$$K = \frac{-p}{\text{constant}} \qquad \text{constant} = \hbar$$

$$T(\delta) = 1 + iK\delta + \dots$$

$$T(\vec{\delta}) = 1 - i\frac{\vec{p}}{\hbar}\cdot\vec{\delta} + O(\delta^2)$$

$$[K, x] = +i$$

$$\left[-\frac{\hat{p}}{\hbar}, \hat{x}\right] = +i \Rightarrow [\hat{p}, \hat{x}] = -i\hbar$$

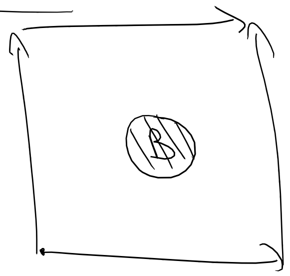
$$\langle(\Delta p)^2\rangle\langle(\Delta x)^2\rangle \geq \langle[p,x]^2\rangle\frac{1}{4} = \hbar^2/4$$

$$[\hat{x}, \hat{y}] = 0$$
$$[\hat{x}, \hat{z}] = 0$$
$$[\hat{y}, \hat{z}] = 0$$

$$\begin{aligned}
T(\delta_y\hat{y})T(\delta_x\hat{x})|\vec{r}\rangle &= T(\delta_y\hat{y})\,|\vec{r} + \delta_x\hat{x}\rangle \\
&= |\vec{r} + \delta_x\hat{x} + \delta_y\hat{y}\rangle \\
&= |\vec{r} + \delta_y\hat{y} + \delta_x\hat{x}\rangle \\
&= T(\delta_x\hat{x})\,|\vec{r} + \delta_y\hat{y}\rangle \\
&= T(\delta_x\hat{x})T(\delta_y\hat{y})|\vec{r}\rangle.
\end{aligned}$$

$$\Rightarrow [T(\delta_y\hat{y}), T(\delta_x\hat{x})] = 0$$

Exception



Aharanov-Bohm effect

$$0 = \left[1 - \frac{i}{\hbar}p_x \delta_x , 1 - \frac{i p_y \delta_y}{\hbar}\right] = \left(\frac{-i}{\hbar}\right)^2 \delta_x \delta_y [p_x, p_y] = 0$$

$$\Rightarrow [p_x, p_y] = 0$$

$$\boxed{\begin{aligned} &[x_i, x_j] = 0 \qquad i,j = 1,2,3 \;\; (x,y,z) \\ &[p_i, p_j] = 0 \\ &[p_i, x_j] = -i\hbar\,\delta_{ij} \end{aligned}}$$

$$T(\vec{\delta}) = 1 - \frac{i}{\hbar}\vec{\delta}\cdot\vec{p} + \mathcal{O}(\delta^2)$$

$$T(\vec{\Delta}) = T\left(\frac{\vec{\Delta}}{2}\right)T\left(\frac{\vec{\Delta}}{2}\right) = \overbrace{T\left(\frac{\vec{\Delta}}{N}\right)\cdots T\left(\frac{\vec{\Delta}}{N}\right)}^{N \text{ times}}$$

$$T(\vec{\Delta}) = \lim_{N\to\infty} T\left(\frac{\vec{\Delta}}{N}\right)^N = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\vec{\delta}\cdot\vec{p} + \mathcal{O}(\delta^2)\right)^N$$

$$\vec{\delta} \equiv \frac{\vec{\Delta}}{N}$$

$$\ln(1+x) = x + \mathcal{O}(x^2)$$

$$= \lim_{N\to\infty} \exp\left\{\ln\left(1 - \frac{i}{\hbar}\vec{\delta}\cdot\vec{p} + \mathcal{O}(\delta^2)\right)\right\}^N$$

$$= \lim_{N\to\infty} \exp\left\{-\frac{i}{\hbar}\vec{\delta}\cdot\vec{p} + \mathcal{O}(\delta^2)\right\}^N$$

$$= \lim_{N\to\infty} \exp\left\{-\frac{i}{\hbar}\underbrace{(N\vec{\delta})}_{\vec{\Delta}}\cdot\vec{p} + \frac{N^2}{N}\mathcal{O}(\delta^2)\right\}$$

$$T(\vec{\Delta}) = \lim_{N\to\infty} \exp\left\{\underbrace{-\frac{i}{\hbar}\vec{\Delta}\cdot\vec{p}} + \frac{1}{N}\mathcal{O}\left((N\delta)^2\right)\right\}$$

$$\boxed{T(\vec{\Delta}) = \exp\left\{-\frac{i}{\hbar}\vec{\Delta}\cdot\vec{p}\right\}}$$

