$$\begin{cases} |S_x+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right) \\ |S_x-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{i\delta_1}|-\rangle\right) \end{cases}$$

$$S_x = \left(\frac{\hbar}{2}\right)|S_x+\rangle\langle S_x+| + \left(-\frac{\hbar}{2}\right)|S_x-\rangle\langle S_x-|$$

$$\Rightarrow = \frac{\hbar}{2}|+\rangle\langle -|e^{-i\delta_1} + \frac{\hbar}{2}|-\rangle\langle +|e^{i\delta_1}$$

"pure ensemble"

$$S_z = \frac{\hbar}{2}|+\rangle\langle +| + \left(-\frac{\hbar}{2}\right)|-\rangle\langle -|$$

$$S_x|-\rangle = \frac{\hbar}{2}e^{-i\delta_1}|+\rangle$$

$$S_x|+\rangle = \frac{\hbar}{2}e^{i\delta_1}|-\rangle$$

$$\begin{cases} |S_y+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_2}|-\rangle\right) \\ |S_y-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{i\delta_2}|-\rangle\right) \end{cases}$$

$$|\langle S_x+|S_y+\rangle|^2 = 50\%$$

$$|\underbrace{\langle S_z+}_{\langle +|}|S_y+\rangle|^2 = \frac{1}{2}$$

$$|S_x +\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right)$$

$$|S_y +\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_2}|-\rangle\right)$$

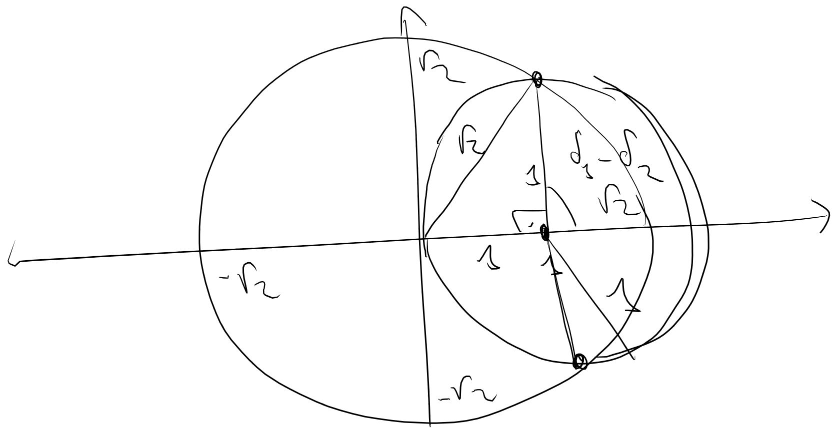
$$\langle S_y + | S_x + \rangle = \frac{1}{2}\left(\langle + | + e^{-i\delta_2}\langle - |\right)\left(|+\rangle + e^{i\delta_1}|-\rangle\right)$$

$$= \frac{1}{2}\left(1 + e^{-i\delta_2}\,0 + e^{i\delta_1}\,0 + e^{i(\delta_1-\delta_2)}\right)$$

$$= \frac{1}{2}\left(1 + e^{i(\delta_1-\delta_2)}\right)$$

$$\frac{1}{2}\left|1 + e^{i(\delta_1-\delta_2)}\right| = \frac{1}{\sqrt{2}}$$

$$\left|1 + e^{i(\delta_1-\delta_2)}\right| = \sqrt{2}$$



$$\delta_1 - \delta_2 = \pm 5($$

$$e^{i(\delta_1-\delta_2)} = \pm i$$

$$\boxed{e^{i\delta_2} = \pm i e^{i\delta_1}}$$

$$|S_x+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right)$$

$$|S_x-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{i\delta_1}|-\rangle\right)$$

$$|S_y+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + ie^{i\delta_1}|-\rangle\right)$$

$$|S_x-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - ie^{i\delta_1}|-\rangle\right)$$

$$|S_x+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \qquad |S_y+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$$

$$|S_x-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \qquad |S_y-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)$$

