$$H = \frac{p^2}{2m} + V(r) \qquad [H, J_i] = 0$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$e^{-\frac{i}{\hbar} L_z \theta} |xyz\rangle = |x'y'z'\rangle$$

$$z' = z$$
$$x' = x\cos\theta - y\sin\theta$$
$$y' = y\cos\theta + x\sin\theta$$

$$\langle \hat{n} | lm \rangle \equiv Y_{lm}(\hat{n}) = Y_{lm}(\theta, \phi)$$

$$\psi(\vec{r}) = \underbrace{\frac{R(r)}{r}}\underbrace{N(\theta, \phi)}$$

$$N(\theta, \phi) = Y_{lm}(\theta, \phi)$$

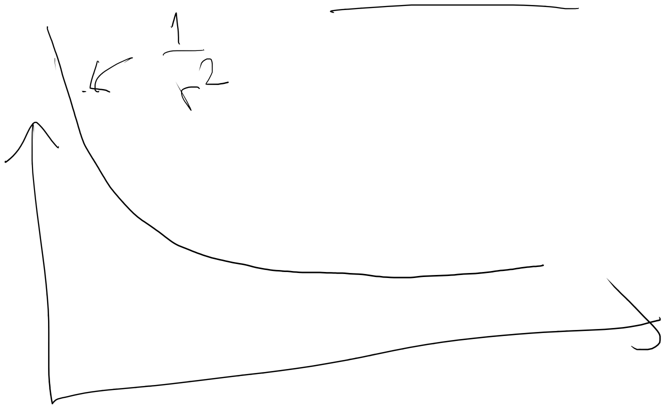
$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}R(r) + \left[\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + V(r)\right]R = R(r)E$$

$$0 < r < \infty$$

$$R(r) \xrightarrow{r \to 0} r^{l+1}$$

$$\psi(\vec{r}) \xrightarrow{r \to 0} r^{l}$$

$$V_{eff}(r) = \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + V(r)$$



Example Free particle

$$\psi = \frac{R(r)}{r} Y_{lm}(\theta,\phi)$$

$$-\frac{\hbar^2}{2m}\frac{d^2R}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} R = E R$$

$E > 0 \qquad E = \frac{\hbar^2 k^2}{2m}$

for $l=0$ $\qquad -\frac{\hbar^2}{2m} R'' = \frac{\hbar^2}{2m} k^2 R$

$$\Rightarrow R = \sin(kr+\delta)$$

$\psi = \frac{R(r)}{r} \Rightarrow R(r=0) = 0 \Rightarrow R = \sin(kr)$

$$\psi(\vec{r}) = N \frac{\sin(kr)}{r}$$

$$E = \frac{\hbar^2 k^2}{2m}$$

For $l \neq 0$

$$-\frac{\hbar^2}{2m} \frac{d^2 R}{dr^2} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} R = E R$$

$R_1(r) = r\, j_l(kr)$ or $R_2 = r\, n_l(kr)$

$R_1(r) \xrightarrow{r \to 0} r^{l+1}$ $\quad R_2 \xrightarrow{r \to 0} \frac{1}{r^l}$

$$\psi(\vec{r}) = N_{lm} j_l(kr)\, Y_{lm}(\theta, \phi)$$

alternative basis

$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} = \sum N_{lm} j_l(r)\, Y_{lm}(\theta, \phi)$$

$$N_{lm} = \int d^3r\, e^{i\vec{k}\cdot\vec{r}}\, j_l(r)\, Y_{lm}(\theta, \phi)$$

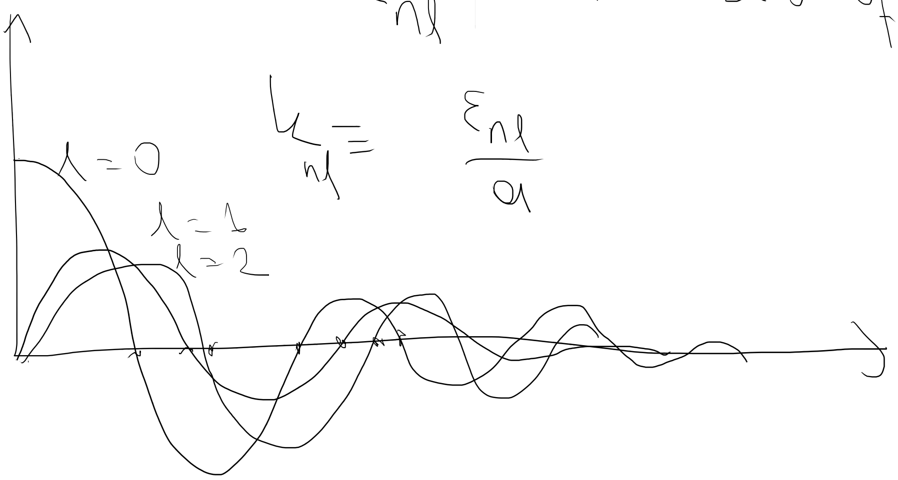
Example particle inside a hard spherical well

