COEFFICIENTS OF FOLDING POLYNOMIALS ATTACHED TO LIE ALGEBRAS OF RANK TWO

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

MUHAMMED AYDOĞDU

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

JULY 2021

Approval of the thesis:

COEFFICIENTS OF FOLDING POLYNOMIALS ATTACHED TO LIE ALGEBRAS OF RANK TWO

submitted by **MUHAMMED AYDOĞDU** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics** ______________

Assoc. Prof. Dr. Ömer Küçüksakallı
Supervisor, **Mathematics Department, METU** ______________

Examining Committee Members:

Assoc. Prof. Dr. Müfit Sezer
Mathematics Department, Bilkent University ______________

Assoc. Prof. Dr. Ömer Küçüksakallı
Mathematics Department, METU ______________

Assoc. Prof. Dr. İbrahim Ünal
Mathematics Department, METU ______________

Date: 09.07.2021

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname: Muhammed Aydoğdu

Signature :

ABSTRACT

COEFFICIENTS OF FOLDING POLYNOMIALS ATTACHED TO LIE ALGEBRAS OF RANK TWO

Aydoğdu, Muhammed
M.S., Department of Mathematics
Supervisor: Assoc. Prof. Dr. Ömer Küçüksakallı

July 2021, 68 pages

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbf{F}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ forms a root system. Reflections in the hyperplanes orthogonal to the simple roots of $\mathfrak{h}^*$ generate the Weyl group of $\mathfrak{g}$. If h is the generalized cosine function associated with the Weyl group of $\mathfrak{g}$, then for a nonnegative integer k, the generalized Chebyshev polynomial associated with $\mathfrak{g}$ is defined by $P_{\mathfrak{g}}^k(h(\mathbf{x})) = h(k\mathbf{x})$. In this thesis, general formulae for $P_{B_2}^k$ and $P_{G_2}^k$ will be found and some algebraic properties of the coefficients of generalized Chebyshev polynomials attached to Lie algebras of rank two will be investigated.

Keywords: Lie Algebra, Root Systems, Weyl Group, Folding Polynomials.

ÖZ

İKİ BOYUTLU LIE CEBİRLERİ ÜZERİNDEKİ KATLAMA POLİNOMLARININ KATSAYILARI

Aydoğdu, Muhammed
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi: Doç. Dr. Ömer Küçüksakallı

Temmuz 2021 , 68 sayfa

$\mathfrak{g}$, $\mathbf{F}$ cismi üzerinde bir Lie cebiri ve $\mathfrak{h}$, $\mathfrak{g}$'nin bir Cartan alt cebiri olmak üzere, $\mathfrak{h}$'nin dual uzayı $\mathfrak{h}^*$ bir kök sistemi belirtmektedir. $\mathfrak{h}^*$'ın basit köklerine dik olan hiper düzlemlere göre yansımalar, $\mathfrak{g}$'nin Weyl grubunu üretmektedir. h, $\mathfrak{g}$'nin Weyl grubuna karşılık gelen genelleştirilmiş cosinüs fonksiyonu iken, negatif olmayan k tam sayısı için, $\mathfrak{g}$ üzerindeki genelleştirilmiş Chebyshev polinomu, $P_{\mathfrak{g}}^k(h(\mathbf{x})) = h(k\mathbf{x})$ eşitliği ile tanımlanır. Bu tezde $P_{B_2}^k$ ve $P_{G_2}^k$ polinomlarının genel formülleri bulunacak ve iki boyutlu Lie cebirlerine karşılık gelen genelleştirilmiş Chebyshev polinomlarının katsayılarının bir takım cebirsel özellikleri incelenecektir.

Anahtar Kelimeler: Lie Cebir, Kök Sistemleri, Weyl Grup, Katlama Polinomları.

To my beloved mother

ACKNOWLEDGMENTS

First, I would like to express my sincere gratitude to my thesis supervisor Assoc. Prof. Dr. Ömer Küçüksakallı for giving me the opportunity to do research and providing invaluable guidance throughout the process of this thesis. It was a great privilege and honor to work under his supervision.

I also would like to thank Assoc. Prof. Dr. İbrahim Ünal and Assoc. Prof. Dr. Müfit Sezer for being on my thesis committee.

I would like to thank my instructors in the Department of Mathematics at METU. Especially I want to thank Prof. Dr. Yıldıray Ozan and Dr. Muhiddin Uğuz for providing me with a productive work environment. I also want to thank Prof. Dr. Zafer Nurlu, Prof. Dr. Hurşit Önsiper, Prof. Dr. Murat Yurdakul, Assoc. Prof. Dr. Mehmetcik Pamuk and Assoc. Prof. Dr. Ebru Solak who taught me at various stages. Their presence has always been a motivating factor throughout my years of study in the department.

I also would like to thank my teachers Barış Demir for his helpful comments and Zeynep Nur İşler for everything she has taught me throughout my year of study in preparatory school at METU.

I want to thank my whole family for their love and support throughout my life. A very special 'thank you' is for the person who has made me the person I am now: To my beloved mother Dudu Aydoğdu. Words cannot express how grateful I am to her for all the sacrifices that she has made on my behalf.

Last but not least, I consider myself to be extremely fortunate to have my beloved wife Zeynep Aydoğdu by my side. Without her encouragement, patience, and support, this thesis would not be possible. A special word of thanks also goes to her dear family for their sincere appreciation.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

1.1 History and Motivation

The Chebyshev polynomial $T_n(x)$ of the first kind is defined by

$$T_n(x) = \cos(n \arccos x) \tag{1.1}$$

where n is a nonnegative integer, and $0 \leq \arccos x \leq \pi$. [12, Chapter 1].

These polynomials were first introduced by the Russian mathematician P.L.Chebyshev in 1854 in his paper on hinge mechanisms [10]. From that time, many important properties of these polynomials were discovered and they have a wide range of use in mathematics from numerical analysis to differential equations.

These polynomials have several remarkable properties. We will give some of these properties of our interest.

Firstly, these polynomials satisfy the composition property:

$$T_{mn}(x) = T_m(x) \circ T_n(x)$$

for any positive integers m and n. Moreover, together with the monomials x^n these are the only polynomials; up to conjugation, satisfying this property [12, Chapter 4].

Secondly, these polynomials can be thought of as arising from the stretching and folding the interval $[0, \pi]$. More precisely the function $\cos^{-1} \circ T_n \circ \cos$ stretches the interval by a factor of n and returns it to itself by folds at integer multiples of π [5]. Therefore these polynomials have the folding property.

They also satisfy a certain recurrence relation [12]. This can be described as follows:

We are familiar with the following elementary trigonometric formulae:

$$\begin{aligned}
\cos 0\theta &= 1, \\
\cos 1\theta &= \cos\theta, \\
\cos 2\theta &= 2\cos^2\theta - 1, \\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta, \\
\cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1.
\end{aligned}$$

If we combine these identities with Equation 1.1, the first few Chebyshev polynomials are given by

$$\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, \\
T_4(x) &= 8x^4 - 8x^2 + 1.
\end{aligned}$$

However it is not beneficial to work out further $T_n(x)$ by definition. If we combine the definition with the following trigonometric identity

$$\cos n\theta = 2\cos\theta \cdot \cos{(n-1)\theta} - \cos{(n-2)\theta},$$

then we obtain the following recurrence relation:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

where $T_0(x) = 1$ and $T_1(x) = x$.

Moreover we have an explicit expression for $T_n(x)$ given in [12]:

$$T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} x^{n-2k}(1-x^2)^k.$$

It follows from this formula that these polynomials have the following properties:

1. The coefficients of $T_n(x)$ are integers.
2. The sum of the coefficients of $T_n(x)$ is 1.
3. $T_n(-x) = (-1)^n T_n(x)$ and in particular $T_n(-1) = (-1)^n$.

The study of multivariate Chebyshev polynomials is rather new but has a significant place in history. These polynomials were studied first by Bourbaki [1] as certain elements of polynomial algebra attached to certain exponential invariants. They gave the relation between these invariants and the root systems of Lie algebras. They also showed that the invariants of the Weyl group of a root system are isomorphic to a polynomial algebra over integers. Later, Lidl and Wells [13] gave the explicit expression for the multivariate Chebyshev polynomials associated with the root system of A_n. Koornwinder [7] studied general orthogonal polynomials in two variables containing the bivariate Chebyshev polynomials as special cases. Ricci [11] was the first to observe that these polynomials satisfy the composition property. Hoffmann and Withers [5] gave a geometric approach to multivariate Chebyshev polynomials by defining them as folding polynomials. They also showed that foldable figures are in one to one correspondence with the Weyl groups of root systems. Withers [14] studied the dynamics of folding polynomials later on and showed that the function $h_{\mathfrak{g}}^{-1} \circ P_{\mathfrak{g}}^{k} \circ h_{\mathfrak{g}}$ stretches and folds the fundamental region of the root system of a simple complex Lie algebra $\mathfrak{g}$ by a factor of k and returns it to itself by folding. Here $h_{\mathfrak{g}}$ is the generalized cosine function and $P_{\mathfrak{g}}^{k}$ is the k^{th} generalized Chebyshev polynomial attached to $\mathfrak{g}$.

1.2 The Goals of the Thesis

In our thesis, we concentrate on understanding the coefficients of generalized Chebyshev polynomials attached to simple complex Lie algebras of rank 2, namely A_2, B_2, G_2.

In [13], the general formula for $P_{A_2}^{k}$ is given by Lidl and Wells. In this thesis, we find the general formula for $P_{B_2}^{k}$ and $P_{G_2}^{k}$ with the help of Waring's formula.

In [14], Withers gives recurrence relations satisfied by these polynomials without proof. We will give a proof of these relations by using the theory of symmetric polynomials.

We compute the multidegrees of these polynomials and give an alternative proof of the fact that the coefficients of these polynomials are integers.

In [8], Küçüksakallı showed that $P_{\mathfrak{g}}^{q}(x,y) \equiv (x^q, y^q) \pmod{p}$ if q is a power of a prime p for $\mathfrak{g} = B_2$ and $\mathfrak{g} = G_2$. We will give a different proof of this fact.

1.3 Organization of the Thesis

The organization of this thesis is as follows:

In Chapter 2, we give some theoretical background related to the theory of Lie algebras. We first give the related definitions and introduce the fundamental concepts including the ideals and homomorphisms. We also mention the classification theorem of simple finite dimensional complex Lie algebras due to Killing and Cartan.

In Chapter 3, we describe the notions of Cartan subalgebra and root space decomposition. Then we explain the root systems in detail including the construction of rank 2 root systems geometrically. We also describe the notions of coroots and fundamental weights together with the Cartan matrix. In the last section, we introduce the Weyl group and its action on lattice generated by fundamental weights.

In Chapter 4, we review the exponential invariants of Bourbaki. We will give their proof of the fact that the subalgebra of invariant elements under the action of the Weyl group on lattice of fundamental weights is isomorphic to a certain polynomial algebra. Then we compute the generalized cosine function of rank 2 root systems. We conclude this chapter by giving examples of generalized Chebyshev polynomials.

In Chapter 5, we find the general formulae of generalized Chebyshev polynomials associated with the root systems of B_2 and G_2. Withers [14] gives the recurrence relations satisfied by these polynomials but he did not mention where the coefficients come from. We will explain why these polynomials satisfy these recurrence relations. Bourbaki [1] proved that the coefficients of these polynomials are integers. We will give an alternative proof of the same result. We will also prove that these polynomials reduce to Frobenius map over finite fields.

CHAPTER 2

PRELIMINARIES

In this chapter, we provide the theoretical background related to the theory of Lie algebras. We first give the related definitions and introduce the fundamental concepts including the ideals and homomorphisms. We also mention the classification theorem of simple finite dimensional complex Lie algebras due to Killing and Cartan. For details, see [1] and [6].

2.1 Definitions and First Examples

Definition 2.1. *A vector space* $\mathfrak{g}$ *over a field* $\mathbf{F}$*, with an operation* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, denoted* $(x, y) \mapsto [x, y]$ *and called the bracket of* x *and* y*, is called a Lie algebra over* $\mathbf{F}$ *if the following axioms are satisfied:*

(L1) The bracket operation is bilinear.

