

$$\{|\alpha\rangle\}$$

$$A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$$

$$\{x_j, p_i\} = +ik\delta_{ij} \Rightarrow \langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4}$$

$$\{x(t), p(t)\}$$

$$|\alpha, t_0; t\rangle$$

$$|\alpha, t_0; t=t_0\rangle = |\alpha\rangle$$

$$|\alpha, t_0; t\rangle = U(t, t_0)|\alpha, t_0; t_0\rangle$$

$$\langle\alpha, t_0; t|\alpha, t_0; t\rangle = \langle\alpha, t_0; t_0|\alpha, t_0; t_0\rangle$$

$$\underbrace{\langle\alpha, t_0; t|}_{\langle\alpha, t_0; t|} U^\dagger(t, t_0) \underbrace{U(t, t_0)|\alpha\rangle}_{|\alpha, t_0; t\rangle} = \langle\alpha|\alpha\rangle$$

$$U^\dagger(t, t_0)U(t, t_0) = \mathbb{1} \Rightarrow U \text{ is a unitary operator}$$

$$|\alpha, t_0; t_0\rangle \xrightarrow{U(t_0 + \delta t_1, t_0)} |\alpha, t_0; t_0 + \delta t_1\rangle \xrightarrow{U(t_0 + \delta t_1 + \delta t_2, t_0 + \delta t_1)} |\alpha, t_0; t_0 + \delta t_1 + \delta t_2\rangle$$

$$U(t_0 + \delta t_1 + \delta t_2, t_0) = U(t_0 + \delta t_1 + \delta t_2, t_0 + \delta t_1)U(t_0 + \delta t_1, t_0)$$

$$U(t_0 + \delta t_1 + \delta t_2, t_0) = U(t_0 + \delta t_1 + \delta t_2, t_0 + \delta t_1)U(t_0 + \delta t_1, t_0)$$

$$|\alpha, t_0; t_0\rangle = U(t_0, t_0)|\alpha, t_0; t_0\rangle$$

$$U(t_0, t_0) = \mathbb{1}$$

$$U(t_0 + dt, t_0) = \mathbb{1} - i\Omega dt + \mathcal{O}(dt^2)$$

$$\begin{aligned} \mathbb{1} &= U^\dagger(t_0 + dt, t_0) U(t_0 + dt, t_0) = (\mathbb{1} + i\Omega^\dagger dt)(\mathbb{1} - i\Omega dt) \\ &= \mathbb{1} - i(\Omega - \Omega^\dagger) dt + \mathcal{O}(dt^2) \end{aligned}$$

$$\Omega = \Omega^\dagger$$

$$\begin{aligned} U(t_0 + dt_1 + dt_2, t_0) &= U(t_0 + dt_1 + dt_2, t_0 + dt_1) U(t_0 + dt_1, t_0) \\ \mathbb{1} - i\Omega(dt_1 + dt_2) &\stackrel{?}{=} (\mathbb{1} - i\Omega dt_2)(\mathbb{1} - i\Omega dt_1) \\ &= \mathbb{1} - i\Omega(dt_1 + dt_2) + \mathcal{O}(dt^2) \end{aligned}$$

$$[\Omega] = [w] = \left[\frac{\tilde{E}}{t_0} \right]$$

$$\Omega = \frac{H}{\hbar}$$

$$\frac{dx}{dt} = \frac{p}{m}$$

Time Evolution by a finite time

Case i H is independent of time

$$U(t_0 + \Delta t, t_0) = \underbrace{U(t_0 + \Delta t, t_0 + \Delta t - dt) \dots U(t_0 + 2dt, t_0 + dt) U(t_0 + dt, t_0)}_{N \text{ terms}} \quad dt = \frac{\Delta t}{N}$$

$$= \lim_{N \rightarrow \infty} \underbrace{\left(1 - \frac{i}{\hbar} H dt\right) \dots \left(1 - \frac{i}{\hbar} H dt\right)}_{N \text{ terms}}$$

$$= \lim_{N \rightarrow \infty} \left(\exp \left\{ -\frac{i}{\hbar} H dt \right\} \right)^N$$