Wavefunctions in Position Space

$$\int d\vec{r}\, |\vec{r}\rangle\langle\vec{r}| = \mathbb{1}$$

$$|\alpha\rangle = \int d^3\vec{r}\, |\vec{r}\rangle\langle\vec{r}|\alpha\rangle$$

$$\langle\vec{r}|\alpha\rangle \equiv \psi_\alpha(\vec{r}) : \text{wave function}$$

$$|\alpha\rangle = \int d^3r\, \psi_\alpha(\vec{r})\, |\vec{r}\rangle$$

$$\langle\beta|\alpha\rangle = \int d^3r\, \underbrace{\langle\beta|\vec{r}\rangle}_{(\langle\vec{r}|\beta\rangle)^*} \overbrace{\langle\vec{r}|\alpha\rangle}^{\psi_\alpha(\vec{r})}$$

$$\psi_\beta(\vec{r})$$

$$\langle\beta|\alpha\rangle = \int d^3r\, \psi_\beta(\vec{r})^*\, \psi_\alpha(\vec{r})$$

$$1 = \langle\alpha|\alpha\rangle = \int d^3r\, \psi_\alpha(\vec{r})^*\, \psi_\alpha(\vec{r})$$

$$1 = \int d^3r\, |\psi_\alpha(\vec{r})|^2$$

$$|\alpha\rangle = \int d^3\vec{r}\, \psi_\alpha(\vec{r})\, |\vec{r}\rangle$$

$$T(\vec{\Delta})\, |\alpha\rangle = \int d^3\vec{r}\ \psi_\alpha(\vec{r})\ |\vec{r}+\vec{\Delta}\rangle$$

$$\vec{r} = \vec{r}\,' - \vec{\Delta}$$
$$d^3\vec{r} = d^3\vec{r}\,'$$

$$T(\vec{\Delta})\, |\alpha\rangle = \int d^3 r'\ \psi_\alpha(\vec{r}\,' - \vec{\Delta})\, |\vec{r}\,'\rangle$$

$$\langle\vec{r}\,|T(\vec{\Delta})|\,\alpha\rangle = \int d^3 r'\ \psi_\alpha(\vec{r}\,' - \vec{\Delta})\ \underbrace{\langle\vec{r}\,|\,\vec{r}\,'\rangle}_{\delta^3(\vec{r}-\vec{r}\,')}$$

$$\langle\vec{r}\,|\,T(\vec{\Delta})\,|\alpha\rangle = \psi_\alpha(\vec{r} - \vec{\Delta}) \Leftarrow$$

$$T(\vec{\Delta})\,|\vec{r}\rangle = |\vec{r} + \vec{\Delta}\rangle$$

$$|\alpha\rangle = \sum c_i\, |a_i\rangle$$

$$\langle\vec{r}\,|\,\alpha\rangle = \sum c_i\, \langle\vec{r}\,|\,a_i\rangle$$

$$\boxed{\psi_\alpha(\vec{r}) = \sum c_i\ \psi_{a_i}(\vec{r})}$$

$$\langle\vec{r}\,|\,T(\vec{\delta})\,|\,\alpha\rangle = \psi_\alpha(\vec{r} - \vec{\delta})$$

$$\langle\vec{r}\,|\,1 - \frac{i}{\hbar}\vec{p}\cdot\vec{\delta}\,|\,\alpha\rangle = \psi_\alpha(\vec{r}) - \vec{\delta}\cdot\vec{\nabla}\psi_\alpha(\vec{r})$$

$$\cancel{\underbrace{\langle\vec{r}\,|\,\alpha\rangle}_{\psi_\alpha(\vec{r})}} - \frac{i}{\hbar}\langle\vec{r}\,|\,\vec{p}\,|\,\alpha\rangle\cdot\vec{\delta} = \cancel{\psi_\alpha(\vec{r})} - \vec{\delta}\cdot\vec{\nabla}\psi_\alpha(\vec{r})$$

$$-\frac{i}{\hbar}\langle\vec{r}\,|\,\vec{p}\,|\,\alpha\rangle = -\vec{\nabla}\,\psi_\alpha(\vec{r})$$

$$\langle \vec{r}|\vec{p}|\alpha\rangle = \left(\frac{\hbar}{i}\vec{\nabla}\right)\langle \vec{r}|\alpha\rangle \qquad AB = \mathbb{1}A\mathbb{1}B\mathbb{1}\mathbb{1}$$

$$\langle \beta|A|\alpha\rangle = \int d^3r\, d^3r' \,\langle\beta|\vec{r}\rangle\langle\vec{r}|A|\vec{r}'\rangle\langle\vec{r}'|\alpha\rangle$$