$$\psi(\vec{r}) = N_{lm} \, j_l(kr) \, Y_{lm}(\theta, \phi)$$

$$\psi(\vec{r})\Big|_{r=a} = 0$$

$$\Rightarrow \quad j_l(ka) = 0$$

$$ka = \varepsilon_{nl} = n^{th} \text{ zero of } j_l(x)$$



$$k_{nl} = \frac{\varepsilon_{nl}}{a}$$

Example Spherical oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m w^2 r^2$$

$$r^2 = x^2 + y^2 + z^2$$

$$H = H_x + H_y + H_y$$

$$H_i = \frac{1}{2m} p_i^2 + \frac{1}{2} m w^2 x_i^2$$

$$E_{n_x n_y n_z} = \hbar w \left(n_x + n_y + n_z + \frac{3}{2} \right)$$

$$E = \hbar w \left(1 + \frac{3}{2} \right)$$

$$1 = \underset{n_x}{1} + \underset{n_y}{0} + \underset{n_z}{0} = 0 + 1 + 0 = 0 + 0 + 1$$

$$\psi = \frac{R(r)}{r} Y_{lm}$$

$$-\frac{\hbar^2}{2m} R'' + \left[\frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} + \frac{1}{2} m w^2 r^2 \right] R = E R$$

$$R(r) \xrightarrow{r \to 0} r^l$$

$\underline{r \to \infty}$

$$-\frac{\hbar^2}{\cancel{2}m} R'' + \frac{1}{\cancel{2}} m\omega^2 r^2 R = 0$$

$$R'' + \left(\frac{m\omega}{\hbar}\right)^2 r^2 R = 0$$

$$\left[\frac{R}{r^2}\right] \qquad \left(\frac{m\omega}{\hbar}\right)^2 = \frac{1}{a^4}$$

$r = au$ $\quad$ u is dinensionless

$$\frac{d^2R}{dr^2} = \frac{1}{a^2}\frac{d^2R}{du^2}$$

$$-\frac{1}{\cancel{a^2}}\frac{d^2R}{du^2} + \frac{1}{\cancel{a^4}}\cancel{a^2} u^2 R = 0$$

$$\frac{d^2R}{du^2} - u^2 R = 0 \qquad (\text{as } u \to \infty)$$

$$\Rightarrow R = e^{-u^2/2} \qquad (\text{as } u \to \infty)$$

$$R' = -uR$$

$$R'' = -R + (-u)(-u)R$$

$$R'' - u^2R = -R$$

negligable

$$R(r) = \underline{r^{l+1}}\, e^{-r^2/2a^2}\, w(r)$$

$$0 = u\frac{d^2w}{du^2} + 2\left[(l+1) - u^2\right]\frac{dw}{du} + \left[\lambda - (2(l+1))\right]uw$$

$$E = \frac{\hbar\omega}{2}\lambda$$

$$w(r) = \sum a_n r^n$$

$$a_{n+2} = \frac{2n+2l+3-\lambda}{(n+2)(n+2l+3)}\, a_n$$

$$a_{n+2} \simeq \frac{1}{n} a_n \Rightarrow w(r) \xrightarrow{r\to\infty} e^{r^2/a^2}$$

$$\lambda = 2N + 3 \qquad N \geq l$$

$$N = n_0 + l = 2q + l$$

$$E = \frac{1}{2}\hbar\omega\lambda = \frac{1}{2}\hbar\omega(2N+3)$$

$$E = \hbar\omega\left(N + \frac{3}{2}\right) \qquad N \geq l$$

For $\underline{N=1}$, $l = 1 \Rightarrow \left.\begin{array}{l} m = +1 \\ m = 0 \\ m = -1 \end{array}\right\}$