$$\boxed{[S_i, S_j] = i\frac{\hbar}{2}\varepsilon_{ijk} S_k}$$

$\boxed{\text{measure } A}$ $\boxed{B}$ $\boxed{C}$

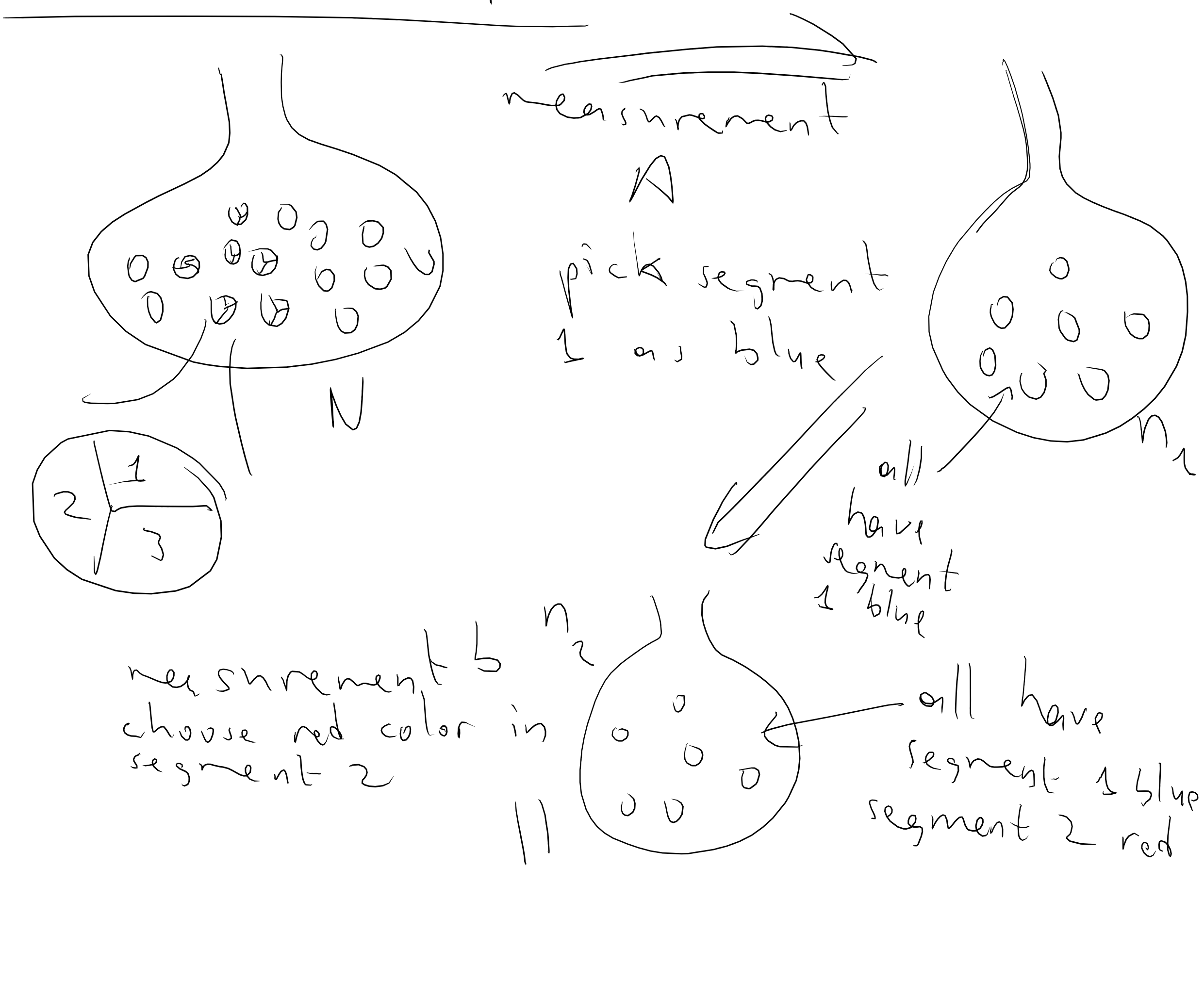
$\{|a_i\rangle\}$ $\{|b_i\rangle\}$ $\{|c_i\rangle\}$

given $a \longrightarrow b \longrightarrow c$

$$P_{a\to b\to c} = |\langle c|b\rangle|^2 |\langle b|a\rangle|^2$$

$$\sum_{b} P_{a\to b\to c} = P_{a\to c} \quad \left(\begin{array}{c}\text{classical}\\ \text{expectation}\end{array}\right)$$

