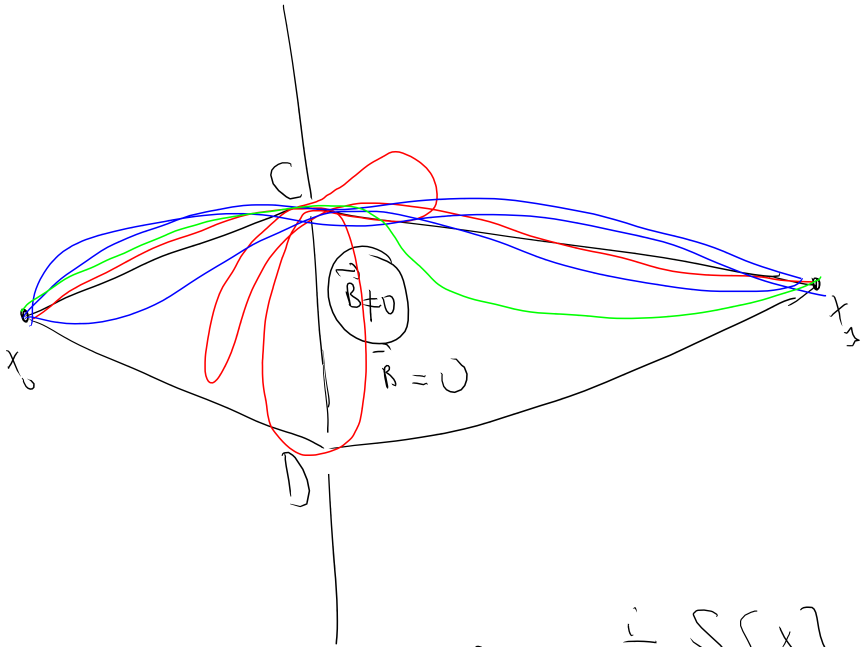
Aharonov-Bohm Effect



$$A(x_0, t_0 \to x_1, t_1) = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x]}$$

$$= \underbrace{\int \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x]}}_{\text{path goes through the slit C}} + \underbrace{\int \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S}}_{\text{paths that go through the slit D.}}$$

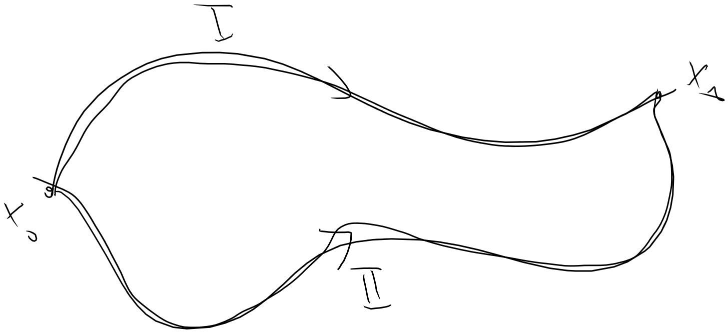
$$S = \int_{t_0}^{t_1} dt\, L$$

$$H = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} \longrightarrow L = L^{(0)} + \frac{q}{c}\vec{A}\cdot\frac{d\vec{x}}{dt}$$

$L^{(0)}$ is the lagrangian in the absence of $\vec{A}$

$$S = \int dt\, L = \int dt \left(L^{(0)} + \frac{q}{c}\vec{A}\cdot\frac{d\vec{x}}{dt} \right)$$

$$S = S^{(0)} + \frac{q}{c}\int_{x_0}^{x_1} \vec{A}\cdot\frac{d\vec{x}}{\cancel{dt}}\,\cancel{dt}$$



$$\int_{x_0,\,\mathrm{I}}^{x_1}\vec{A}\cdot d\vec{x} - \int_{x_0,\,\mathrm{II}}^{x_1}\vec{A}\cdot d\vec{x} = \int_{x_0,\,\mathrm{I}}^{x_1}\vec{A}\cdot d\vec{x} + \int_{x_1,\,\mathrm{II}}^{x_0}\vec{A}\cdot d\vec{x}$$

