$$\Rightarrow \boxed{\langle x_N\, t_N | x_1\, t_1 \rangle}$$

$$= \lim_{\Delta t \to 0} \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{(N-1)/2} \int dx_2 \int dx_3 \cdots \int dx_{N-2}\, e^{\frac{i}{\hbar} S}$$

$$\equiv \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar} \int_{t_1}^{t_2} L_{cl}(x(t), \dot{x}(t))\, dt}$$

$$\Rightarrow S = \sum_i \Delta t \cdot S(t_i, t_{i-1})$$

$x(t_1) = x_1$

$x(t_N) = x_N$

$$\Rightarrow S^{free}(t_i, t_{i-1}) = \frac{1}{2} m \frac{(x_i - x_{i-1})^2}{\Delta t^2}$$

$x_i = x(t_i)$

$$\int_{-\infty}^{\infty} dx_i \exp\left\{ \frac{i}{\hbar} \Delta \left[S(t_{i+1}, t_i) + S(t_i, t_{i-1}) \right] \right\}$$

$$= \int_{-\infty}^{\infty} dx_i \exp\left\{ \frac{i}{\hbar} \cancel{\Delta t} \frac{1}{2} m \left[\frac{(x_{i+1} - x_i)^2}{\Delta t^2} + \frac{(x_i - x_{i-1})^2}{\Delta t^2} \right] \right\}$$

$$= \int_{-\infty}^{\infty} dx_i \exp\left\{ \frac{im}{2\hbar} \frac{1}{\Delta t} \left(2x_i^2 - 2x_i(x_{i+1} + x_{i-1}) + x_{i+1}^2 + x_{i-1}^2 \right. \right.$$

$$\left. \left. 2\frac{1}{4}(x_{i+1} + x_{i-1})^2 - 2\frac{1}{4}(x_{i+1} + x_{i-1})^2 \right) \right\}$$

$$= \int_{-\infty}^{\infty} dx_i \; \exp\left\{ \frac{im}{2\hbar\Delta t}\left[2\left(x_i - \frac{1}{2}(x_{i+1}+x_{i-1})\right)^2 + x_{i+1}^2 + x_{i-1}^2 - \frac{1}{2}(x_{i+1}+x_{i-1})^2\right]\right.$$

$$y_i = x_i - \frac{1}{2}(x_{i+1}+x_{i-1})$$

$$= \int_{-\infty}^{\infty} dy_i \; \exp\left\{ \frac{im}{2\hbar\Delta t}\left[2y_i^2 + \frac{1}{2}(x_{i+1}-x_{i-1})^2\right]\right\}$$

$$\alpha = -\frac{im}{\hbar\Delta t}$$

$$= \sqrt{\frac{\pi}{\alpha}} \; \exp\left\{ \frac{im}{2\cdot\underset{3}{2}\hbar\Delta t}(x_{i+1}-x_{i-1})^2\right\}$$

$$= \underbrace{\sqrt{\frac{i\pi\hbar\Delta t}{m}}} \; \exp\left\{\frac{i}{\underset{m}{(1+1)\hbar}}\Delta t \underbrace{\left(\frac{1}{2}m\frac{(x_{i+1}-x_{i-1})^2}{(\Delta t)^2}\right)}\right.$$

$$\Downarrow$$

$$\int_{-\infty}^{\infty} dx_i \int_{-\infty}^{\infty} dx_{i+1} \; \exp\left\{\frac{i}{\hbar}\left(S(i+2,i+1) + S(i+1,i) + S(i,i-1)\right)\right\}$$

$$= \int_{-\infty}^{\infty} dx_{i+1}\left(\frac{i\pi\hbar\Delta t}{m}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\Delta t\frac{1}{2}m\left(\frac{x_{i+2}-x_{i+1}}{\Delta t}\right)^2 \leftarrow \right.$$

$$\left. + \underbrace{\frac{i}{n\hbar}\Delta t\,\frac{1}{2}m\,\frac{(x_{i+1}-x_{i-1})^2}{\Delta t^2}}\right\}$$

