Electromagnetic interactions

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

$$\boxed{H = T + V}$$

$$\left.\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \\ \vec{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \end{aligned}\right\} \quad \Longleftarrow \quad \left\{\begin{aligned} \vec{A} &\to \vec{A} + \vec{\nabla}\Lambda \\ \phi &\to \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \end{aligned}\right\} \begin{aligned} \vec{B} &\to \vec{B} \\ \vec{E} &\to \vec{E} \end{aligned}$$

$$\vec{F} = \frac{d\vec{\Pi}}{dt}$$

$\vec{\Pi}$: mechanical momentum

$\vec{\Pi} = m\vec{v}$ (classically)

$$\frac{d\vec{\Pi}}{dt} = q\left[-\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{\vec{v}}{c} \times (\vec{\nabla} \times \vec{A})\right]$$

$$\vec{v} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}$$

$$= \vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}$$

$$\frac{d\vec{\Pi}}{dt} = q\left\{-\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{1}{c}\left[\vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}\right]\right\}$$

$$\frac{d\vec{\Pi}}{dt} = -\vec{\nabla}\left\{q\left(\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right)\right\} - \frac{1}{c}q\left(\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}\right)$$

$$\frac{d\vec{A}(\vec{r}(t),t)}{dt} = \frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{A}}{\partial x_i}\frac{\partial x_i}{\partial t}$$

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$$

$$\frac{d\Pi}{dt} = -\nabla\left[q\left(\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right)\right]$$

$$- \frac{1}{c} q \frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}\left(\vec{\Pi} + \frac{q}{c}\vec{A}\right) = -\nabla\left[q\left(\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right)\right]$$

$$\frac{d\vec{p}}{dt} = -\nabla V_{eff}$$

$$V_{eff} = q\left(\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right)$$

$$\vec{p} = \vec{\Pi} + \frac{q}{c}\vec{A}$$: canonical momentum

$$H = \frac{\vec{\Pi}^2}{2m} + q\phi$$

$$\boxed{H = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} + q\phi}$$

$$\frac{dx}{dt} = \frac{1}{i\hbar}[H, x]$$

$$= \frac{1}{i\hbar}\frac{1}{2m}\left[\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2, x\right]$$

$$= \frac{1}{i\hbar}\frac{1}{2m}\left[\left(p_x - \frac{q}{c}A_x\right)^2, x\right]$$

$$= \frac{1}{i\hbar}\frac{1}{2m}\left\{\left(p_x - \frac{q}{c}A_x\right)\left[p_x - \frac{q}{c}A_x, x\right] + \left[p_x - \frac{q}{c}A_x, x\right]\left(p_x - \frac{q}{c}A_x\right)\right.$$

$$= \frac{1}{\cancel{i\hbar}}\frac{1}{\cancel{2}m}\left(p_x - \frac{q}{c}A_x\right)\cancel{2i\hbar}$$

$$\frac{dx}{dt} = \frac{1}{m}\left(p_x - \frac{q}{c}A_x\right)$$

$$\boxed{m\frac{dx}{dt} = \Pi_x}$$

Particle in a magnetic field

$\phi = 0 \qquad \vec{A}, \quad \vec{A}' = \vec{A} + \vec{\nabla}\Lambda(\vec{r})$

$\underline{\phi = 0}, \quad \phi' = 0$

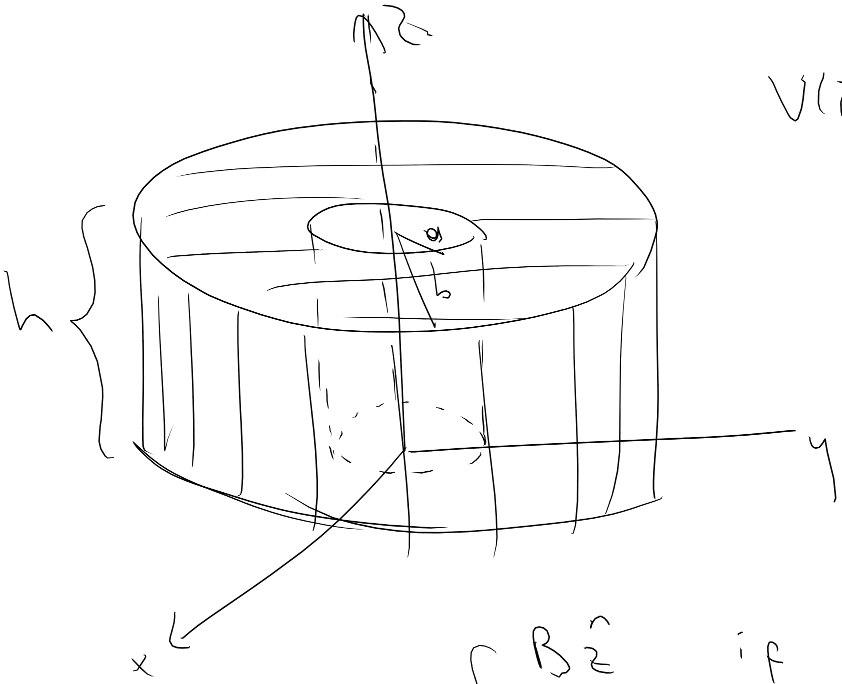
$$H = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} = \frac{1}{2m}\vec{p}^{\,2} + \frac{1}{2m}\frac{q^2}{c^2}\vec{A}^2 - \frac{q}{2mc}\left(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right)$$

