$$[J_x, J_y] = i\hbar J_z$$

$$J_\pm = J_x \pm i J_y$$

$$[J_\pm, J_z] = \mp \hbar J_\pm$$

$$[J^2, J_i] = 0$$

$$[H, a] = -\hbar a$$

$$[H, a^\dagger] = \hbar a^\dagger$$

$$J^2 |a\, b\rangle = a|a, b\rangle \qquad J_z |a, b\rangle = b|a, b\rangle$$

$$a > b^2 \Leftarrow J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$$

$$\langle J^2 \rangle \geq \langle J_z^2 \rangle$$

$$a \geq b^2 \Rightarrow -a \leq b \leq a$$

$$a = b_{max}(b_{max} + \hbar)$$

$$b_{min} = -b_{max}$$

$$b = \hbar m$$

$$b_{max} = \hbar l$$

$$-l \leq m \leq l$$

$$a = \hbar^2 l(l+1)$$

$$\left.\begin{array}{l} |l\, l\rangle \\ |l\, l-1\rangle \\ \vdots \\ |l, -l\rangle \end{array}\right\} 2l+1 \text{ steps} \Rightarrow 2l+1 = n : \text{a integer}$$

$$l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots$$

$$J_\pm = J_x \pm i J_y \Rightarrow J_x = \frac{J_+ + J_-}{2}$$

$$J_y = \frac{J_+ - J_-}{2i}$$

$$J_+ |l m\rangle = C_+(l,m) |l\; m+1\rangle$$

$$J_- |l m\rangle = C_-(l,m) |l\; m-1\rangle$$

$$J^2 = \frac{1}{2}\left(J_+ J_- + J_- J_+\right) + J_z^2$$

$$[J_+, J_-] = [J_x + i J_y, J_x - i J_y]$$

$$= -i i J_z \hbar + i(-i\hbar J_z) = 2\hbar J_z$$

$$J_+ J_- = 2\hbar J_z + J_- J_+$$

$$J^2 = J_- J_+ + \hbar J_z + J_z^2$$

$$\langle lm|J^2|lm\rangle = \hbar^2 l(l+1)$$

$$\langle lm|\, J_- J_+ + \hbar J_z + J_z^2\, |lm\rangle = \hbar^2 l(l+1)$$

$$\langle lm|\, J_+ J_+\, |lm\rangle + \hbar^2 m + \hbar^2 m^2 = \hbar^2 l(l+1)$$

$$|\, J_+|lm\rangle|^2 = \hbar^2 \left[l(l+1) - m(m+1) \right]$$

$$|\, C_+(l,m)|l\, m+1\rangle|^2 = \hbar^2 \left[l(l+1) - m(m+1) \right]$$

$$\Rightarrow |C_+(lm)| = \hbar\sqrt{l(l+1) - m(m+1)}$$

$$C_+(lm) = \hbar\sqrt{(l-m)(l+m+1)}$$

$$J_+|lm\rangle = \underline{\hbar\sqrt{(l-m)(l+m+1)}}\,|l\, m+1\rangle$$

$$J_-|lm\rangle = \hbar\sqrt{(l+m)(l-m+1)}\,|l\, m+1\rangle$$

$\underline{l = 1}$

$$J_+|11\rangle = 0$$

$$J_+|10\rangle = \hbar\sqrt{(1-0)(1+0+1)}\,|11\rangle = \sqrt{2}\hbar|11\rangle$$

$$J_+|1\,{-1}\rangle = \hbar\sqrt{(1-(-1))(1+(-1)+1)}\,|10\rangle = \sqrt{2}\hbar\,|10\rangle$$

$$|11\rangle \doteq \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |10\rangle \doteq \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |1\,{-1}\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$J_+ = \begin{pmatrix}0 & \sqrt{2}\hbar & 0\\ 0 & 0 & \sqrt{2}\hbar\\ 0 & 0 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & \sqrt{2}\hbar & 0\\ 0 & 0 & \sqrt{2}\hbar\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}1\\0\\0\end{pmatrix} \checkmark$$

$$J_- |1\,1\rangle = \hbar\sqrt{(1+1)(1-1+1)}\,|1\,0\rangle = \sqrt{2}\hbar|1\,0\rangle$$

$$J_- |1\,0\rangle = \hbar\sqrt{(1+0)(1-0+1)}\,|1\,-1\rangle = \sqrt{2}\hbar|1\,-1\rangle$$

$$J_- |1\,-1\rangle = 0$$

$$J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix} \underset{|1\,0\rangle}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = \underset{\sqrt{2}\hbar|1\,-1\rangle}{\begin{pmatrix} 0 \\ 0 \\ \sqrt{2}\hbar \end{pmatrix}}$$

