$$a = \frac{1}{\sqrt{\hbar\omega}}\left(\sqrt{\frac{m}{2}}\,\omega x + \frac{ip}{\sqrt{2m}}\right)$$

$$a^{\dagger} = \frac{1}{\sqrt{\hbar\omega}}\left(\sqrt{\frac{m}{2}}\,\omega x - \frac{ip}{\sqrt{2m}}\right)$$

$$H = \left(a^{\dagger}a + \frac{1}{2}\right)\hbar\omega$$

$$[H, a] = -\hbar\omega a$$

$$[H, a^{\dagger}] = \hbar\omega a^{\dagger}$$

$$H\left(a|E_\alpha\rangle\right) = \left(E_\alpha - \hbar\omega\right)\left(a|E_\alpha\rangle\right)$$

$$H\left(a^{\dagger}|E_\alpha\rangle\right) = \left(E_\alpha + \hbar\omega\right)\left(a^{\dagger}|E_\alpha\rangle\right)$$

$$\langle H\rangle_\alpha \geq 0$$

$$\boxed{a|0\rangle = 0}$$

$$H = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)$$

$$H|0\rangle = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)|0\rangle = \frac{\hbar\omega}{2}|0\rangle$$

$$H(a^+|0\rangle) = \left(\hbar\omega + \frac{\hbar\omega}{2}\right)(a^+|0\rangle)$$

$$H(a^+)^n|0\rangle = \left(n\hbar\omega + \frac{\hbar\omega}{2}\right)(a^+)^n|0\rangle$$

$$= \left(n+\frac{1}{2}\right)\hbar\omega \; (a^+)^n|0\rangle$$

$$(a^+)^n|0\rangle = |\tilde{n}\rangle \Leftarrow$$

$$N = a^+a \Rightarrow H = \hbar\omega\left(N+\frac{1}{2}\right)$$

$$H|\tilde{n}\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|\tilde{n}\rangle$$

$$\Rightarrow N|\tilde{n}\rangle = n|\tilde{n}\rangle$$

$$\langle\tilde{n}|N|\tilde{n}\rangle = n\langle\tilde{n}|\tilde{n}\rangle$$

$$\langle\tilde{n}|a^+a|\tilde{n}\rangle = n\langle\tilde{n}|\tilde{n}\rangle$$

$$a^+a = aa^+ + (a^+a - aa^+)$$

$$= aa^+ + \underbrace{[a^+, a]}_{-1}$$

$$n\langle\tilde{n}|\tilde{n}\rangle = \langle\tilde{n}|aa^+ - 1|\tilde{n}\rangle$$

$$n\langle\tilde{n}|\tilde{n}\rangle = \underbrace{(a|\tilde{n}\rangle)^{+}}\ \underbrace{(a^{+}|\tilde{n}\rangle)}$$

$$-\langle\tilde{n}|\tilde{n}\rangle$$

$$= \langle\widetilde{n+1}|\widetilde{n+1}\rangle$$

$$-\langle\tilde{n}|\tilde{n}\rangle$$

$$\Downarrow$$

$$\boxed{\langle\widetilde{n+1}|\widetilde{n+1}\rangle = (n+1)\langle\tilde{n}|\tilde{n}\rangle}$$

$$\langle\tilde{0}|\tilde{0}\rangle = 1 \Rightarrow |0\rangle = |\tilde{0}\rangle$$

$$\langle\tilde{1}|\tilde{1}\rangle = \langle 0|0\rangle = 1 \Rightarrow |1\rangle = |\tilde{1}\rangle$$

$$\langle\tilde{2}|\tilde{2}\rangle = 2\langle\tilde{1}|\tilde{1}\rangle \Rightarrow |2\rangle = \frac{1}{\sqrt{2}}|\tilde{2}\rangle$$

$$\langle\tilde{3}|\tilde{3}\rangle = 3\langle\tilde{2}|\tilde{2}\rangle \Rightarrow |3\rangle = \frac{1}{\sqrt{3}\sqrt{2}}|\tilde{3}\rangle$$