classical example

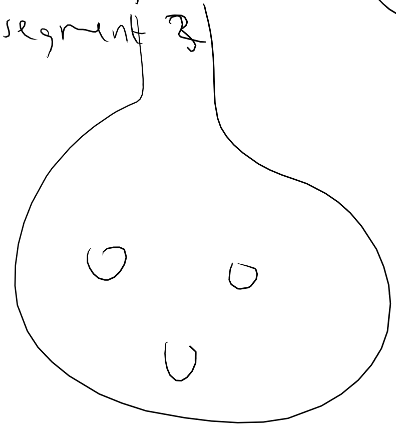


measurement C $\Big\Downarrow$

choose green in segment 3

$$\left(\frac{n_{2br}}{n_{1r}}\right)\left(\frac{n_{3brg}}{n_{2br}}\right)$$

n_3



all have
segment 1 blue
segment 2 red
segment 3 green

$$\sum_{b} \underbrace{P_{a\to b\to c}}_{|\langle c|b\rangle|^2 |\langle b|a\rangle|^2} = P_{a\to c} \quad \left(\begin{matrix}\text{classical}\\ \text{expectation}\end{matrix}\right)$$

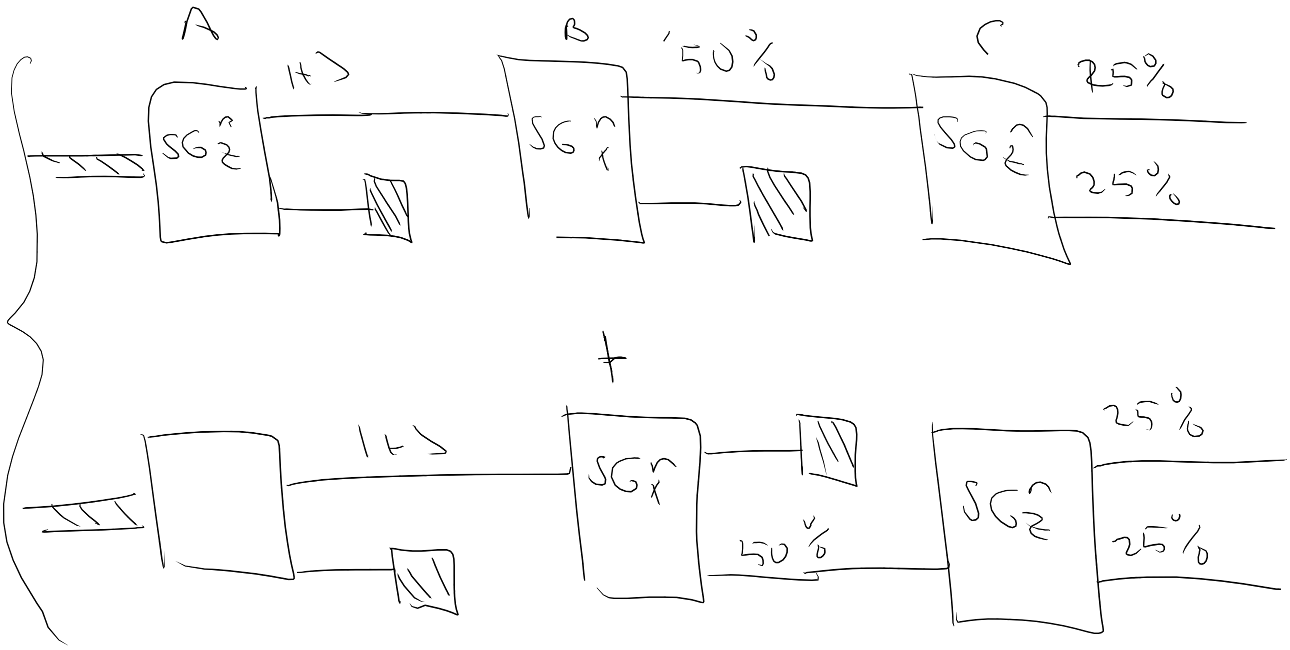
$$\Rightarrow \boxed{\sum_b |\langle c|b\rangle|^2 |\langle b|a\rangle|^2} \overset{?}{=} |\langle c|a\rangle|^2$$

