$$a_{m+2} = \frac{(2m+1-\varepsilon)\, a_m}{(m+1)(m+2)}$$

$$\boxed{\psi = e^{-u^2/2}\, \phi(u)}$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\phi(u) = \sum_{n=0}^{\infty} a_n u^n$$

$$x = x_0 u$$

$$\boxed{\frac{d^2\psi}{du^2} + (\varepsilon - u^2)\psi = 0}$$

$$E = \frac{\hbar\omega}{2}\varepsilon$$

$$\psi(u) \xrightarrow{u\to\infty} e^{\mp u^2/2}$$

$$\int_{-\infty}^{\infty} |\psi|^2 dx < \infty$$

$$\frac{d^2\phi}{du^2} - 2u\frac{d\phi}{du} + (\varepsilon - 1)\phi = 0$$

$$\phi = \sum_{n=0}^{\infty} a_n u^n$$

$$= \underbrace{\sum_{n=0}^{N} a_n u^n}_{\text{polynomial}} + \sum_{n=N+1}^{\infty} a_n u^n$$

$$\boxed{a_{m+2} = \frac{(2m+1-\varepsilon)\, a_m}{(m+1)(m+2)}}$$

assume

$a_{m_0+2} = 0$

$$\boxed{\varepsilon = 2m_0+1}$$

$m_0 = 0, 1, 2 \dots$

$n > N \qquad a_{m+2} \simeq \frac{2}{m} a_m$

$b_m = a_{2m}$

$c_m = a_{2m+1}$

$a_{n+2} \simeq \frac{2}{m} a_m$

$a_{2n+2} \simeq \frac{2}{2m} a_{2m}$

$$\boxed{b_{n+1} = \frac{1}{m} b_m}$$

$a_{2n+1+2} \simeq \frac{2 a_{2n+1}}{2n+1}$

$c_{n+1} \simeq \frac{1}{n} c_m$

$$b_{m+1} \simeq \frac{1}{m(n-1)(m-2)\dots\left(\frac{N}{2}\right)} b_{N/2}$$

$$b_{m+1} = \frac{1}{m!} \left(\frac{N}{2}-1\right)!\, b_{N/2}$$

define

$$b_{n+1} = \frac{B}{m!}$$

$$\frac{b_{n+1}}{b_m} = \frac{\frac{B}{m!}}{\frac{B}{(n-1)!}} = \frac{(n-1)!}{m!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{m}$$

$$\phi = \sum_{n=0}^{N} a_n u^n + \sum_{n=N+1}^{\infty} a_n u^n$$

$$= \sum_{n=0}^{N} a_n u^n + \sum_{m>N/2} b_m u^{2m} + \sum_{m>\frac{N}{2}} c_m u^{2m+1}$$

$$= \sum_{n=0}^{N} a_n u^n + \sum_{m>N/2} \frac{B}{m!} u^{2m} + \sum_{m>N/2} \frac{C}{m!} u^{2m+1}$$

$$= \sum_{m=0}^{\infty} \frac{B}{m!} u^{2m} + \sum_{m=0}^{\infty} \frac{C}{m!} u^{2m+1} \left. \begin{array}{l} + \sum_{n=0}^{N} a_n u^n - \sum_{m=0}^{N/2} \frac{B}{m!} u^{2m} \\ - \sum_{m=0}^{N/2} \frac{C}{m!} u^{2m+1} \end{array} \right\} \text{polynomial}$$

$$\phi(u) = \text{polynomial} + Be^{u^2} - Cue^{u^2}$$

$$\phi(u) \xrightarrow[u\to\infty]{} e^{u^2} \quad \text{unless} \quad B = 0 \text{ and } C = 0$$

$$\psi(u) \xrightarrow[u\to\infty]{} e^{-u^2/2} e^{u^2} \simeq e^{u^2/2}$$

$$a_{n+2} = \frac{(2n+1-\varepsilon)\, a_n}{(n+1)(n+2)}$$

assume $a_{n_0+2} = 0$

$\Rightarrow \varepsilon = 2n_0 + 1$

$n_0 = 0, 1$

Explicit solns

$\varepsilon = 1$

$$a_{n+2} = \frac{2n}{(n+1)(n+2)} a_n \qquad E = \frac{\hbar\omega}{2}$$

$\phi_0 = 1$

$a_2 = a_{0+2} = 0$

ϕ_n corresponds to $\varepsilon = 2n+1$

other soln:

$a_1 \neq 0$

$a_3 = \frac{2}{6} a_1 = \frac{1}{3} a_1 \neq 0$

$a_5 \neq 0 \; \ldots$

$$\tilde{\phi}_1(u) \xrightarrow[u\to\infty]{} e^{u^2}$$

$\underline{\varepsilon = 3}$

$$a_{n+2} = \frac{2n-2}{(n+1)(n+2)} a_n$$

$n=0$: $a_2 = -a_0$

$n=2$ $a_4 = \frac{2}{3\cdot 4} a_2 = -\frac{1}{6} a_0$

$\vdots$

non-normalizable soln.

