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GENERATING THE TWIST SUBGROUP BY INVOLUTIONS

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ABSTRACT. For a nonorientable surface, the twist subgroup is an index 2 subgroup of the mapping class group. It is generated by Dehn twists about two-sided simple closed curves. In this paper, we study involution generators of the twist subgroup. We give generating sets of involutions with the smallest number of elements our methods allow.

1. INTRODUCTION

Let N_g denote a closed connected nonorientable surface of genus g. The mapping class group of N_g is defined to be the group of the isotopy classes of all diffeomorphisms of N_g. Throughout the paper this group will be denoted by $\mathrm{Mod}(N_g)$. Let Σ_g denote a closed connected orientable surface of genus g. The mapping class group of Σ_g is the group of the isotopy classes of orientation preserving diffeomorphisms and is denoted by $\mathrm{Mod}(\Sigma_g)$.

In the orientable case, it is a classical result that $\mathrm{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves [3, 5, 10]. The study of algebraic properties of mapping class group, finding small generating sets, generating sets with particular properties, is an active one leading to interesting developments. Wajnryb [19] showed that $\mathrm{Mod}(\Sigma_g)$ can be generated by two elements given as a product of Dehn twists. As the group is not abelian, this is the smallest possible. Later, Korkmaz [7] showed that one of these generators can be taken as a Dehn twist, he also proved that $\mathrm{Mod}(\Sigma_g)$ can be generated by two torsion elements. Recently, the third author showed that $\mathrm{Mod}(\Sigma_g)$ is generated by two torsions of small orders [20].

Generating $\mathrm{Mod}(\Sigma_g)$ by involutions was first considered by McCarthy and Papadopoulus [13]. They showed that the group can be generated by infinitely many conjugates of a single involution (element of order two) for $g \geq 3$. In terms of generating by finitely many involutions,

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Luo [12] showed that any Dehn twist about a nonseparating simple closed curve can be written as a product six involutions, which in turn implies that $\mathrm{Mod}(\Sigma_g)$ can be generated by $12g+6$ involutions. Brendle and Farb [1] obtained a generating set of six involutions for $g \geq 3$. Following their work, Kassabov [6] showed that $\mathrm{Mod}(\Sigma_g)$ can be generated by four involutions if $g \geq 7$. Recently, Korkmaz [8] showed that $\mathrm{Mod}(\Sigma_g)$ is generated by three involutions if $g \geq 8$ and four involutions if $g \geq 3$. Also, the third author improved his result showing that it is generated by three involutions if $g \geq 6$ [21].

Compared to orientable surfaces less is known about $\mathrm{Mod}(N_g)$. Lickorish [9, 11] showed that it is generated by Dehn twists about two-sided simple closed curves and a so-called Y-homeomorphism (or a crosscap slide). Chillingworth [2] gave a finite generating set for $\mathrm{Mod}(N_g)$ that linearly depends on g. Szepietowski [18] proved that $\mathrm{Mod}(N_g)$ is generated by three elements and by four involutions.

The twist subgroup $\mathcal{T}_g$ of $\mathrm{Mod}(N_g)$ is the group generated by Dehn twists about two-sided simple closed curves. The group $\mathcal{T}_g$ is a subgroup of index 2 in $\mathrm{Mod}(N_g)$ [11]. Chillingworth [2] showed that $\mathcal{T}_g$ can be generated by finitely many Dehn twists. Stukow [16] obtained a finite presentation for $\mathcal{T}_g$ with $(g+2)$ Dehn twist generators. Later Omori [14] reduced the number of Dehn twist generators to $(g+1)$ for $g \geq 4$. If it is not required that all generators are Dehn twists, Du [4] obtained a generating set consisting of three elements, two involutions and an element of order $2g$ whenever $g \geq 5$ and odd. Recently, Yoshihara [22] was interested in the problem of finding generating sets for $\mathcal{T}_g$ consisting of only involutions. He proved that $\mathcal{T}_g$ can be generated by six involutions for $g \geq 14$ and by eight involutions if $g \geq 8$.

Our aim in this paper is to generate $\mathcal{T}_g$ with fewer number of involutions. It is known that any group generated by two involutions is isomorphic to a quotient of a dihedral group. Hence, $\mathcal{T}_g$ cannot be generated by two involutions. We are not sure whether $\mathcal{T}_g$ can be generated by three involutions. Based on the approach of [8], we obtain the following result:

Main Theorem. *The twist subgroup $\mathcal{T}_g$ of $\mathrm{Mod}(N_g)$ is generated by*

(1) *four involutions if $g \geq 12$ and even,*
(2) *four involutions if $g = 4k+1 \geq 5$,*
(3) *five involutions if $g = 4k+3 \geq 11$.*

We also prove that the twist subgroup $\mathcal{T}_g$ can be generated by

(4) *five involutions if $g = 8, 10$,*
(5) *six involutions if $g = 6, 7$.*

Note that if a group is generated by involutions, then its first integral homology group should consist of elements of order 2. For the twist subgroup $\mathcal{T}_g$, this is the case when $g \geq 5$ [15].

The paper is organized as follows. In Section 2, we recall some basic results on $\mathrm{Mod}(N_g)$ and its subgroup $\mathcal{T}_g$. We work with nonorientable surfaces of even genus in Section 3 and nonorientable surfaces of odd genus in Section 4.

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2. Background and Results on Mapping Class Groups

Let N_g be a closed connected nonorientable surface of genus g. Note that the *genus* for a nonorientable surface is the number of projective planes in a connected sum decomposition. The *mapping class group* $\mathrm{Mod}(N_g)$ of the surface N_g is defined to be the group of the isotopy classes of diffeomorphisms $N_g \to N_g$. Throughout the paper we do not distinguish a diffeomorphism from its isotopy class. For the composition of two diffeomorphisms, we use the functional notation; if g and h are two diffeomorphisms, the composition gh means that h acts on N_g first.

A simple closed curve on a nonorientable surface N_g is said to be *one-sided* if a regular neighbourhood of it is homeomorphic to a Möbius band. It is called *two-sided* if a regular neighbourhood of it is homeomorphic to an annulus. If a is a two-sided simple closed curve on N_g, to define the Dehn twist t_a, we need to fix one of two possible orientations on a regular neighbourhood of a (as we did for the curve a_1 in Figure 1). Following [8] the right-handed Dehn twist t_a about a will be denoted by the corresponding capital letter A.

Recall the following properties of Dehn twists: let a and b be two-sided simple closed curves on N_g and let $f \in \mathrm{Mod}(N_g)$.

- **Commutativity:** If a and b are disjoint, then $AB = BA$.
- **Conjugation:** If $f(a) = b$, then $fAf^{-1} = B^s$, where $s = \pm 1$ depending on whether f is orientation preserving or orientation reversing on a neighbourhood of a with respect to the chosen orientation.

Consider the surface N_g shown in Figure 1. The Dehn twist generators of Omori can be given as follows (note that we do not have the curve d_r when g is odd).

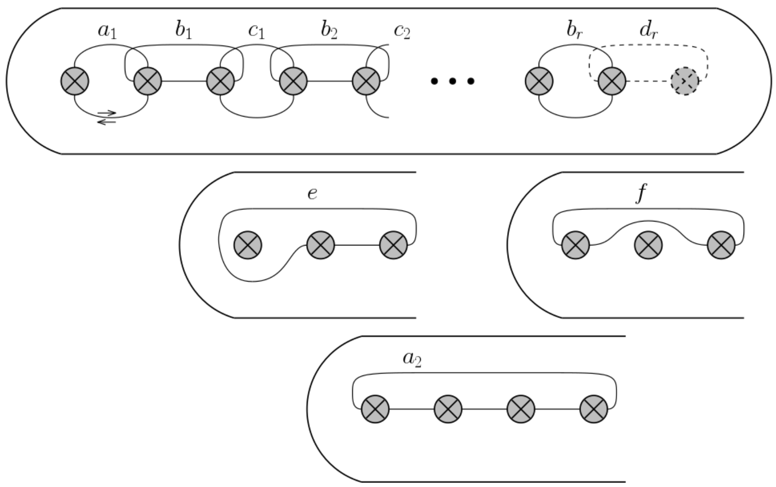

FIGURE 1. The curves a_1, a_2, b_i, c_i, e and f on the surface N_g.

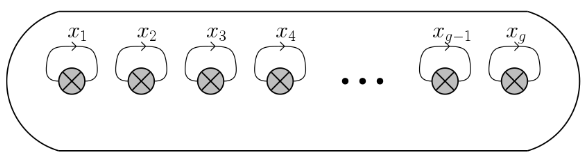

FIGURE 2. Generators of $H_1(N_g;\mathbb{R})$.

Theorem 2.1. [14] *The twist subgroup $\mathcal{T}_g$ is generated by the following $(g+1)$ Dehn twists*

(1) $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}$ *and* E *if* $g = 2r+1$ *and*
(2) $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}, D_r$ *and* E *if* $g = 2r+2$.