$$= \int_{-\infty}^{\infty} dx_{i+1}\left(\frac{i\pi\hbar\Delta t}{m}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\Delta t\frac{1}{2}m\frac{1}{(\Delta t)^2}\frac{1}{n}\left[n(x_{i+2}-x_{i+1})^2 + (x_{i+1}-x_{i-1})^2\right]\right\}$$

$$= \int_{-\infty}^{\infty} dx_{i+1} \left(\frac{i\pi\hbar\Delta t}{m}\right)^{1/2} \exp\left\{ \frac{i}{\hbar}\,\frac{1}{2}m\,\frac{1}{\Delta t}\,\frac{1}{n}\right.$$

$$\left[(n+1)\; x_{i+1}^2 - 2x_{i+1}\left(n x_{i+2} + x_{i-1}\right) + n x_{i+2}^2 + x_{i-1}^2 \right.$$

$$\left.\left. + (n+1)\left(\frac{n x_{i+2} + x_{i-1}}{n+1}\right)^2 - (n+1)\left(\frac{n x_{i+2} + x_{i-1}}{n+1}\right)^2\right]\right\}$$

$$= \int_{-\infty}^{\infty} dx_{i+1} \left(\frac{i\pi\hbar\Delta t}{m}\right)^{1/2} \exp\left\{ \frac{i}{\hbar}\,\frac{1}{2}m\,\frac{1}{\Delta t}\,\frac{1}{n}\right.$$

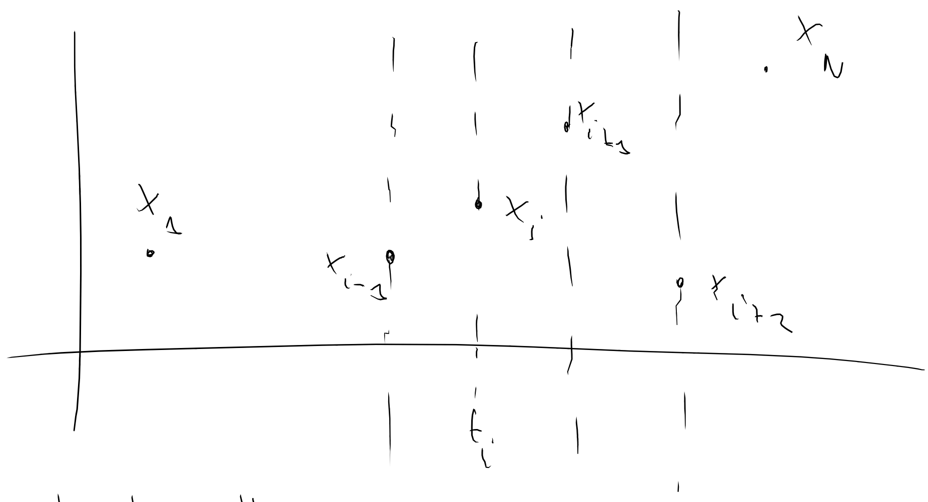
$$\left[(n+1)\Big(\underbrace{x_{i+1} - \frac{n x_{i+2} + x_{i-1}}{n+1}}_{y_i}\Big)^2 + n x_{i+2}^2 + x_{i-1}^2 \left.\left. - \frac{n^2 x_{i+2}^2 + x_{i-1}^2 + 2n x_{i+2} x_{i-1}}{n+1}\right]\right\}\right.$$

$$= \left(\frac{i\pi\hbar\Delta t}{m}\right)^{1/2} \left[\frac{\pi}{-\frac{i}{\hbar}\,\frac{1}{2}m\,\frac{1}{\Delta t}\left(\frac{n+1}{n}\right)}\right]^{1/2}$$