$$= \oint \vec{A}\cdot d\vec{x}$$

$$= \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

$$\int_{x_0,\,\mathrm{I}}^{x_1}\vec{A}\cdot d\vec{x} - \int_{x_0,\,\mathrm{II}}^{x_1}\vec{A}\cdot d\vec{x} = \int \vec{B}\cdot d\vec{S}$$

$$\int_{\substack{\text{path goes}\\\text{through the}\\\text{slit C (D)}}} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x]} = \int_{\substack{\text{goes through}\\\text{slit C (or D)}}} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S^0 + \frac{i}{\hbar}\frac{q}{c}\underline{\int\vec{A}\cdot d\vec{l}}}$$

$$= \Bigg(\int_{\substack{\text{goes through}\\\text{slit C (or D)}}} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S^0}\Bigg)\, e^{\frac{i}{\hbar}\frac{q}{c}\int_{\substack{\text{through}\\\text{slit C}}}\vec{A}\cdot d\vec{l}}$$

$$\underbrace{}_{\langle x_0 t_0 | x_1 t_1\rangle^0_{\text{C (or D)}}}\; e^{\frac{i}{\hbar}\frac{q}{c}\int_{\substack{\text{through}\\\text{slit C or D}}}\vec{A}\cdot d\vec{l}}$$

$$\langle x_0 t_0 | x_1 t_1 \rangle = \langle x_0 t_0 | x_1 t_1 \rangle^0_C \exp\left\{ \frac{i}{\hbar} \frac{q}{c} \int_{\substack{\text{through}\\ \text{slit } C}} \vec{A} \cdot d\vec{l} \right\}$$

$$+ \langle x_0 t_0 | x_1 t_1 \rangle^0_D \exp\left\{ \frac{i}{\hbar} \frac{q}{c} \int_{\substack{\text{through}\\ \text{slit } D}} \vec{A} \cdot d\vec{l} \right\}$$

$$= \exp\left\{ \frac{i}{\hbar} \frac{q}{c} \int_{\substack{\text{through}\\ \text{slit } C}} \vec{A} \cdot d\vec{l} \right\}$$

$$\Bigg[\langle x_0 t_0 | x_1 t_1 \rangle^0_C + \langle x_0 t_0 | x_1 t_1 \rangle^0_D \underbrace{\exp\left\{ \frac{i}{\hbar} \frac{q}{c} \left(\int_{\substack{\text{through}\\ \text{slit } D}} \vec{A} \cdot d\vec{l} - \int_{\substack{\text{through}\\ \text{slit } C}} \vec{A} \cdot d\vec{l} \right) \right\}}_{= \exp\left\{ \frac{i}{\hbar} \frac{q}{c} \Phi_B \right\}} \Bigg]$$

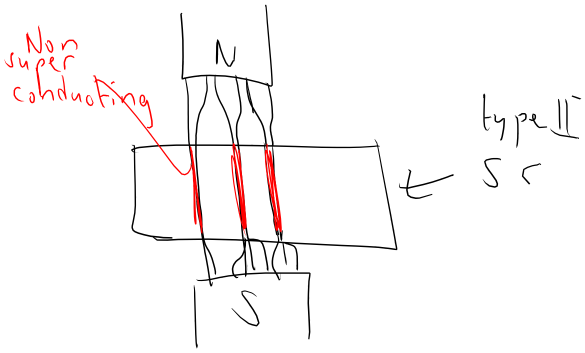
$$= \langle x_0 t_0 | x_1 t_1 \rangle$$

$$\frac{q}{\hbar c} \Phi_B = 2\pi n$$

or $\Phi_B = n \Phi_B^0$

$$\boxed{\Phi_B^0 = 2\pi \frac{\hbar c}{|q|}}$$

por an electron $\frac{2\pi \hbar c}{|e|} = 4 \times 10^{-7}\ \text{G cm}^2$



Magnetic Monopoles and Charge quantization

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_E \qquad \left(\rho_E/\varepsilon_0\right)$$

in SI

$$\vec{\nabla}\cdot\vec{B} = 4\pi\,\rho_M$$

$$\boxed{\vec{B} = \frac{e_M}{r^2}\hat{r}}$$

$\vec{B} \neq \vec{\nabla}\times\vec{A}$ b.c. otherwise if $\vec{B} = \vec{\nabla}\times\vec{A}$