$$N = \underset{n_r}{0} + \overset{l}{1}$$

$$\neq 1 + 0 \Leftarrow$$

not even

$$N = n_x + n_y + n_z \qquad |n_x n_y n_z\rangle$$

$$l \overset{?}{=} l(n_x, n_y, n_z) \qquad |N l m\rangle$$

$$m \overset{?}{=} m(n_x, n_y, n_z)$$

<u>Example</u> Coulomb Potential

$$V(r) = -\frac{\alpha}{r} \qquad \alpha = Ze^2$$

$$\psi(\vec{r}) = \frac{R(r)}{r} Y_{lm} \Leftarrow$$

$$\frac{R(r)}{r} = r^l e^{-r/a_0} \underline{w(r)}$$

Example Angular momentum + spin-$\frac{1}{2}$

$$\psi(\vec{r}) = \widetilde{\psi_+(\vec{r})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_-(\vec{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

$$\vec{J} = \vec{L} + \vec{S} \Leftarrow$$

$$[L_i, S_j] = 0 \qquad Y_{lm}(\theta, \phi) \otimes |\pm\rangle$$

$$|l s; m m_s\rangle$$

$$L^2 |l s; m m_s\rangle = \hbar^2 l(l+1) |l s; m m_s\rangle$$
$$S^2 |l s; m m_s\rangle = \hbar^2 s(s+1) |l s; m m_s\rangle$$
$$L_z |l s; m m_s\rangle = \hbar m |l s; m m_s\rangle$$
$$S_z |l s; m m_s\rangle = \hbar m_s |l s; m m_s\rangle$$

$$[J_i, L_j] = [L_i + S_i, L_j] = [L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$$
$$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$$
$$[J_i, L^2] = L_j [J_i, L_j] + [J_i, L_j] L_j$$

$$= L_j\, i\hbar \varepsilon_{ijk} L_k + i\hbar \varepsilon_{ijk} L_k L_j$$

$$= i\hbar \varepsilon_{ijk} (L_j L_k + L_k L_j) = 0$$

$$[J_i, L^2] = [L_i + S_i, L^2] = 0$$

$$[J_i, S^2] = [L_i + S_i, S^2] = 0$$

$\{J^2, J_z, L^2, S^2\}$ maximal set of commuting operators.

$$\{|ls; j m_j\rangle\} \qquad \{|ls; m m_s\rangle\}$$

$$\underline{|ls; j m_j\rangle} = \sum_{m m_s} C^{lsj}_{m m_s m_j} \underline{|ls; m m_s\rangle}$$

Clebsh Gordon Coefficients

$$C^{lsj}_{m m_s m_j} = \langle ls; m m_s | ls; j m_j \rangle$$

$\langle l s; m m_s | l s; j m_j \rangle = 0$ unless $m + m_s = m_j$

$$\langle l s; m m_s | J_z | l s; j m_j \rangle = \hbar m_j \langle l s; m m_s | l s; j m_j \rangle$$

$$\langle l s; m m_s | L_z + S_z | l s; j m_j \rangle = \hbar (m + m_s) \langle l s; m m_s | l s; j m_j \rangle$$

either $m_j = m + m_s$ or $\langle l s; m m_s | l s; j m_j \rangle = 0$

$$| l s; m m_s \rangle = | l m \rangle \otimes | s m_s \rangle$$

$$\vec{J} = \vec{L} + \vec{S} = \vec{L} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}$$

$$J_z = L_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z$$

$$L_z \otimes \mathbb{1} \, | l m \rangle \otimes | s m_s \rangle = (L_z | l m \rangle) \otimes (\mathbb{1} | s m_s \rangle)$$

$$L_z \otimes \mathbb{1} \, | l m; j m_j \rangle = \ldots$$

j_{max} = maximum possible value of m_j

= maximum " " " $(m + m_s)$

$$j_{max} = l + s$$

Example $\quad l = 1 \qquad s = \frac{1}{2}$

$$J_{max} = 1 + \frac{1}{2} = \frac{3}{2}$$

$l = 1$: 3 states
$s = \frac{1}{2}$: 2 states

$\Big\}$ 6 members in the basis

$J = \frac{3}{2}$: 4 states

$J = \frac{1}{2} \Leftarrow 2$ states

$|l\, s; m_l\, m_s\rangle^{ls} \qquad\qquad |l\, s; J\, m_J\rangle^{J}$

$$|1\, \tfrac{1}{2}; 1\, \tfrac{1}{2}\rangle^{ls} = |1\, \tfrac{1}{2}; \tfrac{3}{2}\, \tfrac{3}{2}\rangle^{J}$$