$$\exp\left\{ \frac{i}{\hbar}\,\frac{1}{2}m\,\frac{1}{\Delta t}\,\frac{1}{n}\left[x_{i+2}^2\Big(\overbrace{n - \frac{n^2}{n+1}}^{\frac{n}{n+1}}\Big) + x_{i-1}^2\Big(\overbrace{1 - \frac{1}{n+1}}^{\frac{n}{n+1}}\Big) - 2n\,\frac{x_{i+2} x_{i-1}}{n+1}\right]\right\}$$

$$= \left(\frac{i\pi\hbar\Delta t}{m}\right)^{1/2} \left(\frac{i\pi\hbar}{m}\,\frac{2n}{n+1}\,\Delta t\right)^{1/2}$$

$$\exp\left\{ \frac{i}{\hbar}\,\frac{1}{2}m\,\frac{1}{\Delta t}\,\frac{1}{n+1}\left(x_{i+2} - x_{i-1}\right)^2\right\}$$



evaluate all N-2 integrals

$$\langle x_N t_N | x_1 t_1\rangle = \exp\left\{ \frac{i}{\hbar}\,\frac{1}{2}m\,\frac{1}{\Delta t}\,\frac{1}{(N-1)}\,(x_N - x_1)^2 \right\}$$

$$\left\{\left(\frac{2i\pi\hbar\Delta t}{m}\right)^{\cancel{\frac{N-2}{2}}\,-1/2} \left(\frac{\cancel{1}}{\cancel{1+1}}\;\frac{\cancel{2}}{\cancel{2+1}}\;\frac{\cancel{3}}{\cancel{3+1}}\cdots\frac{(N-\cancel{1})}{N-1+1}\right)^{1/2}\right.$$

$$\cancel{\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{N-1/2}}$$

$$\lim_{\substack{N\to\infty\\ \Delta t\to 0\\ N\Delta t = t_N - t_1}} (\;) = \exp\left\{\frac{i}{\hbar}\,\frac{1}{2}m\,\frac{(x_N-x_1)^2}{(t_N-t_1)}\right\}\sqrt{\frac{m}{2i\pi\hbar(t_N-t_1)}}$$

$$\boxed{\langle x_N t_N | x_1 t_1\rangle = \sqrt{\frac{m}{2i\pi\hbar(t_N-t_1)}}\;\exp\left\{\frac{im}{2\hbar}\,\frac{(x_N-x_1)^2}{(t_N-t_1)}\right\}}$$

$$\langle x_N t_N | x_1 t_1 \rangle$$

$$= \lim_{\Delta t \to 0} \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{(N-1)/2} \int dx_2 \int dx_3 \dots \int dx_{N-2} \, e^{\frac{i}{\hbar} S}$$

$$\equiv \int \mathcal{D}[x(t)] \, e^{\frac{i}{\hbar} \int_{t_1}^{t_2} L_{cl}(x(t), \dot{x}(t)) \, dt}$$

$$S[x] = S[x_{cl} + \eta]$$

$$= S[x_{cl}] + \underbrace{\left.\frac{\delta S}{\delta \eta}\right|_{x = x_{cl}}}_{= 0} \eta + \left.\frac{\delta^2 S}{\delta \eta^2}\right|_{x = x_{cl}} \eta^2 + \dots$$

$$\psi = \sqrt{\rho} \, e^{\frac{i}{\hbar} S}$$

$$S \simeq S_{cl}$$

$\tilde{E} = -\nabla\phi$ $\qquad$ $\phi \to \phi + \Lambda_0$

$H = \frac{\vec{p}^2}{2m} + \phi$ $\qquad$ $\tilde{H} = \frac{\vec{p}^2}{2m} + \phi + \Lambda_0$

$\langle x \rangle, \langle p \rangle, \ldots$ should be same for both systems.