(L2) $[x, x] = 0$ *for all* x *in* $\mathfrak{g}$.

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* x, y, z *in* $\mathfrak{g}$.

Axiom *(L3)* is called the Jacobi identity. By applying the axioms *(L1)* and *(L2)* to $[x + y, x + y]$, we obtain

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Therefore we have

$$[x, y] = -[y, x]. \qquad \text{(L2}'\text{)}$$

Conversely if char$\mathbf{F} \neq 2$ then *(L2′)* will imply the axiom *(L2)*.

Definition 2.2. *A Lie algebra* $\mathfrak{g}$ *is called abelian if* $[x, y] = [y, x]$ *for all* $x, y \in \mathfrak{g}$.

Note that in the view of axiom *(L2′)*, a Lie algebra $\mathfrak{g}$ is abelian if and only if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Example 2.3. *Let* V *be a vector space over* $\mathbf{F}$*. Define* $[x, y] = 0$ *for all* $x, y \in V$*, then* V *becomes a Lie algebra over* $\mathbf{F}$ *and this is the abelian Lie algebra structure on* V*.*

Example 2.4. *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be two Lie algebras over a field* $\mathbf{F}$*. Then the direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *with the bracket operation*

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, x_2], [y_1, y_2])$$

becomes a Lie algebra over $\mathbf{F}$*. Note that the vector space here is simply* $\mathfrak{g}_1 \times \mathfrak{g}_2$*.*

Example 2.5. *Let* V *be a finite dimensional vector space over a field* $\mathbf{F}$*, and denote by* $\mathfrak{gl}(V)$ *the set of all linear transformations from* V *to* V*. Note that* $\mathfrak{gl}(V)$ *is also a finite dimensional vector space over* $\mathbf{F}$*. In fact* $\dim(\mathfrak{gl}(V)) = n^2$ *where* $n = \dim(V)$*.* $\mathfrak{gl}(V)$ *becomes a Lie algebra over* $\mathbf{F}$ *with the bracket operation*

$$[T, U] := T \circ U - U \circ T$$

where $T, U \in \mathfrak{gl}(V)$*, and* $\circ$ *denotes the composition of maps.*

Example 2.6. *Consider* $\mathfrak{gl}(n, \mathbf{F})$*, the vector space of all* $n \times n$ *matrices over* $\mathbf{F}$*. If we define the bracket operation* $[A, B] = A \cdot B - B \cdot A$ *where* $\cdot$ *denotes the usual product of the matrices* A *and* B*, then* $\mathfrak{gl}(n, \mathbf{F})$ *becomes a Lie algebra over* $\mathbf{F}$*. Moreover* $\mathfrak{gl}(n, \mathbf{F})$ *has a basis consisting of the matrices* e_{ij} *given by*

$$e_{ij} = \begin{cases} 1 & \text{in the } (i,j) \text{ position} \\ 0 & \text{otherwise.} \end{cases}$$

The action of the bracket operation on the basis elements is given by

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$$

where δ_{ij} *is the Kronecker-delta function.*

Note that this example is a matrix version of Example 2.5 because if V is an n dimensional vector space over $\mathbf{F}$ and if we fix a basis for V, then matrix representation of any linear transformation in $\mathfrak{gl}(V)$ with respect to the fixed basis corresponds to an $n \times n$ matrix in $\mathfrak{gl}(n, \mathbf{F})$.

Definition 2.7. *A vector subspace* $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$ *is called a Lie subalgebra of* $\mathfrak{g}$ *if* $[x, y] \in \mathfrak{h}$ *whenever* $x, y \in \mathfrak{h}$.

Example 2.8. *Let* V *be a vector space over a field* $\mathbf{F}$ *with* $\dim(V) = l + 1$. *The set of endomorphisms of* V *having trace zero is denoted by* $\mathfrak{sl}(V)$ *or* $\mathfrak{sl}(l + 1, \mathbf{F})$. *If* $x, y \in \mathfrak{sl}(l + 1, \mathbf{F})$, *then*

$$tr([x, y]) = tr(xy - yx) = tr(xy) - tr(yx) = 0. \tag{2.1}$$

Therefore $[x, y] \in \mathfrak{sl}(l + 1, \mathbf{F})$. *Hence* $\mathfrak{sl}(V)$ *is a subalgebra of* $\mathfrak{gl}(V)$. *It is called the special linear algebra. The standart basis for* $\mathfrak{sl}(V)$ *is the set*

$$\{e_{ij} : i \neq j\} \cup \{h_i : h_i = e_{ii} - e_{i+1,i+1}\}$$

where $1 \leq i \leq l$. *The former set has* $(l + 1)^2 - (l + 1)$ *elements and the letter set has* l *elements. Thus,* $\dim(\mathfrak{sl}(V)) = (l + 1)^2 - (l + 1) + l = l^2 + 2l$.

Example 2.9. *Consider the following subspaces of* $\mathfrak{gl}(n, \mathbf{F})$:

$$\begin{aligned}
\mathfrak{t}(n, \mathbf{F}) &:= \{[a_{ij}] \in \mathfrak{gl}(n, \mathbf{F}) : a_{ij} = 0 \ \textit{ if } \ i > j\}, \\
\mathfrak{n}(n, \mathbf{F}) &:= \{[a_{ij}] \in \mathfrak{gl}(n, \mathbf{F}) : a_{ij} = 0 \ \textit{ if } \ i \geq j\}, \\
\mathfrak{d}(n, \mathbf{F}) &:= \{[a_{ij}] \in \mathfrak{gl}(n, \mathbf{F}) : a_{ij} = 0 \ \textit{ if } \ i \neq j\},
\end{aligned}$$

the set of upper triangular, strictly upper triangular and diagonal matrices respectively. It is easily verified that each of these is closed under the bracket operation given in Example 2.6 and hence is a subalgebra of $\mathfrak{gl}(n, \mathbf{F})$.

2.2 Ideals, Homomorphisms and Representations

Definition 2.10. *A subspace* I *of a Lie algebra* $\mathfrak{g}$ *is called an ideal of* $\mathfrak{g}$ *if* $[x, y] \in I$ *whenever* $x \in \mathfrak{g}$ *and* $y \in I$.

Example 2.11. *If* $\mathfrak{g}$ *is a Lie algebra, then* $\{0\}$ *and* $\mathfrak{g}$ *itself will always be ideals of* $\mathfrak{g}$. *These are called the trivial ideals.*

Example 2.12. *Let* $x \in \mathfrak{gl}(n, \mathbf{F})$ *and* $y \in \mathfrak{sl}(n, \mathbf{F})$ *be two arbitrary elements. Then we see that* $[x, y] \in \mathfrak{sl}(n, \mathbf{F})$ *by Equation 2.1. Hence* $\mathfrak{sl}(n, \mathbf{F})$ *is an ideal of* $\mathfrak{gl}(n, \mathbf{F})$.

Example 2.13. *Referring to Example 2.9, if* $x \in \mathfrak{t}(n, \mathbf{F})$ *and* $y \in \mathfrak{n}(n, \mathbf{F})$ *then we have* $[x, y] = xy - yx \in \mathfrak{n}(n, \mathbf{F})$. *Therefore the set of strictly upper triangular matrices is an ideal of the set of upper triangular matrices.*

An ideal is always a subalgebra but a subalgebra need not be an ideal. For example $\mathfrak{t}(n, \mathbf{F})$ is a subalgebra of $\mathfrak{gl}(n, \mathbf{F})$. However if $n \geq 2$, it is not an ideal, because $e_{11} \in \mathfrak{t}(n, \mathbf{F})$ and $e_{21} \in \mathfrak{gl}(n, \mathbf{F})$, but $[e_{21}, e_{11}] = e_{21} \notin \mathfrak{t}(n, \mathbf{F})$.

Example 2.14. *For a Lie algebra* $\mathfrak{g}$, *the subset*

$$Z(\mathfrak{g}) := \{z \in \mathfrak{g} : [x, z] = 0 \textit{ for all } x \in \mathfrak{g}\}$$

is called the center of $\mathfrak{g}$ *and it is an ideal of* $\mathfrak{g}$.

Now we will introduce the concept of simplicity for a Lie algebra $\mathfrak{g}$. This concept will play an important role in our further discussions.

Definition 2.15. *A Lie algebra* $\mathfrak{g}$ *is called simple if it is non-abelian and it has no ideals except* $\{0\}$ *and itself.*

Example 2.16. *The Lie algebra* $\mathfrak{sl}(2, \mathbf{F})$ *where char*$\mathbf{F} \neq 2$ *is simple because if we take the standart basis* β *for* $\mathfrak{sl}(2, \mathbf{F})$ *given by*

$$\beta = \left\{ x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\},$$

then we have $[x, y] = h$, $[h, x] = 2x$, *and* $[h, y] = -2y$. *Let* I *be a nonzero ideal of* $\mathfrak{sl}(2, \mathbf{F})$ *and let* $ax + by + ch$ *be a nonzero element of* I. *We have* $[x, ax + by + ch] = bh - 2cx \in I$. *Moreover* $[x, bh - 2cx] = -2bx \in I$. *Similarly* $[y, ax + by + ch] = 2cy - ah \in I$, *and thus* $[y, 2cy - ah] = -2ay \in I$. *Therefore if* a *or* b *is nonzero then either* $y \in I$ *or* $x \in I$ *and it follows that* $I = \mathfrak{sl}(2, \mathbf{F})$. *On the other hand if* $a = b = 0$, *then* $h \in I$ *and it follows that* $I = \mathfrak{sl}(2, \mathbf{F})$.

Definition 2.17. *Let* $\mathfrak{g}$ *and* $\mathfrak{l}$ *be two Lie algebras over* $\mathbf{F}$, *and let* $\varphi : \mathfrak{g} \to \mathfrak{l}$ *be a linear transformation. Then* φ *is called a homomorphism if*

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

for all $x, y \in \mathfrak{g}$, *and the set* $\mathrm{Ker}\varphi := \{x \in \mathfrak{g} : \varphi(x) = 0\}$ *is called the kernel of* φ.

We can observe that Kerφ is an ideal of $\mathfrak{g}$, because if $x \in$ Kerφ and $y \in \mathfrak{g}$, then $\varphi([x,y]) = [\varphi(x), \varphi(y)] = [0, \varphi(y)] = 0$, thus $[x,y] \in$ Kerφ.

Definition 2.18. *Let $\mathfrak{g}$ be a Lie algebra and let V be a vector space over a field $\mathbf{F}$. A representation of $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$.*

In the following example, we will see an extremely important representation of a Lie algebra $\mathfrak{g}$ which will play a crucial role in our further discussions.

Example 2.19. *Consider the map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $(adx)(y) = [x,y]$. Then*

$$\begin{aligned} ad([x,y])(z) &= [[x,y],z] \\ &= [x,[y,z]] + [[x,z],y] \\ &= [x,[y,z]] - [y,[x,z]] \\ &= adx([y,z]) - ady([x,z]) \\ &= (adx \circ ady)(z) - (ady \circ adx)(z) \\ &= [adx, ady](z). \end{aligned}$$

Therefore ad is a representation. It is called the adjoint representation.

We observe that the kernel of this map is $Z(\mathfrak{g})$. Moreover if $\mathfrak{g}$ is simple then $Z(\mathfrak{g}) = 0$ and the map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a monomorphism. Therefore any simple Lie algebra is isomorphic to a linear Lie algebra. Before we give the classification theorem, it is useful to introduce the following four types of families:

Definition 2.20. *The classical Lie algebras are finite dimensional Lie algebras over a field $\mathbf{F}$ which can be classified into four types A_n, B_n, C_n and D_n. Here $\mathfrak{gl}(n, \mathbf{F})$ is the general linear Lie algebra and I_n is the $n \times n$ identity matrix.*

$$\begin{aligned} &A_n := \mathfrak{sl}(n+1, \mathbf{F}) = \{x \in \mathfrak{gl}(n+1, \mathbf{F}) : tr(x) = 0\}, \\ &B_n := \mathfrak{so}(2n+1, \mathbf{F}) = \{x \in \mathfrak{gl}(2n+1, \mathbf{F}) : x + x^T = 0\}, \\ &C_n := \mathfrak{sp}(2n, \mathbf{F}) = \left\{ x \in \mathfrak{gl}(2n, \mathbf{F}) : J_n x + x^T J_n = 0, J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right\}, \\ &D_n := \mathfrak{so}(2n, \mathbf{F}) = \{x \in \mathfrak{gl}(2n, \mathbf{F}) : x + x^T = 0\}. \end{aligned}$$

The classification of simple finite dimensional complex Lie algebras was first given by Killing and later completed by Cartan in his Ph.D thesis [2]. The following theorem is due to Killing and Cartan.