$$\vdots$$

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^{n}|0\rangle}$$

$$a^{+}|n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^{n+1}|0\rangle$$

$$= \frac{\sqrt{n+1}}{\sqrt{(n+1)!}} (a^+)^{n+1} |0\rangle$$

$$\boxed{a^+|n\rangle = \sqrt{n+1}\, |n+1\rangle}$$

$$a|n\rangle = \frac{1}{\sqrt{n!}}\, a(a^+)^n |0\rangle$$

$$= \frac{1}{\sqrt{n!}} \left[a(a^+)^n - (a^+)^n a\right] |0\rangle$$

$$a|n\rangle = \frac{1}{\sqrt{n!}} \left[a, (a^+)^n\right] |0\rangle$$

$$[a, a^+] = 1 = \frac{\partial a^+}{\partial a^+}$$

$$[a, (a^+)^2] = a^+ \underbrace{[a, a^+]}_{1} + \underbrace{[a, a^+]}_{1} a^+$$

$$[a, (a^+)^2] = 2a^+ = \frac{\partial (a^+)^2}{\partial a^+}$$

$$[a, (a^+)^3] = [a, a^+ (a^+)^2]$$

$$= \underbrace{[a, a^+]}_{1} (a^+)^2 + a^+ \underbrace{[a, (a^+)^2]}_{2a^+}$$

$$[a,(a^\dagger)^3] = 3(a^\dagger)^2 = \frac{\partial (a^\dagger)^3}{\partial a^\dagger}$$

$$[a,(a^\dagger)^n] = \frac{\partial (a^\dagger)^n}{\partial a^\dagger} = n(a^\dagger)^{n-1}$$

assume

$$[a,(a^\dagger)^{n-1}] = (n-1)(a^\dagger)^{n-2}$$

$$[a,(a^\dagger)^n] = [a, a^\dagger (a^\dagger)^{n-1}]$$

$$= \underbrace{[a, a^\dagger]}(a^\dagger)^{n-1} + a^\dagger [a,(a^\dagger)^{n-1}]$$

$$= (a^\dagger)^{n-1} + a^\dagger (n-1)(a^\dagger)^{n-2}$$

$$[a,(a^\dagger)^n] = n(a^\dagger)^{n-1} \quad \checkmark$$

$$a|n\rangle = \frac{1}{\sqrt{n!}}[a,(a^\dagger)^n]|0\rangle$$

$$= \frac{1}{\sqrt{n!}}\, n \underbrace{(a^\dagger)^{n-1}|0\rangle}_{\sqrt{(n-1)!}\,|n-1\rangle}$$

$$a|n\rangle = \frac{n}{\sqrt{n}}|n-1\rangle = \sqrt{n}\,|n-1\rangle$$

$$\begin{cases} a|n\rangle = \sqrt{n}\,|n-1\rangle \\ a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases}$$

$$\langle m|x|n\rangle = ?$$

$$(a+a^\dagger) = \frac{1}{\sqrt{\hbar\omega}}\sqrt{2m}\,\omega x$$

$$= \sqrt{\frac{2m\omega}{\hbar}}\,x \Rightarrow \boxed{x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)$$

$$a-a^\dagger = \frac{1}{\sqrt{\hbar\omega}}\left(\frac{2i}{\sqrt{2m}}\,p\right) = i\sqrt{\frac{2}{\hbar\omega m}}\,p$$

$$\boxed{p = \sqrt{\frac{\hbar\omega m}{2}}\,\frac{(a-a^\dagger)}{i}}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger) \qquad p = \sqrt{\frac{\hbar\omega m}{2}}\,\frac{(a-a^\dagger)}{i}$$

$$\langle n|x|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|a+a^\dagger|m\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|\left(\sqrt{m}\,|m-1\rangle + \sqrt{m+1}\,|m+1\rangle\right)$$