$\int d^3r\, |\vec{r}\rangle\langle\vec{r}|$ $\qquad \int d^3r'\, |\vec{r}'\rangle\langle\vec{r}'|$

$$\langle\beta|A|\alpha\rangle = \int d^3r\, d^3r'\, \psi_\beta^*(\vec{r})\underbrace{\langle\vec{r}|A|\vec{r}'\rangle}\psi_\alpha(\vec{r}')$$

Example

$A = A(\hat{\vec{r}})$

$$\langle\vec{r}|A(\hat{\vec{r}})|\vec{r}'\rangle = \langle\vec{r}|A(\vec{r}')|\vec{r}'\rangle = A(\vec{r}')\langle\vec{r}|\vec{r}'\rangle$$
$$= A(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')$$

$$\boxed{\langle\beta|A(\hat{\vec{r}})|\alpha\rangle = \int d^3r\, \psi_\beta^*(\vec{r})\, A(\vec{r})\,\psi_\alpha(\vec{r})}$$

$$A(\hat{\vec{r}}) = \sum a_n (\hat{\vec{r}})^n$$

$$(\hat{\vec{r}})^n|\vec{r}'\rangle = \underbrace{\hat{\vec{r}}\cdots\hat{\vec{r}}}_{n\text{-times}}|\vec{r}'\rangle = \underbrace{\hat{\vec{r}}\cdots\hat{\vec{r}}}_{n\text{-}1\text{ times}}\overbrace{(\hat{\vec{r}}|\vec{r}'\rangle)}^{\vec{r}'|\vec{r}'\rangle}$$

$$= \vec{r}'\,\underbrace{\hat{\vec{r}}\cdots\hat{\vec{r}}}_{n\text{-}2\text{ times}}\underbrace{\hat{\vec{r}}|\vec{r}'\rangle}_{\vec{r}'|\vec{r}'\rangle}$$

$$= (\vec{r}')^n|\vec{r}'\rangle$$

$$A(\hat{\vec{r}})|\vec{r}'\rangle = \sum a_n(\hat{\vec{r}})^n|\vec{r}'\rangle = \underbrace{\sum a_n(\vec{r}')^n}_{A(\vec{r}')}|\vec{r}'\rangle$$
$$A(\vec{r}')\,|\vec{r}'\rangle$$

Example

$$\langle\alpha|\vec{p}|\beta\rangle = \int d^3r\, d^3r'\, \psi_\alpha(\vec{r})^* \langle\vec{r}|\vec{p}|\vec{r}'\rangle \psi_\beta(\vec{r}')$$

$$\langle\vec{r}|\vec{p}|\vec{r}'\rangle = \frac{\hbar}{i}\vec{\nabla}_r \underbrace{\langle\vec{r}|\vec{r}'\rangle}_{\delta^3(\vec{r}-\vec{r}')}$$

$$\langle\alpha|\vec{p}|\beta\rangle = \int d^3r\, d^3r'\, \psi_\alpha(\vec{r})^* \frac{\hbar}{i}\vec{\nabla}_r\, \delta^3(\vec{r}-\vec{r}')\psi_\beta(\vec{r}')$$

$$= \int d^3r\, \psi_\alpha(\vec{r})^* \frac{\hbar}{i}\vec{\nabla}_r \int d^3r'\, \delta^3(\vec{r}-\vec{r}')\psi_\beta(\vec{r}')$$

$$\boxed{\langle\alpha|\vec{p}|\beta\rangle = \int d^3r\, \psi_\alpha(\vec{r})^* \frac{\hbar}{i}\vec{\nabla}\psi_\beta(\vec{r})}$$

$$\{|\vec{r}\rangle\} \rightarrow \{|\vec{r}\rangle, |\vec{r}_1\vec{r}_2\rangle, |\vec{r}_1\vec{r}_2\vec{r}_3\rangle, \dots\}$$

Momentum Wave Functions

$$\hat{\vec{p}}|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$$

$$\int d^3\vec{p}\,|\vec{p}\rangle\langle\vec{p}| = \mathbb{1}$$

$$\langle\vec{p}'|\vec{p}\rangle = \delta^3(\vec{p}-\vec{p}')$$

$$|\alpha\rangle = \int d^3\vec{p}\,|\vec{p}\rangle \underbrace{\langle\vec{p}|\alpha\rangle}_{\substack{\psi_\alpha(\vec{p}) \\ \phi_\alpha(\vec{p})}}$$

$$U = \sum_i |\alpha_i\rangle\langle\beta_i| \rightarrow \langle\alpha|\beta\rangle$$