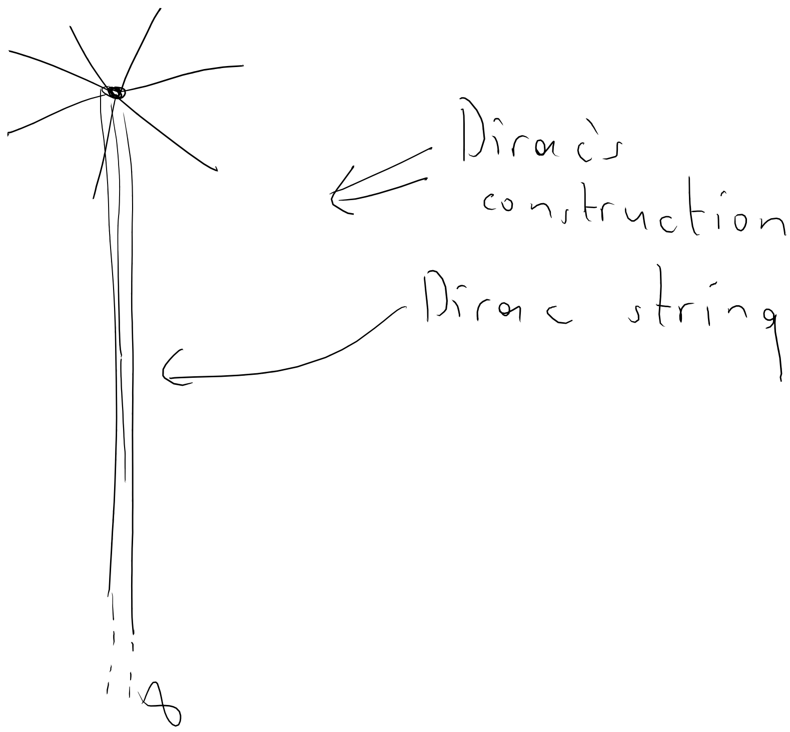
$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

$$\vec{A}^{(I)} = \left(e_M\frac{(1-\cos\theta)}{r\sin\theta}\right)\hat{\phi} \quad \theta > \varepsilon \quad \vec{\nabla}\times\vec{A}^{(I)} = \vec{\nabla}\times\vec{A}^{(II)} = \frac{e_M}{r^2}\hat{r}$$

$$\vec{A}^{(II)} = -\frac{e_M(1+\cos\theta)}{r\sin\theta}\hat{\phi} \quad \theta < \pi - \varepsilon$$

$$\vec{A}^{(I)} = \frac{e_M\left[\cancel{1} + (\cancel{1}\cdot)\cancel{2}\cos^2\frac{\theta}{2}\right]}{r\,\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}\hat{\phi}$$

$$\left.\begin{aligned} \vec{A}^{(I)} &= \frac{e_M\cos\frac{\theta}{2}}{r\sin\frac{\theta}{2}}\hat{\phi} \quad \pi > \theta > \varepsilon \\ \vec{A}^{II} &= \frac{e_M}{r}\frac{\sin\theta/2}{\cos\frac{\theta}{2}}\hat{\phi} \quad \pi-\varepsilon > \theta > 0 \end{aligned}\right\} \pi - \varepsilon > \theta > \varepsilon$$



$$A^{(II)} - A^{(I)} = -\frac{2e_M}{r\sin\theta}\hat{\phi} = \vec{\nabla}\Lambda$$

$$= \hat{r}\frac{\partial\Lambda}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Lambda}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial\phi}$$

$$\frac{\partial\Lambda}{\partial\phi} = -2e_M \qquad \frac{\partial\Lambda}{\partial\theta} = 0 \qquad \frac{\partial\Lambda}{\partial r} = 0$$

$$\Rightarrow \Lambda = -2e_M\phi$$

$$\psi^{II} = e^{\frac{i}{\hbar}\frac{q}{c}\Lambda}\psi^{I} = e^{-\frac{i}{\hbar}\frac{q}{c}2e_M\phi}\psi^{I}$$