Theorem 2.21. *Every simple finite dimensional complex Lie algebra is isomorphic to one of the following classical Lie algebras*

$$\begin{aligned} \mathfrak{sl}(n+1,\mathbf{C}) &\quad n \geq 1 \quad (A_n), \\ \mathfrak{so}(2n+1,\mathbf{C}) &\quad n \geq 2 \quad (B_n), \\ \mathfrak{sp}(2n,\mathbf{C}) &\quad n \geq 3 \quad (C_n), \\ \mathfrak{so}(2n,\mathbf{C}) &\quad n \geq 4 \quad (D_n), \end{aligned}$$

or one of the five exceptional Lie algebras E_6, E_7, E_8, F_4, G_2.

Proof. The proof relies on the fact that for each simple finite dimensional complex Lie algebra, there exists a corresponding Dynkin diagram, and classifying simple finite dimensional complex Lie algebras is equivalent to classifying their Dynkin diagrams. For details see [1, p.201] □

Note that, in Theorem 2.21, the Lie algebras $\mathfrak{so}(3,\mathbf{C}), \mathfrak{sp}(2,\mathbf{C}), \mathfrak{sp}(4,\mathbf{C}), \mathfrak{so}(2,\mathbf{C})$, $\mathfrak{so}(4,\mathbf{C})$ and $\mathfrak{so}(6,\mathbf{C})$ are not mentioned because we have the following well known isomorphisms:

$$\begin{aligned} &\mathfrak{so}(3,\mathbf{C}) \simeq \mathfrak{sp}(2,\mathbf{C}) \simeq \mathfrak{sl}(2,\mathbf{C}) \\ &\mathfrak{sp}(4,\mathbf{C}) \simeq \mathfrak{so}(5,\mathbf{C}) \\ &\mathfrak{so}(4,\mathbf{C}) \simeq \mathfrak{sl}(2,\mathbf{C}) \oplus \mathfrak{sl}(2,\mathbf{C}) \\ &\mathfrak{so}(6,\mathbf{C}) \simeq \mathfrak{sl}(4,\mathbf{C}). \end{aligned}$$

Note also that, $\mathfrak{so}(2,\mathbf{C})$ is one dimensional and abelian. Therefore it is not simple.

CHAPTER 3

ROOT SYSTEMS OF SEMISIMPLE LIE ALGEBRAS

In this chapter, we describe the notions of Cartan subalgebra of a Lie algebra and root space decomposition. Then we explain the root systems. The main connection between the Lie algebras and root systems is that the dual of the Cartan subalgebra of a semisimple Lie algebra satisfies the axioms of being a root system. A geometric construction of a root system of $\mathfrak{sl}(3, \mathbf{C})$ will be introduced. We also describe the notions of coroots and fundamental weights together with the Cartan matrix and Weyl chamber. For details, see [1], [6] and [4]. In the last section 3.4, we will see the Weyl groups of rank 2 root systems and compute the orbits of the fundamental weights under the action of this group.

3.1 Cartan Subalgebra and Root Space Decomposition

Before we give the definitions of Cartan subalgebra and root space decomposition, we will start with a motivating example to understand these notions better.

Example 3.1. *We know that*

$$\mathfrak{sl}(3, \mathbf{C}) = \text{span}\left\{ e_{ij}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

where

$$e_{ij} = \begin{cases} 1 & \text{in the } (i,j) \text{ position} \\ 0 & \text{otherwise} \end{cases}$$

for each $i, j \in \{1, 2, 3\}$ *with* $i \neq j$*. Now let* $\mathfrak{h}$ *be the* 2 *dimensional subalgebra of* $\mathfrak{sl}(3, \mathbf{C})$ *spanned by the diagonal matrices. That is*

$$\mathfrak{h} = \text{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}.$$

Now if $h \in \mathfrak{h}$ *has diagonal entries* a_1, a_2, a_3*, then we have*

$$adh(e_{ij}) = [h, e_{ij}] = (a_i - a_j)e_{ij}. \tag{3.1}$$

Therefore $ad\mathfrak{h} = \{(a_i - a_j) \cdot e_{ij} : h \in \mathfrak{h}\}$*, and the elements* e_{ij} *for* $i \neq j$ *are common eigenvectors for the elements of* $ad\mathfrak{h}$*. If we consider the functionals*

$$\begin{aligned} \varepsilon_i : \mathfrak{h} &\to \mathbf{C} \\ h &\mapsto a_i, \end{aligned}$$

then Equation 3.1 can be written in terms of these functionals as

$$adh(e_{ij}) = (\varepsilon_i - \varepsilon_j)(h)e_{ij}. \tag{3.2}$$

If we let $\mathfrak{g}_{ij} := \{x \in \mathfrak{sl}(3, \mathbf{C}) : adh(x) = (\varepsilon_i - \varepsilon_j)(h)x \text{ for all } h \in \mathfrak{h}\}$*, then by Equation 3.2, we have* $\mathfrak{g}_{ij} = \text{span}\{e_{ij}\}$ *for* $i \neq j$*. Hence there is a direct sum decomposition given by*

$$\mathfrak{sl}(3, \mathbf{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij}.$$

To see the existence of this decomposition more abstractly, we need the following definitions:

Definition 3.2. *A Lie algebra that can be written as a direct sum of finitely many simple Lie algebras is called semisimple. On the other hand, an element* x *in* $\mathfrak{g}$ *is called semisimple if* adx *is a diagonalizable endomorphism of* $\mathfrak{g}$ *with respect to a suitable basis of* $\mathfrak{g}$ *as a vector space.*

Definition 3.3. *A Lie subalgebra* $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$ *is called Cartan subalgebra if* $\mathfrak{h}$ *is abelian and every element* $h \in \mathfrak{h}$ *is semisimple.*

Definition 3.4. *A root is a functional* $\alpha : \mathfrak{h} \to \mathbf{F}$ *so that the set*

$$\big\{x \in \mathfrak{g} : (adh)x = \alpha(h)x \ \textit{for all} \ \ h \in \mathfrak{h}\big\}$$

is a nonzero subspace of $\mathfrak{g}$. *The set of all nonzero roots is denoted by* Φ.

Definition 3.5. *A root space of a Lie algebra* $\mathfrak{g}$ *is a nonzero subspace of* $\mathfrak{g}$ *of the form*

$$\mathfrak{g}_\alpha := \big\{x \in \mathfrak{g} : (adh)x = \alpha(h)x \ \textit{for all} \ \ h \in \mathfrak{h}\big\}$$

for each root $\alpha \in \mathfrak{h}^*$.

Theorem 3.6. *Let* $\mathfrak{h}$ *be a Cartan subalgebra of a semisimple Lie algebra* $\mathfrak{g}$, *then*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Proof. Since the Cartan subalgebra $\mathfrak{h}$ is abelian, the elements of $ad\mathfrak{h}$ are simultaneously diagonalizable. In other words, $\mathfrak{g}$ is the direct sum of subspaces $\mathfrak{g}_\alpha$. Since $\mathfrak{g}_0$ contains $\mathfrak{h}$, theorem follows. For details, see [6, p.35]. □

Definition 3.7. *The direct sum decomposition given in Theorem 3.6 is called the root space decomposition.*

Example 3.8. *In the language of roots, the decomposition for* $\mathfrak{sl}(3, \mathbf{C})$ *in Example 3.1 can be written as*

$$\mathfrak{sl}(3, \mathbf{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

where $\Phi = \{\varepsilon_i - \varepsilon_j : i, j \in \{1, 2, 3\} \ \textit{with} \ \ i \neq j\}$ *consisting of 6 roots.*

3.2 Root Systems

Definition 3.9. *Let $\mathbf{E}$ be a finite dimensional vector space over $\mathbf{R}$ equipped with an inner product $(-,-)$. A reflection in $\mathbf{E}$ is an invertible linear transformation leaving some hyperplane (subspace of codimension one) pointwise fixed and sending any vector orthogonal to that hyperplane into its negative. That is if $\alpha \in \mathbf{E}$ is a nonzero vector, the reflection s_α, with reflecting hyperplane $H_\alpha = \{\beta \in \mathbf{E} : (\alpha, \beta) = 0\}$, is the map $s_\alpha : \mathbf{E} \to \mathbf{E}$ such that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\beta) = \beta$ for all $\beta \in H_\alpha$. In fact an explicit formula for $s_\alpha(x)$ where x is an arbitrary vector in $\mathbf{E}$ is given by*

$$s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha \tag{3.3}$$

Definition 3.10. *A subset Φ of a Euclidean space $\mathbf{E}$ is called a root system if it satisfies the following axioms:*

1. *Φ is finite, it spans $\mathbf{E}$, and it does not contain 0.*
2. *If $\alpha \in \Phi$ then the only scalar multiples of α in Φ are $\pm\alpha$.*
3. *If $\alpha \in \Phi$ then the reflection s_α permutes the elements of Φ.*
4. *If $\alpha, \beta \in \Phi$ then $\langle \alpha, \beta \rangle := \dfrac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbf{Z}$.*

Now we will see, how we pass from a Lie algebra to a root system. The following proposition will enable us to understand the main connection between Lie algebras and root systems.

Proposition 3.11. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{R}$ and $\mathfrak{h}$ be its Cartan subalgebra. Let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$. Then $\mathfrak{h}^*$ is a Euclidean space, i.e. a finite dimensional vector space over $\mathbf{R}$ endowed with a positive definite symmetric bilinear form and the subset Φ consisting of the roots of $\mathfrak{h}^*$ is a root system.*

Proof. The proof relies on the following argument:

The dual space $\mathfrak{h}^*$ is an $\mathbf{R}$-vector space and the Killing form κ defined by

$$\begin{aligned} \kappa : \mathfrak{g} \times \mathfrak{g} &\to \mathbf{R} \\ (x,y) &\mapsto \operatorname{tr}(adx \circ ady) \end{aligned}$$

induces an isomorphism φ from $\mathfrak{h}$ to $\mathfrak{h}^*$. Therefore we may define a positive definite symmetric bilinear form $B(-,-)$ on $\mathfrak{h}^*$ by

$$\begin{aligned} B : \mathfrak{h}^* \times \mathfrak{h}^* &\to \mathbf{R} \\ (\alpha, \beta) &\mapsto \kappa\left(\varphi^{-1}(\alpha), \varphi^{-1}(\beta)\right). \end{aligned}$$

This allows us to see $\mathfrak{h}^*$ as a Euclidean space. Moreover, the subset Φ of $\mathfrak{h}^*$ consisting of roots satisfies the axioms given in Definition 3.10. For details see [6, Section 8] □

In the remaining of this section, we study the geometric constructions of rank 2 root systems corresponding to the Lie algebras $A_1 \times A_1, A_2, B_2$ and G_2.

When we turn back to Definition 3.10, the fourth condition limits the possible angles that can occur between pairs of roots. The possible angles are determined by the following lemma:

Lemma 3.12. *Suppose that Φ is a root system in the Euclidean space $\mathbf{E}$. Let α and β be two linearly independent roots in Φ. Then $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$.*

Proof. We know that the inner product of the vectors α and β in $\mathbf{R}$ is given by $(\alpha, \beta) = \|\alpha\| \cdot \|\beta\| \cdot \cos\theta$. Therefore we have

$$\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \cdot \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\|\alpha\| \cdot \|\beta\| \cdot \cos\theta}{(\beta, \beta)} \cdot \frac{2\|\beta\| \cdot \|\alpha\| \cdot \cos\theta}{(\alpha, \alpha)} = 4\cos^2\theta.$$

Since $0 \leq \cos^2\theta \leq 1$ and $4\cos^2\theta \in \mathbf{Z}$, we must have $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$. Note that the case $4\cos^2\theta = 4$ cannot occur because otherwise $\theta = k\pi$ for some $k \in \mathbf{Z}$, making α and β linearly dependent contrary to our assumption. □

Corollary 3.13. *There are only few possibilities for the angle θ by Lemma 3.12. These values are given in Table 3.1.*

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\dfrac{\|\beta\|}{\|\alpha\|}$
0	0	$\pi/2$	*undetermined*
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
-1	-2	$3\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-3	$5\pi/6$	$\sqrt{3}$

Table 3.1: Possible Values of θ.