$$J_x = \frac{J_+ + J_-}{2} = \frac{1}{2}\left[\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$[H, J^2] = 0$$

$$[H, J_i] = 0$$

$$^n\langle l m' | l m\rangle$$

$$|l m'\rangle^n = \mathcal{D}(R)|l m\rangle$$

$$A(|lm\rangle \to |lm'\rangle^n) \propto {}^n\langle lm'|lm\rangle = \langle lm'|\mathcal{D}(R)^\dagger|lm\rangle$$

$$[H, J_i] = 0$$

Orbital Angular Momentum

$$\vec{L} = \vec{r}\times\vec{p}$$

$$(xp)^\dagger = xp$$

$$(xp)_{cl} \Rightarrow \frac{1}{2}(xp+px)_{qm}$$

$$L_x = yp_z - p_y z$$

$$L_y = zp_x - p_z x$$

$$L_z = xp_y - yp_x$$

$$[L_i, L_j] = i\hbar\, \varepsilon_{ijk} L_k$$

$$[L_x, L_y] = [yp_z - p_y z,\ zp_x - p_z x]$$

$$= yp_x\,[p_z, z] + p_y x\,[z, p_z]$$

$$= y p_x(-i\hbar) + p_y x(i\hbar)$$

$$= +i\hbar(p_y x - y p_x) = i\hbar L_z$$

$$|x\,y\,z\rangle^R = \left(1 - \frac{i}{\hbar} L_z\,\delta\phi\right)|x\,y\,z\rangle$$

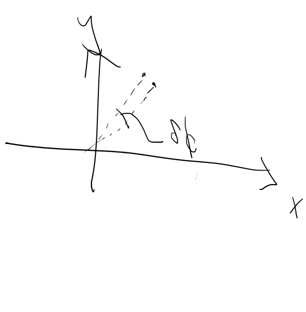
$$= \left(1 - \frac{i}{\hbar}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)\,\delta\phi\right)|x\,y\,z\rangle$$

$$= \underbrace{\left(1 - \frac{i\delta\phi}{\hbar} x p_y + \frac{i\delta\phi}{\hbar} y p_x\right)}|x\,y\,z\rangle$$

$$|x\,y\,z\rangle^R = \left(1 - \frac{i}{\hbar}\vec{p}\cdot\delta\vec{r}\right)|x\,y\,z\rangle$$

$$\delta\vec{r} = -y\,\delta\phi\,\hat{x} + x\,\delta\phi\,\hat{y}$$

$$D(R_z(\delta\phi))|x\,y\,z\rangle = |x - y\,\delta\phi,\ y + x\,\delta\phi,\ z\rangle$$



$$D(R_z(\phi))|x\,y\,z\rangle = |x\cos\phi - y\sin\phi,\ y\cos\phi + x\sin\phi,\ z\rangle$$

$$D(R_z(2\pi))|x\,y\,z\rangle = |x\,y\,z\rangle \Rightarrow l = 0, 1, 2, \ldots$$

$$|xyz\rangle = |r\,\theta,\phi\rangle$$

$$D_z(\gamma)\,|r\,\theta\,\phi\rangle = |r\,\theta\,(\phi+\gamma)\rangle$$

$$\Leftarrow$$

$$\langle r\,\theta\,\phi|\left(1 - \frac{i}{\hbar}\delta\phi L_z\right)|\psi\rangle$$

$$= \langle r\,\theta\,\phi|\psi\rangle - \frac{i}{\hbar}\delta\phi\,\langle r\,\theta\,\phi|L_z|\psi\rangle$$

$$= \psi(r,\theta,\phi) - \frac{i}{\hbar}\delta\phi\,\langle r\,\theta\,\phi|L_z|\psi\rangle$$

$$\langle r\,\theta\,\phi|\left(1 - \frac{i}{\hbar}\delta\phi L_z\right)|\psi\rangle$$

$$= \langle r\,\theta\,(\phi-\delta\phi)|\psi\rangle \Leftarrow$$

$$= \psi(r,\theta,\phi-\delta\phi) = \psi(r,\theta,\phi) + \frac{\partial\psi}{\partial\phi}(-\delta\phi)$$

$$\langle r\,\theta\,\phi|L_z|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\langle r\,\theta\,\phi|\psi\rangle$$

$$\langle xyz|xp_y - yp_x|\psi\rangle = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\langle xyz|\psi\rangle$$

$$\frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\langle\vec{r}|L^2|\psi\rangle = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\psi(\vec{r})$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L^2 = \varepsilon_{ijk} x_j p_k \varepsilon_{ilm} x_l p_m$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(x_j p_k x_l p_m)$$

