$$A(t) = e^{-\frac{i}{\hbar}Ht} A e^{\frac{i}{\hbar}Ht}$$

$$e^{-i\lambda B} A e^{i\lambda B} = f(\lambda) = \frac{f_0}{0!} + \frac{f_1}{1!}\lambda + \frac{f_2}{2!}\lambda^2 + \cdots$$

$$f_n = \left(\frac{d^n f(\lambda)}{d\lambda^n}\right)\Big|_{\lambda=0}$$

$$f(0) = A$$

$$\frac{df}{d\lambda} = -iB e^{-i\lambda B} A e^{i\lambda B} + e^{-i\lambda B} A e^{i\lambda B} iB$$

$$\frac{df}{d\lambda}\Big|_{\lambda=0} = -i(BA - AB) = -i[B,A]$$

$$\frac{df}{d\lambda} = -i e^{-i\lambda B} [B,A] e^{i\lambda B}$$

$$\frac{d^2f}{d\lambda^2} = (-i)^2 e^{-i\lambda B} [B,[B,A]] e^{i\lambda B}$$

$$\frac{d^nf}{d\lambda^n} = (-i)^n e^{-i\lambda B} \underbrace{[B,\ldots,[B,A]]}_{n\ B's} e^{i\lambda B}$$

$$e^{-i\lambda B} A e^{i\lambda B} = A + (-i)[B,A]\lambda + (-i)^2[B,[B,A]]\frac{\lambda^2}{2!} + \cdots (-i)^n [B,[B,\ldots,[B,A]]]\frac{\lambda^n}{n!} + \cdots$$

$$a(t) = e^{-i\frac{t}{\hbar}H} a e^{\frac{i}{\hbar}tH}$$

$$[H,a] = \hbar\omega a$$

$$\overbrace{[H,[H,\ldots[H,a]]]}^{n\ \text{times}} = (-\hbar\omega)^n a$$

$$(Ha - aH)|E_\alpha\rangle$$

$$= H\,a|E_\alpha\rangle - E_\alpha a|E_\alpha\rangle$$

$$= [(E_\alpha - \hbar\omega) - E_\alpha]\,a|E_\alpha\rangle$$

$$a(t) = \sum_{n=0}^{\infty} \left(-i\frac{t}{\hbar}\right)^n \frac{(-\hbar\omega)^n}{n!} a$$

$$= \left(\sum_{n=0}^{\infty} \frac{(i\omega t)^n}{n!}\right) a$$

$$a = (\;)^{c_1} x - i(\;)^{c_2} p$$

$$a^\dagger = (\;) x + i(\;) p$$

$$\boxed{a(t) = \exp(i\omega t)\, a(0)}$$

$$a^\dagger(t) = \exp(-i\omega t)\, a^\dagger(0)$$

$$x(t) = (\;)\left(a(t) + a^\dagger(t)\right)$$

$$= (\;)\Big[e^{i\omega t}\left(c_1 x(0) - i c_2 p(0)\right)$$

$$+ e^{-i\omega t}\left(c_1 x(0) + i c_2 p(0)\right)\Big]$$

$$= (\;)\left[x(0)\, c_1 2\cos\omega t - c_2 p(0) 2\sin(\omega t)\right]$$

$$\langle x(t)\rangle = (\;)\left[\langle x(0)\rangle c_1 2\cos\omega t - 2c_2 \langle p(0)\rangle \sin(\omega t)\right]$$

Time Dependent Schroedinger Eqn.