$H|\psi\rangle = E|\psi\rangle$ $\qquad$ $\tilde{H}|\tilde{\psi}\rangle = \tilde{E}|\tilde{\psi}\rangle$

$|\tilde{\psi}\rangle = U|\psi\rangle$, $U = ?$

$\langle \tilde{\psi}|\tilde{\psi}\rangle = 1 = \langle \psi|\psi\rangle$

$\langle x \rangle_\psi = \langle x \rangle_{\tilde{\psi}} \Rightarrow \boxed{x = U^\dagger x U}$

$\tilde{E} = E + \Lambda_0$ $\qquad$ $\tilde{H} = H + \Lambda_0$

$|\tilde{\psi}\rangle_{t=0} = |\psi\rangle_{t=0}$

$$|\tilde{\psi}\, t\rangle = e^{-\frac{i}{\hbar}\tilde{H}t} |\tilde{\psi}\, t=0\rangle$$

$$= e^{-\frac{i}{\hbar}Ht} e^{-\frac{i}{\hbar}\Lambda_0 t} |\psi\, t=0\rangle$$

$$\boxed{|\tilde{\psi}\, t\rangle = e^{-\frac{i}{\hbar}\Lambda_0 t} |\psi\, t\rangle}$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Et}\,|\psi(t=0)\rangle$$

$$|\tilde{\psi}\; t\rangle = e^{-\frac{i}{\hbar}(E+\Lambda_0)t}\,|\psi(t=0)\rangle$$

$$\boxed{|\tilde{\psi}(t)\rangle = e^{-\frac{i}{\hbar}(E+\Lambda_0)}\,|\tilde{\psi}(t=0)\rangle}$$

$$\boxed{\tilde{E} = E+\Lambda_0}$$

$$\tilde{\psi}(\vec{r},t) = \int d\vec{r}'\,\widetilde{\langle \vec{r}\,t|\vec{r}'\,0\rangle}\,\tilde{\psi}(\vec{r}',0)$$

$$= \int d\vec{r}'\,\widetilde{\langle \vec{r}\,t|\vec{r}'\,0\rangle}\,\psi(\vec{r},0)$$

$$\widetilde{\langle \vec{r}\,t|\vec{r}'\,0\rangle} = \int \mathcal{D}[x(t)]\,e^{\frac{i}{\hbar}\int dt\,\tilde{L}}$$

$$\tilde{L} = T - \tilde{\phi} = T - (\phi+\Lambda_0) = L - \Lambda$$

$$\widetilde{\langle \vec{r}\,t|\vec{r}'\,0\rangle} = \int \mathcal{D}[x(t)]\,e^{\frac{i}{\hbar}\int dt\,L - \frac{i}{\hbar}\int dt\,\Lambda_0}$$