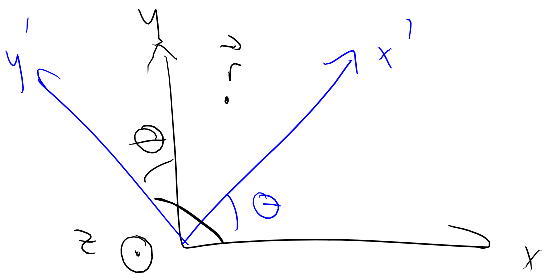
$$\psi^{II} = e^{-\frac{i}{\hbar}\frac{q}{c}2e_M\phi}\psi^{I}$$

$$\frac{2qe_M}{\hbar c} = N \qquad N = 0, \pm1, \pm2, \ldots$$

$$q = \left(\frac{\hbar c}{2e_M}\right)N$$

Symmetries

Rotations



$$\vec{r} = x\hat{x} + y\hat{y}$$
$$= x'\hat{x}' + y'\hat{y}'$$

$$x' = \hat{x}'\cdot\vec{r}$$
$$= x\,\hat{x}'\cdot\hat{x} + y\,\hat{x}'\cdot\hat{y}$$
$$= x\cos\theta + y\cos\left(\frac{\pi}{2}-\theta\right)$$

$$\boxed{x' = x\cos\theta + y\sin\theta}$$

$$y' = \hat{y}'\cdot\vec{r} = x\,\hat{y}'\cdot\hat{x} + y\,\hat{y}'\cdot\hat{y}$$
$$= x\cos\left(\frac{\pi}{2}+\theta\right) + y\cos(\theta)$$

$$\boxed{y' = -x\sin\theta + y\cos\theta}$$

$$\boxed{z' = z}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_z(\theta)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_z(\theta)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$R_z(\varepsilon) = \begin{pmatrix} 1 & \varepsilon & 0 \\ -\varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(\varepsilon^2)$$

$$R_z(\varepsilon) = \mathbb{1} + \varepsilon \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{-iM_z} + \mathcal{O}(\varepsilon^2)$$

$$\boxed{M_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

M_z: generator of rotations around the z-axis.

$$R_z(\varepsilon) = \mathbb{1} - i\varepsilon M_z + \mathcal{O}(\varepsilon^2)$$

$$R_z(\theta) = R_z\left(\frac{\theta}{N}\right)^N$$

$$= \lim_{N\to\infty} R_z\left(\frac{\theta}{N}\right)^N$$

$$= \lim_{\substack{N\to\infty \\ \varepsilon = \frac{\theta}{N}}} \left(\mathbb{1} - i\varepsilon M_z + \mathcal{O}(\varepsilon^2)\right)^N$$

$$= \lim_{\substack{N\to\infty \\ \varepsilon = \frac{\theta}{N}}} \left(\exp\{-i\varepsilon M_z + \mathcal{O}(\varepsilon^2)\}\right)^N$$

$$= \lim_{\substack{N\to\infty \\ \varepsilon = \frac{\theta}{N}}} \exp\{-i\underbrace{(\varepsilon N)}_{=\theta} M_z + \underbrace{(\varepsilon N)}_{=\theta} \mathcal{O}(\varepsilon)\}$$

$$\boxed{R_z(\theta) = \exp\{-i\theta M_z\}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_z^{-1}(\theta) = \exp\{i\theta M_z\} = R_z(\theta)^\dagger = R_z(\theta)^T$$

Under general rotations

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

$$= (x' \; y' \; z') \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$= \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}^T \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$= (x \; y \; z) \, R^T R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\equiv (x \; y \; z) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$R^T R = \mathbb{1}$

R is an orthogonal matrix.

$$M_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$[M_x, M_y] = M_x M_y - M_y M_y$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} - (\quad)(\quad)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[M_x, M_y] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -i M_z$$

$$[M_x, M_y] = -i M_z \qquad [M_i, M_j] = -i \varepsilon_{ijk} M_k$$

$$M_x^2 + M_y^2 + M_z^2 = \underbrace{1 (1+1)}_{l(l+1)} = 2 \mathbb{1}$$