$n=1$ $a_1 \neq 0$
$a_3 = 0$
$a_5 = 0$
$a_7 = 0$
$\vdots$

$\phi_1(u) = 2u$

$\underline{\varepsilon = 5}$

$$a_{n+2} = \frac{2n-4}{(n+1)(n+2)} a_n$$

$\phi_2 = 1 - 2u$

$n=0$: $a_2 = \frac{-4}{1\cdot 2} a_0 = -2a_0$

$u = \frac{x}{x_0}$

$n=2$: $a_4 = 0$, $a_6 = 0, \ldots$

$\phi_0 = 1$
$\phi_1 = u$
$\phi_2 = 1 - 2u$

$$\psi_0(x) = N_0 e^{-\frac{x^2}{2x_0^2}}$$

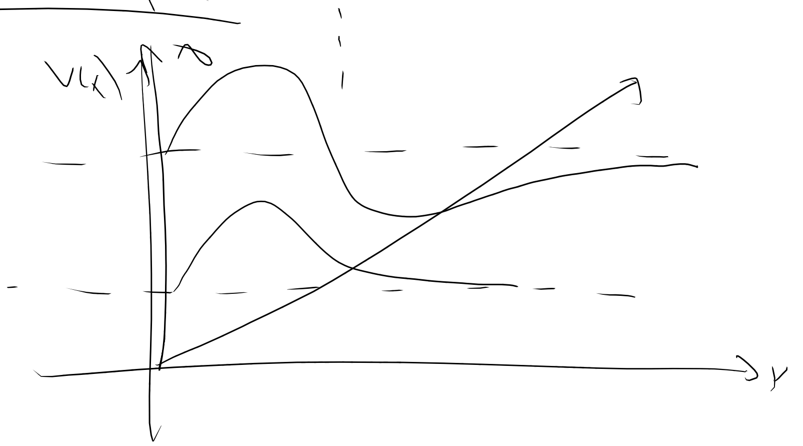
$$\psi_1(x) = N_1 \left(\frac{x}{x_0}\right) e^{-x^2/2x_0^2}$$

$$\psi_2(x) = N_2 \left(1 - 2\frac{x}{x_0}\right) e^{-\frac{x^2}{2x_0^2}}$$

$$\left.\begin{array}{l}\phi_0 = 1 \\ \phi_1 = y \\ \phi_2 = 1-2y \\ \vdots\end{array}\right\}$$ Hermite Polynomials

$$\frac{\Delta E}{E} = \frac{\hbar\omega}{\hbar\omega\left(n+\frac{1}{2}\right)} \xrightarrow[n\to\infty]{} 0$$

Example



$$V(x) = \begin{cases} mgx & x > 0 \\ \infty & x < 0 \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$\psi(x) = 0$ if $x < 0$

$\psi(x) = ?$ for $x > 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + mgx\psi = E\psi$$

$\psi(x=0) = 0$

$\psi(x) \xrightarrow[x\to\infty]{} 0$

$$\left[\frac{\hbar^2}{2mx^2}\right] = [mgx]$$

$$[x^3] = \left[\frac{\hbar^2}{2m^2 g}\right]$$

$$x_0 = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3}$$

$$x = x_0 u$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{x_0^2\,du^2} + mgx_0 u\psi = E\psi$$

$$\frac{d^2\psi}{du^2} - \underbrace{\left(\frac{2m^2 g\, x_0^3}{\hbar^2}\right)}_{1} u\psi = \overbrace{-\frac{2mE}{\hbar^2}x_0^2}^{\varepsilon}\psi$$

$$\boxed{\frac{d^2\psi}{du^2} + (\varepsilon - u)\psi = 0}$$

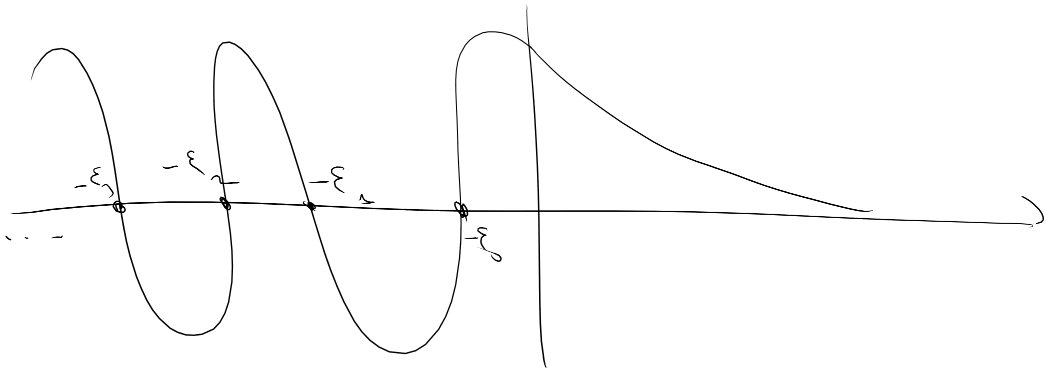
$$\boxed{E = \frac{\hbar^2}{2mx_0^2}\varepsilon}$$