$$|\alpha\, t_0; t\rangle = U(t,t_0)|\alpha\, t_0\rangle$$

$$\frac{d}{dt}|\alpha\, t_0; t\rangle = \underbrace{\left[\frac{dU}{dt}(t,t_0)\right]|\alpha\, t_0\rangle}_{\frac{-i}{\hbar} H\, U |\alpha\, t_0\rangle}$$

$$\boxed{\frac{\hbar}{i}\frac{d}{dt}|\alpha\, t_0; t\rangle = H|\alpha\, t_0; t\rangle}$$

$$\langle x|\, i\hbar \frac{d}{dt}|\alpha\, t_0; t\rangle = \langle x| \frac{p^2}{2m} + V(\hat{x})|\alpha\, t_0, t\rangle$$

$$i\hbar \frac{\partial}{\partial t}\langle x|\alpha\, t_0; t\rangle = \frac{-\hbar^2}{2m}\nabla^2\langle x|\alpha\, t_0, t\rangle + V(x)\langle x|\alpha\, t_0, t\rangle$$

$$\hat{x}|x\rangle = x|x\rangle$$

$$\langle x|\alpha\, t_0; t\rangle \equiv \psi(\vec{r}, t)$$

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r},t)}$$

$$\boxed{-\frac{\hbar}{i}\frac{d}{dt}|\alpha\, t_0; t\rangle = H|\alpha\, t_0; t\rangle}$$

Time Independent Schroedinger's Eqn.

if $|\alpha\, t_0\rangle$ is a stationary state

$$|\alpha\, t_0; t\rangle = e^{-\frac{i}{\hbar}E_\alpha(t-t_0)}|\alpha\, t_0\rangle$$

$$\frac{d}{dt}|\alpha\, t_0; t\rangle = -\frac{i}{\hbar}E_\alpha|\alpha\, t_0; t\rangle$$

$$\langle x|E_\alpha|\alpha\, t_0; t\rangle = \langle x|H|\alpha\, t_0; t\rangle$$

$$\boxed{E_\alpha \psi_\alpha(\vec{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi_\alpha(\vec{r})}$$

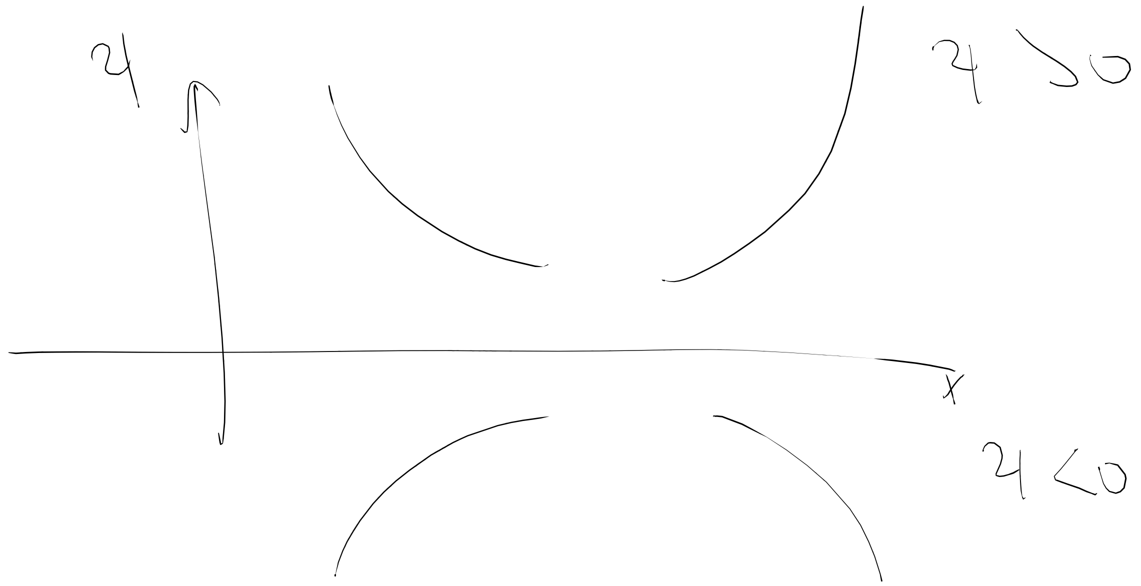
$$\psi_\alpha(\vec{r}) = \langle x|\alpha\rangle$$

$$\boxed{\nabla^2\psi_\alpha = \frac{2m}{\hbar}(V(\vec{r}) - E_\alpha)\psi_\alpha}$$