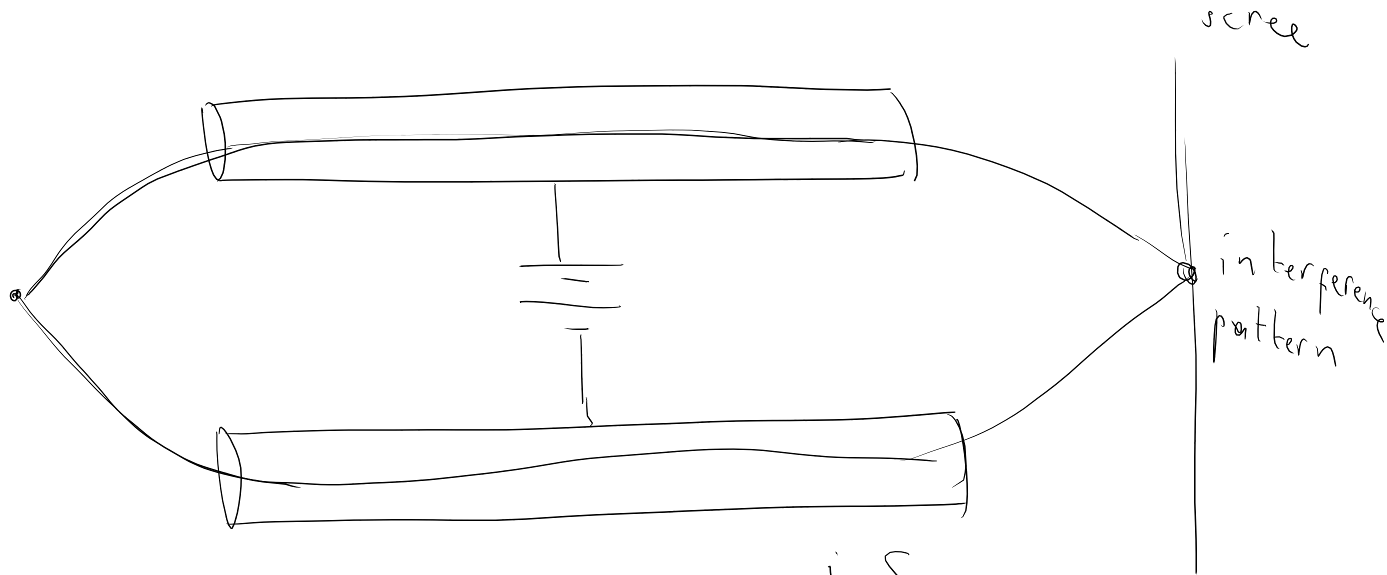
$$\widetilde{\langle \vec{r}\,t|\vec{r}'\,0\rangle} = e^{-\frac{i}{\hbar}\int dt\,\Lambda_0}\int \mathcal{D}[x(t)]\,e^{\frac{i}{\hbar}\int dt\,L}$$

$$\widetilde{\langle \vec{r}\,t|\vec{r}'\,0\rangle} = e^{-\frac{i}{\hbar}\int dt\,\Lambda_0}\,\langle \vec{r}\,t|\vec{r}'\,0\rangle$$

$$\tilde{\psi}(\vec{r},t) = \int d^3\vec{r}\,' \langle \vec{r}\, t | \vec{r}\,' 0 \rangle \psi(\vec{r}\,',0)$$

$$= e^{-\frac{i}{\hbar}\int dt \Lambda_0} \underbrace{\int d^3\vec{r}\,' \langle \vec{r}\, t | \vec{r}\,' 0 \rangle \psi(\vec{r}\,',0)}_{\psi(\vec{r},t)}$$

$$\boxed{\tilde{\psi}(\vec{r},t) = e^{-\frac{i}{\hbar}\int dt \Lambda_0} \psi(\vec{r},0)}$$



$$\langle x_f t_f | x_i t_i \rangle = \int \mathcal{D}[x]\, e^{\frac{i}{\hbar} S}$$

$$= \int \mathcal{D}[x]_{above}\, e^{\frac{i}{\hbar} S_{above}}$$

$$+ \int \mathcal{D}[x]_{below}\, e^{\frac{i}{\hbar} S_{below}}$$

$$S_{above} = \int dt\,(\cancel{L} - \phi^{above}) = S^0 - \phi^{above}(t_f - t_i)$$

$$S_{below} = S^0 - \phi^{below}(t_f - t_i)$$

$$\langle x_f t_f | x_i t_i \rangle = \langle x_f t_f | x_i t_i \rangle^0_{above} e^{-\frac{i}{\hbar}(t_f - t_i)\phi^{above}}$$

$$+ \langle x_f t_f | x_i t_i \rangle^0_{below} e^{-\frac{i}{\hbar}(t_f - t_i)\phi^{below}}$$

$$= \left[A_1 + A_2 e^{-\frac{i}{\hbar}(t_f - t_i)(\phi^{below} - \phi^{above})} \right]$$