By using Table 3.1, we can obtain all possible root systems of rank 2 as follows:

1. When the dimension of $\mathbf{E}$ is one, in the view of the second condition of Definition 3.10, there is only one possibility. This root system is labeled as A_1, and can be described by the following picture:

Figure 3.1: A_1 Root system

2. If we start with two roots α and β perpendicular to each other and with no restriction on the ratio of the length of α to the length of β, we would get the following root system:

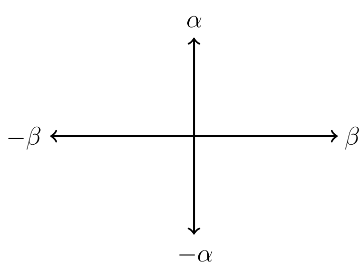
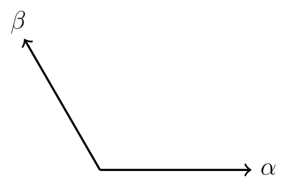
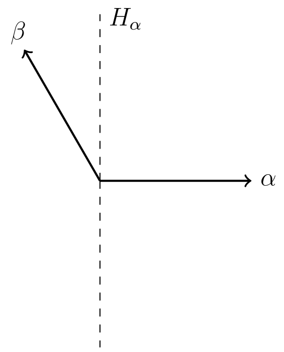


Figure 3.2: $A_1 \times A_1$ Root system

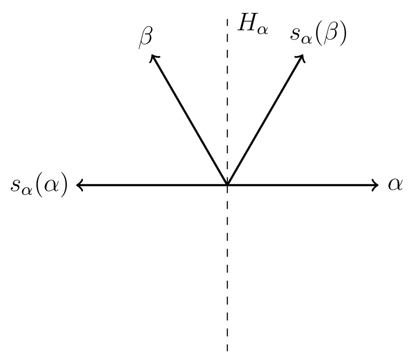
3. Now we will construct the root system A_2 step by step as follows. We start with two roots α and β in $\mathbf{R}^3$ such that $\|\beta\| = \|\alpha\|$ and that the angle between them is $2\pi/3$.



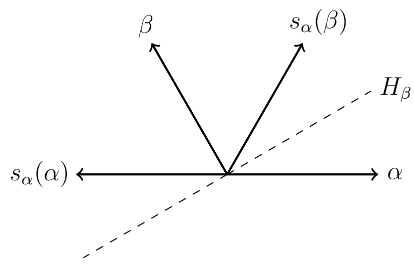
Next we consider the hyperplane H_α.



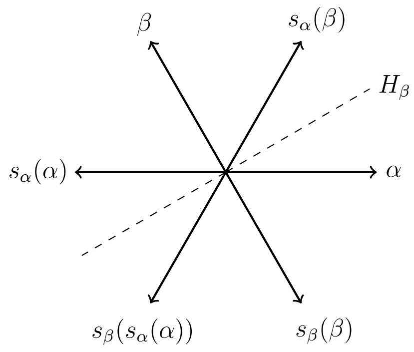
If we take the reflection of our roots in this hyperplane, we obtain



Similarly we consider the hyperplane H_β.



If we take the reflection of our roots in this hyperplane, we obtain



Observe that we cannot obtain more roots no matter what hyperplane we choose. In other words, this is a closed system under the reflections. By using Equation 3.3, we obtain the following picture:

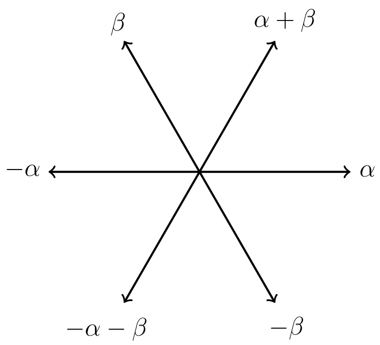


Figure 3.3: A_2 Root system.

4. If we start with two roots α and β in $\mathbf{R}^2$ such that $\|\beta\| = \sqrt{2}\|\alpha\|$ and that the angle between them is $3\pi/4$ and apply the same steps as in the previous case we obtain the root system called B_2 root system. See Figure 3.4.

5. Another important root system, called G_2, is obtained by starting with the two roots α and β in $\mathbf{R}^3$ such that $\|\beta\| = \sqrt{3}\|\alpha\|$ and that the angle between them is $5\pi/6$. See Figure 3.5.

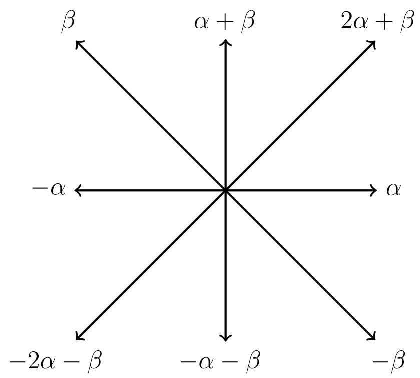


Figure 3.4: B_2 Root system

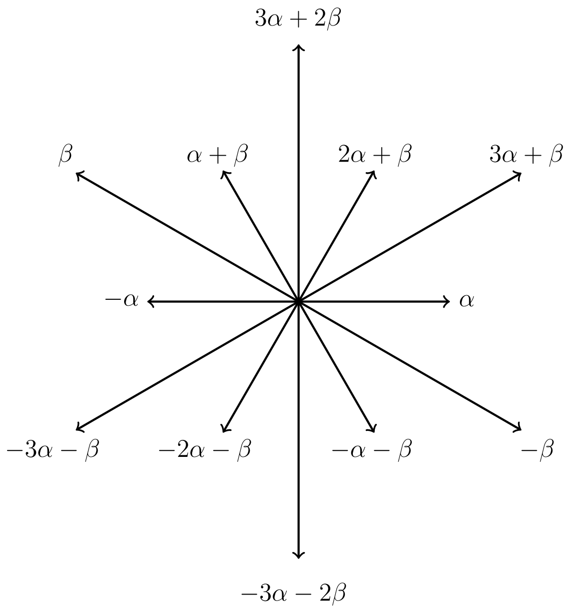


Figure 3.5: G_2 Root system

3.3 More about Root Systems

Definition 3.14. *Let Φ be a root system in a Euclidean space $\mathbf{E}$. A subset B of Φ is called a base if the following two conditions are satisfied:*

(B1) B *is a basis of* $\mathbf{E}$*,*

(B2) Each root β can be written as $\beta = \sum k_\alpha \alpha$, $\alpha \in B$ with integer coefficients k_α all nonnegative or all nonpositive.

The roots in B are called simple roots, and the reflections $s_\alpha(\alpha \in B)$ are called simple reflections. Moreover the cardinality of B is called the rank of Φ.

We will now introduce the bases of the root systems A_2, B_2 and G_2 according to the choice of Humphreys in [6]. Throughout the following example let $\{e_1, e_2, e_3\}$ be the orthonormal basis of $\mathbf{R}^3$.

Example 3.15.

1. *In the case of A_2, basis elements are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, and*
$$\Phi = \big\{ \pm \alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\big\}.$$
2. *In the case of B_2, basis elements are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, and*
$$\Phi = \big\{ \pm \alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2)\big\}.$$
3. *In the case of G_2, basis elements are $\alpha_1 = e_1 - e_2$, $\alpha_2 = -2e_1 + e_2 + e_3$, and*
$$\Phi = \big\{ \pm \alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\big\}.$$

Since the expression for β in condition *(B2)* is unique, we may define a partial order $\geq$ on Φ. If $B = (\alpha_i)_{1\leq i\leq n}$ is a base of Φ, then for $\gamma, \mu \in \Phi$ we say $\gamma \geq \mu$ if and only if $\gamma - \mu$ is a linear combination of the base elements α_i with positive coefficients. This partial order will allow us to define the lattices of coroots and fundamental weights.

Definition 3.16. *Let Φ be a root system. For each $\alpha \in \Phi$, we have the coroot*
$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}.$$

The set of coroots $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ forms a root system in $\mathbf{E}$ and called the dual of Φ.

Definition 3.17. *Let Φ be a root system with base $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$. Then $\Phi^\vee$ has a base $B^\vee = \{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_n^\vee\}$. The set $Q(\Phi^\vee) := \mathbf{Z}\alpha_1^\vee \oplus \mathbf{Z}\alpha_2^\vee \oplus \cdots \oplus \mathbf{Z}\alpha_n^\vee$ together with the partial order $\geq$ is called the coroot lattice of Φ.*

The importance of the coroot lattice appears in the following definition:

Definition 3.18. *The set* $\mathcal{P}(\Phi) = \{x \in \mathbf{E} : (x, s) \in \mathbf{Z}$ *for all* $s \in Q(\Phi^\vee)\}$ *together with the partial order* $\geq$ *is called the weight lattice of* Φ *and it has generators* $\omega_1, \omega_2, ..., \omega_n$ *called the fundamental weights defined by*

$$(\omega_j, \alpha_k^\vee) := \delta_{jk}$$

Example 3.19. *In the following examples we will find coroots and fundamental weights of the root systems* A_2*,* B_2 *and* G_2 *respectively.*

1. *Working with the base given in Example 3.15 and solving the system of equations* $(\omega_j, \alpha_k^\vee) = \delta_{jk}$ *gives that*

   $\alpha_1^\vee = \alpha_1,\ \alpha_2^\vee = \alpha_2,\ \textit{and}\ \omega_1 = (2/3)\alpha_1 + (1/3)\alpha_2,\ \omega_2 = (1/3)\alpha_1 + (2/3)\alpha_2.$

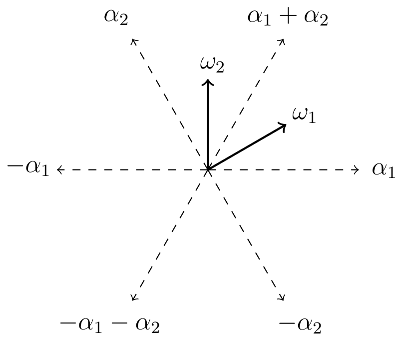


Figure 3.6: Fundamental weights of A_2.

2. *Working with the base given in Example 3.15 and solving the system of equations* $(\omega_j, \alpha_k^\vee) = \delta_{jk}$ *gives that*

$$\alpha_1^\vee = \alpha_1,\ \alpha_2^\vee = 2\alpha_2\ \textit{and}\ \omega_1 = \alpha_1 + \alpha_2,\ \omega_2 = (1/2)\alpha_1 + \alpha_2.$$

3. *Working with the base given in Example 3.15 and solving the system of equations* $(\omega_j, \alpha_k^\vee) = \delta_{jk}$ *gives that*

$$\alpha_1^\vee = \alpha_1,\ \alpha_2^\vee = (1/3)\alpha_2\ \textit{and}\ \omega_1 = 2\alpha_1 + \alpha_2,\ \omega_2 = 3\alpha_1 + 2\alpha_2.$$

While finding fundamental weights of the root systems above we see that, for example in the case A_2,

$$\omega_1 = (2/3)\alpha_1 + (1/3)\alpha_2 \text{ and } \omega_2 = (1/3)\alpha_1 + (2/3)\alpha_2.$$

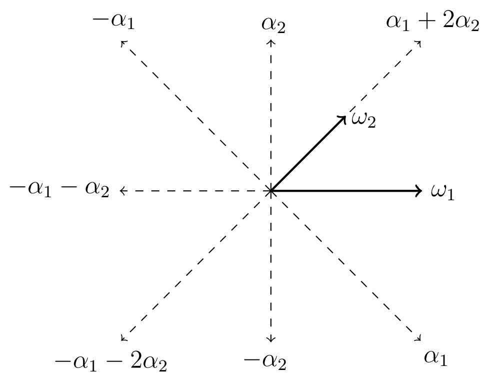


Figure 3.7: Fundamental weights of B_2.