Consider a basis $\{x_1, x_2 \ldots, x_{g-1}\}$ for $H_1(N_g;\mathbb{R})$ such that the curves x_i are one-sided and disjoint as in Figure 2. It is known that every diffeomorphism $f : N_g \to N_g$ induces a linear map $f_* : H_1(N_g;\mathbb{R}) \to H_1(N_g;\mathbb{R})$. Therefore, one can define a homomorphism $D : \mathrm{Mod}(N_g) \to \mathbb{Z}_2$ by $D(f) = \det(f_*)$. The following lemma from [9] tells when a mapping class falls into the twist subgroup $\mathcal{T}_g$.

Lemma 2.2. *Let $f \in \mathrm{Mod}(N_g)$. Then $D(f) = 1$ if $f \in \mathcal{T}_g$ and $D(f) = -1$ if $f \notin \mathcal{T}_g$.*

3. THE EVEN CASE

For $g = 2r+2$, we work with the models in Figure 3. This surface is obtained from a genus r orientable surface by deleting the interiors of two disjoint disks and identifying the antipodal points on the boundary.

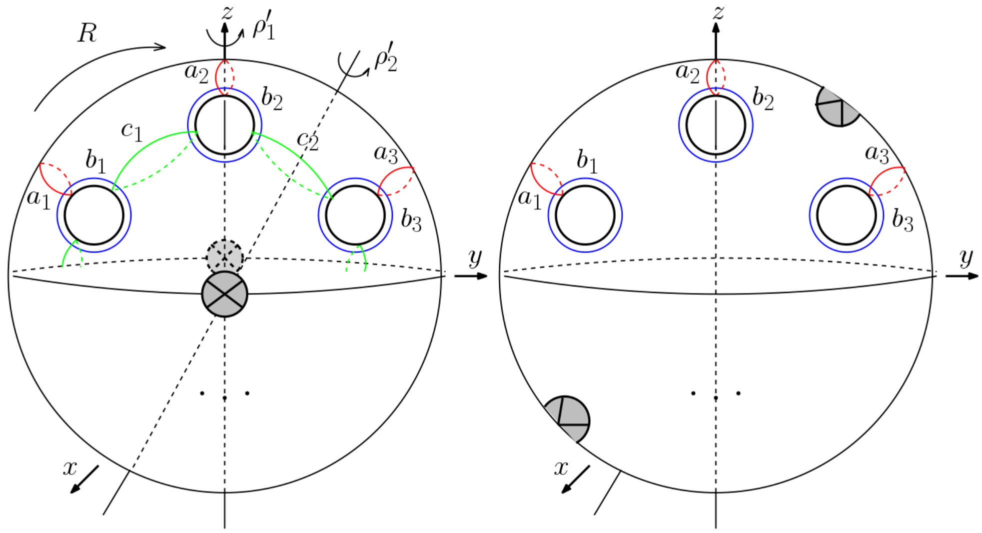

FIGURE 3. The models for N_g if $g = 2r + 2$.

Moreover, the genus r surface minus two disks is embedded in $\mathbb{R}^3$ in such a way that each genus is in a circular position with the second genus on the $+z$-axis and the rotation R by $\frac{2\pi}{r}$ about x-axis maps the curve b_i to b_{i+1} for $i = 1, \ldots, r-1$ and b_r to b_1.

We use the explicit homeomorphism constructed in [15, Section 3] to identify the models in Figure 1 and 3. On the left hand side of Figure 3, one of the crosscaps is centered on the $+x$-axis and the other one is obtained by rotating the first one by π about the z-axis. The model on the right hand side is obtained from the model on the left hand side by sliding crosscaps via a diffeomorphism, say ϕ.

Let τ be the blackboard reflection of N_g for the model in the left hand side in Figure 3. If r is odd, we consider the reflection τ and if r is even, we consider the reflection $\phi\tau\phi^{-1}$. The surface N_g is invariant under the reflections τ and $\phi\tau\phi^{-1}$. Abusing the notation, we keep writing τ instead of $\phi\tau\phi^{-1}$. Note that $D(\tau) = -1$.

Note that the surface N_g is invariant under the two rotations ρ_1' and ρ_2' where ρ_1' is the rotation by π about z-axis and ρ_2' is the rotation by π about the line $z = tan(\frac{\pi}{r})y$, $x = 0$ as in Figure 3. The rotations ρ_1' and ρ_2' satisfy $D(\rho_1') = D(\rho_2') = -1$, which implies that the twist subgroup $\mathcal{T}_g$ does not contain ρ_1' and ρ_2'. Let $\rho_1 = \rho_1'\tau$ and $\rho_2 = \rho_2'\tau$. Then the involutions ρ_1 and ρ_2 are contained in $\mathcal{T}_g$ by Lemma 2.2. Observe that the rotation $R = \rho_2\rho_1$.

3.1. **Generating sets for the twist subgroup $\mathcal{T}_g$.** Recently, Korkmaz [8] introduced new generating sets for the mapping class group of an orientable surface. We follow the outline of his proofs. Especially, since the curves a_i, b_i and c_i are exactly the same as in [8], statements about these curves follows directly from [8]. Before we state our result, let us recall the above mentioned theorem of Korkmaz. Recall that A_i, B_i, C_i, E and F represent the Dehn twists about the corresponding lower case letters in Figure 1 and 3.

Theorem 3.1. [8] *Let Σ_g denote a closed connected oriented surface of genus g. Then, if $g \geq 3$, $\mathrm{Mod}(\Sigma_g)$ is generated by the four elements $R, A_1A_2^{-1}, B_1B_2^{-1}$ and $C_1C_2^{-1}$.*

Using the above theorem, we give a generating set for $\mathcal{T}_g$ when g is even.

Theorem 3.2. *Let $r \geq 3$ and $g = 2r+2$. Then the twist subgroup $\mathcal{T}_g$ is generated by the elements $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, D_r$ and E if $g = 2r+2$.*

Proof. Let G be the subgroup of $\mathcal{T}_g$ generated by the set

$$\{R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, D_r, E\}$$

if $g = 2r+2$.

Let $\mathcal{S}$ denote the set of isotopy classes of two-sided non-separating simple closed curves on N_g. Define a subset $\mathcal{G}$ of $\mathcal{S} \times \mathcal{S}$ as

$$\mathcal{G} = \{(a,b) : AB^{-1} \in G\}.$$

The set $\mathcal{G}$ defines an equivalence relation on $\mathcal{S}$ which satisfies G-invariance property, that is,

$$\text{if } (a,b) \in \mathcal{G} \text{ and } H \in G \text{ then } (H(a), H(b)) \in \mathcal{G}.$$

Then it follows from the proof of Theorem 3.1 that the Dehn twists A_i and B_i for $i = 1, \ldots, r$ are contained in G. Also, G contains C_j for $j = 1, \ldots, r-1$. Since all generators given in Theorem 2.1 are contained in the group G. We conclude that $G = \mathcal{T}_g$. □

3.2. **Involution generators.** We consider the surface N_g where g-crosscaps are distributed on the sphere as in Figure 4. If $g = 2r+2$ and $r \geq 3$, there is a reflection, σ, of the surface N_g in the xy-plane such that

- $\sigma(f) = a_1$, $\sigma(b_r) = d_r$,
- $\sigma(x_2) = x_3$, $\sigma(x_4) = x_5$ $\sigma(x_{g-2}) = x_g$ and
- $\sigma(x_i) = x_i$ if $i = 6, \ldots, g-3$ or $i = 1, g-1$.

with reverse orientation. (Recall that x_i's are the generators of $H_1(N_g;\mathbb{R})$ as shown in Figure 2.)

The linear map D associated to σ satisfies $D(\sigma) = 1$ if g is even.

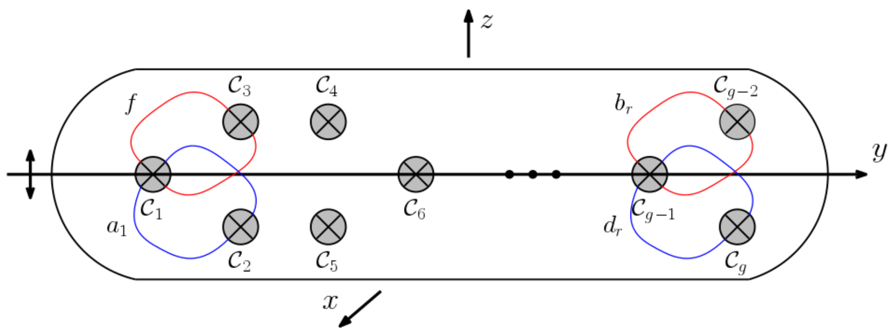

FIGURE 4. The involution σ if $g = 2r + 2$.

This implies that the involution σ is contained in $\mathcal{T}_g$ if g is even.