$$x_0 = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3}$$

$$z = u - \varepsilon$$

$$\frac{d^2\psi}{du^2} - z\psi = 0$$

$\psi(z) = Ai(z)$ $\qquad$ other soln $Bi(z)$



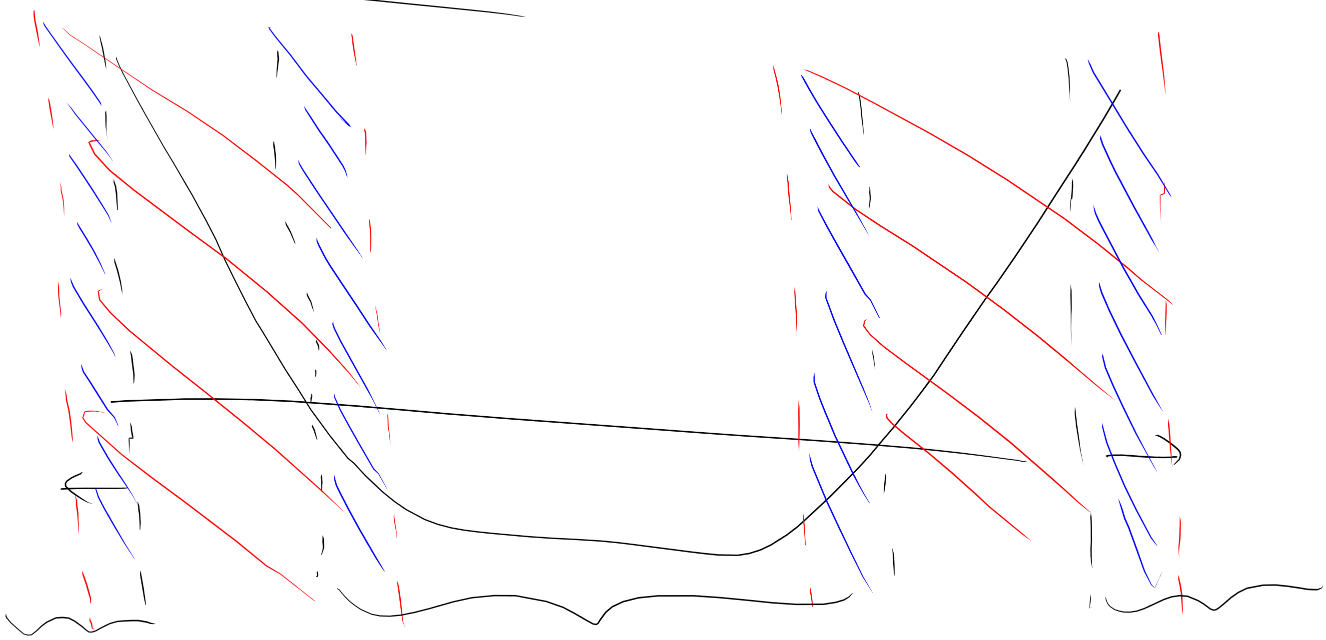
$$\psi(u) = Ai(u-\varepsilon)$$

B.C. $\psi(u=0) = 0$

$$\boxed{Ai(-\varepsilon) = 0}$$

$$\boxed{\psi_n(u) = Ai(u-\varepsilon_n)}$$

WKB Methods



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \underbrace{\frac{2m}{\hbar^2}\left(E - V(x)\right)}_{k(x)^2}\psi = 0$$

$$\boxed{k(x)^2 = \frac{2m}{\hbar^2}\left(E - V(x)\right)}$$

$$\frac{d^2\psi}{dx^2} + k(x)^2\psi = 0$$

$$\psi = e^{\frac{i}{\hbar}W(x)}$$

$$\frac{d\psi}{dx} = \frac{i}{\hbar}W'\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{i}{\hbar}W''\psi + \left(\frac{i}{\hbar}W'\right)^2\psi$$

$$-\frac{1}{\hbar^2}W'^2\cancel{\psi} + \frac{i}{\hbar}W''\cancel{\psi} + k(x)^2\cancel{\psi} = 0$$