$$= \exp \left\{ -\frac{i}{\hbar} H \Delta t \right\}$$

Case II $H(t)$ depends on time, but $\mathcal{O} = [H(t), H(t')]$

$$U(t_0 + \Delta t, t_0) = \left(1 - \frac{i}{\hbar} H(t_0 + \Delta t - dt) dt\right) \dots \left(1 - \frac{i}{\hbar} H(t_0) dt\right)$$

$$= \mathbb{1} - \frac{i}{\hbar} \sum_{n=0}^{N-1} dt H(t_0 + n dt) + \mathcal{O}(dt^2)$$

$$H(t_0 + n dt) \cong H_n$$

$$U(t_0 + \Delta t, t_0) = \left(1 - \frac{i}{\hbar} H_{N-1} dt\right) \left(1 - \frac{i}{\hbar} H_{N-2} dt\right) \dots \left(1 - \frac{i}{\hbar} H_2 dt\right) \left(1 - \frac{i}{\hbar} H_1 dt\right) \left(1 - \frac{i}{\hbar} H_0 dt\right)$$

$$= \mathbb{1} - \frac{i}{\hbar} H_0 dt - \frac{i}{\hbar} H_1 dt - \frac{i}{\hbar} H_2 dt + \dots - \frac{i}{\hbar} H_{N-2} dt$$

$$- \frac{i}{\hbar} H_{N-1} dt + (dt)^2 \left(\frac{N(N-1)}{2} \text{ terms} \right) + \mathcal{O}(dt^3)$$

$$= \mathbb{1} - \frac{i}{\hbar} \int_{t_0}^{t_0 + \Delta t} dt' H(t') + \dots$$

$$\begin{aligned}
U(t_0 + \Delta t, t_0) &= \left(1 - \frac{i}{\hbar} H_{n-1} dt\right) \left(1 - \frac{i}{\hbar} H_{n-2} dt\right) \dots \left(1 - \frac{i}{\hbar} H_2 dt\right) \left(1 - \frac{i}{\hbar} H_1 dt\right) \left(1 - \frac{i}{\hbar} H_0 dt\right) \\
&= \exp\left\{-\frac{i}{\hbar} H_{n-1} dt\right\} \exp\left\{-\frac{i}{\hbar} H_{n-2} dt\right\} \dots \exp\left\{-\frac{i}{\hbar} H_0 dt\right\} \\
&= \exp\left\{-\frac{i}{\hbar} \sum_{n=0}^{N-1} H(t_0 + n dt) dt + (dt) H(t) (N \text{ terms})\right\} \\
&= \exp\left\{-\frac{i}{\hbar} \int_{t_0}^{t_0 + \Delta t} dt' H(t')\right\}
\end{aligned}$$

$$\exp\{A\} \exp\{B\} \neq \exp\{A+B\} \quad \text{if } [A, B] \neq 0$$

case III $[H(t), H(t')] \neq 0$ if $t \neq t'$

$$U(t_1, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t_1} dt' H(t') + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t_1} dt' \int_{t_0}^{t'} dt'' H(t') H(t'') + \dots$$

Energy Eigenkets

$$[H, A] = 0$$

$$A |a_i\rangle = a_i |a_i\rangle$$

$$H |a_i\rangle = E_{a_i} |a_i\rangle$$

$$H = \sum_i E_{a_i} |a_i\rangle \langle a_i|$$

$$\begin{aligned}
U(t_1, t_0) &= \exp\left\{-\frac{i}{\hbar} H(t_1 - t_0)\right\} \\
&= \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar} (t_1 - t_0)\right)^n \frac{1}{n!} H^n \quad \Leftarrow
\end{aligned}$$

$$H^2 = \left(\sum_i E_{a_i} |a_i\rangle \langle a_i| \right) \left(\sum_j E_{a_j} |a_j\rangle \langle a_j| \right)$$

$$= \sum_{i,j} E_{a_i} E_{a_j} |a_i\rangle \underbrace{\langle a_i | a_j \rangle}_{\delta_{ij}} \langle a_j|$$