Theorem 3.3. *The twist subgroup $\mathcal{T}_{12}$ is generated by the involutions $\rho_1, \rho_2, \rho_1 A_1 B_2 C_4 A_3$ and σ.*

Proof. Consider the surface N_{12} as in Figure 3. Since

$$\rho_1(a_1) = a_3, \rho_1(b_2) = b_2 \text{ and } \rho_1(c_4) = c_4,$$

and τ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we get

- $\rho_1 A_1 \rho_1 = A_3^{-1}$,
- $\rho_1 B_2 \rho_1 = B_2^{-1}$ and
- $\rho_1 C_4 \rho_1 = C_4^{-1}$.

It is easy to verify that $\rho_1 A_1 B_2 C_4 A_3$ is an involution. Let $E_1 = A_1 B_2 C_4 A_3$ and let H be the subgroup of $\mathcal{T}_{12}$ generated by the set

$$\{\rho_1, \rho_2, \rho_1 E_1, \sigma\}.$$

Note that the rotation R is in the subgroup H. By Theorem 3.2, we need to show that the elements $A_1 A_2^{-1}, B_1 B_2^{-1}, C_1 C_2^{-1}, D_5$ and E are contained in H.

Let $E_2 = R E_1 R^{-1} = A_2 B_3 C_5 A_4$. It can be easily shown that

$$E_2 E_1(a_2, b_3, c_5, a_4) = (b_2, a_3, c_5, a_4),$$

so that $E_3 = B_2 A_3 C_5 A_4$ is in H.

Let

$$E_4 = R^2 E_1 R^{-2} = A_3 B_4 C_1 A_5.$$

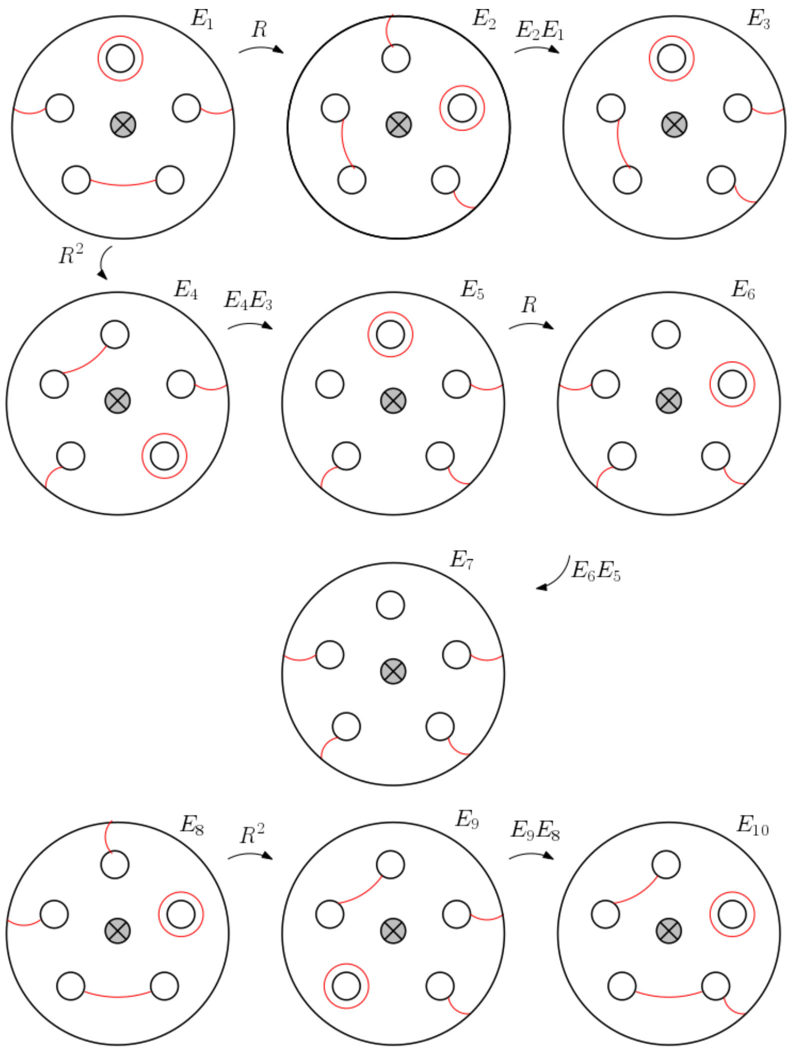

FIGURE 5. The proof of Theorem 3.3.

It is easy to show that

$$E_4E_3(a_3, b_4, c_1, a_5) = (a_3, a_4, b_2, a_5)$$

so that $E_5 = A_3A_4B_2A_5$ are contained in H. Hence,

$$E_5E_3^{-1} = A_5C_5^{-1} \in H.$$

One can easily see that the elements $A_iC_i^{-1}$ are contained in H by conjugating $A_5C_5^{-1}$ with powers of R.
Let

$$E_6 = RE_5R^{-1} = A_4A_5B_3A_1.$$

One can easily show that

$$E_6E_5(a_4, a_5, b_3, a_1) = (a_4, a_5, a_3, a_1)$$

so that $E_7 = A_4A_5A_3A_1$ is in H. Therefore,

$$E_7E_6^{-1} = A_3B_3^{-1} \in H.$$

By conjugating with powers of R, we get $A_iB_i^{-1} \in H$. Hence,

$$B_5C_5^{-1} = (B_5A_5^{-1})(A_5C_5^{-1}) \in H.$$

Again by conjugating with powers of R, the elements $B_iC_i^{-1}$ are contained in H.
Let

$$E_8 = (A_2B_2^{-1})(B_3A_3^{-1})E_1 = A_1A_2C_4B_3$$

and

$$E_9 = R^2F_8R^{-2} = A_3A_4C_1B_5.$$

It can also be shown that

$$E_9E_8(a_3, a_4, c_1, b_5) = (b_3, a_4, c_1, c_4)$$

so that $E_{10} = B_3A_4C_1C_4$. Hence,

$$E_9E_{10}^{-1}B_3A_3^{-1} = B_5C_4^{-1} \in H.$$

The conjugation of this with powers of R implies that $B_{i+1}C_i^{-1} \in H$. Hence

- $A_1A_2^{-1} = (A_1C_1^{-1})(C_1B_2^{-1})(B_2A_2^{-1})$,
- $B_1B_2^{-1} = (B_1C_1^{-1})(C_1B_2^{-1})$ and
- $C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1})$

are contained in H. Also it follows from the fact that

$$\sigma(a_1) = f \text{ and } \sigma(b_5) = d_5$$

with a choice of orientations of regular neighbourhoods of the curves, the element D_5 and F are contained in H. By the fact that $A_1(f) = e$, E is in H. We conclude that $H = T_{12}$. □

Theorem 3.4. *For $g = 2r + 2$, the twist subgroup $\mathcal{T}_g$ is generated by the involutions $\rho_1, \rho_2, \rho_1A_1B_2C_{\frac{r+3}{2}}A_3$ and σ if $r \geq 7$ and odd.*

Proof. Consider the surface N_g as in Figure 3. We have

$$\rho_1(a_1) = a_3, \rho_1(b_2) = b_2 \text{ and } \rho_1(c_{\frac{r+3}{2}}) = c_{\frac{r+3}{2}}.$$

Since τ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we get

- $\rho_1A_1\rho_1 = A_3^{-1}$

- $\rho_1 B_2 \rho_1 = B_2^{-1}$ and
- $\rho_1 C_{\frac{r+3}{2}} \rho_1 = C_{\frac{r+3}{2}}^{-1}$.

It can be shown that $\rho_1 A_1 B_2 C_{\frac{r+3}{2}} A_3$ is an involution. Let $G_1 = A_1 B_2 C_{\frac{r+3}{2}} A_3$ and let K be the subgroup of $\mathcal{T}_g$ generated by the set

$$\{\rho_1, \rho_2, \rho_1 G_1, \sigma\}.$$

Note that the rotation R is in K. By Theorem 3.2, we need to show that the elements $A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, D_r$ and E are contained in K. It follows from

- $G_2 = RG_1R^{-1} = A_2B_3C_{\frac{r+5}{2}}A_4 \in K$,
- $G_3 = (G_2G_1)G_2(G_2G_1)^{-1} = B_2A_3C_{\frac{r+5}{2}}A_4 \in K$,
- $G_4 = RG_3R^{-1} = B_3A_4C_{\frac{r+7}{2}}A_5 \in K$,
- $G_5 = (G_4G_3)G_4(G_4G_3)^{-1} = A_3A_4C_{\frac{r+7}{2}}A_5 \in K$

that

$$G_4G_5^{-1} = B_3A_3^{-1} \in K.$$

Hence, the elements $B_iA_i^{-1}$ are contained in K by conjugating $B_3A_3^{-1}$ with powers of R. Let

- $G_6 = R^{\frac{r-3}{2}}G_4R^{\frac{3-r}{2}} = B_{\frac{r+3}{2}}A_{\frac{r+5}{2}}C_2A_{\frac{r+7}{2}} \in K$,
- $G_7 = (G_6G_4)G_6(G_6G_4)^{-1} = B_{\frac{r+3}{2}}A_{\frac{r+5}{2}}B_3A_{\frac{r+7}{2}} \in K$ if $r > 7$, ($G_7 = A_5A_6B_3A_7 \in K$ if $g = 7$).