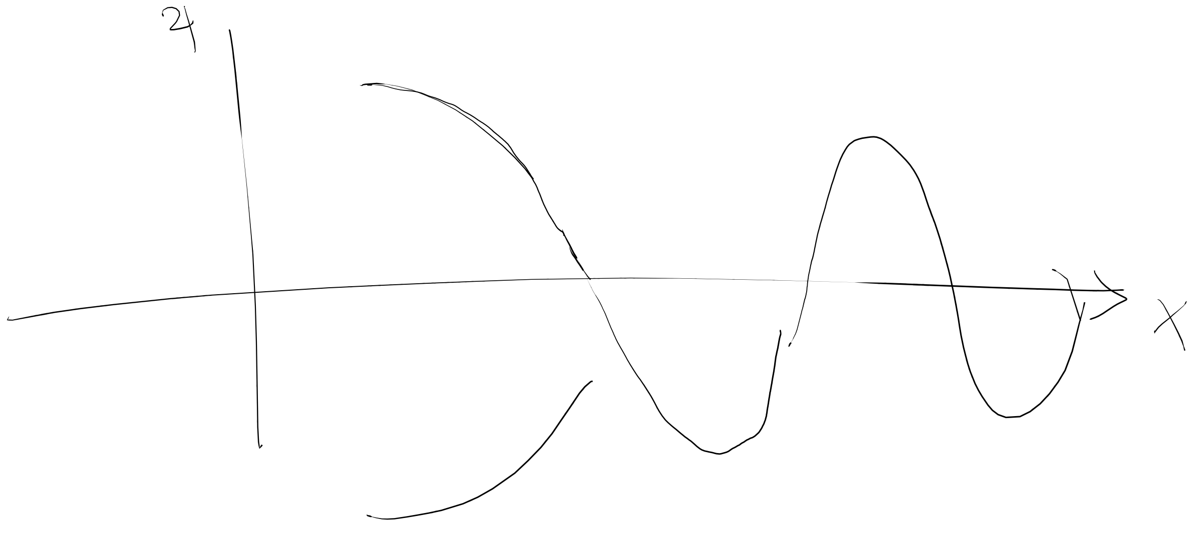
in 1D

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar}\underbrace{(V(x) - E)}\psi$$