$$|\langle c|a\rangle|^2 = \langle c\overset{\mathbb{1}}{|}a\rangle\langle a\overset{\mathbb{1}}{|}c\rangle$$

$$\mathbb{1} = \sum_b |b\rangle\langle b|$$

$$|\langle c|a\rangle|^2 = \sum_{bb'} \langle c|b\rangle\langle b|a\rangle\langle a|b'\rangle\langle b'|c\rangle$$

$$\sum_b |\langle c|b\rangle|^2 |\langle b|a\rangle|^2 = \sum_b \underbrace{\langle c|b\rangle \overbrace{\langle b|a\rangle\langle a|b\rangle}^{|\langle a|b\rangle|^2}\langle b|c\rangle}_{|\langle c|b\rangle|^2}$$



eliminate B measurement

A measurement

after B measurement

$$|+\rangle \longrightarrow |S_x \pm\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$$

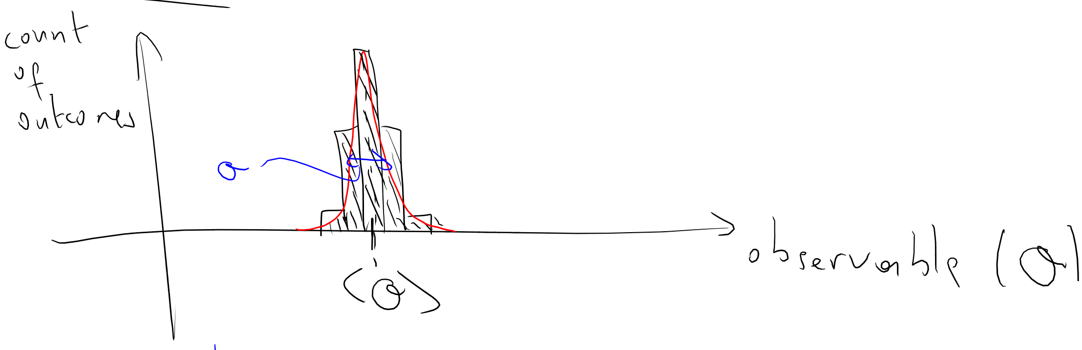
$$\frac{1}{\sqrt{2}}\left(|S_x+\rangle + |S_x-\rangle\right)$$

measurement C

50% $|+\rangle$

50% $|-\rangle$

Averages



value $= \langle O \rangle \mp \sigma$

$$\langle O \rangle = \sum (\text{value})(\text{probability of value})$$

$$\Rightarrow \left\{ |S_x+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \Leftarrow \right.$$

$$\langle S_z \rangle \equiv \langle S_x + | S_z | S_x + \rangle = \left(\frac{\hbar}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{\hbar}{2}\right)\left(\frac{1}{2}\right) = 0$$

$$S_z |S_x+\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

$$\langle \alpha | O | \alpha \rangle \equiv \langle O \rangle \equiv \langle O \rangle_\alpha$$

$$O \quad \{|\alpha_i\rangle\} \quad O|\alpha_i\rangle = \alpha_i |\alpha_i\rangle$$