Apart from the explicit calculations, there is a reason behind this. Observe that in terms of matrices these equalities can be written as

$$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

and the matrix on the right hand side has a special name and will play an important role in our further discussions.

Definition 3.20. *Let Φ be a root system and let $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be an ordered base of Φ. The matrix $[\langle \alpha_i, \alpha_j \rangle]$ is called the Cartan matrix of Φ, where $\langle \alpha_i, \alpha_j \rangle$ is as it is defined in condition 4 of Definition 3.10.*

Observe that each entry $\langle \alpha_i, \alpha_j \rangle$ of the Cartan matrix is an integer and these entries are called the Cartan integers. As we have seen, the Cartan matrix transforms the fundamental weights into the simple roots.

In the next example, we will give the Cartan matrices of rank 2 root systems. However, there is a point that we need to be careful about. The Cartan matrix depends on the choice of B, the chosen ordering on B and it also depends on the lengths of the roots in B. The notion of Dynkin diagram is used to overcome this difficulty.

Definition 3.21. *Let Φ be a root system and let $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a base of Φ. A Dynkin diagram of a root system Φ is a graph having the following properties:*

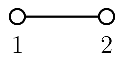
1. *The vertices are labeled by the indices of the basis elements of* Φ.

2. *Between the vertices, labeled by* i *and* j*, there are* $\langle \alpha_i, \alpha_j \rangle \cdot \langle \alpha_j, \alpha_i \rangle \in \{0, 1, 2, 3\}$ *edges.*

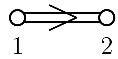
3. *If* α_i *and* α_j *have different lengths and are not orthogonal, there is an arrow pointing from the longer root to the shorter root.*

Example 3.22. *As a tradition, Dynkin diagrams of root systems* A_2, B_2 *and* G_2 *are given as follows:*

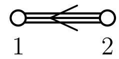
A_2 :



B_2 :



G_2 :



Example 3.23. *Let* $C(\Phi)$ *denotes the Cartan matrix of the root system* Φ*. According to the Dynkin diagrams in Example 3.22, we have*

$$C(A_2) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \; C(B_2) = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}, \; C(G_2) = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

There is one more notion we will introduce related to the root systems.

Definition 3.24. *The hyperplanes* $H_\alpha, (\alpha \in \Phi)$ *partition* $\mathbf{E}$ *into finitely many regions. The connected components of* $\mathbf{E} \setminus \bigcup\limits_{\alpha} H_\alpha$ *are called the (open) Weyl chambers of* $\mathbf{E}$.

Definition 3.25. *The fundamental Weyl chamber associated to a base* B *is the set*

$$C := \{x \in \mathbf{E} : (x, \alpha) > 0 \;\text{ for all }\; \alpha \in B\}.$$

The closure of the fundamental Weyl chamber

$$\overline{C} = \{x \in \mathbf{E} : (x, \alpha) \geq 0 \;\text{ for all }\; \alpha \in B\}$$

is called the fundamental region of Φ.

Example 3.26. *In Figure 3.8 the shaded region is the fundamental Weyl chamber of A_2 associated to the base $B = \{\alpha_1, \alpha_2\}$.*

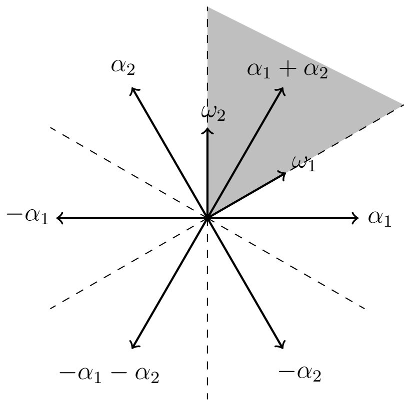


Figure 3.8: Fundamental Weyl Chamber of A_2.

3.4 Weyl Group of a Root System

Definition 3.27. *Let Φ be a root system in $\mathbf{E}$. The subgroup W of $GL(\mathbf{E})$ generated by the reflections s_α where $\alpha \in \Phi$ is called the Weyl group of Φ.*

Since the reflection s_α leaves Φ invariant, we can say that W permutes the elements of Φ and since Φ is finite we can identify W with a subgroup of the symmetric group on Φ. In particular the Weyl group W is finite.

Proposition 3.28. *Let B be a base of a root system Φ. Then the Weyl group of Φ, $W(\Phi)$, is generated by the simple reflections.*

Proof. See [6, Section 10.3] $\square$

Example 3.29. *In this example, we will give the Weyl groups of rank 2 root systems.*

1. $\Phi_{A_1} = \{\pm\alpha\}$. *We can take $B_{A_1} = \{\alpha\}$ as the set of simple roots. Therefore $W(A_1) = \langle s_\alpha \rangle$. Since s_α^2 is the identity reflection $W(A_1) = \big\langle s_\alpha \big\rangle \simeq \mathbf{Z}_2$. Hence $W(A_1 \times A_1) = \mathbf{Z}_2 \times \mathbf{Z}_2$.*

2. $\Phi_{A_2} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$. *We can take* $B_{A_2} = \{\alpha, \beta\}$ *as the set of simple roots. Therefore* $W(A_2) = \big\langle s_\alpha, s_\beta \big\rangle$. *We also observe the following representation for* $W(A_2)$.

$$W(A_2) = \big\{ s_\alpha, s_\beta : (s_\alpha)^2 = (s_\beta)^2 = (s_\alpha \circ s_\beta)^3 = 1 \big\} \simeq S_3 \simeq D_3.$$

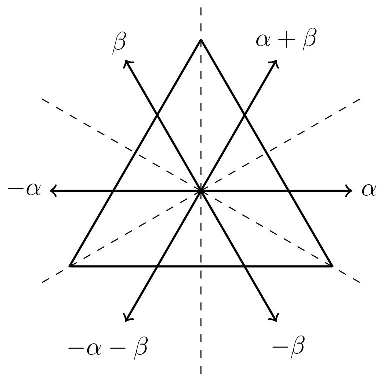


Figure 3.9: Weyl group of A_2 is the symmetry group of an equilateral triangle.

3. $\Phi_{B_2} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$. *We can take* $B_{B_2} = \{\alpha, \beta\}$ *as the set of simple roots. Therefore* $W(B_2) = \langle s_\alpha, s_\beta \rangle$. *Moreover*

$$W(B_2) = \{ s_\alpha, s_\beta : (s_\alpha)^2 = (s_\beta)^2 = (s_\alpha \circ s_\beta)^4 = 1 \} \simeq D_4$$

symmetry group of the square.

4. $\Phi_{G_2} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$. *If we take* $B_{G_2} = \{\alpha, \beta\}$ *to be the set of simple roots, then* $W(G_2) = \langle s_\alpha, s_\beta \rangle$.

$$W(G_2) = \{ s_\alpha, s_\beta : (s_\alpha)^2 = (s_\beta)^2 = (s_\alpha \circ s_\beta)^6 = 1 \} \simeq D_6$$

symmetry group of the hexagon.

Let Φ be a rank n root system with the base $B = \{\alpha_1, ..., \alpha_n\}$, and let $\mathcal{P}(\Phi)$ be lattice of fundamental weights with base $\{\omega_1, ..., \omega_n\}$. By Equation 3.3 we have a natural action of $W(\Phi)$ on $\mathcal{P}(\Phi)$ given by

$$s_{\alpha_i}(\omega_j) = \omega_j - \frac{2(\omega_j, \alpha_i)}{(\alpha_i, \alpha_i)}. \tag{3.4}$$

We will be interested in finding the orbits of fundamental weights under the action of the Weyl group. Let us denote the orbits of ω_1 and ω_2 by ϕ_1 and ϕ_2 respectively.

Example 3.30 ($\Phi = A_1$)**.** *By using Equation 3.4, we have*

$$\begin{aligned}\phi_1 &= \big\{w \cdot \omega_1 : w \in W(A_1)\big\}\\ &= \big\{\omega_1, -\omega_1\big\}.\end{aligned}$$

Example 3.31 ($\Phi = A_2$)**.** *By using Equation 3.4, we have*

$$\begin{aligned}\phi_1 &= \big\{w \cdot \omega_1 : w \in W(A_2)\big\}\\ &= \big\{\omega_1, -\omega_2, -\omega_1 + \omega_2\big\}.\end{aligned}$$

$$\begin{aligned}\phi_2 &= \big\{w \cdot \omega_2 : w \in W(A_2)\big\}\\ &= \big\{\omega_2, -\omega_1, \omega_1 - \omega_2\big\}.\end{aligned}$$

Example 3.32 ($\Phi = B_2$)**.** *By using Equation 3.4, we have*

$$\begin{aligned}\phi_1 &= \big\{w \cdot \omega_1 : w \in W(B_2)\big\}\\ &= \big\{\omega_1, -\omega_1, \omega_1 - 2\omega_2, -\omega_1 + 2\omega_2\big\}.\end{aligned}$$

$$\begin{aligned}\phi_2 &= \big\{w \cdot \omega_2 : w \in W(B_2)\big\}\\ &= \big\{\omega_2, -\omega_2, \omega_1 - \omega_2, -\omega_1 + \omega_2\big\}.\end{aligned}$$

Example 3.33 ($\Phi = G_2$)**.** *By using Equation 3.4, we have*

$$\begin{aligned}\phi_1 &= \big\{w \cdot \omega_1 : w \in W(G_2)\big\}\\ &= \big\{\omega_1, -\omega_1, \omega_1 - \omega_2, -\omega_1 + \omega_2, 2\omega_1 - \omega_2, -2\omega_1 + \omega_2\big\}.\end{aligned}$$

$$\begin{aligned}\phi_2 &= \big\{w \cdot \omega_2 : w \in W(G_2)\big\}\\ &= \big\{\omega_2, -\omega_2, 3\omega_1 - \omega_2, -3\omega_1 + \omega_2, 3\omega_1 - 2\omega_2, -3\omega_1 + 2\omega_2\big\}.\end{aligned}$$

If the rank of the underlying root system increases, then our calculations will be more challenging. Therefore we will try to understand the action of the Weyl group of a root system on the lattice of fundamental weights by matrix multiplication. To do this, we will see the Weyl group of a root system as a subgroup of a matrix group.

First of all, we know that if Φ is a root system with base $B = \{\alpha_1, ..., \alpha_n\}$, then $\Phi^\vee$ is generated by $B^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\}$. Moreover the Weyl group of $\Phi^\vee$ is isomorphic to the Weyl group of Φ. Therefore generators of the Weyl group of a root system can be considered as the matrix representation of s_{α_i} with respect to the base of the coroots where $i = 1, 2, ..., n$. To find the matrix representation of s_{α_i} with respect to the base of the coroots, we will use the fact that the lattice $\mathcal{P}(\Phi)$ and $\Phi^\vee$ are dual to each other in the sense that $(\omega_j, \alpha_k^\vee) = \delta_{jk}$. This argument allows us to see that

$$\left[s_{\alpha_i}\right]_{\{\alpha_1^\vee,...,\alpha_n^\vee\}} = \left[s_{\alpha_i}\right]^{\mathrm{T}}_{\{\omega_1,...,\omega_n\}}. \tag{3.5}$$

We know that

$$s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha, \;\; \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \;\text{ and }\; (\omega_j, \alpha_k^\vee) = \delta_{jk}.$$

By using these definitions algebraically we obtain

$$s_{\alpha_i}(\omega_j) = \omega_j - \delta_{ij}\alpha_i. \tag{3.6}$$

Then Equation 3.6 gives us that

$$s_{\alpha_i}(\omega_j) = \begin{cases} \omega_j & \text{if } j \neq i \\ \omega_i - \alpha_i & \text{if } j = i \end{cases} \tag{3.7}$$

At this point, we need to express α_i in terms of fundamental weights. We will do this by using the Cartan matrix. Since the Cartan matrix expresses the change of basis from fundamental weights to the simple roots we have

$$\sum_{j=1}^{n} \langle \alpha_i, \alpha_j \rangle \, \omega_j = \alpha_i. \tag{3.8}$$

By using Equations 3.5, 3.7, and 3.8 we get

$$\left[s_{\alpha_i}\right]_{\{\alpha_1^\vee,\ldots,\alpha_n^\vee\}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ -\langle\alpha_i,\alpha_1\rangle & -\langle\alpha_i,\alpha_2\rangle & \cdots & 1-\langle\alpha_i,\alpha_i\rangle & \cdots & -\langle\alpha_i,\alpha_n\rangle \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

Moreover we see that

$$\left[s_{\alpha_i}\right]_{\{\alpha_1^\vee,\ldots,\alpha_n^\vee\}} = I_n - i^{th}\text{row of the Cartan matrix.} \tag{3.9}$$

Using this argument, we can identify each generator of the Weyl group with a matrix with integer entries. We also know that the Weyl group is generated by s_{α_i} therefore if $w \in W(\Phi)$, then we have $w = s_{\alpha_1}^{k_1} \circ \cdots \circ s_{\alpha_n}^{k_n}$ where each $k_i \in \mathbf{Z}$. Since each s_{α_i} is linear, an arbitrary element $w \in W(\Phi)$ can be identified with the matrix T_w where

$$T_w = \left(\left[s_{\alpha_1}\right]_{\{\alpha_1^\vee,\ldots,\alpha_n^\vee\}}\right)^{k_1} \cdots \left(\left[s_{\alpha_n}\right]_{\{\alpha_1^\vee,\ldots,\alpha_n^\vee\}}\right)^{k_n}. \tag{3.10}$$

This argument allows us to identify the Weyl group of a root system with a subgroup of invertible matrices with integer entries. More precisely, the map

$$\varphi : W \longrightarrow GL_n(\mathbf{Z})$$
$$w \longmapsto T_w$$

is a group homomorphism. Moreover if $T_w = I_n$ for a $w \in W$, then this means w fixes the coroots. Then it must fix the roots as well. Hence w must be the identity element of W. Thus the kernel of φ contains only the identity element. Then by the first isomorphism theorem Weyl group is isomorphic to a subgroup of $GL_n(\mathbf{Z})$.