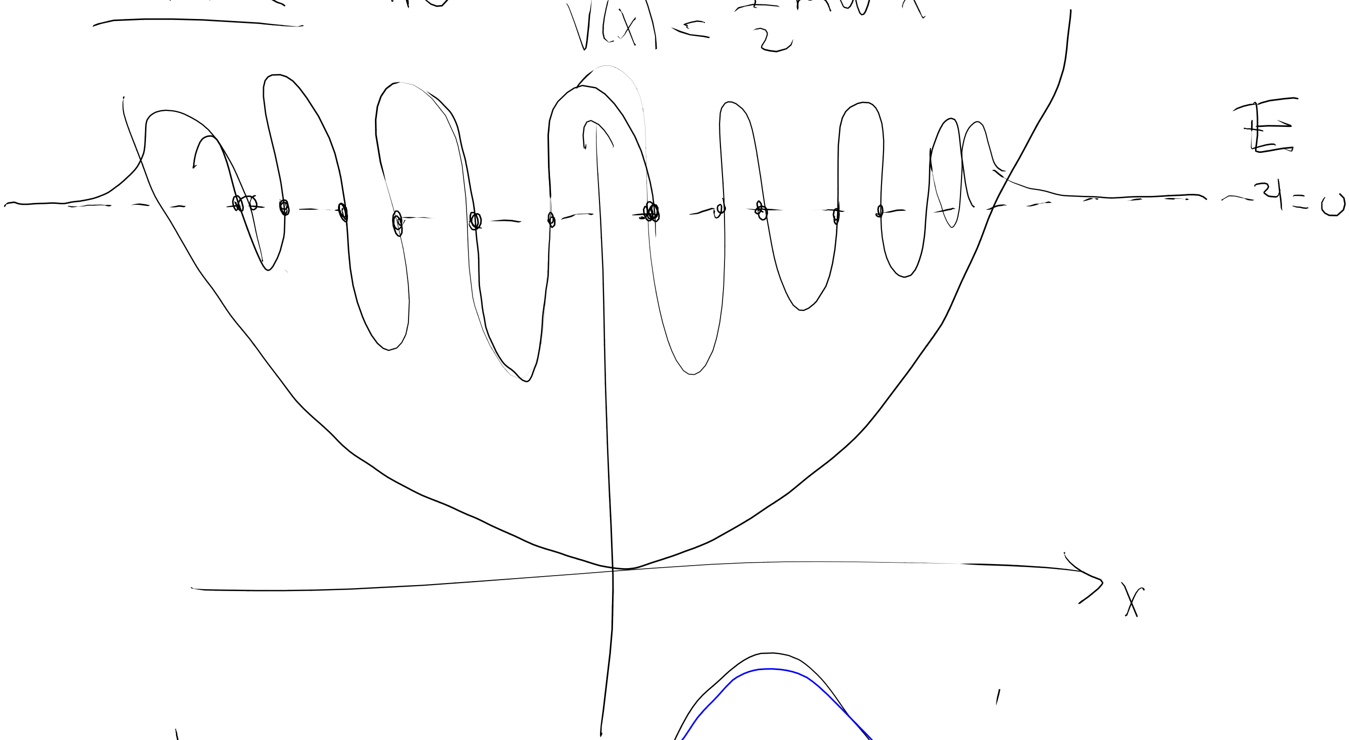
if $\underline{V(x) > E}$ no oscillations