$$\langle n|x|m\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{m}\,\delta_{n,m-1} + \sqrt{m+1}\,\delta_{n,m+1}\right)$$

$$\langle n|x|n\rangle = 0$$

$$\langle n|p|m\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}\,\langle n|\,a - a^{+}|m\rangle$$

$$= -i\sqrt{\frac{\hbar\omega m}{2}}\,\langle n|\left(\sqrt{m}\,|m-1\rangle - \sqrt{m+1}\,|m+1\rangle\right)$$

$$\langle n|p|m\rangle = -i\sqrt{\frac{\hbar\omega m}{2}}\left(\sqrt{m}\,\delta_{n,m-1} - \sqrt{m+1}\,\delta_{n,m+1}\right)$$

$$\langle n|p|n\rangle = 0$$

$$\langle n|(\Delta x)^2|n\rangle = \langle n|x^2|n\rangle - \underbrace{\langle n|x|n\rangle^2}_{=0}$$

$$= \langle n|x^2|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|(a+a^{+})^2|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|a^2 + a^{+2} + aa^{+} + a^{+}a|n\rangle$$

$$= \frac{\hbar}{2m\omega}\left\{\underbrace{\langle n|a^2|n\rangle}_{=0} + \underbrace{\langle n|(a^{+})^2|n\rangle}_{=0} + \underbrace{\langle n|a a^{+}|n\rangle}_{=n+1} + \underbrace{\langle n|a^{+}a|n\rangle}_{=n}\right\}$$

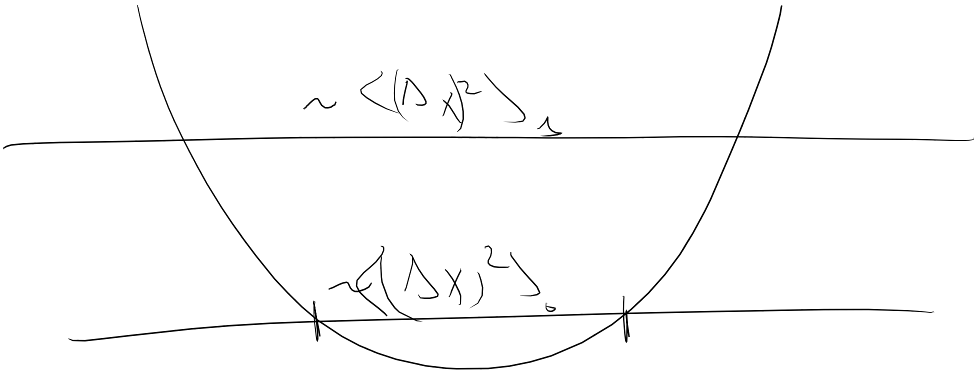
$$\langle n|a^2|n\rangle \propto \langle n|n-2\rangle = 0$$

$a^\dagger a = N$: number operators

$$a a^\dagger = a^\dagger a + \underbrace{a a^\dagger - a^\dagger a}_{[a, a^\dagger] = 1}$$

$$a a^\dagger = N + \mathbb{1}$$

$$\boxed{\langle n | (\Delta x)^2 | n \rangle = \frac{\hbar}{2m\omega}(2n+1)}$$



$$\langle n | (\Delta p)^2 | n \rangle = \langle n | p^2 | n \rangle - (\underbrace{\langle n | p | n \rangle}_{=0})^2$$

$$= -\frac{\hbar \omega m}{2} \langle n | (a - a^\dagger)^2 | n \rangle$$

$$= -\frac{\hbar \omega m}{2} \langle n | a^2 + a^{\dagger 2} - \underbrace{a a^\dagger}_{=N+1} - \underbrace{a^\dagger a}_{=N} | n \rangle$$