Then

$$G_7G_6^{-1} = B_3C_2^{-1} \in K \text{ if } r > 7$$

and

$$G_7G_6^{-1}B_5A_5^{-1} = B_3C_2^{-1} \in K \text{ if } r = 7.$$

Therefore, the elements $B_{i+1}C_i^{-1}$ are contained in the group K by conjugating $B_3C_2^{-1}$ with powers of R. Let

- $G_8 = R^{\frac{r-1}{2}}G_4R^{\frac{1-r}{2}} = B_{\frac{r+5}{2}}A_{\frac{r+7}{2}}C_3A_{\frac{r+9}{2}} \in K$ if $r > 7$, ($G_8 = B_6A_7C_3A_1 \in K$ if $r = 7$),
- $G_9 = (G_8G_4)G_8(G_8G_4)^{-1} = B_{\frac{r+5}{2}}A_{\frac{r+7}{2}}B_3A_{\frac{r+9}{2}} \in K$ if $r > 7$. ($G_9 = B_6A_7B_3A_1 \in K$ if $r = 7$).

Then

$$G_9G_8^{-1} = B_3C_3^{-1} \in K \text{ if } r \geq 7.$$

This implies that the subgroup K contains $B_iC_i^{-1}$ by conjugating $B_3C_3^{-1}$ with powers of R. The rest of the proof is very similar to the proof of Theorem 3.3. ∎

Theorem 3.5. *For $g = 2r + 2$, the twist subgroup $\mathcal{T}_g$ is generated by the involutions $\rho_1, \rho_2, \rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}}$ and σ if $r \geq 6$ and even.*

Proof. Consider the surface N_g as in Figure 3. The involution ρ_1 satisfies

$$\rho_1(a_2) = a_2, \rho_1(b_{\frac{r+4}{2}}) = b_{\frac{r+4}{2}} \text{ and } \rho_1(c_{\frac{r}{2}}) = c_{\frac{r+6}{2}}.$$

Since τ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have

- $\rho_1 A_2 \rho_1 = A_2^{-1}$
- $\rho_1 B_{\frac{r+4}{2}} \rho_1 = B_{\frac{r+4}{2}}^{-1}$ and
- $\rho_1 C_{\frac{r}{2}} \rho_1 = C_{\frac{r+6}{2}}^{-1}$.

It can be shown that $\rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}}$ is an involution. Let $H_1 = A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}}$ and let K be the subgroup of $\mathcal{T}_g$ generated by the set

$$\{\rho_1, \rho_2, \rho_1 H_1, \sigma\}.$$

Note that the rotation R is in K. By Theorem 3.2, we need to show that the elements $A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, D_r$ and E are contained in K. Let

- $H_2 = RH_1R^{-1} = A_3 C_{\frac{r+2}{2}} B_{\frac{r+6}{2}} C_{\frac{r+8}{2}} \in K$,
- $H_3 = (H_2H_1)H_2(H_2H_1)^{-1} = A_3 B_{\frac{r+4}{2}} C_{\frac{r+6}{2}} C_{\frac{r+8}{2}} \in K$,
- $H_4 = RH_3R^{-1} = A_4 B_{\frac{r+6}{2}} C_{\frac{r+8}{2}} C_{\frac{r+10}{2}} \in K$,
- $H_5 = (H_4H_3)H_4(H_4H_3)^{-1} = A_4 C_{\frac{r+6}{2}} C_{\frac{r+8}{2}} C_{\frac{r+10}{2}} \in K$.

Then, we get

$$H_4H_5^{-1} = B_{\frac{r+6}{2}} C_{\frac{r+6}{2}}^{-1} \in K$$

and

$$H_2H_3^{-1}\Big(C_{\frac{r+6}{2}} B_{\frac{r+6}{2}}^{-1}\Big) = C_{\frac{r+2}{2}} B_{\frac{r+4}{2}}^{-1} \in K.$$

By conjugating the elements $B_{\frac{r+6}{2}} C_{\frac{r+6}{2}}^{-1}$ and $C_{\frac{r+2}{2}} B_{\frac{r+4}{2}}^{-1}$ with powers of R, we conclude that $B_iC_i^{-1}$ and $C_iB_{i+1}^{-1}$ are contained in K.
Let

- $H_6 = (B_{\frac{r+6}{2}} C_{\frac{r+6}{2}}^{-1})(B_{\frac{r}{2}} C_{\frac{r}{2}}^{-1})H_1 = B_{\frac{r+6}{2}} B_{\frac{r}{2}} A_2 B_{\frac{r+4}{2}} \in K$,
- $H_7 = R^{\frac{r-4}{2}} H_6 R^{\frac{4-r}{2}} = A_{\frac{r}{2}} B_{r-2} B_r B_1 \in K$,
- $H_8 = (H_7H_6)H_7(H_7H_6)^{-1} = B_{\frac{r}{2}} B_{r-2} B_r B_1 \in K$.

Then

$$H_8H_7^{-1} = B_{\frac{r}{2}} A_{\frac{r}{2}}^{-1} \in K.$$

By conjugating with powers of R, K contains $B_iA_i^{-1}$. The rest of the proof is very similar to the proof of Theorem 3.3. □

In the rest of this section, we introduce involution generators for $\mathcal{T}_g$ for $g = 6, 8$ and 10.

We consider the models for the surface N_{10}, where 10-crosscaps are

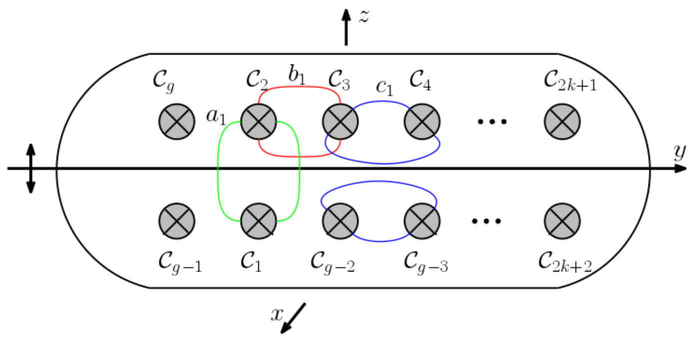

FIGURE 6. The involution δ_1 for $g = 4k + 2$.

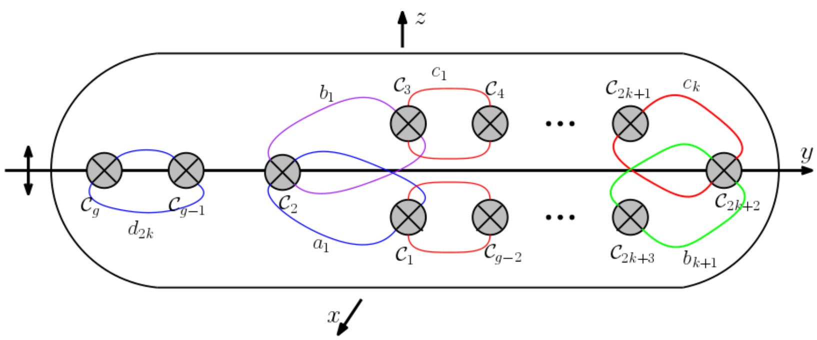

FIGURE 7. The involution δ_2 for $g = 4k + 2$.

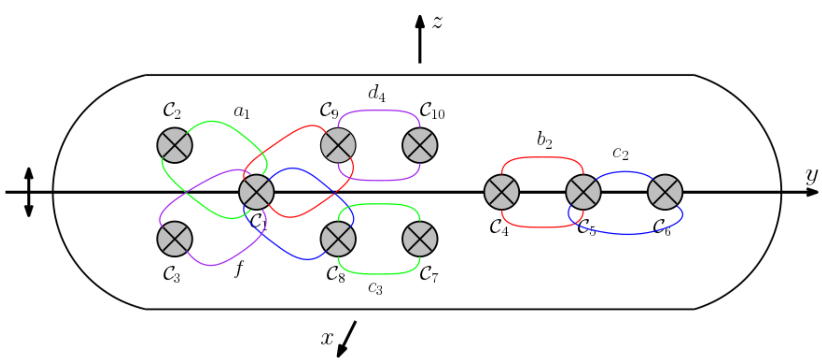

FIGURE 8. The involution δ_3 for $g = 10$.

distributed on the sphere as in Figure 6, 7 and 8. There are reflections, δ_1, δ_2 and δ_3, of the surface N_{10} in the xy-plane such that

- $\delta_1(x_i) = x_{i+1}$ if $i = 1, 5, 9$,
  $\delta_1(x_3) = x_8$, $\delta_1(x_4) = x_7$,

- $\delta_2(x_i) = x_i$ if $i = 2, 6, 9, 10$,
  $\delta_2(x_1) = x_3$, $\delta_2(x_4) = x_8$, $\delta_2(x_5) = x_7$ and
- $\delta_3(x_i) = x_i$ if $i = 1, 4, 5, 6$,
  $\delta_3(x_2) = x_3$, $\delta_3(x_8) = x_9$, $\delta_3(x_7) = x_{10}$.

