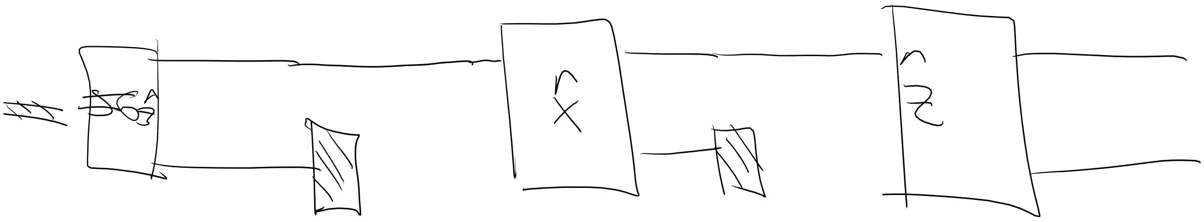
$[A, B] = 0$ compatible observables



$[S_x, S_z] \neq 0$

$|S_x +\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$\langle S_z \rangle = 0$

$\Delta S_z = S_z - \langle S_z \rangle$

A, $\langle A \rangle = \langle | A | \rangle$

$\Delta A = A - \langle A \rangle$

$\langle \Delta A \rangle = \langle A - \langle A \rangle \rangle = \langle A \rangle - \langle \langle A \rangle \rangle = 0$

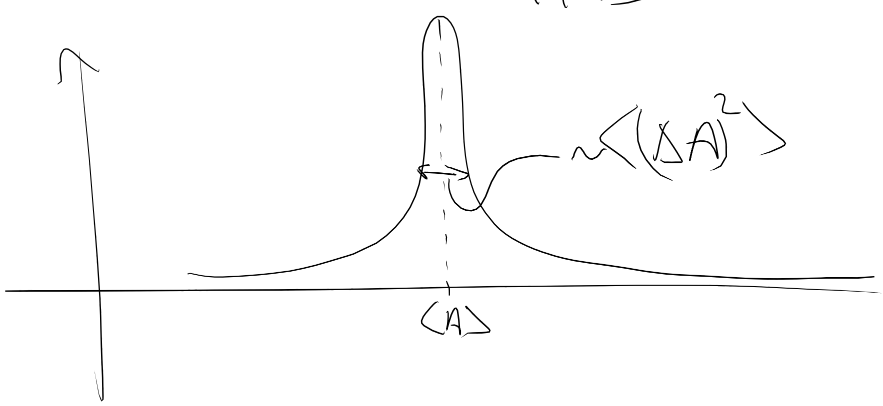
$\langle (\Delta A)^2 \rangle = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 - 2A\langle A \rangle + \langle A \rangle^2 \rangle$

$= \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2$

$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$

Assume $A|\alpha\rangle = \alpha|\alpha\rangle$

$\langle A \rangle_\alpha = \langle \alpha | \underbrace{A | \alpha \rangle}_{\alpha|\alpha\rangle} = \alpha$



$$\langle A \rangle_\alpha = \alpha$$

$$\langle A^2 \rangle_\alpha = \langle \alpha | A \underbrace{A | \alpha \rangle}_{\alpha | \alpha \rangle} = \alpha^2 \langle \alpha | \alpha \rangle = \alpha^2$$

$$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 = \alpha^2 - (\alpha)^2 = 0$$

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \langle [A, B] \rangle^2$$

$$|\vec{\alpha}|^2 |\vec{\beta}|^2 \geq |\vec{\alpha} \cdot \vec{\beta}|^2$$

<u>Claim</u> $\quad \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$

$$\big|\, |\alpha\rangle + \lambda |\beta\rangle \big|^2 = \left(\langle \alpha | + \lambda^* \langle \beta | \right) \left(|\alpha\rangle + \lambda |\beta\rangle \right) \geq 0$$

$$\frac{\partial}{\partial \lambda} \left(\langle \alpha | + \lambda^* \langle \beta | \right) \left(|\alpha\rangle + \lambda |\beta\rangle \right) = \left(\langle \alpha | + \lambda^* \langle \beta | \right) |\beta\rangle$$