$$H^2 = \sum_i E_{a_i}^2 |a_i\rangle \langle a_i|$$

$$H^n = \sum_i E_{a_i}^n |a_i\rangle \langle a_i|$$

$$U(t_1, t_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} (t_1 - t_0) \right)^n \frac{1}{n!} \sum_j E_{a_j}^n |a_j\rangle \langle a_j|$$

$$= \sum_j \left(\sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} (t_1 - t_0) \right)^n \frac{E_{a_j}^n}{n!} \right) |a_j\rangle \langle a_j|$$

$$= \sum_j e^{-\frac{i}{\hbar} (t_1 - t_0) E_{a_j}} |a_j\rangle \langle a_j|$$

$$U(t_1, t_0) = \sum_j e^{-\frac{i}{\hbar} (t_1 - t_0) E_{a_j}} |a_j\rangle \langle a_j|$$

$$|\alpha(t=t_0)\rangle = \sum_i c_i |a_i\rangle$$

$$|\alpha(t)\rangle = U(t, t_0) |\alpha(t=t_0)\rangle$$

$$= \sum_j e^{-\frac{i}{\hbar} (t-t_0) E_{a_j}} |a_j\rangle \underbrace{\langle a_j | \alpha(t=t_0) \rangle}_{c_j}$$

$$|\alpha(t)\rangle = \sum_j e^{\frac{i}{\hbar} (t-t_0) E_{a_j}} c_j |a_j\rangle$$

$$|\alpha(t=t_0)\rangle = \sum_j c_j |a_j\rangle$$

$$|\alpha(t)\rangle = \sum_j c_j(t) |a_j\rangle$$

$$c_j(t) = e^{-\frac{i}{\hbar}(t-t_0)E_{a_j}} c_j(t_0)$$

$\frac{d}{dt} \|c_j(t)\| = 0 \iff$ only if $\{|a_i\rangle\}$ are energy eigenkets.

time dependence of expectation values

$$[B, H] \neq 0$$

$$t_0 = 0$$

$$\begin{aligned} \langle B \rangle_\alpha &= \langle \alpha(t) | B | \alpha(t) \rangle \\ &= \left(\sum_i e^{\frac{i}{\hbar} t E_{a_i}} c_i^* \langle a_i | \right) B \left(\sum_j e^{-\frac{i}{\hbar} t E_{a_j}} c_j | a_j \rangle \right) \end{aligned}$$

$$\langle B \rangle_\alpha(t) = \sum_{ij} c_i^* c_j e^{-\frac{i}{\hbar} t (E_{a_i} - E_{a_j})} \langle a_i | B | a_j \rangle$$

Assume $c_{i_0} = 1$, $c_i = 0$ if $i \neq i_0$

$$|\alpha(t)\rangle = e^{\frac{i}{\hbar} t E_{a_{i_0}}} |a_{i_0}\rangle$$

$$\langle B \rangle_\alpha(t) = \langle a_{i_0} | B | a_{i_0} \rangle \quad \text{independent of time}$$

$|a_i\rangle$ are stationary states

stationary state is an energy eigenstate.

Example

$$H = -\vec{\mu} \cdot \vec{B}$$

$$= -\mu_z B$$

$$= -\frac{g}{\hbar m_e} S_z B$$

$$H = -\mu B S_z$$

$$\vec{B} = B \hat{z}$$

$$\begin{aligned} [H, S_z] &= [-\mu B S_z, S_z] \\ &= -\mu B [S_z, S_z] = 0 \end{aligned}$$

$$H|+\rangle = -\frac{\mu B \hbar}{2} |+\rangle \equiv -\frac{E}{2} |+\rangle$$

$$H|-\rangle = +\frac{\mu B \hbar}{2} |-\rangle \equiv \frac{E}{2} |-\rangle$$

$$\boxed{E = \mu B \hbar}$$

$$|\alpha(t_0)\rangle = a|+\rangle + b|-\rangle$$

$$\Rightarrow |\alpha(t)\rangle = a e^{-\frac{i}{\hbar} t (-\frac{E}{2})} |+\rangle + b e^{-\frac{i}{\hbar} t (\frac{E}{2})} |-\rangle$$