We have

$$w \cdot \omega_j = \omega_j \cdot T_w$$

where the operation on the left hand side is the action of W on $\mathcal{P}(\Phi)$ given in Equation 3.4 and the operation on the right hand side is a matrix multiplication for which ω_j is a row vector for the basis $\{\omega_1, \ldots \omega_n\}$ whose j^{th} component is 1, and T_w is as it is given in Equation 3.10.

Example 3.34 ($\Phi = A_2$). *By using Equation 3.9, we find the generators and hence*

$$W(A_2) \simeq \langle \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \rangle$$
$$= \left\{ \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right\}.$$

Example 3.35 ($\Phi = B_2$). *By using Equation 3.9, we find the generators and hence*

$$W(B_2) \simeq \langle \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \rangle$$
$$= \left\{ \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right\}.$$

Example 3.36 ($\Phi = G_2$). *By using Equation 3.9, we find the generators and hence*

$$W(G_2) \simeq \langle \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \rangle$$
$$= \left\{ \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \right\}.$$

CHAPTER 4

GENERALIZED CHEBYSHEV POLYNOMIALS

In this chapter, in section 4.1, we will follow [1] and review the exponential invariants of Bourbaki. We will prove that the subalgebra of invariant elements under the action of the Weyl group on lattice of fundamental weights is isomorphic to a certain polynomial algebra. In section 4.2, we will follow [5] and introduce the concept of the generalized cosine function. We will also compute the generalized cosine function of the rank 2 root systems. In the last section 4.3, we give several examples of generalized Chebyshev polynomials from [14].

4.1 Exponential Invariants

Let A be a commutative ring with a unit element and let P be a free $\mathbf{Z}$-module of finite rank l. We denote by $A[P]$ the group algebra of the additive group of P over A. For any $p \in P$, denote by e^p the corresponding element of $A[P]$. Then the elements $(e^p)_{p\in P}$ form a basis of the A-module $A[P]$, and for any $p, p' \in P$, we have

$$e^p e^{p'} = e^{p+p'}, \quad (e^p)^{-1} = e^{-p}, \quad e^0 = 1.$$

Example 4.1. *Let Φ be a root system of rank n in a Euclidean space $\mathbf{E}$. The lattice $\mathcal{P}(\Phi)$ is a free $\mathbf{Z}$-module of rank n with basis consisting of the fundamental weights. The group algebra $\mathbf{Z}[\mathcal{P}(\Phi)]$ is free $\mathbf{Z}$-module with basis $(e^p)_{p\in\mathcal{P}(\Phi)}$ and consists of the elements of the form*

$$x = \sum_{p\in\mathcal{P}(\Phi)} x_p e^p$$

with $x_p \in \mathbf{Z}$.

Definition 4.2. *Let* $x = \sum_{p \in \mathcal{P}(\Phi)} x_p e^p$ *be an element of* $\mathbf{Z}[\mathcal{P}(\Phi)]$. *The set* S *of* $p \in \mathcal{P}(\Phi)$ *such that* $x_p \neq 0$ *is called the support of* x *and the set* X *of maximal elements of* S *is called the maximal support of* x. *A term* $x_p e^p$ *is then called a maximal term of* x.

Here, maximal elements are determined with respect to the partial order $\geq$. That is if $p, p' \in \mathcal{P}(\Phi)$, $p \geq p'$ if and only if $p - p'$ is a linear combination of the basis elements α_i with positive coefficients.

Example 4.3. *Consider the root system* A_2 *and the action of its Weyl group* W *on the lattice* $\mathcal{P}(A_2)$. *We found in Example 3.31 that the orbit of* ω_1 *is given by*

$$\phi_1 = \{\omega_1, -\omega_2, -\omega_1 + \omega_2\}.$$

Now consider the characteristic function $\chi_{\phi_1} : \mathcal{P}(A_2) \to \mathbf{R}$ *defined by*

$$\chi_{\phi_1}(p) = \begin{cases} 1 & \text{if } p \in \phi_1 \\ 0 & \text{if } p \notin \phi_1 \end{cases} .$$

If we take the element $x = \sum_{p \in \mathcal{P}(A_2)} \chi_{\phi_1}(p) e^p$ *of* $\mathbf{Z}[\mathcal{P}(A_2)]$ *then we see that support* S *of* x *is equal to the orbit* ϕ_1. *Moreover if we write the elements of* S *in terms of the simple roots* α_1 *and* α_2 *we get*

$$S = \{(2/3)\alpha_1 + (1/3)\alpha_2, -(1/3)\alpha_1 - (2/3)\alpha_2, -(1/3)\alpha_1 + (1/3)\alpha_2\}.$$

Therefore the only maximal element is $(2/3)\alpha_1 + (1/3)\alpha_2 = \omega_1$. *Hence* $X = \{\omega_1\}$ *is the maximal support of* x. *This tells us that* $\chi_{\phi_1}(\omega_1)e^{\omega_1} = e^{\omega_1}$ *is the unique maximal term of* x.

Lemma 4.4. *Let* $x \in \mathbf{Z}[\mathcal{P}(\Phi)]$ *and let* $(x_p e^p)_{p \in X}$ *be the family of maximal terms of* x. *Let* $q \in \mathcal{P}(\Phi)$ *and let* $y \in \mathbf{Z}[\mathcal{P}(\Phi)]$ *be such that* e^q *is the unique maximal term of* y. *Then the family of maximal terms of* xy *is* $(x_p e^{p+q})_{p \in X}$.

Proof. We will give the idea of the proof. Let $x = \sum_{p \in \mathcal{P}(\Phi)} x_p e^p$, $y = \sum_{r \in \mathcal{P}(\Phi)} y_r e^r$ and $xy = \sum_{t \in \mathcal{P}(\Phi)} z_t e^t$. Since e^q is the unique maximal term of y, $r \leq q$ for all $r \in \mathcal{P}(\Phi)$ such that $y_r \neq 0$ and $z_t = \sum_{t = p + r} x_p y_r$. Then it is shown that $X + q$ is the maximal support of xy. For details see Bourbaki [1, p.195]. $\square$

Now let Φ be a root system with base $B = (\alpha_i)_{1\leq i\leq l}$ and $\overline{C}$ be its fundamental region. Then we see that the elements of $\mathcal{P}(\Phi) \cap \overline{C}$ can be written uniquely in the form $n_1\omega_1 + \cdots + n_l\omega_l$ where $n_i \in \mathbf{N}$ for each $i = 1, 2, ..., l$.

We have two important lemmas about the set $\mathcal{P}(\Phi) \cap \overline{C}$.

Lemma 4.5. *Every orbit of W in $\mathcal{P}(\Phi)$ meets $\mathcal{P}(\Phi) \cap \overline{C}$ in exactly one point.*

Proof. See [1, p.166, Theorem 2 (ii)] □

Example 4.6. *We found in Example 3.31 that the orbit of ω_1 is given by*

$$\phi_1 = \{\omega_1, -\omega_2, -\omega_1 + \omega_2\}.$$

As we see in Figure 4.1 the orbit ϕ_1 meets $\mathcal{P}(A_2) \cap \overline{C}$ in exactly one point.

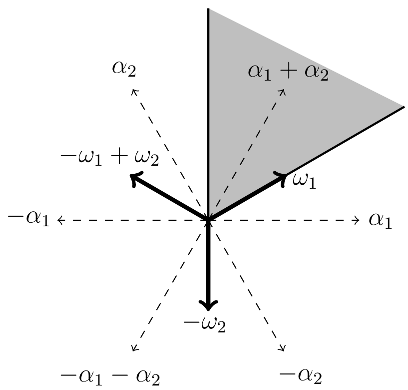


Figure 4.1: Orbit of ω_1 meets $\mathcal{P}(A_2) \cap \overline{C}$ in ω_1 only.

Lemma 4.7. *If $p \in \mathcal{P}(\Phi) \cap \overline{C}$ then $p \geq w(p)$ for all $w \in W$.*

Proof. See [1, p.171, Proposition 18] □

Definition 4.8. *Let $x = \sum_{p\in\mathcal{P}(\Phi)} x_p e^p$ be an arbitrary element of $\mathbf{Z}[\mathcal{P}(\Phi)]$. Then x is said to be invariant under W if $x_{w(p)} = x_p$ for all $p \in \mathcal{P}(\Phi)$ and for all $w \in W$.*

Example 4.9. *Let Φ be a root system and $p \in \mathcal{P}(\Phi)$. Let us denote by $W \cdot p$ the orbit of p under the action of W. Consider the element $S(e^p) = \sum_{q\in W\cdot p} e^q$. Then we have*

$$S(e^{w(p)}) = \sum_{q \in W \cdot w(p)} e^q = \sum_{q \in W \cdot p} e^q = S(e^p).$$

Therefore the elements $S(e^p)$ for $p \in \mathcal{P}(\Phi)$ are invariant under the action of W. Moreover if $p \in \mathcal{P}(\Phi) \cap \overline{C}$ then by Lemma 4.7, the term e^p will be the unique maximal term of $S(e^p)$.

Proposition 4.10. *Let $\mathbf{Z}[\mathcal{P}(\Phi)]^W$ be the subalgebra of $\mathbf{Z}[\mathcal{P}(\Phi)]$ consisting of the elements invariant under W. Then $S(e^p)$ for $p \in \mathcal{P}(\Phi) \cap \overline{C}$ form a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathcal{P}(\Phi)]^W$.*

Proof. We know by Lemma 4.1 every orbit of W in $\mathcal{P}(\Phi)$ meets $\mathcal{P}(\Phi) \cap \overline{C}$ in exactly one point. It follows from here that if $x = \sum_{p \in \mathcal{P}(\Phi)} x_p e^p$ is an arbitrary element of $\mathbf{Z}[\mathcal{P}(\Phi)]^W$ then $x = \sum_{p \in \mathcal{P}(\Phi) \cap \overline{C}} x_p S(e^p)$. □

Observe that for $p \in \mathcal{P}(\Phi) \cap \overline{C}$, $S(e^p)$ is an element of $\mathbf{Z}[\mathcal{P}(\Phi)]^W$ with unique maximal term e^p, and we have seen that the family $(S(e^p))_{p \in \mathcal{P}(\Phi) \cap \overline{C}}$ forms a basis for $\mathbf{Z}[\mathcal{P}(\Phi)]^W$. More generally, we have

Proposition 4.11. *For any $p \in \mathcal{P}(\Phi) \cap \overline{C}$, let x_p be an element of $\mathbf{Z}[\mathcal{P}(\Phi)]^W$ with unique maximal term e^p. Then the family $(x_p)_{p \in \mathcal{P}(\Phi) \cap \overline{C}}$ is a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathcal{P}(\Phi)]^W$.*

Proof. See [1, p.199, Proposition 3] □

Theorem 4.12. *Let $\omega_1, ..., \omega_l$ be the fundamental weights corresponding to the chamber C, and for $1 \leq i \leq l$, let x_i be an element of $\mathbf{Z}[\mathcal{P}(\Phi)]^W$ with e^{ω_i} as its unique maximal term. Let*

$$\varphi : \mathbf{Z}[X_1, ..., X_l] \to \mathbf{Z}[\mathcal{P}(\Phi)]^W$$

be the homomorphism from the polynomial algebra $\mathbf{Z}[X_1, ..., X_l]$ to $\mathbf{Z}[\mathcal{P}(\Phi)]^W$ that takes X_i to x_i. Then, the map φ is an isomorphism.