$$|\alpha\rangle = \sum a_i |\alpha_i\rangle$$

$$\langle O \rangle = \sum_j \alpha_j |a_j|^2$$

$$= \sum_j \alpha_j \left| \langle \alpha_j | \alpha \rangle \right|$$

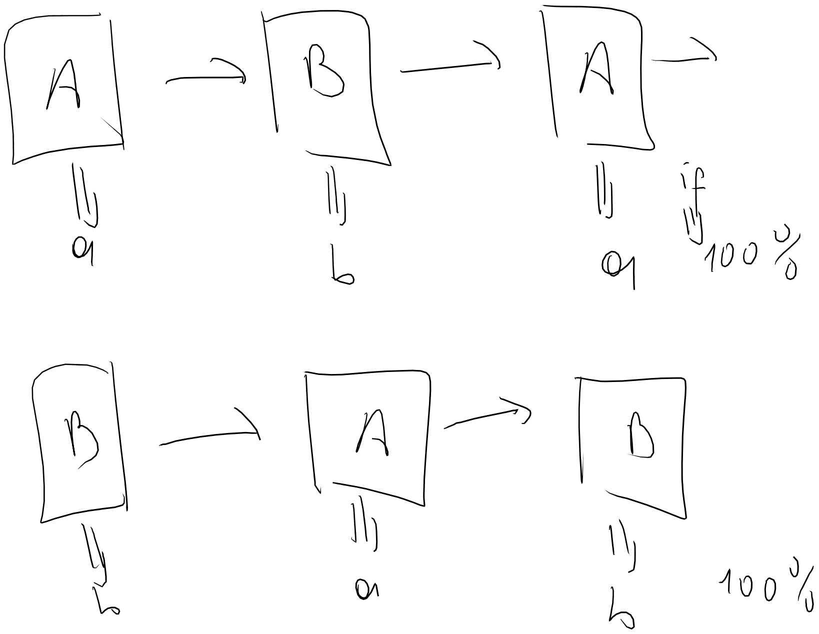
$$= \sum_j \alpha_j \langle\alpha|\alpha_j\rangle\langle\alpha_j|\alpha\rangle$$

$$= \langle\alpha| \underbrace{\left(\sum_j \alpha_j |\alpha_j\rangle\langle\alpha_j| \right)}_{O} |\alpha\rangle$$

$$\boxed{\langle O\rangle = \langle\alpha| O |\alpha\rangle}$$

Compatible and Incompatible Observables

A , B : two observables



A & B are compatible observables

$$A|a\,b\rangle = a|a\,b\rangle$$
$$B|a\,b\rangle = b|a\,b\rangle$$

$$AB|a\,b\rangle = A(b|a\,b\rangle)$$
$$= bA|a\,b\rangle$$
$$= ba|a\,b\rangle$$
$$= ab|a\,b\rangle$$
$$= aB|a\,b\rangle$$
$$= Ba|a\,b\rangle$$

$$AB|ab\rangle = BA|ab\rangle \Leftarrow$$

$$\Rightarrow \quad AB = BA \Rightarrow [A,B] = 0$$

<u>Claim</u>: If $[A,B]=0$ A & B can be simultaneously diagonalized.

<u>Proof</u>

A $\{|\alpha_i\rangle\}$ no degeneracy

$$0 = \langle\alpha_j|[A,B]|\alpha_i\rangle$$
$$= \langle\alpha_j|AB-BA|\alpha_i\rangle$$
$$= \langle\alpha_j|A^{+}B|\alpha_i\rangle - \langle\alpha_j|BA|\alpha_i\rangle$$
$$= \alpha_j\langle\alpha_j|B|\alpha_i\rangle - \alpha_i\langle\alpha_j|B|\alpha_i\rangle$$
$$0 = (\alpha_j - \alpha_i)\langle\alpha_j|B|\alpha_i\rangle$$
$$\{\langle\alpha_j|B|\alpha_i\rangle = 0 \quad \text{if} \quad \alpha_i \neq \alpha_j \Rightarrow i \neq j$$

$$\langle \alpha_j | B | \alpha_i \rangle \propto \delta_{ij}$$

$$[L^2, L_z] = 0$$

$$L^2 = \begin{array}{ccc} l=0 & l=1 & l=2 \end{array}$$
$$\begin{pmatrix} (\ )_{1\times1} & & 0 \\ & (\ )_{3\times3} & \\ 0 & & (\ )_{5\times5} \end{pmatrix}$$

$$L_z = \hbar \begin{pmatrix} 0 & & 0 \\ & \begin{pmatrix} +1 & & 0 \\ & 0 & \\ 0 & & -1 \end{pmatrix} & \\ 0 & & \begin{pmatrix} +2 & & & & 0 \\ & +1 & & & \\ & & 0 & & \\ & & & -1 & \\ 0 & & & & -2 \end{pmatrix} \end{pmatrix} \subseteq$$

$$[L^2, L_x] = 0$$

$$L_x = \begin{pmatrix} (0) & & 0 \\ & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{l=1} & \\ 0 & & (\ )_{l=5} \end{pmatrix}$$

<u>Incompatible observables</u> $[A, B] \neq 0$

$|\alpha\rangle$

ΔA

ΔB

$\Delta A \, \Delta B \geq$