$$\langle \vec{r} | H | \alpha t_0 ; t \rangle = \langle \vec{r} | i\hbar \frac{\partial}{\partial t} | \alpha t_0 ; t \rangle$$

$$\langle \vec{r} | \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} | \alpha t_0 ; t \rangle = i\hbar \frac{\partial}{\partial t} \psi(t)$$

$$\frac{\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)}{2m} \cdot \langle \vec{r} | \vec{p} - \frac{q}{c}\vec{A} | \alpha t_0 ; t \rangle = i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t)$$

$$\boxed{\left(-i\hbar\vec{\nabla} \cdot \frac{q}{c}\vec{A}\right) \cdot \left[\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right) \psi(\vec{r}, t)\right] = i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t)}$$



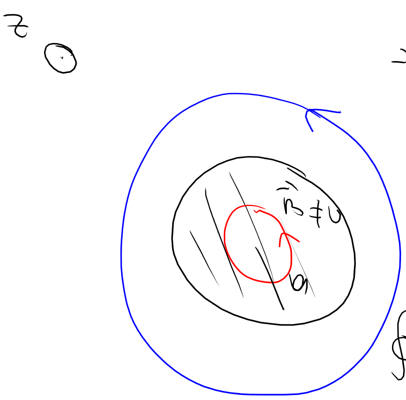
$$V(\vec{r}) = \begin{cases} 0 & \text{if } 0<z<h,\ a<\rho<b \\ \infty & \text{otherwise} \end{cases}$$

$$\vec{B}(\vec{r}) = \begin{cases} B\hat{z} & \text{if } \rho<a \\ 0 & \text{if } \rho>a \end{cases}$$

$$\vec{\nabla}\times\vec{A} = \vec{B} \implies \int d\vec{S}\cdot(\vec{\nabla}\times\vec{A}) = \int d\vec{S}\cdot\vec{B}$$

$$\boxed{\oint d\vec{l}\cdot\vec{A} = \Phi_B}$$

$$\vec{A} = \hat{\phi}\, A(\rho)$$



$\vec{B} = 0$

$$d\vec{l} = dl\,\hat{\phi}$$

$$d\vec{l}\cdot\vec{A} = A(\rho)\,dl$$

$$\oint d\vec{l}\cdot\vec{A} = \oint A(\rho)\,dl = 2\pi\rho\, A(\rho)$$

$$\Phi_B = B\pi\rho^2$$

$$A(\rho) = \frac{B\pi\rho^2}{2\pi\rho} = \boxed{\frac{1}{2}B\rho = A(\rho) \quad \rho<a}$$

$$\Phi_B = B\pi a^2 \Rightarrow A(\rho) = \frac{B\pi a^2}{2\pi\rho} = \frac{Ba^2}{2}\frac{1}{\rho} \quad \rho>a$$

$$\boxed{\vec{A}(\rho) = \frac{B a^2}{2} \frac{1}{\rho} \hat{\phi} \qquad \rho > a}$$

$$\phi_B = B \pi a^2$$

$$-i\hbar \vec{\nabla} - \frac{q}{c} \vec{A}$$

$$= -i\hbar \left[\frac{\partial}{\partial \rho} \hat{\rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \right] - \frac{q}{c} \frac{B a^2 \pi}{2\pi \rho} \hat{\phi}$$

$$= -i\hbar \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \underbrace{\left(\frac{\partial}{\partial \phi} - \frac{i}{\hbar} \frac{q}{c} \frac{\phi_B}{2\pi} \right)} + \hat{z} \frac{\partial}{\partial z} \right]$$