Proof. Consider the monomial $X_1^{n_1} \cdots X_l^{n_l}$. Then

$$\begin{aligned}\varphi(X_1^{n_1} \cdots X_l^{n_l}) &= \varphi(X_1^{n_1}) \cdots \varphi(X_l^{n_l}) \\ &= x_1^{n_1} \cdots x_l^{n_l}.\end{aligned}$$

We know by assumption that each of x_i for $1 \leq i \leq l$ is an element of $\mathbf{Z}[\mathcal{P}(\Phi)]^W$ with e^{ω_i} as its unique maximal term. By Lemma 4.4 the maximal term of the image $x_1^{n_1} \cdots x_l^{n_l}$ is $e^{n_1\omega_1 + \cdots + n_l\omega_l}$. We also know that the elements of $\mathcal{P}(\Phi) \cap \overline{C}$ are of the form $n_1\omega_1 + \cdots + n_l\omega_l$. This means the image $x_1^{n_1} \cdots x_l^{n_l}$ is an element of the form x_p with unique maximal term e^p for some $p \in \mathcal{P}(\Phi) \cap \overline{C}$. Therefore by Proposition 4.11 the elements $x_1^{n_1} \cdots x_l^{n_l}$ form a basis for $\mathbf{Z}[\mathcal{P}(\Phi)]^W$. Hence φ is an isomorphism. $\square$

Example 4.13. *Let $\omega_1, ..., \omega_l$ be the fundamental weights of a root system Φ corresponding to the chamber C. For each $1 \leq i \leq l$, the element*

$$S(e^{\omega_i}) = \sum_{q \in W \cdot \omega_i} e^q$$

has e^{ω_1} as its unique maximal term and it is an element of $\mathbf{Z}[\mathcal{P}(\Phi)]^W$. Then the map

$$\begin{aligned}\varphi : \mathbf{Z}[X_1, ..., X_l] &\to \mathbf{Z}[\mathcal{P}(\Phi)]^W \\ X_i &\mapsto S(e^{\omega_i})\end{aligned}$$

is an isomorphism.

4.2 Generalized Cosine Function

Hoffmann and Withers [5] use notions from analysis to study the polynomials coming from exponential invariants of Bourbaki. In this section, we recall their treatment and give some examples.

Definition 4.14. *Let $f : \mathcal{P}(\Phi) \to \mathbf{R}$ have finite support and be invariant under W. Then the Fourier transform of f is defined to be*

$$\mathscr{F}(f)(\mathbf{x}) = \sum_{\mathbf{r} \in \mathcal{P}(\Phi)} f(\mathbf{r}) e^{-2\pi i (\mathbf{r}, \mathbf{x})}.$$

Let Ψ denote the set of functions $f : \mathcal{P}(\Phi) \to \mathbf{R}^n$ with finite support and invariant under W. Then Ψ forms a vector space and a natural basis for this space is the set of characteristic functions

$$\chi_\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \phi \\ 0 & \text{if } \mathbf{x} \notin \phi \end{cases}$$

where ϕ is the orbit of some point in $\mathcal{P}(\Phi)$ under the action of the Weyl group W.

Definition 4.15. *For $k = 1, 2, , ..., n$ let ϕ_k be the orbit generated by the fundamental weight ω_k under the action of the Weyl group and let y_k be the Fourier transform of χ_{ϕ_k}. The generalized cosine function $h : \mathbf{R}^n \to \mathbf{R}^n$ associated with the Weyl group W is defined to be*

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x}), ..., y_n(\mathbf{x})).$$

In the following examples we will calculate the generalized cosine function for the root systems of Lie algebras A_1, A_2, B_2 and G_2.

Example 4.16 ($\Phi = A_1$)**.** *In Example 3.30 we found the orbit of ω_1 to be*

$$\phi_1 = \{\omega_1, -\omega_1\}.$$

Therefore

$$y_1(\mathbf{x}) = \sum_{\mathbf{r} \in \mathcal{P}(\Phi)} \chi_{\phi_1}(\mathbf{r}) e^{-2\pi i (\mathbf{r}, \mathbf{x})} = e^{-2\pi i (-\omega_1, \mathbf{x})} + e^{-2\pi i (\omega_1, \mathbf{x})} = e^{2\pi i u_1} + e^{-2\pi i u_1}$$

where $\mathbf{x} = u_1 \alpha_1^\vee$. From here we obtain

$$h(\mathbf{x}) = y_1(\mathbf{x}) = e^{2\pi i u_1} + e^{-2\pi i u_1} = 2\cos 2\pi u_1.$$

Example 4.17 ($\Phi = A_2$)**.** *In Example 3.31, we found the orbits of the fundamental weights as*

$$\phi_1 = \{\omega_1, -\omega_2, -\omega_1 + \omega_2\},$$
$$\phi_2 = \{\omega_2, -\omega_1, \omega_1 - \omega_2\}.$$

Therefore we have

$$\begin{aligned}
y_1(\mathbf{x}) = \mathscr{F}(\chi_{\phi_1})(\mathbf{x}) &= \sum_{\mathbf{r}\in\mathcal{P}(\Phi)} \chi_{\phi_1}(\mathbf{r}) e^{-2\pi i(\mathbf{r},\mathbf{x})} \\
&= e^{-2\pi i(\omega_1,\mathbf{x})} + e^{-2\pi i(-\omega_2,\mathbf{x})} + e^{-2\pi i(-\omega_1+\omega_2,\mathbf{x})} \\
&= e^{-2\pi i u_1} + e^{2\pi i u_2} + e^{2\pi i(u_1-u_2)}, \\
y_2(\mathbf{x}) = \mathscr{F}(\chi_{\phi_2})(\mathbf{x}) &= \sum_{\mathbf{r}\in\mathcal{P}(\Phi)} \chi_{\phi_2}(\mathbf{r}) e^{-2\pi i(\mathbf{r},\mathbf{x})} \\
&= e^{-2\pi i(\omega_2,\mathbf{x})} + e^{-2\pi i(-\omega_1,\mathbf{x})} + e^{-2\pi i(\omega_1-\omega_2,\mathbf{x})} \\
&= e^{-2\pi i u_2} + e^{2\pi i u_1} + e^{-2\pi i(u_1-u_2)}
\end{aligned}$$

where $\mathbf{x} = u_1\alpha_1^\vee + u_2\alpha_2^\vee$. *Hence*

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x})).$$

Example 4.18 ($\Phi = B_2$)**.** *In Example 3.32, we found the orbits of the fundamental weights as*

$$\begin{aligned}
\phi_1 &= \{\omega_1, -\omega_1, \omega_1 - 2\omega_2, -\omega_1 + 2\omega_2\}, \\
\phi_2 &= \{\omega_2, -\omega_2, \omega_1 - \omega_2, -\omega_1 + \omega_2\}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
y_1(\mathbf{x}) = \mathscr{F}(\chi_{\phi_1})(\mathbf{x}) &= \sum_{\mathbf{r}\in\mathcal{P}(\Phi)} \chi_{\phi_1}(\mathbf{r}) e^{-2\pi i(\mathbf{r},\mathbf{x})} \\
&= e^{-2\pi i(\omega_1,\mathbf{x})} + e^{-2\pi i(-\omega_1,\mathbf{x})} + e^{-2\pi i(\omega_1-2\omega_2,\mathbf{x})} + e^{-2\pi i(-\omega_1+2\omega_2,\mathbf{x})} \\
&= e^{-2\pi i u_1} + e^{2\pi i u_1} + e^{-2\pi i(u_1-2u_2)} + e^{2\pi i(u_1-2u_2)}, \\
y_2(\mathbf{x}) = \mathscr{F}(\chi_{\phi_2})(\mathbf{x}) &= \sum_{\mathbf{r}\in\mathcal{P}(\Phi)} \chi_{\phi_2}(\mathbf{r}) e^{-2\pi i(\mathbf{r},\mathbf{x})} \\
&= e^{-2\pi i(\omega_2,\mathbf{x})} + e^{-2\pi i(-\omega_2,\mathbf{x})} + e^{-2\pi i(\omega_1-\omega_2,\mathbf{x})} + e^{-2\pi i(-\omega_1+\omega_2,\mathbf{x})} \\
&= e^{-2\pi i u_2} + e^{2\pi i u_2} + e^{-2\pi i(u_1-u_2)} + e^{2\pi i(u_1-u_2)}
\end{aligned}$$

where $\mathbf{x} = u_1\alpha_1^\vee + u_2\alpha_2^\vee$. *Hence*

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x})).$$

Example 4.19 ($\Phi = G_2$). *In Example 3.33, we found the orbits of the fundamental weights as*

$$\phi_1 = \{\omega_1, -\omega_1, \omega_1 - \omega_2, -\omega_1 + \omega_2, 2\omega_1 - \omega_2, -2\omega_1 + \omega_2\},$$
$$\phi_2 = \{\omega_2, -\omega_2, 3\omega_1 - \omega_2, -3\omega_1 + \omega_2, 3\omega_1 - 2\omega_2, -3\omega_1 + 2\omega_2\}.$$

Therefore we have

$$\begin{aligned}
y_1(\mathbf{x}) = \mathscr{F}(\chi_{\phi_1})(\mathbf{x}) &= \sum_{\mathbf{r}\in\mathcal{P}(\Phi)} \chi_{\phi_1}(\mathbf{r}) e^{-2\pi i(\mathbf{r},\mathbf{x})} \\
&= e^{-2\pi i(\omega_1,\mathbf{x})} + e^{-2\pi i(-\omega_1,\mathbf{x})} + e^{-2\pi i(\omega_1-\omega_2,\mathbf{x})} \\
&\quad + e^{-2\pi i(-\omega_1+\omega_2,\mathbf{x})} + e^{-2\pi i(2\omega_1-\omega_2,\mathbf{x})} + e^{-2\pi i(-2\omega_1+\omega_2,\mathbf{x})} \\
&= e^{-2\pi i u_1} + e^{2\pi i u_1} + e^{-2\pi i(u_1-u_2)} + e^{2\pi i(u_1-u_2)} + e^{-2\pi i(2u_1-u_2)} + e^{2\pi i(2u_1-u_2)}, \\
y_2(\mathbf{x}) = \mathscr{F}(\chi_{\phi_2})(\mathbf{x}) &= \sum_{\mathbf{r}\in\mathcal{P}(\Phi)} \chi_{\phi_2}(\mathbf{r}) e^{-2\pi i(\mathbf{r},\mathbf{x})} \\
&= e^{-2\pi i(\omega_2,\mathbf{x})} + e^{-2\pi i(-\omega_2)\mathbf{x}} + e^{-2\pi i(3\omega_1-\omega_2,\mathbf{x})} \\
&\quad + e^{-2\pi i(-3\omega_1+\omega_2,\mathbf{x})} + e^{-2\pi i(3\omega_1-2\omega_2,\mathbf{x})} + e^{-2\pi i(-3\omega_1+2\omega_2,\mathbf{x})} \\
&= e^{-2\pi i u_2} + e^{2\pi i u_2} + e^{-2\pi i(3u_1-u_2)} + e^{2\pi i(3u_1-u_2)} + e^{-2\pi i(3u_1-2u_2)} + e^{2\pi i(3u_1-2u_2)}.
\end{aligned}$$

where $\mathbf{x} = u_1\alpha_1^\vee + u_2\alpha_2^\vee$. *Hence*

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x})).$$

Now we will see a very important theorem that will allow us to understand where the definition of generalized Chebyshev polynomials comes from. First, we need the following lemma:

Lemma 4.20. *If* $f : \mathcal{P}(\Phi) \to \mathbf{R}$ *has finite support and is invariant under* W, *then* $\mathscr{F}(f)$ *has also finite support and is invariant as a function of* $\mathbf{x}$ *under* W.