$$= \langle \alpha | \beta \rangle + \lambda^* \langle \beta | \beta \rangle = 0$$

$$\Rightarrow \lambda^* = - \frac{\langle \alpha | \beta \rangle}{\langle \beta | \beta \rangle}$$

$$\lambda = - \frac{\langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle}$$

$$\left(\langle \alpha | - \frac{\langle \alpha | \beta \rangle \langle \beta |}{\langle \beta | \beta \rangle} \right) \left(|\alpha\rangle - \frac{\langle \beta | \alpha \rangle |\beta\rangle}{\langle \beta | \beta \rangle} \right) \geq 0$$

$$\langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle} - \cancel{\frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}}$$

$$+ \cancel{\frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle\cancel{\langle\beta|\beta\rangle}}{\langle\beta|\beta\rangle\cancel{\langle\beta|\beta\rangle}}} \geq 0$$

$$\langle\alpha|\alpha\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle} \geq 0$$

$$\boxed{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2}$$

$|\alpha\rangle \equiv \Delta A|\,\rangle$ $\qquad \langle\alpha|\alpha\rangle = \langle(\Delta A)^2\rangle$

$|\beta\rangle \equiv \Delta B|\,\rangle$ $\qquad \langle\beta|\beta\rangle = \langle(\Delta B)^2\rangle$

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle|\Delta A\Delta B|\rangle|^2$$

$$\Delta A\Delta B = \frac{1}{2}(\Delta A\Delta B - \Delta B\Delta A) + \frac{1}{2}(\Delta A\Delta B + \Delta B\Delta A)$$

$$= \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}$$

$$[\Delta A, \Delta B] = [A - \langle A\rangle, B - \langle B\rangle] = [A, B]$$

$$[C,D]^\dagger = (CD - DC)^\dagger = D^\dagger C^\dagger - C^\dagger D^\dagger$$

$$= DC - CD$$

$$= -(CD - DC)$$

$$= -[C,D]$$

$\Rightarrow$ eigenvalues of $[C,D]$ are purely imaginary

$\langle [C,D] \rangle$ is purely imaginary

$$\{C,D\}^\dagger = (CD + DC)^\dagger = D^\dagger C^\dagger + C^\dagger D^\dagger$$

$$= DC + CD$$

$$= CD + DC$$

$$= \{C,D\}$$

$\Rightarrow$ $\{C,D\}$ is hermitian

$\Rightarrow$ all eigenvalues are real

$\Rightarrow$ $\langle \{C,D\} \rangle$ $\Rightarrow$ is a real number.

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq |\langle|\Delta A \Delta B|\rangle|^2$$

$$\langle|\Delta A \Delta B|\rangle = \frac{1}{2}\langle|\{\Delta A, \Delta B\}|\rangle + \frac{1}{2}\langle|[\Delta A, \Delta B]|\rangle$$

$$= \underbrace{\frac{1}{2}\langle|\{\Delta A, \Delta B\}|\rangle}_{\text{purely real}} + \underbrace{\frac{1}{2}\langle|[A, B]|\rangle}_{\text{purely imaginary}}$$

$$|\langle|\Delta A \Delta B|\rangle|^2 = \frac{1}{4}|\langle|\{\Delta A, \Delta B\}|\rangle|^2 + \frac{1}{4}|\langle|[A,B]|\rangle|^2$$

$$\geq \frac{1}{4}|\langle|[A,B]|\rangle|^2$$

$$\boxed{\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4}|\langle|[A,B]|\rangle|^2}$$

$$[S_x, S_y] = i\frac{\hbar}{2} S_z$$

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle \geq \left(\frac{\hbar}{2}\right)^2 \frac{1}{4}|\langle S_z \rangle|^2$$

$$\langle (\Delta p_x)^2 \rangle \langle (\Delta x)^2 \rangle \geq \frac{\hbar^2}{4}$$