$$\phi_\alpha(\vec{p}) = \langle \vec{p} | \alpha \rangle$$
$$= \langle \vec{p} | \mathbb{1} | \alpha \rangle$$
$$= \langle \vec{p} | \int d^3r \, |\vec{r}\rangle \langle \vec{r} | \alpha \rangle$$
$$= \int d^3r \, \underbrace{\langle \vec{p} | \vec{r} \rangle}_{?} \, \psi_\alpha(\vec{r})$$

$$\langle \vec{p} | \vec{r} \rangle = ?$$

$$\langle \vec{r} | \vec{p} \rangle$$

$$\langle \vec{r} | \hat{\vec{p}} | \vec{p} \rangle \overset{\checkmark}{=} \langle \vec{r} | \vec{p} | \vec{p} \rangle = \vec{p} \langle \vec{r} | \vec{p} \rangle$$

$$\hat{\vec{p}} \, |\vec{p}\rangle = \vec{p} \, |\vec{p}\rangle$$

operator

not operator

$$\underbrace{\langle \vec{r} | \hat{\vec{p}} | \alpha \rangle} = \frac{\hbar}{i} \vec{\nabla} \langle \vec{r} | \alpha \rangle \qquad \text{for any } |\alpha\rangle$$

Take $|\alpha\rangle = |\vec{p}\rangle$

$$\langle \vec{r} | \hat{\vec{p}} | \vec{p} \rangle = \frac{\hbar}{i} \vec{\nabla} \langle \vec{r} | \vec{p} \rangle$$

$$\langle \vec{r} | \hat{\vec{p}} | \vec{p} \rangle = \vec{p} \langle \vec{r} | \vec{p} \rangle$$

$$\frac{\hbar}{i} \vec{\nabla} \underbrace{\langle \vec{r} | \vec{p} \rangle}_{\psi_{\vec{p}}(\vec{r})} = \vec{p} \langle \vec{r} | \vec{p} \rangle \implies \psi_{\vec{p}}(\vec{r}) = N e^{\frac{i}{\hbar} \vec{p} \cdot \vec{r}}$$

$$\boxed{\langle \vec{r} | \vec{p} \rangle = N e^{\frac{i}{\hbar} \vec{p} \cdot \vec{r}}}$$

$$\langle \vec{p} | \vec{r} \rangle = N^* e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{r}} \quad \leftarrow$$

$$\langle \vec{p}' | \vec{p} \rangle = \delta^3(\vec{p} - \vec{p}')$$

$$\int d^3r \langle \vec{p}' | \vec{r} \rangle \langle \vec{r} | \vec{p} \rangle = \delta^3(\vec{p} - \vec{p}')$$

$$= \int d^3r \left(N^* e^{-\frac{i}{\hbar}\vec{p}' \cdot \vec{r}} \right) N e^{\frac{i}{\hbar}\vec{p} \cdot \vec{r}}$$

$$\int_{-\infty}^{\infty} dx\, e^{ik \cdot x} = 2\pi \delta(k)$$

$$= \int d^3r\, |N|^2 e^{\frac{i}{\hbar}(\vec{p} - \vec{p}') \cdot \vec{r}}$$

$$= |N|^2 (2\pi\hbar)^3 \delta^3(\vec{p} - \vec{p}') = \delta^{(3)}(\vec{p} - \vec{p}')$$

$$\Rightarrow |N| = \frac{1}{(2\pi\hbar)^{3/2}} \Rightarrow N = \frac{1}{(2\pi\hbar)^{3/2}}$$

$$\psi_\alpha(\vec{r}) = \langle \vec{r} | \alpha \rangle = \int d^3p \langle \vec{r} | \vec{p} \rangle \langle \vec{p} | \alpha \rangle$$

$$\boxed{\psi_\alpha(\vec{r}) = \int d^3r \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar}\vec{p} \cdot \vec{r}} \phi_\alpha(\vec{p})}$$

$$\phi_\alpha(\vec{p}) = \langle \vec{p} | \alpha \rangle = \int d^3r \langle \vec{p} | \vec{r} \rangle \langle \vec{r} | \alpha \rangle$$

$$\boxed{\phi_\alpha(\vec{p}) = \int \frac{d^3r}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}\vec{p} \cdot \vec{r}} \psi_\alpha(\vec{r})}$$

$$\left.\begin{aligned} |\alpha\rangle &= \sum_{\vec{r}} \psi_\alpha(\vec{r}) |\vec{r}\rangle \\ &= \sum_{\vec{p}} \phi_\alpha(\vec{p}) |\vec{p}\rangle \end{aligned}\right\}$$