$$\left(-i\hbar \vec{\nabla} - \frac{q}{c} \vec{A} \right)^2$$

$$= -\hbar^2 \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \left(\frac{\partial}{\partial \phi} - \frac{i}{\hbar} \frac{q \phi_B}{c 2\pi} \right)^2 + \frac{\partial^2}{\partial z^2} \right]$$

$$\psi(\vec{r}) = R(\rho)\, \Phi(\phi)\, Z(z)$$

$$\left. \begin{array}{ll} \psi(\vec{r})\big|_{z=0} = 0 & \psi(\vec{r})\big|_{z=h} = 0 \\ \psi(\vec{r})\big|_{\rho=a} = 0 & \psi(\vec{r})\big|_{\rho=b} = 0 \end{array} \right\} \begin{array}{l} Z(0) = Z(h) = 0 \\ R(a) = R(b) = 0 \end{array}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\phi}-\frac{i}{\hbar}\frac{q\Phi_B}{c2\pi}\right)^2+\frac{\partial^2}{\partial z^2}\right]\psi = E\psi$$

$$\Downarrow$$

$$E = -\frac{\hbar^2}{2m}\left[\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}(\rho R') + \frac{1}{\rho^2}\frac{1}{\Phi}\left(\frac{d}{d\phi}-\frac{i}{\hbar}\frac{q\Phi_B}{c2\pi}\right)^2\Phi + \underbrace{\frac{Z''}{Z}}_{= -k_n^2}\right]$$

$$\boxed{p = \hbar k}$$

$$\frac{Z''}{Z} = \text{const} = -k^2 \Rightarrow Z(z) = A\sin(kz)$$

$$kh = n\pi \qquad n = 1, 2, 3, \dots$$

$$k_n = n\frac{\pi}{h}$$

$$\boxed{Z_n(z) = A_n \sin(k_n z)}$$

$$\Downarrow$$

$$\left(E - \frac{\hbar^2 k_n^2}{2m}\right) = -\frac{\hbar^2}{2m}\left[\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}(\rho R')\right] - \frac{\hbar^2}{2m}\frac{1}{\rho^2}\underbrace{\left\{\frac{1}{\Phi}\left(\frac{d}{d\phi}-\frac{i}{\hbar}\frac{q\Phi_B}{2\pi c}\right)^2\Phi\right\}}$$

$$\frac{1}{\Phi}\left(\frac{d}{d\phi}-\frac{i}{\hbar}\frac{q\Phi_B}{2\pi c}\right)^2\Phi = \text{const} = -\alpha^2$$

$$\Phi(\phi) = Be^{im\phi} \qquad m = 0, \pm1, \pm2, \pm3, \dots$$

$$-\alpha^2 = \frac{1}{\cancel{Be^{in\phi}}}\left(im - \frac{i}{\hbar}\frac{q\phi_B}{2\pi c}\right)^2 \cancel{Be^{in\phi}}$$

$$\boxed{\alpha_m^2 = \left(m - \frac{q\phi_B}{2\pi c\hbar}\right)^2}$$

$$E - \frac{\hbar^2 k_n^2}{2m} = -\frac{\hbar^2}{2m}\left[\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho R'\right) - \frac{1}{\rho^2}\alpha_m^2\right]$$

$$\left[\frac{1}{R}\frac{1}{\rho}\left(\rho R'' + R'\right) - \frac{1}{\rho^2}\alpha_n^2\right]\frac{\hbar^2}{2m} + \left(E - \frac{\hbar^2 k_n^2}{2m}\right) = 0$$

$$\boxed{\rho^2 R'' + \rho R' + \frac{2m}{\hbar^2}\left(E - \frac{\hbar^2 k_n^2}{2m}\right)\rho^2 R - \alpha_n^2 R = 0}$$

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2$$

$$H\psi = E\psi$$

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$

$$H \to H'$$

$$\psi_E \to \psi'_E$$

$$\boxed{\psi'_E(\vec{r}) = U_\Lambda(\vec{r})\,\psi_E(\vec{r})}$$

$$\boxed{\begin{aligned}&\left(-i\hbar\vec{\nabla} - \vec{A}'\right) U_\Lambda \\ &= U_\Lambda\left(-i\hbar\vec{\nabla} - \vec{A}\right)\end{aligned}}$$

$$\left(-i\hbar\vec{\nabla} - \vec{A}\right)\psi(\vec{r}) \longrightarrow \left(-i\hbar\vec{\nabla} - \vec{A}'\right) U\,\psi(\vec{r})$$

$$= U(\vec{r})\left(-i\hbar\vec{\nabla} - \vec{A}\right)\psi(\vec{r})$$

$$H'\psi'_E = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}'\right)^2\psi'(\vec{r})$$