$$= x_l p_m x_l p_m - x_m \overset{\frown}{p_l x_l} p_m$$

$$= x_l (x_l p_m + \underset{\substack{\shortparallel \\ p_m x_l - x_l p_m = -i\hbar\delta_{ml}}}{[p_m, x_l]}) p_m - x_m [x_l p_l + \underset{\substack{\shortparallel \\ -i\hbar 3}}{[p_l, x_l]}] p_m$$

$$= r^2 p^2 - i\hbar \vec{r}\cdot\vec{p} - x_l x_m p_l p_m + 3i\hbar \vec{r}\cdot\vec{p}$$

$$L^2 = r^2 p^2 + 2i\hbar \vec{r}\cdot\vec{p} - x_l (p_l x_m + \underbrace{[x_m, p_l]}_{i\hbar\delta_{ml}}) p_m$$

$$= r^2 p^2 + 2i\hbar \vec{r}\cdot\vec{p} - (\vec{r}\cdot\vec{p})^2 - i\hbar \vec{r}\cdot\vec{p}$$

$$L^2 = r^2 p^2 + i\hbar \vec{r}\cdot\vec{p} - (\vec{r}\cdot\vec{p})^2$$

$$p^2 = \frac{L^2}{r^2} + \frac{(\vec{r}\cdot\vec{p})^2}{r^2} - i\hbar \frac{\vec{r}\cdot\vec{p}}{r^2}$$

$$\langle \vec{r} | p^2 | \psi \rangle = -\hbar^2 \nabla^2 \psi(\vec{r})$$

$$\langle \vec{r} | \vec{r} \cdot \vec{p} | \psi \rangle = \vec{r} \cdot \frac{\hbar}{i} \vec{\nabla} \psi = \frac{\hbar}{i} r \frac{\partial}{\partial r} \psi$$

$$\langle \vec{r} | (\vec{r} \cdot \vec{p})^2 | \psi \rangle = -\hbar^2 r \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \psi \right)$$

$$-\hbar^2 \nabla^2 \psi = \frac{1}{r^2} L^2 \psi - \frac{\hbar^2}{r^2} r \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \psi \right)$$

$$- \frac{i\hbar}{r^2} \frac{\hbar}{i} r \frac{\partial}{\partial r} \psi$$

$$\boxed{-\frac{\hbar^2}{2m} \psi = -\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \psi \right) + \frac{1}{r} \frac{\partial}{\partial r} \psi + \frac{1}{\hbar^2} \frac{L^2}{r^2} \psi \right]}$$

Spherically symmetric potentials

$$H = \frac{p^2}{2m} + V(r) \Rightarrow [H, L^2] = 0$$

$$H | E l m \rangle = E | E l m \rangle$$

Possible Degeneracy.

$$[H, J_\pm] = 0$$

$$H(\underbrace{J_\pm |Elm\rangle}_{\neq |Elm\rangle}) = E(J_\pm |Elm\rangle)$$

$$\langle \vec{r} | Elm\rangle = \psi(\vec{r}) = \psi_{Elm}(r)\, Y_{lm}(\theta)$$

$$|\vec{r}\rangle = |r\,\theta\,\phi\rangle = |r\rangle \otimes |\theta\,\phi\rangle$$

$$|Elm\rangle = |E\rangle^{lm} \otimes |lm\rangle$$

$$\langle \theta\,\phi | lm\rangle \equiv Y_{lm}(\theta, \phi)$$

$$\langle \theta\,\phi | L_z | lm\rangle = \hbar m \langle \theta\,\phi | lm\rangle$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial \phi} \langle \theta\,\phi | lm\rangle$$

$$\frac{\hbar}{i}\frac{\partial}{\partial \phi} Y_{lm} = \hbar m\, Y_{lm}$$

$$\Rightarrow Y_{lm} \propto e^{+im\phi} f(\theta)$$

$$\langle \theta \phi | L_+ | l l \rangle = 0$$

$$-i\hbar e^{i\phi}\left(+i \frac{\partial}{\partial \theta} - \cot\theta \frac{\partial}{\partial \phi}\right) Y_{ll} = 0$$

$$H = \frac{p^2}{2m} + V(r)$$

$$H \psi_{Elm}(\vec{r}) = -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\psi\right) + \frac{1}{r}\frac{\partial}{\partial r}\psi + \frac{1}{\hbar^2}\frac{L^2}{r^2}\psi\right] + V(r)\psi$$

$$= -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\psi\right) + \frac{1}{r}\frac{\partial}{\partial r}\psi\right]$$