Recall that x_i's are the generators of $H_1(N_g; \mathbb{R})$ as shown in Figure 2. Note that the involutions δ_1, δ_2 and δ_3 reverse the orientation of a neighbourhood of a two-sided simple closed curve. Since $D(\delta_i) = 1$, the involutions δ_i are in $\mathcal{T}_{10}$ for $i = 1, 2, 3$.

Theorem 3.6. *The twist subgroup $\mathcal{T}_{10}$ is generated by five involutions $\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \delta_3$.*

Proof. Let K be the subgroup of $\mathcal{T}_{10}$ generated by the set

$$\{\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \delta_3\}.$$

It is clear that $\delta_2\delta_1\delta_2A_2$ and δ_1A_1 are involutions. It follows from

- $A_1 = \delta_1(\delta_1A_1)$ and
- $A_2 = (\delta_2\delta_1\delta_2)(\delta_2\delta_1\delta_2A_2)$

that the elements A_1 and A_2 are in K. Also, It follows from

- $\delta_2(a_1) = b_1$
- $\delta_2\delta_1(b_i) = c_i$ for $i = 1, 2, 3$ and
- $\delta_2\delta_1(c_i) = b_{i+1}$ for $i = 1, 2$

that B_i, C_i are contained in K for $i = 1, 2, 3$. Moreover, since

- $\delta_3(c_3) = d_4$,
- $\delta_1\delta_2\delta_3\delta_1\delta_3(c_1) = b_4$ and
- $A_1\delta_3(a_1) = e$

then the elements D_4, B_4 and E are in K. We conclude that $K = \mathcal{T}_{10}$ by Theorem 2.1. □

We consider the models for the surface N_8, where 8-crosscaps are distributed on the sphere as in Figure 9, 10 and 11. There are reflections, λ_1, λ_2 and λ_3, of the surface N_8 in the xy-plane such that

- $\lambda_1(x_i) = x_i$ if $i = 7, 8$,
  $\lambda_1(x_i) = x_{i+1}$ if $i = 1, 4$ and $\lambda_1(x_3) = x_6$,
- $\lambda_2(x_i) = x_i$ if $i = 2, 5$,
  $\lambda_2(x_1) = x_3$, $\lambda_2(x_4) = x_6$, $\lambda_2(x_7) = x_8$, and
- $\lambda_3(x_i) = x_i$ if $i = 1, 4$,
  $\lambda_3(x_2) = x_3$, $\lambda_3(x_5) = x_8$ and $\lambda_3(x_6) = x_7$.

Note that the involutions λ_i reverse the orientation of a neighbourhood of a two-sided simple closed curve for $i = 1, 2, 3$. Since $D(\delta_i) = 1$, the involutions δ_i are contained in $\mathcal{T}_8$ for $i = 1, 2, 3$.

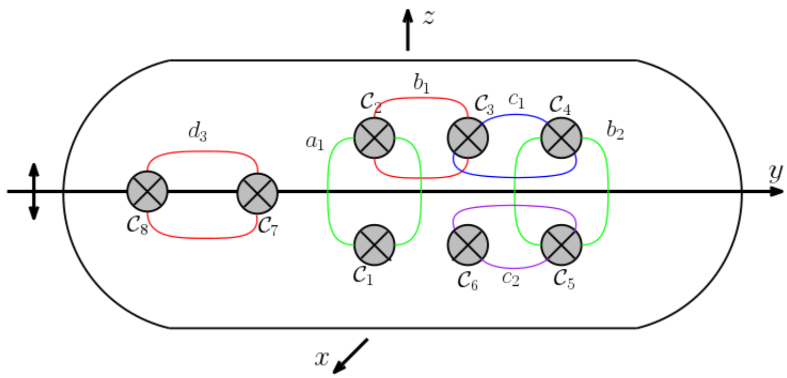

FIGURE 9. The involution λ_1 for $g = 8$.

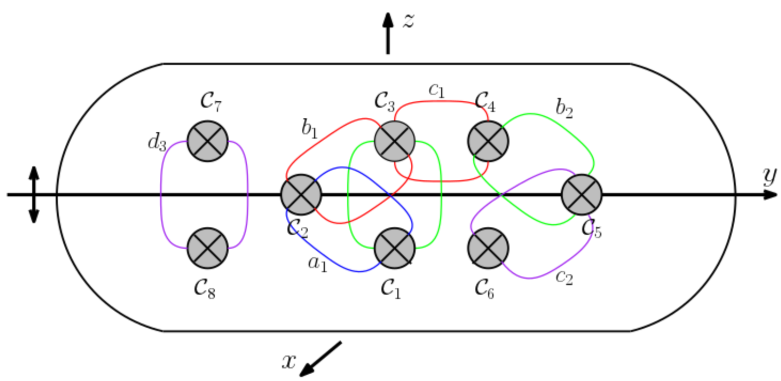

FIGURE 10. The involution λ_2 for $g = 8$.

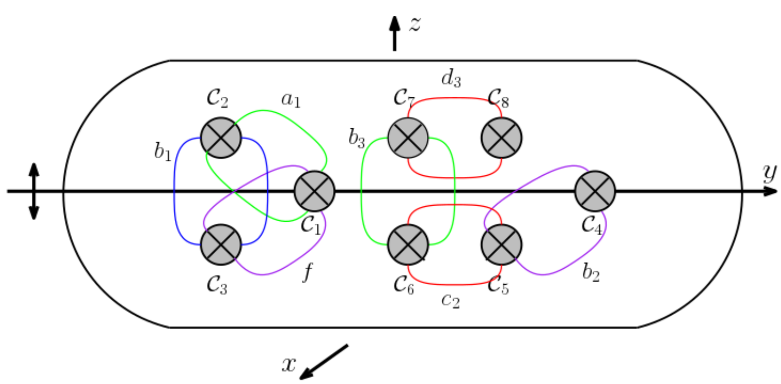

FIGURE 11. The involution λ_3 for $g = 8$.

Theorem 3.7. *The twist subgroup $\mathcal{T}_8$ is generated by five involutions $\lambda_1, \lambda_2, \lambda_2\lambda_1\lambda_2 A_2, \lambda_1 A_1$ and λ_3.*

Proof. Let K be the subgroup of $\mathcal{T}_8$ generated by the set

$$\{\lambda_1, \lambda_2, \lambda_2\lambda_1\lambda_2 A_2, \lambda_1 A_1, \lambda_3\}.$$

It is clear that $\lambda_2\lambda_1\lambda_2 A_2$ and $\lambda_1 A_1$ are involutions. It follows from

- $A_1 = \lambda_1(\lambda_1 A_1)$ and
- $A_2 = (\lambda_2\lambda_1\lambda_2)(\lambda_2\lambda_1\lambda_2 A_2)$

that the elements A_1 and A_2 are in K. Also, It follows from

- $\lambda_2\lambda_1(a_1) = b_1$,
- $\lambda_2\lambda_1(b_i) = c_i$ for $i = 1, 2$,
- $\lambda_2\lambda_1(c_1) = b_2$,
- $\lambda_3(c_2) = d_3$,
- $\lambda_1\lambda_2\lambda_3\lambda_1\lambda_3(c_1) = b_3$ and
- $A_1\lambda_3(a_1) = e$

that all generators of $\mathcal{T}_8$ given in Theorem 2.1 are contained in K. This completes the proof. □

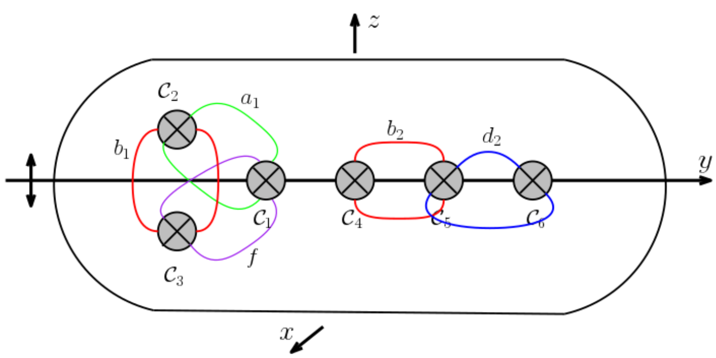

FIGURE 12. The involution ξ_1 for $g = 6$.