$$= -\frac{\hbar \omega m}{2} \left(-(n+1) - n \right)$$

$$\langle (\Delta p)^2 \rangle_n = \frac{\hbar m \omega}{2} (2n+1)$$

$$\langle(\Delta p)^2\rangle_n \langle(\Delta x)^2\rangle_n = \frac{\hbar m\omega}{2}(2n+1)\,\frac{\hbar}{2m\omega}(2n+1)$$

$$\boxed{\langle(\Delta p)^2\rangle_n \langle(\Delta x)^2\rangle_n = \frac{\hbar^2}{4}(2n+1)^2}$$

$$\boxed{\langle(\Delta p)^2\rangle_0 \langle(\Delta x)^2\rangle_0 = \frac{\hbar^2}{4}}$$

$a|\alpha\rangle = \alpha|\alpha\rangle$: Coherent states

$$\langle x\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|a+a^\dagger|\alpha\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left\{\langle\alpha|\underbrace{(a|\alpha\rangle)}_{\alpha|\alpha\rangle} + \underbrace{(a|\alpha\rangle)^\dagger}_{(\alpha|\alpha\rangle)^\dagger}|\alpha\rangle\right\}$$

$$\langle x\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)$$

$$\boxed{\langle x\rangle_\alpha = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\alpha}$$

$$\langle x^2\rangle_\alpha = \frac{\hbar}{2m\omega}\langle\alpha|(a+a^\dagger)^2|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\alpha|a^2+a^{\dagger 2}+2\underbrace{N}_{a^\dagger a}+1|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}\left\{\langle\alpha|(a^2|\alpha\rangle) + (a^2|\alpha\rangle)^\dagger|\alpha\rangle + 2(a|\alpha\rangle)^\dagger(a|\alpha\rangle) + 1\right\}$$

$$= \frac{\hbar}{2m\omega}\left\{\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1\right\}$$

$$= \frac{\hbar}{2m\omega}\left[\left(\alpha+\alpha^*\right)^2 + 1\right]$$

$$\langle x^2\rangle_\alpha = \underbrace{\left[\sqrt{\frac{\hbar}{2m\omega}}\left(\alpha+\alpha^*\right)\right]^2}_{\langle x\rangle_\alpha} + \frac{\hbar}{2m\omega}$$

$$\langle x^2\rangle_\alpha = \langle x\rangle_\alpha^2 + \frac{\hbar}{2m\omega} \Rightarrow \langle(\Delta x)^2\rangle_\alpha = \frac{\hbar}{2m\omega}$$

Exercise

$$\langle(\Delta p)^2\rangle_\alpha = \frac{\hbar m\omega}{2}$$

$$\langle(\Delta x)^2\rangle_\alpha \langle(\Delta p)^2\rangle_\alpha = \frac{\hbar^2}{4}$$

$$\langle H\rangle_\alpha = \hbar\omega\langle\alpha|\, a^\dagger a + \tfrac{1}{2}\,|\alpha\rangle$$

$$\boxed{\langle H\rangle_\alpha = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right)}$$

$$N|n\rangle = n|n\rangle \qquad N^\dagger = N$$

$$a|\alpha\rangle = \alpha|\alpha\rangle \qquad a^\dagger \neq a$$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \qquad c_n = ?$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n \underbrace{a|n\rangle}_{\sqrt{n}\,|n-1\rangle}$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle$$

$$= \sum_{n=1}^{\infty} c_n \sqrt{n}\,|n-1\rangle$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\,|n\rangle$$

$$= \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

$$\Rightarrow c_{n+1}\sqrt{n+1} = \alpha\, c_n$$

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n = \frac{\alpha}{\sqrt{n+1}} \frac{\alpha}{\sqrt{n}} c_{n-1}$$

$$\Rightarrow \underline{c_{n+1}} = \frac{\alpha}{\sqrt{n+1}} \frac{\alpha}{\sqrt{n}} \frac{\alpha}{\underline{\sqrt{n-1}}} \underline{c_{n-2}} =$$

$$= \frac{\alpha^{n+1}}{\sqrt{n+1}\,\sqrt{n}\cdots\sqrt{1}} c_0$$

$$c_{n+1} = \frac{\alpha^{n+1}}{\sqrt{(n+1)!}} c_0 \qquad n = 0$$

$$c_1 = \alpha c_0$$

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\boxed{c_n = \frac{\alpha^n}{\sqrt{n!}} c_0}$$