$$\langle S_z \rangle = \langle \alpha(t) | S_z | \alpha(t) \rangle$$

$$= \langle \alpha(t) | \left(\frac{\hbar}{2} \right) (|+\rangle \langle +| - |-\rangle \langle -|) | \alpha(t) \rangle$$

$$= \frac{\hbar}{2} |\langle + | \alpha(t) \rangle|^2 - \frac{\hbar}{2} |\langle - | \alpha(t) \rangle|^2$$

$$\boxed{\langle S_z \rangle = \frac{\hbar}{2} |a|^2 - \frac{\hbar}{2} |b|^2} \quad [H, S_z]$$

$$S_x = \frac{\hbar}{2} (|+\rangle \langle -| + |-\rangle \langle +|)$$

$$\langle S_x \rangle_\alpha = \langle \alpha(t) | \left(\frac{\hbar}{2} \right) (|+\rangle \langle -| + |-\rangle \langle +|) | \alpha(t) \rangle$$

$$= \frac{\hbar}{2} \left(\langle \alpha(t) | + \rangle \langle - | \alpha(t) \rangle + \langle \alpha(t) | - \rangle \langle + | \alpha(t) \rangle \right)$$

$$= \hbar \operatorname{Re} \left(\langle \alpha(t) | + \rangle \langle - | \alpha(t) \rangle \right)$$

$$= \hbar \operatorname{Re} \left(a^* e^{-\frac{i}{\hbar} t \frac{E}{2}} \frac{\hbar}{2} b e^{-\frac{i}{\hbar} t \frac{E}{2}} \right)$$

$$\langle S_x \rangle = \hbar \operatorname{Re} (a^* b e^{-\frac{i}{\hbar} t E})$$

assume $|\alpha(t=0)\rangle = |S_x +\rangle \Rightarrow a = b = \frac{1}{\sqrt{2}}$

$$\langle S_x \rangle = \hbar \operatorname{Re} \left(\frac{1}{2} e^{-\frac{i}{\hbar} t E} \right) = \frac{\hbar}{2} \cos \left(\frac{E}{\hbar} t \right)$$

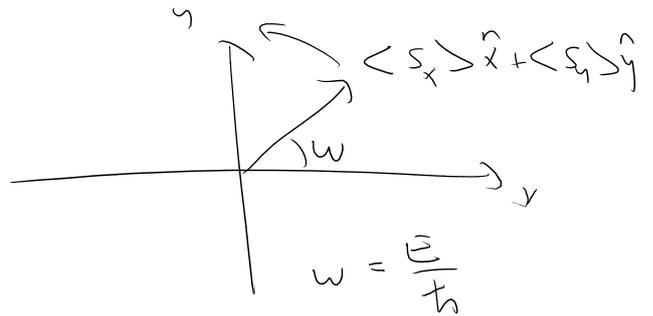
$$S_y = -i \frac{\hbar}{2} (|+\rangle \langle -| - |-\rangle \langle +|)$$

$$\begin{aligned} \langle S_y \rangle &= \hbar \operatorname{Im} (\langle \alpha(t) | |+\rangle \langle -| \alpha(t) \rangle) \\ &= \hbar \operatorname{Im} (a^* b e^{-\frac{i}{\hbar} t E}) \end{aligned}$$

$$|\alpha(t=0)\rangle = |S_x +\rangle$$

$$\langle S_y \rangle = \hbar \operatorname{Im} \left(\frac{1}{2} e^{-\frac{i}{\hbar} t E} \right) = \frac{\hbar}{2} \sin \left(\frac{E}{\hbar} t \right)$$

$$\langle S_x \rangle^2 + \langle S_y \rangle^2 = \frac{\hbar^2}{4}$$



$$H|a\rangle = E|a\rangle$$

$$H|b\rangle = E|b\rangle$$

$$H(\alpha|a\rangle + \beta|b\rangle) = E(\alpha|a\rangle + \beta|b\rangle)$$