$$|1\, 1; \tfrac{3}{2}\, \tfrac{1}{2}\rangle^{J} \propto J_-\, |1\, \tfrac{1}{2}; \tfrac{3}{2}\, \tfrac{3}{2}\rangle^{J}$$

$$\| $$

$$(L_- + S_-)\, |1\, \tfrac{1}{2}; 1\, \tfrac{1}{2}\rangle^{ls}$$

$$= \hbar \sqrt{(1+1)(1-1+1)}\, |1\, \tfrac{1}{2}; 0\, \tfrac{1}{2}\rangle^{ls}$$

$$+ \hbar \sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)\left(\tfrac{1}{2}-\tfrac{1}{2}+1\right)}\, |1\, \tfrac{1}{2}; 1\, -\tfrac{1}{2}\rangle^{ls}$$

$$\Downarrow$$

$$\Big\{ |1\, 1; \tfrac{3}{2}\, \tfrac{1}{2}\rangle^{J} = \frac{1}{\sqrt{3}}\left(\sqrt{2}\, |1\, \tfrac{1}{2}; 0\, \tfrac{1}{2}\rangle^{ls} + |1\, \tfrac{1}{2}; 1\, -\tfrac{1}{2}\rangle^{ls}\right) \Leftarrow$$

$$|11;\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle^{J} \propto J_{-}\,|11;\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \ldots$$

$$|11;\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle^{J} \propto J_{-}\,|11;\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle^{J} = \ldots$$

$$|1\,1\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle^{J}$$

$$m_s + m = \tfrac{1}{2} \Rightarrow (m_s, m) = (\tfrac{1}{2}, 0)$$

or

$$(m_s, m) = (-\tfrac{1}{2}, 1)$$

$$|1\,1\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle^{J} = a\,|1\,\tfrac{1}{2};0\,\tfrac{1}{2}\rangle^{ls} + b\,|1\,\tfrac{1}{2},1\,{-\tfrac{1}{2}}\rangle^{ls}$$

$$\underline{{}^{J}\langle 1\,\tfrac{1}{2},\tfrac{3}{2}\,\tfrac{1}{2}|}\,1\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle^{J} = 0$$

$$= \sqrt{2}\,a + b = 0 \Rightarrow b = -\sqrt{2}\,a$$

$$|1\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle^{J} = \frac{1}{\sqrt{3}}\left(|1\,\tfrac{1}{2};0\,\tfrac{1}{2}\rangle^{ls} - \sqrt{2}\,|1\,\tfrac{1}{2},1\,{-\tfrac{1}{2}}\rangle^{ls}\right)$$

$$|1\,\tfrac{1}{2};\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle^{J} \propto J_{-}\,|1\,\tfrac{1}{2};\tfrac{1}{2}\,{+\tfrac{1}{2}}\rangle^{J}$$

$$|j_1 j_2; m_1 m_2\rangle$$

$$|j_1-j_2| \leq J \leq j_1+j_2$$

$$J,\ 2(j_1+j_2)+1 \left\{ \begin{array}{l} |j_1 j_2; \overset{J}{\widetilde{j_1+j_2}}; \overset{m}{\widetilde{j_1+j_2}}\rangle^J = |j_1 j_2; j_1 j_2\rangle^{j_1 j_2} \\ |j_1 j_2; j_1+j_2, j_1+j_2-1\rangle^J \propto J_- \underbrace{|j_1 j_2\, j_1+j_2, j_1+j_2\rangle^J} \\ \quad \vdots \\ |j_1 j_2; j_1+j_2, -j_1-j_2\rangle^J \end{array} \right.$$

$$\{|j_1 j_2; j_1+j_2-1, j_1+j_2-1\rangle^J$$

$$= \underline{a}\,|j_1 j_2\, j_1\, j_2-1\rangle + \underline{b}\,|j_1 j_2; j_1-1\, j_2\rangle$$

$$\boxed{|j_1-j_2| \leq J \leq j_1+j_2}$$