda |a_i\rangle)$$

$$|a_i\rangle \longrightarrow \frac{|a_i\rangle}{\sqrt{\langle a_i | a_i\rangle}}$$