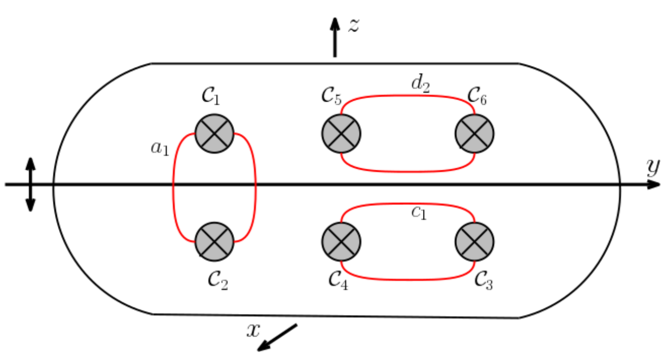

FIGURE 13. The involution ξ_2 for $g = 6$.

We consider the models for the surface N_6, where 6-crosscaps are distributed on the sphere as in Figure 6, 7, 12 and 13. There are reflections $\delta_1, \delta_2, \xi_1$ and ξ_2 such that

- $\delta_1(x_i) = x_{i+1}$ if $i = 1, 3, 5$,
- $\delta_2(x_i) = x_i$ if $i \neq 1, 3$ and $\delta_2(x_1) = x_3$,
- $\xi_1(x_i) = x_i$ if $i \neq 2, 3$ and $\xi_1(x_2) = x_3$ and
- $\xi_2(x_i) = x_{i+1}$ if $i = 1, 4$ and $\xi_2(x_3) = x_6$.

Note that the involutions δ_i and ξ_i reverse the orientation of a neighbourhood of a two-sided simple closed curve for $i = 1, 2$. We obtain that $D(\delta_i) = D(\xi_i) = 1$, the twist subgroup $\mathcal{T}_8$ contains the involutions δ_i and ξ_i for $i = 1, 2$.

Theorem 3.8. *The twist subgroup $\mathcal{T}_6$ is generated by six involutions $\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \xi_1$ and ξ_2.*

Proof. Let K be the subgroup of $\mathcal{T}_6$ generated by the set

$$\{\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \xi_1, \xi_2\}.$$

It is clear that $\delta_2\delta_1\delta_2A_2$ and δ_1A_1 are involutions. It follows from

- $A_1 = \delta_1(\delta_1A_1)$ and
- $A_2 = (\delta_2\delta_1\delta_2)(\delta_2\delta_1\delta_2A_2)$

that the elements A_1 and A_2 are in K. Also, It follows from

- $\delta_2(a_1) = b_1$,
- $\delta_2\delta_1(b_1) = c_1$,
- $\delta_1\delta_2\xi_2(b_1) = b_2$,
- $\xi_2(c_1) = d_2$ and
- $A_1\xi_1(a_1) = e$

that all generators of $\mathcal{T}_6$ given in Theorem 2.1 are contained in K. This completes the proof. □

4. The odd case

For $g = 4k + 1$, we work with two models for N_g: one is on the left hand side of Figure 14, the other one is depicted in Figure 15. The model in Figure 14 is the nonorientable surface obtained from $\mathbb{S}^2$ embedded in $\mathbb{R}^3$ and by deleting the interiors of g-disjoint disks and identifying the antipodal points on the boundary of each removed disks, say $\mathcal{C}_i$. Moreover, each crosscap $\mathcal{C}_i$ is in a circular position with the second crosscap $\mathcal{C}_2$ on the $+z$-axis and the rotation T by $\frac{2\pi}{g}$ about x-axis maps the crosscap C_i to C_{i+1}. The model in Figure 15 is obtained from a genus r orientable surface by deleting the interior of a disk and identifying the antipodal points on the boundary. Moreover, the genus r surface minus a disk is embedded in $\mathbb{R}^3$ in such a way that

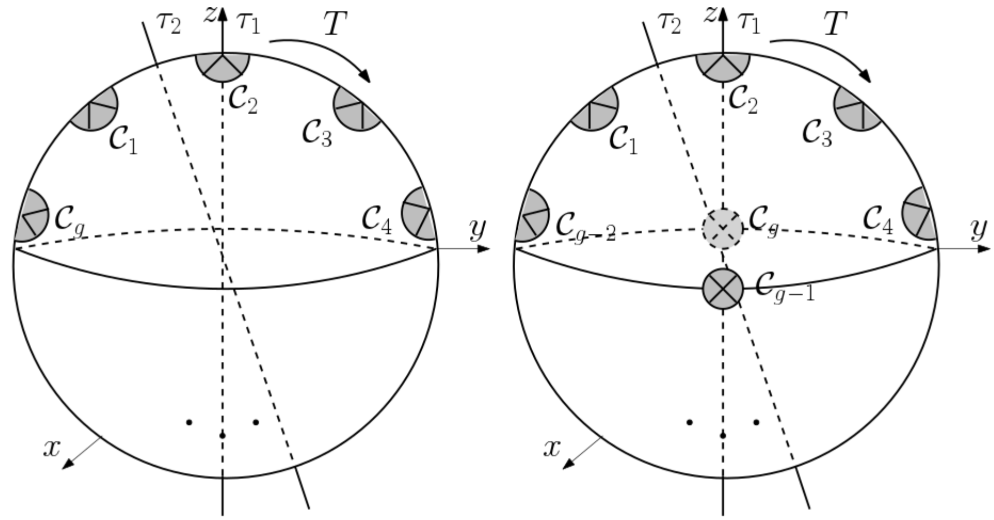

FIGURE 14. The involutions τ_1 and τ_2.

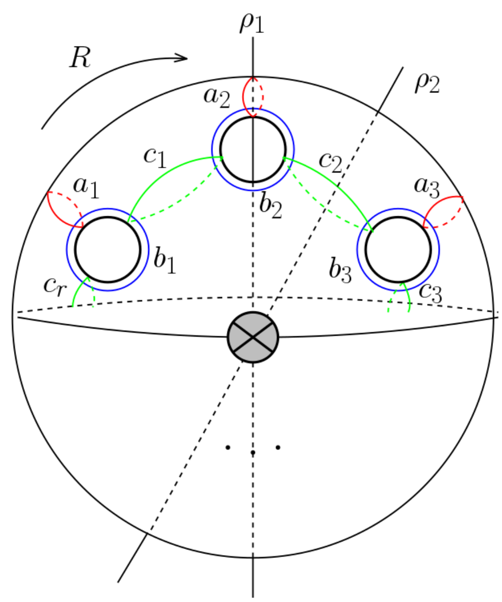

FIGURE 15. The involutions ρ_1 and ρ_2 for $g = 2r + 1$.

each genus is in a circular position with the second genus on the $+z$-axis and the rotation R by $\frac{2\pi}{r}$ about x-axis maps the curve b_i to b_{i+1} for $i = 1, \dots, r-1$ and b_r to b_1.

We use the explicit homeomorphism constructed in [15, Section 3] to identify the models in Figure 1 and Figure 15. In Figure 15, one crosscap is on the $+x$-axis. Note that the surface N_g is invariant under the two involutions ρ_1 and ρ_2 where ρ_1 is the reflection in the xz-plane and ρ_2 is the reflection in the plane $z = tan(\frac{\pi}{r})y$ as in Figure 15. The rotations ρ_1 and ρ_2 satisfy $D(\rho_1) = D(\rho_2) = 1$ if $g = 4k + 1$. In this case, the twist subgroup $\mathcal{T}_g$ contains ρ_1 and ρ_2. Observe that the rotation $R = \rho_2\rho_1$.

For $g = 4k + 3$, we work with the model on the right hand side of Figure 14. This surface is a genus-g nonorientable surface obtained from $\mathbb{S}^2$ embedded in $\mathbb{R}^3$ and by deleting the interiors of g-disjoint disks and identifying the antipodal points on the boundary of each removed disks, say $\mathcal{C}_i$. Moreover, each crosscap $\mathcal{C}_i$ for $i = 1, \ldots, g-2$ is in a circular position with the second crosscap $\mathcal{C}_2$ on the $+z$-axis, the rotation T by $\frac{2\pi}{g-2}$ about x-axis maps the crosscap $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$ for $i = 1, \ldots, g-3$. The crosscap $\mathcal{C}_{g-1}$ is on the $+x$-axis and $\mathcal{C}_g$ is obtained by rotating $\mathcal{C}_{g-1}$ by π about $+z$-axis. Note that the surface N_g is invariant under the two reflections τ_1 and τ_2 where τ_1 is the reflection in the z-axis and τ_2 is the reflection in the plane $z = tan(\frac{\pi}{r})y$ as in Figure 14. The reflections τ_1 and τ_2 satisfy $D(\tau_1) = D(\tau_2) = 1$ if r is even, which implies that τ_1 and τ_2 are contained in the twist subgroup $\mathcal{T}_g$.

Recall that in Theorem 3.2 we give a generating set for $\mathcal{T}_g$ when g is even. We have the following generators when g is odd.