$$= \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}'\right)\cdot\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}'\right)\psi'$$

$$= \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}'\right) U\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\psi$$

$$= \frac{1}{2m} U\left(-i\hbar\vec{\nabla} - \vec{A}\right)^2\psi$$

$$= U H\psi$$

$$= U E\psi$$

$$\boxed{H'\psi'_E = E\psi'}$$

$$\Rightarrow \left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}'\right) U_n = U_n\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)$$

$$\Rightarrow \boxed{\vec{A}' = \vec{A} + \vec{\nabla}\Lambda}$$

$$\vec{\nabla}(U_n\psi) = U_n(\vec{\nabla}\psi) + (\vec{\nabla}U_n)\psi$$

$$\vec{\nabla}U_n = U_n\vec{\nabla} + (\vec{\nabla}U_n)$$

$$\cancel{-i\hbar U_n\vec{\nabla}} - i\hbar(\vec{\nabla}U_n) - \cancel{\frac{q}{c}\vec{A}U_n} + \frac{q}{c}(\vec{\nabla}\Lambda)U_n$$

$$= \cancel{-i\hbar U_n\vec{\nabla} - \frac{q}{c}U_n\vec{A}}$$

$$\Rightarrow -i\hbar(\vec{\nabla}U_n) = \frac{q}{c}(\vec{\nabla}\Lambda)U_n$$

$$\boxed{U_n = e^{\frac{qi}{c\hbar}\Lambda}}$$

$$\phi = 0$$

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$$

$$\psi \rightarrow e^{\frac{qi}{c\hbar}\Lambda}\psi$$

$$\langle\vec{p}\rangle = \int d\vec{r}\,\psi^*(\vec{r})\left(-i\hbar\vec{\nabla}\right)\psi$$

$$\Rightarrow \int d\vec{r}\,\psi^* e^{-\frac{qi}{c\hbar}\Lambda}\left(-i\hbar\vec{\nabla}\right)e^{\frac{i}{\hbar}\frac{q}{c}\Lambda}\psi$$

$$= \int d\vec{r}\,\psi^* \cancel{e^{-\frac{qi}{c\hbar}\Lambda}}\,\cancel{e^{\frac{qi}{c\hbar}\Lambda}}\left[\frac{q}{c}(\vec{\nabla}\Lambda) - i\hbar\vec{\nabla}\right]\psi$$

$$\langle\vec{p}\rangle \Rightarrow \langle\vec{\nabla}\Lambda\rangle\frac{q}{c} + \langle\vec{p}\rangle$$

$$\vec{\Pi} = \vec{p} - \frac{q}{c}\vec{A}$$

$$\langle \vec{\Pi} \rangle = \langle \vec{p} \rangle - \frac{q}{c}\langle \vec{A} \rangle$$

$$\Rightarrow \cancel{\langle \vec{\nabla}\Lambda \rangle \frac{q}{c}} + \langle \vec{p} \rangle - \frac{q}{c}\left(\langle \vec{A} \rangle + \cancel{\langle \vec{\nabla}\Lambda \rangle}\right)$$

$$= \langle \vec{p} - \frac{q}{c}\vec{A} \rangle$$

$$\langle \vec{\Pi} \rangle = \langle \vec{\Pi} \rangle$$

$$\vec{\Pi} = m\frac{d\vec{x}}{dt}$$ in Heisenberg picture.

$$\rho(\vec{r}) = \sqrt{\psi^*\psi} \longrightarrow \sqrt{\psi^* \underbrace{U^\dagger U}_{1} \psi} = \sqrt{\psi^*\psi}$$

$\rho(\vec{r})$ is Gauge invariant.

$\vec{j}(\vec{r}) \propto \mathrm{Im}\, \psi^* \nabla \psi$ is not gauge invariant.

$\Uparrow$ is valid only if $H = \frac{p^2}{2m} + V(\vec{r})$

Now $H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2$ $\qquad \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$

$$\vec{J}(\vec{r}) \propto \mathrm{Im}\, \psi^* \left(\vec{\nabla} - \frac{i}{\hbar}\frac{q}{c}\vec{A}\right)\psi \Leftarrow$$

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r}) \longrightarrow H = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} + V(\vec{r})$$

$$\vec{p} \longrightarrow \vec{p} - \frac{q}{c}\vec{A}$$

$$\vec{\nabla} \longrightarrow \vec{\nabla} - \frac{i}{\hbar}\frac{q}{c}\vec{A}$$

$$\boxed{\langle \vec{\pi} \rangle = m \frac{d\langle \vec{x} \rangle}{dt}}$$