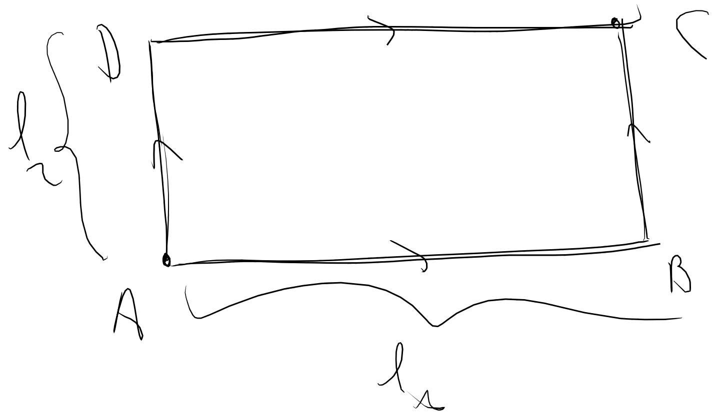
overall phase $\Longrightarrow$ $e^{-\frac{i}{\hbar}(t_f - t_i)\phi^{above}}$

$$L = \frac{1}{2} m v^2 - m g z$$

$$S = \int L \, dt$$

$$\delta S = 0 \Rightarrow \delta \int \left(\frac{1}{2} v^2 - g z \right) dt = 0$$

$$H = \frac{p^2}{2m} + m g z$$



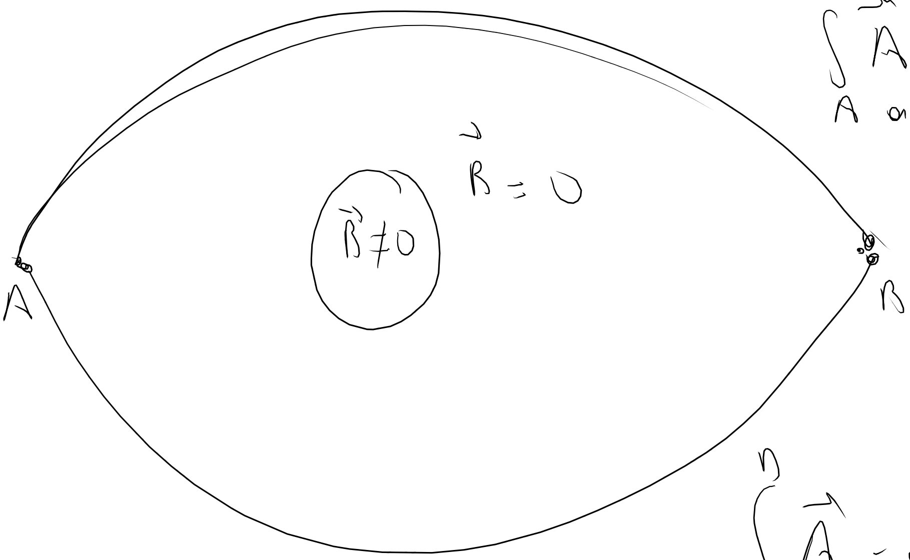
$$\Delta \phi = \frac{\sin \delta \, m^2 g \, l_1 l_2 \lambda}{\hbar^2}$$

$$L = (T - U) - \frac{e}{c}\vec{v}\cdot\vec{A}$$

$$S^{\vec{A}} = S^{\vec{A}=0} - \frac{e}{c}\int \vec{A}\cdot\underbrace{\vec{v}\,dt}_{d\vec{l}}$$

$$= S^{\vec{A}=0} - \frac{e}{c}\int \vec{A}\cdot d\vec{l}$$

$$\int_{A\text{ above}}^{B} \vec{A}\cdot d\vec{l}$$



$$\int_{A\text{ below}}^{B} \vec{A}\cdot d\vec{l}$$

$$\text{phase difference} = \int_{A\text{ above}}^{B} \vec{A}\cdot d\vec{l} - \int_{A\text{ below}}^{B} \vec{A}\cdot d\vec{l}$$

$$= \oint \vec{A}\cdot d\vec{l} = \int (\underbrace{\vec{\nabla}\times\vec{A}}_{=\vec{B}})\cdot d\vec{S}$$

$$\text{phase difference} = \int \vec{B}\cdot d\vec{S} = \Phi_B \text{ : magnetic flux}$$