$$\boxed{-\frac{1}{\hbar^2}(W')^2 + \frac{i}{\hbar}W'' + k(x)^2 = 0}$$

if $\left|\frac{W''}{\hbar}\right| \ll k(x)^2$ $\Leftarrow$

$$w' = \mp i\sqrt{\hbar^2 k(x)^2}$$

$$w(x) = \mp \hbar \int^x \sqrt{k(x)}\, dx'$$

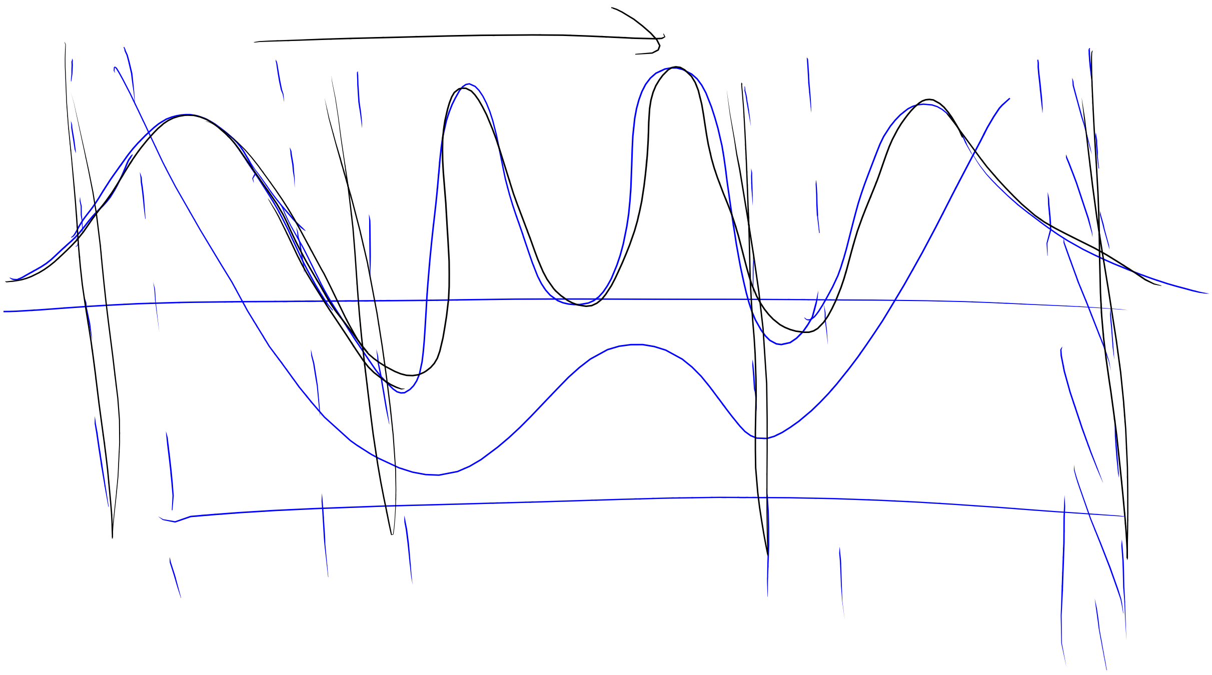
(i) $$\boxed{\psi \simeq e^{\mp i \int^{x'} k(x')\,dx'}}$$

In the classically forbidden region, $k(x)$ is imaginary

$$k(x) = i K(x)$$

(ii) $\psi(x) \simeq e^{-\int^{x'} K(x')\,dx'}$ $\quad x \to \infty$

(iii) $\psi(x) \simeq e^{+\int K(x')\,dx'}$ $\quad x \to -\infty$



$$\boxed{\oint \hbar k(x)\,dx = 2\pi\hbar\left(n+\frac{1}{2}\right)}$$

$$\boxed{\oint k(x)\,dx = 2\pi\left(n+\frac{1}{2}\right)}$$

Quantization condition

$$\Uparrow$$

$$W' = \hbar k(x)$$

$$W'' = \hbar \frac{dk}{dx}$$

$$\frac{W''}{\hbar} \ll k^2$$

$$\left|\frac{1}{k}\frac{dk}{dx}\right| \ll k = \frac{1}{\lambda}$$

$$k = \frac{1}{\hbar}\sqrt{2m(E-V(x))}$$

$$k = \frac{1}{\lambda} \qquad \frac{1}{k}\frac{dk}{dx} = -\lambda\frac{d\lambda}{dx}\frac{1}{\lambda^2} = -\frac{d\lambda}{\lambda dx}$$

$$\left|\frac{1}{\lambda}\frac{d\lambda}{dx}\right| < k$$

$$\left|\frac{1}{k}\frac{dk}{dx}\right| \ll \frac{1}{\lambda} \Rightarrow \lambda \gg \frac{1}{\left|\frac{1}{k}\left(\frac{dk}{dx}\right)\right|}$$