Theorem 4.1. *Let $r \geq 3$ and $g = 2r + 1$. Then the twist subgroup $\mathcal{T}_g$ is generated by the elements $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}$ and E.*

Proof. Let G be the subgroup of $\mathcal{T}_g$ generated by the set

$$\{R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, E\}$$

if $g = 2r + 1$. Let $\mathcal{S}$ denote the set of isotopy classes of two-sided non-separating simple closed curves on N_g. Define a subset $\mathcal{G}$ of $\mathcal{S} \times \mathcal{S}$ as

$$\mathcal{G} = \{(a, b) : AB^{-1} \in G\}.$$

The set $\mathcal{G}$ defines an equivalence relation on $\mathcal{S}$ which satisfies G-invariance property, that is,

$$\text{if } (a, b) \in \mathcal{G} \text{ and } H \in G \text{ then } (H(a), H(b)) \in \mathcal{G}.$$

Then it follows from the proof of Theorem 3.1 that the Dehn twists A_i and B_i for $i = 1, \ldots, r$ are contained in G. Also, G contains C_j for $j = 1, \ldots, r-1$. Since all generators given in Theorem 2.1 are contained in the group G. We conclude that $G = \mathcal{T}_g$. □

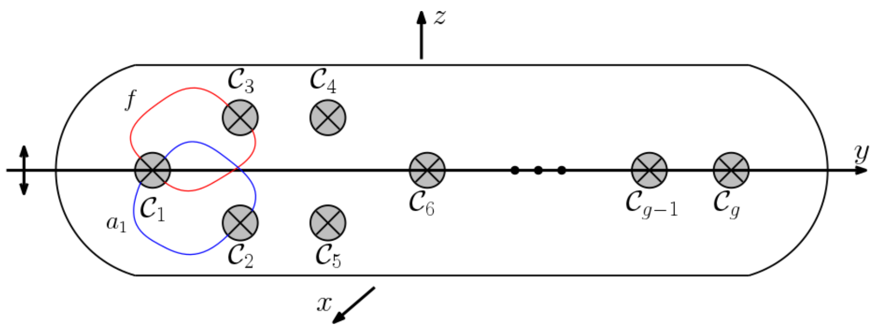

FIGURE 16. The involution β for $g = 2r + 1$.

Let $g = 2r + 1$ and consider the surface N_g, where g-crosscaps are distributed on $\mathbb{S}^2$ as in Figure 16. First, we introduce a reflection β on N_g in the xy-plane such that

- $\beta(a_1) = f$,
- $\beta(x_2) = x_3$, $\beta(x_4) = x_5$ and
- $\beta(x_1) = x_1$, $\beta(x_i) = x_i$ for $i = 6, 7, \ldots, g$.

The involution β reverses the orientation of a neighbourhood of a two-sided simple closed curve. It satisfies $D(\beta) = 1$ and hence β is an element of $\mathcal{T}_g$.
For the remaining generators of the following theorem we refer to Figures 15 and 16.

Theorem 4.2. *For $g = 4k + 1$ and $k \geq 3$, the twist subgroup $\mathcal{T}_g$ is generated by the four involutions $\rho_1, \rho_2, \rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}}$ and β, where $r = 2k$.*

Proof. Consider the surface N_g as in Figure 15. The involution ρ_1 satisfies

$$\rho_1(a_2) = a_2, \rho_1(b_{\frac{r+4}{2}}) = b_{\frac{r+4}{2}} \text{ and } \rho_1(c_{\frac{r}{2}}) = c_{\frac{r+6}{2}}.$$

Since ρ_1 reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have

- $\rho_1 A_2 \rho_1 = A_2^{-1}$
- $\rho_1 B_{\frac{r+4}{2}} \rho_1 = B_{\frac{r+4}{2}}^{-1}$ and
- $\rho_1 C_{\frac{r}{2}} \rho_1 = C_{\frac{r+6}{2}}^{-1}$.

It can be shown that $\rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}}$ is an involution. Let H be the subgroup of $\mathrm{Mod}(N_g)$ generated by the set

$$\{\rho_1, \rho_2, \rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}}, \beta\}.$$

Observe that $R = \rho_1\rho_2 \in K$. By the proof of Theorem 3.5, the elements $A_1A_2^{-1}, B_1B_2^{-1}$ and , $C_1C_2^{-1}$ belong to H. Since $A_1\beta(a_1) = e$, the element E is in H. We conclude that $\mathcal{T}_g = H$ by Theorem 4.1. □

Although we give a generating set of 4 involutions, for completeness of the applications of our method, first we give the following theorem.

Theorem 4.3. *For $g = 4k + 1$ and $k \geq 1$, the twist subgroup $\mathcal{T}_g$ is generated by the five involutions τ_1, τ_2, $\tau_1\tau_2\tau_1A_2$, τ_2A_1 and β.*

Proof. Let K be the subgroup of $\mathcal{T}_g$ generated by the set

$$\{\tau_1, \tau_2, \tau_1\tau_2\tau_1A_2, \tau_2A_1, \beta\}.$$

Note that the rotation $T = \tau_1\tau_2$ is contained in K. It follows from

- $A_1 = \tau_2\tau_2A_1$ and
- $A_2 = (\tau_1\tau_2)\tau_1(\tau_1\tau_2\tau_1)A_2$

that the elements A_1 and A_2 are in K. By conjugating A_1 with powers of T, $\mathcal{T}_g$ contains the elements B_i and C_i. Moreover, it follows from $\beta(a_1) = f$ that the element F is in K. Since $A_1(f) = e$, we get $E \in K$. This finishes the proof by Theorem 2.1. □

In the next theorem, we present four involutions to generate particularly $\mathcal{T}_5$ and $\mathcal{T}_9$. This completes the case $g = 4k+1$ and $k \geq 1$. First, recall that A_1, A_2, B_1, B_2, C_1 and E generate $\mathcal{T}_5$ and $A_1, A_2, B_1, B_2, B_3, B_4$, C_1, C_2, C_3 and E generate $\mathcal{T}_9$. We use the following three involutions $\gamma, S\gamma$ and $S^{2k-2}(S\gamma)S^{2-2k}A_2$ of the generating set given in [18, Theorem 5]. The involution γ is defined as the reflection in the xz-plane where the crosscaps are distributed along the equator on $\mathbb{S}^2$. The map S is defined as the composition $B_{2k}C_{2k-1}B_{2k-1}\cdots C_1B_1A_1$. Note that $D(\gamma) = D(S\gamma) = D(S^{2k-2}(S\gamma)S^{2-2k}A_2) = 1$.

Theorem 4.4. *The twist subgroups $\mathcal{T}_5$ and $\mathcal{T}_9$ can be generated by the involutions $\gamma, S\gamma, S^{2k-2}(S\gamma)S^{2-2k}A_2$ and β for $k = 1, 2$.*

Proof. The generator A_1 can be obtained by S and A_2 [7, Theorem 5]. By conjugating with powers of S, it is easy to see that the elements B_i and C_i belong to $\mathcal{T}_g$. Also, the generator E is contained in $\mathcal{T}_g$ since $A_1\beta(a_1) = e$. □

Now, let $g = 4k + 3$ and consider N_g, where g-crosscaps are distributed over $\mathbb{S}^2$ as in Figure 17. The surface N_g is symmetrical in the xy-plane. Let μ be the reflection in the xy-plane. Note that the linear map associated to the involution μ satisfies $D(\mu) = 1$ if $k \geq 2$. Therefore, the involution μ is in $\mathcal{T}_g$ for $k \geq 2$.

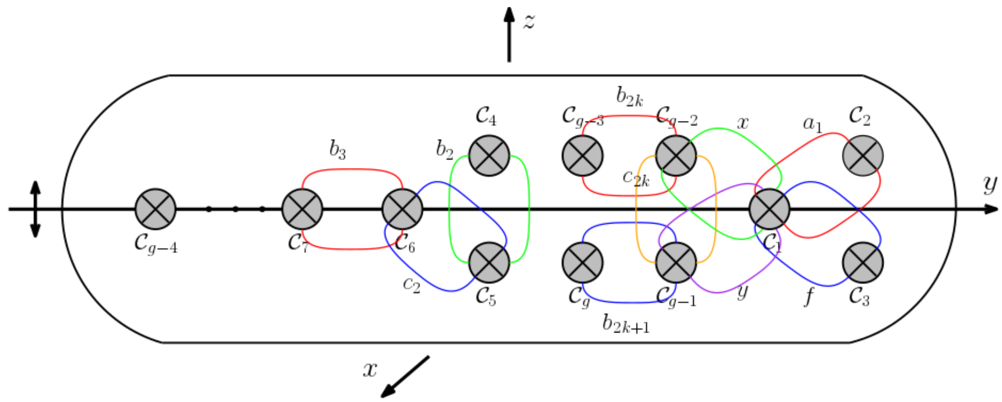

FIGURE 17. The involution μ for $g = 4k + 3$.