<u>Exercises</u>

i) $c_0 = ?$ st $\langle\alpha|\alpha\rangle = 1$

ii) $\langle\alpha|\alpha'\rangle = ?$

iii) (BONUS)

$$\mathbb{1} = \int d\alpha\, d\alpha^* \, |\alpha\rangle\langle\alpha| \, C(\alpha, \alpha^*)$$

$$C(\alpha, \alpha^*) = ?$$

$$\langle x\rangle_t = \langle x\rangle_{t=0} \cos(\omega t) + \frac{\langle p\rangle_{t=0}}{m\omega} \sin(\omega t)$$

$$\langle x | n \rangle = \psi_n(x) = ?$$

$$\langle x | a | 0 \rangle = 0$$

$$\langle x | \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i\hat{p}}{\sqrt{2\hbar\omega m}} | 0 \rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\, x \langle x | 0 \rangle + \frac{i}{\sqrt{2\hbar\omega m}} \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | 0 \rangle = 0$$

$$\Rightarrow \sqrt{\frac{m\omega}{2\hbar}}\, x\, \psi_0(x) + \sqrt{\frac{\hbar}{2m\omega}} \frac{\partial \psi_0}{\partial x} = 0$$

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} x\, \psi_0 \qquad \left[\frac{\hbar}{m\omega}\right] = [L]^2$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\frac{d\psi_0}{dx} = -\frac{x}{x_0^2}\psi_0$$

$$\psi_0(x) = N e^{-\frac{x^2}{2x_0}}$$

$$\langle 0 | 0 \rangle = \int dx\, |\psi_0(x)|^2 = 1$$

$$1 = \int_{-\infty}^{\infty} dx\, |N|^2 e^{-x^2/x_0^2}$$

$$1 = |N|^2 \sqrt{\pi}\, x_0 \Rightarrow N = \frac{1}{\sqrt{\sqrt{\pi}\, x_0}}$$

$$\boxed{\psi_0(x) = \frac{1}{\sqrt{\sqrt{\pi}\, x_0}}\, e^{-x^2/2x_0^2}}$$

$$\psi_n(x) = \langle x | n \rangle = \langle x | \frac{1}{\sqrt{n!}} (a^\dagger)^n | 0 \rangle$$

$$= \langle x | \frac{1}{\sqrt{n!}} \left(\sqrt{\frac{m}{2\hbar\omega}}\, \hat{x} + i \frac{\hat{p}}{\sqrt{2m\hbar\omega}} \right)^n | 0 \rangle$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}} \left(\sqrt{\frac{m}{2\hbar\omega}}\, x + \sqrt{\frac{\hbar}{2m\omega}} \frac{\partial}{\partial x} \right)^n \psi_0(x)$$

$$\psi_1(x) = \overbrace{\left(\sqrt{\frac{m}{2\hbar\omega}}\, x + \sqrt{\frac{\hbar}{2m\omega}} \frac{\partial}{\partial x} \right)} \frac{1}{\sqrt{\sqrt{\pi}\, x_0}}\, e^{-x^2/2x_0^2}$$

$$= \sqrt{\frac{m}{2\hbar\omega}} \frac{x}{\sqrt{\sqrt{\pi}\, x_0}}\, e^{-x^2/2x_0^2} \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$+ \sqrt{\frac{x_0}{2}} \frac{1}{\sqrt{\sqrt{\pi}\, x_0}} \left(- \frac{x}{x_0^2} \right) e^{-x^2/2x_0^2}$$

$$\psi_1(x) = (\quad)\, x\, e^{-x^2/2x_0^2}$$