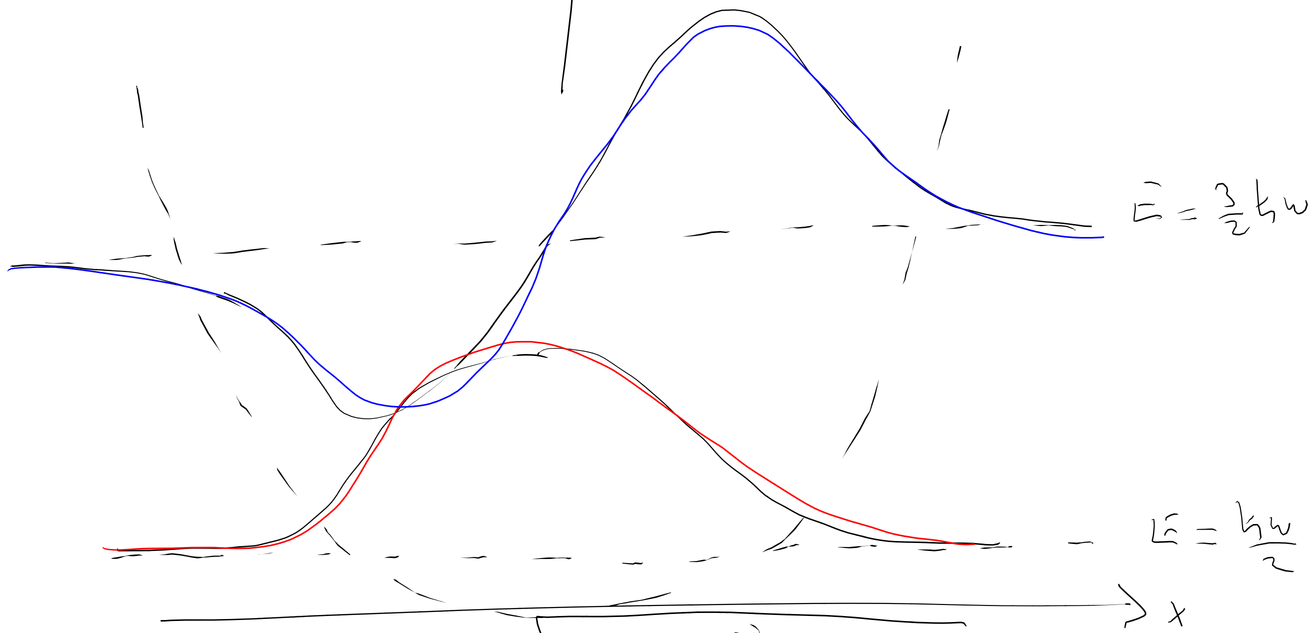


if $\underline{V(x) < E}$: oscillatory solns.



Exercise SHO





$E < \lim_{r \to \infty} V(\vec{r})$

$$\boxed{\psi(\vec{r}) \xrightarrow{r \to \infty} 0}$$

classically allowed I $\boxed{\text{classically forbidden}}$ II classically allowed

$e^- \longrightarrow$

tunnelling

$|\psi(\vec{r},t)|^2$: probability density

$|\psi(\vec{r},t)|^2 d^3r$: probability that the particle will be around the point in a volume d^3r

$$i\hbar\psi^*\frac{\partial\psi}{\partial t} = \psi^* H\psi = -\frac{\psi^*\hbar^2}{2m}\nabla^2\psi + \psi^* V(\vec{r})\psi$$

$$-i\hbar\psi\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + \psi V^*(\vec{r})\psi^*$$

$$H^\dagger = H \Rightarrow V^* = V$$

$$i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right)$$

$$\Downarrow$$

$$i\hbar\frac{\partial}{\partial t}\underbrace{(\psi^*\psi)}_{\rho(\vec{r},t)} = -\frac{\hbar^2}{2m}\vec{\nabla}\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$\frac{\partial\rho}{\partial t} = \frac{i\hbar}{2m}\vec{\nabla}\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$ continuity eqn.

$$\vec{J} = -\frac{i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

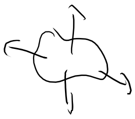
$$\vec{J} = \frac{\hbar}{2m}\mathrm{Im}(\psi^*\nabla\psi)$$

$$\frac{d}{dt}\int \rho\, d^3r = -\int (\vec{\nabla}\cdot\vec{J})\, d^3r$$

$$= -\oint \vec{J}\cdot d\vec{S}$$

$$= 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$



$$|\psi|^2 = \rho \Longrightarrow \psi(\vec{r},t) = \sqrt{\rho(\vec{r},t)}\, e^{\frac{i}{\hbar} S(\vec{r},t)}$$

S is a real function

$$\vec{J} = -\frac{i\hbar}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

$$= -\frac{i\hbar}{2m}\left[\sqrt{\rho}\,\cancel{e^{-\frac{i}{\hbar}S}}\left((\nabla\sqrt{\rho})\,\cancel{e^{\frac{i}{\hbar}S}} + \frac{i}{\hbar}(\nabla \vec{S})\sqrt{\rho}\,\cancel{e^{\frac{i}{\hbar}S}}\right) - c.c.\right]$$

$$= -\frac{i\hbar}{2m}\left[\sqrt{\rho}(\nabla\sqrt{\rho}) + \frac{i}{\hbar}\rho(\vec{\nabla} S) - \sqrt{\rho}\,(\nabla\sqrt{\rho}) + \frac{i}{\hbar}\rho(\vec{\nabla}\vec{S})\right]$$

$$\vec{j} = \cancel{-i}\,\frac{\hbar}{\cancel{2}m}\,\frac{\cancel{2i}\rho}{\cancel{\hbar}}\left(\vec{\nabla} S\right) = \rho\left(\frac{\vec{\nabla} S}{m}\right)$$

$$\vec{j} = \rho\ \text{"}\vec{v}\text{"} \qquad \boxed{\vec{v} = \frac{\vec{\nabla} S}{m}}$$