$$-\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\psi + V(r)\psi = E\psi$$

$$\psi(r, \theta, \phi) = \frac{R(r)}{r} Y_{lm}(\theta, \phi)$$

$$\frac{\partial \psi}{\partial r} = \left(\frac{R'(r)}{r} - \frac{R}{r^2}\right) Y_{lm}$$

$$r\frac{\partial\psi}{\partial r} = \left(R' - \frac{R}{r}\right)Y_{lm}$$

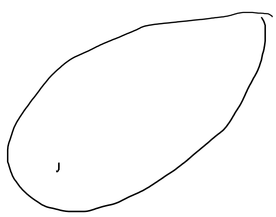
$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \left(R'' - \frac{R'}{r} + \frac{R}{r^2}\right)Y_{lm}$$

$$H\psi = \left\{-\frac{\hbar^2}{2m}\left[\frac{1}{r}\left(R'' - \cancel{\frac{R'}{r}} + \cancel{\frac{R}{r^2}}\right) + \frac{1}{r}\left(\cancel{\frac{R'}{r}} - \cancel{\frac{R}{r^2}}\right) + \frac{l(l+1)}{\hbar^2 r^2}\frac{R}{r}\right] + V(r)\frac{R}{r}\right\}\cancel{Y_{lm}} = E\frac{R}{r}\cancel{Y_{lm}}$$

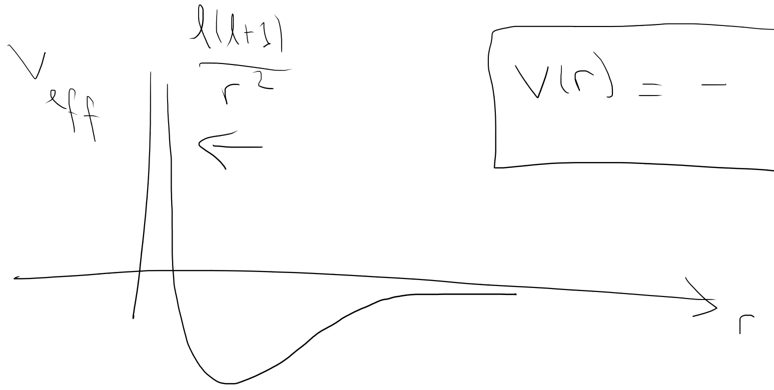
$$\boxed{-\frac{\hbar^2}{2m}R'' + \underbrace{\left[\underset{\color{red}+}{-}\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + V(r)\right]}_{V_{eff}}R = ER}$$

$$\psi = \frac{R(r)}{r}Y_{lm}(\theta,\phi)$$

In classical physics, a particle with $\vec{L} \neq 0$ and $E < \infty$, it can never reach $r = 0$



$r_{min} < r$



$$\boxed{V(r) = -\frac{\alpha}{r}}$$

$$-\frac{\hbar^2}{2m} R'' + \left(\frac{1}{2m}\frac{l(l+1)}{r^2} + V(r)\right) R = E R$$

$r \to 0$

$$-\frac{\hbar^2}{2m} R'' + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} R = 0 \qquad (r \to 0)$$

assume $R = r^\alpha$

$$-\cancel{\frac{\hbar^2}{2m}} \alpha(\alpha-1) \cancel{r^{\alpha-2}} + \cancel{\frac{\hbar^2}{2m}} l(l+1) \cancel{r^{\alpha-2}} = 0$$

$$l(l+1) - \alpha(\alpha-1) = 0$$

$$\Rightarrow \quad \alpha = l+1 \qquad \alpha = -l$$

$$\frac{R(r)}{r} \doteq A r^l + \frac{B}{r^{l+1}}$$

$$J_r \propto \hat{r} \cdot \mathrm{Re}(\psi^* \nabla \psi) = -\mathrm{Re}\left(\psi^* \frac{\partial}{\partial r} \psi\right)$$

$$J_r = + B^2 (l+1) \frac{1}{r^{2l+1}} + \cdots$$

$$\Rightarrow \psi(\vec{r}) \xrightarrow{r \to 0} r^l Y_{lm}(\theta, \phi)$$

as $r \to \infty$

$$-\frac{\hbar^2}{2m} R'' = E R \qquad \left(\text{for a bound state } E < 0 \right)$$

$$K = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$R(r) \xrightarrow{r \to \infty} e^{-Kr}$$

$$\psi(r, \theta, \phi) = r^l e^{-Kr} f(r) Y_{lm}(\theta, \phi)$$

$$f(r) \xrightarrow[r \to \infty]{r \to 0} \text{const}$$