Theorem 4.5. *For $g = 4k + 3$ and $k \geq 2$, the twist subgroup $\mathcal{T}_g$ is generated by the five involutions τ_1, τ_2, $\tau_1\tau_2\tau_1 A_2$, $\tau_2 A_1$ and μ.*

Proof. Let K be the subgroup of $\mathcal{T}_g$ generated by the set

$$\{\tau_1, \tau_2, \tau_1\tau_2\tau_1 A_2, \tau_2 A_1, \mu\}.$$

Note that the rotation $T = \tau_1\tau_2$ is contained in K. It follows from

- $A_1 = \tau_2(\tau_2 A_1)$ and
- $A_2 = (\tau_1\tau_2\tau_1)(\tau_1\tau_2\tau_1 A_2)$

that the elements A_1 and A_2 are in K. By conjugating A_1 with powers of T, $\mathcal{T}_g$ contains the elements B_i for $i = 1, \ldots, 2k$ and C_j for $j = 1, \ldots, 2k - 1$.
Let $T(b_{2k}) = x$ and $\mu(x) = y$. Then the elements X and Y are contained in K by the fact that B_{2k} is in K.
It follows from

- $T^{-1}(y) = c_{2k}$,
- $\mu(b_{2k}) = b_{2k+1}$

that C_{2k} and B_{2k+1} are contained in K. This completes the proof by Theorem 2.1. □

For the surface N_7, we introduce two involutions, σ_1 and σ_2, shown in Figure 18 and 19. In these figures, the surface is symmetric with respect to the xy-plane. Both σ_1 and σ_2 are reflections in the xy-plane and $D(\sigma_1) = D(\sigma_2) = 1$. Hence, both σ_1 and σ_2 belong to $\mathcal{T}_7$. For the remaining generators in the following theorem we refer to the model on the right hand side of Figure 14.

Theorem 4.6. *The twist subgroup $\mathcal{T}_7$ is generated by the six involutions τ_1, τ_2, $\tau_1\tau_2\tau_1 A_2$, $\tau_2 A_1$, σ_1 and σ_2.*

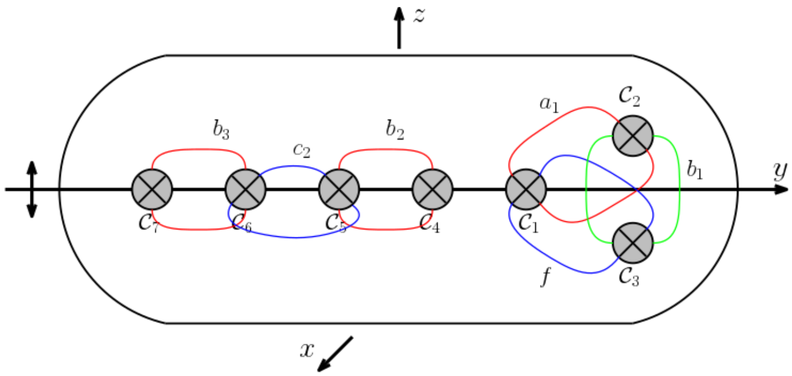

FIGURE 18. The involution σ_1 for $g = 7$.

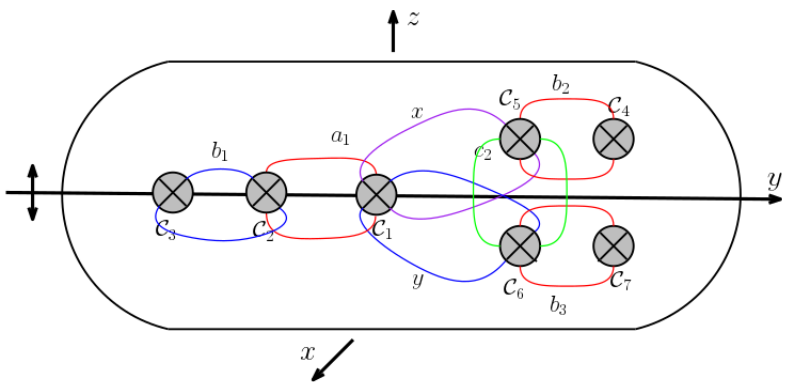

FIGURE 19. The involution σ_2 for $g = 7$.

Proof. Let K be the subgroup of $\mathcal{T}_g$ generated by the set

$$\{\tau_1, \tau_2, \tau_1\tau_2\tau_1 A_2, \tau_2 A_1, \sigma_1, \sigma_2\}.$$

Note that the rotation $T = \tau_1\tau_2$ is contained in K. It follows from

- $A_1 = \tau_2(\tau_2 A_1)$ and
- $A_2 = (\tau_1\tau_2\tau_1)(\tau_1\tau_2\tau_1 A_2)$

that the elements A_1 and A_2 are in K. By conjugating A_1 with powers of T, $\mathcal{T}_g$ contains the elements B_1, C_1 and B_2.
Let $T(b_2) = x$ and $\sigma_2(x) = y$. Then the elements X and Y are contained in K. It follows from

- $T^{-1}(y) = c_2$,
- $\sigma_2(b_2) = b_3$

that C_2 and B_3 are contained in K. Moreover, since $A_1\sigma_1(a_1) = e$, $E \in K$, which completes the proof by Theorem 2.1. □

References

[1] T. E. Brendle, B. Farb: *Every mapping class group is generated by* 6 *involutions*, J. of Algebra **278**, (1) (2004), 187-198.

[2] D. R. J. Chillingworth: *A finite set of generators for the homeotopy group of a non-orientable surface*, Proc. Cambridge Philos. Soc. **65**, (2) (1969), 409-430.

[3] M. Dehn: *The group of mapping classes*, In: Papers on Group Theory and Topology. Springer-Verlag, 1987. Translated from the German by J. Stillwell (Die Gruppe der Abbildungsklassen, Acta Math **69**, (1938), 135-206).

[4] X. Du: *The Torsion generating set of the mapping class groups of non-orientable surfaces*, ArXiv math.GT/04883, v3 19Nov2018.

[5] S. Humphries: *Generators for the mapping class group*, In: Topology of LowDimensional Manifolds, Proc. Second Sussex Conf., Chelwood Gate, (1977), Lecture Notes in Math. **722**, (2) (1979), Springer-Verlag, 44-47.

[6] M. Kassabov: *Generating mapping class groups by involutions*, ArXiv math.GT/0311455, v1 25Nov2003.

[7] M. Korkmaz: *Generating the surface mapping class group by two elements*, Trans. Amer. Math. Soc. **367**, (8) (2005), 3299-3310.

[8] M. Korkmaz: *Mapping class group is generated by three involutions*, ArXiv math. GT/1904.08156, v2 14May2019.

[9] W. B. R. Lickorish: *Homeomorphisms of non-orientable two manifolds*, Proc. Cambridge Philos. Soc. **59**, (2) (1963), 307-317.

[10] W. B. R. Lickorish: *A finite set of generators for the homeotopy group of a* 2*-manifold*, Proc. Cambridge Philos. Soc. **60**, (4) (1964), 769-778.

[11] W. B. R. Lickorish: *On the homeomorphisms of a non-orientable surface*, Proc. Cambridge Philos. Soc. **61**, (1) (1965), 61-64.

[12] F. Luo: *Torsion elements in the mapping class group of a surface*, ArXiv math.GT/0004048, v1 8Apr2000.

[13] J. D. McCarthy, A. Papadopoulos: *Involutions in surface mapping class groups*, Enseign. Math. **33**, (2) (1987), 275-290.

[14] G. Omori: *A small generating set for the twist subgroup of the mapping class group of a non-orientable surface by Dehn twists*, Hiroshima Math. J. **48**, (1) (2018), 81-88.

[15] M. Stukow: *The twist subgroup of the mapping class group of a nonorientable surface*, Osaka J. Math. **46**, (3) (2009), 717-738.

[16] M. Stukow: *A finite presentation for the twist subgroup of the mapping class group of a nonorientable surface*, Bull. Korean Math. Soc. **53**, (2) (2016), 601-614.

[17] B. Szepietowski: *Involutions in mapping class groups of non-orientable surfaces*, Collect. Math. **55**, (3) (2004), 253-260.

[18] B. Szepietowski: *The mapping class group of a nonorientable surface is generated by three elements and by four involutions*, Geom. Dedicata **117**, (1) (2006), 1-9.

[19] B. Wajnryb: *Mapping class group of a surface is generated by two elements*, Topology **35**, (2) (1996), 377-383.

[20] O. Yıldız: *Generating mapping class group by two torsion elements*, in preparation.

[21] O. Yıldız: *Generating mapping class group by three involutions*, in preparation.

[22] K. Yoshihara: *Generating twist subgroup of mapping class group of non-orienatable surface by involutions*, ArXiv math.GT/1902.06842, v1 19Sep2019.

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