—— o ——

$$\psi(\vec{r},t) = \sqrt{\rho(\vec{r},t)}\; e^{\frac{i}{\hbar} S(\vec{r},t)}$$

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\left[\left(\frac{\partial}{\partial t}\sqrt{\rho}\right) e^{\frac{i}{\hbar}S} + \sqrt{\rho}\;\frac{i}{\hbar}\left(\frac{\partial S}{\partial t}\right) e^{\frac{i}{\hbar}S}\right]$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\sqrt{\rho}\,\frac{\partial S}{\partial t}\, e^{\frac{i}{\hbar}S} + O(\hbar)$$

$$\vec{\nabla}\psi = \left(\vec{\nabla}\sqrt{\rho}\right) e^{\frac{i}{\hbar}S} + \frac{i}{\hbar}\left(\vec{\nabla} S\right)\sqrt{\rho}\, e^{\frac{i}{\hbar}S}$$

$$\nabla^2\psi = \nabla^2\left(\sqrt{\rho}\right) e^{\frac{i}{\hbar}S} + \left(\vec{\nabla}\sqrt{\rho}\right)\cdot\left(\vec{\nabla} e^{\frac{i}{\hbar}S}\right) + \frac{i}{\hbar}\left(\vec{\nabla} S\right)\cdot\vec{\nabla}\left(\sqrt{\rho}\, e^{\frac{i}{\hbar}S}\right)$$

$$+ \frac{i}{\hbar}\left(\nabla^2 S\right)\sqrt{\rho}\, e^{\frac{i}{\hbar}S}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = +\frac{\cancel{\hbar^2}}{2m}\left(\frac{\cancel{i}}{\cancel{\hbar}}\right)\left(\vec{\nabla} S\right)\cdot\left(\frac{\cancel{i}}{\cancel{\hbar}}\vec{\nabla} S\right)\psi$$

$$+ O(\hbar)$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = \frac{\left(\vec{\nabla} S\right)^2}{2m}\psi + O(\hbar)$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi + \mathcal{O}(\hbar)$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi$$

$$\Rightarrow -\frac{\partial S}{\partial t}\psi = \frac{(\vec{\nabla} S)^2}{2m}\psi + V(x)\psi + \mathcal{O}(\hbar)$$

$\hbar \to 0$

$$\boxed{\frac{(\vec{\nabla} S)^2}{2m} + \frac{\partial S}{\partial t} + V(\vec{r}) = 0}$$

Hamilton Jacobi eqn in classical physics

S: action in classical physics.

$$\boxed{\vec{\nabla} S = \vec{p}}$$ in classical physics.

$$S = \int L\, dt$$

$$\underbrace{-i\hbar\vec{\nabla}\left(\sqrt{\rho}\, e^{\frac{i}{\hbar}S}\right)}_{\psi} = -i\hbar(\vec{\nabla}\sqrt{\rho})e^{\frac{i}{\hbar}S} - i\hbar\sqrt{\rho}\left(\frac{i}{\hbar}\nabla S\right)e^{\frac{i}{\hbar}S}$$

$$= \underbrace{(\vec{\nabla} S)}_{\vec{p}}\psi + \underbrace{\mathcal{O}(\hbar)}$$

$\Downarrow$

$$\boxed{\vec{p}\psi = -i\hbar\vec{\nabla}\psi}$$ semiclassically

$\mathcal{O}(\hbar)$: of the order of $\hbar$

$$\frac{\partial}{\partial t}(\psi^*\psi) = \left(\frac{\partial \psi^*}{\partial t}\right)\psi + \psi^*\left(\frac{\partial \psi}{\partial t}\right)$$

$$= \left(\frac{1}{i\hbar} H\psi\right)^* \psi + \psi^*\left(\frac{1}{i\hbar} H\psi\right)$$

$$H = \frac{p^2}{2m} + V(x)$$