Theorem 4.21. *For* $j = 1, 2, ..., n$ *let* y_j *be the Fourier transform of* χ_{ϕ_j}. *Then for any* $f \in \Psi, \mathscr{F}(f)$ *can be written as a polynomial in* $\mathbf{y} = (y_1, ..., y_n)$.

Proof. We will see that this theorem is same as Theorem 4.12 written in different notations. First, we know that the characteristic function $\chi_{\phi_j} : \mathcal{P}(\Phi) \to \mathbf{R}$ has finite support and it is invariant under W. Therefore $y_j(\mathbf{x}) \in \Psi$ by Lemma 4.20. Consider the map

$$\begin{aligned} \varphi : \mathbf{Z}[X_1, ..., X_n] &\to \Psi \\ X_j &\mapsto y_j \end{aligned}$$

then we have

$$y_j(\mathbf{x}) = \mathscr{F}(\chi_{\phi_j})(\mathbf{x}) = \sum_{\mathbf{r} \in \mathcal{P}(\Phi)} \chi_{\phi_j}(\mathbf{r}) e^{-2\pi i (\mathbf{r}, \mathbf{x})} = \sum_{\mathbf{r} \in \phi_j} e^{-2\pi i (\mathbf{r}, \mathbf{x})} = \sum_{\mathbf{r} \in \phi_j} e^{-2\pi i \mathbf{r}(\mathbf{x})} \tag{4.1}$$

where the last equality follows from the fact that each $\mathbf{r} \in \phi_j$ is a $\mathbf{Z}$-linear combination of fundamental weights and each fundamental weight ω_j is a functional defined by $\omega_j(\alpha_k^\vee) = \delta_{jk}$. Moreover, $\omega_j(\mathbf{x}) = (\omega_j, \mathbf{x})$ where $\mathbf{x}$ is written in coroot basis. Therefore each $\mathbf{r}$ is of the form $\mathbf{r}(\mathbf{x}) = (\mathbf{r}, \mathbf{x})$. Since Equation 4.1 is true for every $\mathbf{x}$,

$$y_j = \sum_{\mathbf{r} \in \phi_j} e^{-2\pi i \mathbf{r}}$$

Observe that y_j is an element of Ψ with $e^{2\pi i \omega_j}$ as its unique maximal term. Therefore φ is an isomorphism by Theorem 4.12. $\square$

Theorem 4.22. *The function $h(k\mathbf{x})$ is a polynomial in $h(\mathbf{x})$ for integer $k \geq 1$.*

Proof. To see that $h(k\mathbf{x})$ is a polynomial in $h(\mathbf{x})$ for integer $k \geq 1$, we need to show that each component $y_j(k\mathbf{x})$ of $h(k\mathbf{x})$ is a polynomial in $h(\mathbf{x})$. Indeed

$$y_j(k\mathbf{x}) = \sum_{\mathbf{r} \in \mathcal{P}(\Phi)} \chi_{\phi_j}(\mathbf{r}) e^{-2\pi i (\mathbf{r}, k\mathbf{x})} = \sum_{\mathbf{r} \in \mathcal{P}(\Phi)} f(\mathbf{r}) e^{-2\pi i (\mathbf{r}, \mathbf{x})} = \mathscr{F}(f)(\mathbf{x})$$

$$\text{where} \quad f(\mathbf{r}) = \begin{cases} \chi_{\phi_j}(\mathbf{r}/k) & \text{if } \mathbf{r}/k \in \mathcal{P}(\Phi) \\ 0 & \text{otherwise} \end{cases}$$

Since $f \in \Psi$, by Theorem 4.21 $y_j(k\mathbf{x})$ is a polynomial in $h(\mathbf{x})$. $\square$

The fact that $h(k\mathbf{x})$ is a polynomial in $h(\mathbf{x})$ yields to the following definition:

Definition 4.23. *For integer $k \geq 1$, the generalized Chebyshev polynomials (or folding polynomials) associated with the Lie algebra $\mathfrak{g}$ are the polynomial functions $P_{\mathfrak{g}}^k : \mathbf{R}^n \to \mathbf{R}^n$ defined by*

$$P_{\mathfrak{g}}^k(h(\mathbf{x})) = h(k\mathbf{x}).$$

It follows from this definition that $P_{\mathfrak{g}}^{kl} = P_{\mathfrak{g}}^k \circ P_{\mathfrak{g}}^l$. The coefficients of these polynomials are integers because each component of $P_{\mathfrak{g}}^k$ is a polynomial in y_i with integer coefficients for each $i = 1, 2, ..., n$ by Theorem 4.21.

4.3 Examples of Generalized Chebyshev Polynomials

In this section, we will see several examples of generalized Chebyshev polynomials. We will calculate the ones associated with the root systems A_2, B_2 and G_2 for $k = 2$. For simplicity, we will use the notation $\mathcal{A}_k := P_{A_2}^k, \mathcal{B}_k := P_{B_2}^k$, and $\mathcal{G}_k := P_{G_2}^k$.

Example 4.24. *We found in Example 4.17 that the generalized cosine function associated with the Lie algebra A_2 is given by*

$$\begin{aligned} h(\mathbf{x}) &= (y_1(\mathbf{x}), y_2(\mathbf{x})) \\ &= \left(e^{-2\pi i u_1} + e^{2\pi i u_2} + e^{2\pi i(u_1-u_2)}, e^{-2\pi i u_2} + e^{2\pi i u_1} + e^{-2\pi i(u_1-u_2)}\right) \end{aligned}$$

where $\mathbf{x} = u_1\alpha_1^\vee + u_2\alpha_2^\vee$. This gives us that

$$\begin{aligned} h(2\mathbf{x}) &= (y_1(2\mathbf{x}), y_2(2\mathbf{x})) \\ &= \left(e^{-4\pi i u_1} + e^{4\pi i u_2} + e^{4\pi i(u_1-u_2)}, e^{-4\pi i u_2} + e^{4\pi i u_1} + e^{-4\pi i(u_1-u_2)}\right) \\ &= \left(y_1(\mathbf{x})^2 - 2y_2(\mathbf{x}), y_2(\mathbf{x})^2 - 2y_1(\mathbf{x})\right). \end{aligned}$$

Therefore the generalized Chebyshev polynomial of A_2 for $k = 2$ determined from the condition $\mathcal{A}_2(h(\mathbf{x})) = h(2\mathbf{x})$ is given by

$$\mathcal{A}_2(x, y) = (x^2 - 2y, y^2 - 2x).$$

Note that the first few generalized Chebyshev polynomials of A_2 are given by

$$\begin{aligned}
\mathcal{A}_0(x,y) &= (3,3),\\
\mathcal{A}_1(x,y) &= (x,y),\\
\mathcal{A}_2(x,y) &= (x^2-2y, y^2-2x),\\
\mathcal{A}_3(x,y) &= (x^3-3xy+3, y^3-3xy+3),\\
\mathcal{A}_4(x,y) &= (x^4-4x^2y+2y^2+4x, y^4-4y^2x+2x^2+4y),\\
\mathcal{A}_5(x,y) &= (x^5-5x^3y+5xy^2+5x^2-5y, y^5-5y^3x+5yx^2+5y^2-5x).
\end{aligned}$$

Example 4.25. *We know from Example 4.18 that the generalized cosine function associated with the Lie algebra B_2 is given by*

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x}))$$

where

$$\begin{aligned}
y_1(\mathbf{x}) &= e^{-2\pi i u_1} + e^{2\pi i u_1} + e^{-2\pi i(u_1-2u_2)} + e^{2\pi i(u_1-2u_2)},\\
y_2(\mathbf{x}) &= e^{-2\pi i u_2} + e^{2\pi i u_2} + e^{-2\pi i(u_1-u_2)} + e^{2\pi i(u_1-u_2)}
\end{aligned}$$

and $\mathbf{x} = u_1\alpha_1^\vee + u_2\alpha_2^\vee$. Then

$$h(2\mathbf{x}) = (y_1(2\mathbf{x}), y_2(2\mathbf{x}))$$

where

$$\begin{aligned}
y_1(2\mathbf{x}) &= e^{-4\pi i u_1} + e^{4\pi i u_1} + e^{-4\pi i(u_1-2u_2)} + e^{4\pi i(u_1-2u_2)},\\
y_2(2\mathbf{x}) &= e^{-4\pi i u_2} + e^{4\pi i u_2} + e^{-4\pi i(u_1-u_2)} + e^{4\pi i(u_1-u_2)}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
h(2\mathbf{x}) &= (y_1(2\mathbf{x}), y_2(2\mathbf{x}))\\
&= (y_1(\mathbf{x})^2 - 2y_2(\mathbf{x})^2 + 4y_1(\mathbf{x}) + 4, y_2(\mathbf{x})^2 - 2y_1(\mathbf{x}) - 4).
\end{aligned}$$

Therefore the generalized Chebyshev polynomial associated with B_2 for $k=2$ determined from the condition $\mathcal{B}_2(h(\mathbf{x})) = h(2\mathbf{x})$ is given by

$$\mathcal{B}_2(x,y) = (x^2-2y^2+4x+4, y^2-2x-4).$$

Note also that the first few Chebyshev polynomials of B_2 are given by

$$\begin{aligned}
\mathcal{B}_0(x,y) &= (4,4),\\
\mathcal{B}_1(x,y) &= (x,y),\\
\mathcal{B}_2(x,y) &= (x^2-2y^2+4x+4, y^2-2x-4),\\
\mathcal{B}_3(x,y) &= (x^3-3xy^2+6x^2+9x, y^3-3xy-3y),\\
\mathcal{B}_4(x,y) &= (x^4-4x^2y^2+2y^4+8x^3-8xy^2+20x^2-8y^2+16x+4,\\
&\qquad y^4-4xy^2-4y^2+2x^2+8x+4),\\
\mathcal{B}_5(x,y) &= (x^5-5x^3y^2+5xy^4+10x^4-20x^2y^2+35x^3-25xy^2+50x^2+25x,\\
&\qquad y^5-5xy^3-5y^3+5x^2y+15xy+5y).
\end{aligned}$$

Example 4.26. *We know from Example 4.19 that the generalized cosine function associated with the root system G_2 is given by*

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x}))$$

where $\mathbf{x} = u_1\alpha_1^\vee + u_2\alpha_2^\vee$ and

$$y_1(\mathbf{x}) = e^{-2\pi i u_1} + e^{2\pi i u_1} + e^{-2\pi i(u_1-u_2)} + e^{2\pi i(u_1-u_2)} + e^{-2\pi i(2u_1-u_2)} + e^{2\pi i(2u_1-u_2)},$$

$$y_2(\mathbf{x}) = e^{-2\pi i u_2} + e^{2\pi i u_2} + e^{-2\pi i(3u_1-u_2)} + e^{2\pi i(3u_1-u_2)} + e^{-2\pi i(3u_1-2u_2)} + e^{2\pi i(3u_1-2u_2)}.$$

Then $h(2\mathbf{x}) = (y_1(2\mathbf{x}), y_2(2\mathbf{x}))$ where

$$y_1(2\mathbf{x}) = e^{-4\pi i u_1} + e^{4\pi i u_1} + e^{-4\pi i(u_1-u_2)} + e^{4\pi i(u_1-u_2)} + e^{-4\pi i(2u_1-u_2)} + e^{4\pi i(2u_1-u_2)},$$

$$y_2(2\mathbf{x}) = e^{-4\pi i u_2} + e^{4\pi i u_2} + e^{-4\pi i(3u_1-u_2)} + e^{4\pi i(3u_1-u_2)} + e^{-4\pi i(3u_1-2u_2)} + e^{4\pi i(3u_1-2u_2)}.$$

When we write each component of $h(2\mathbf{x})$ in terms of y_1 and y_2 we get

$$h(2\mathbf{x}) = (y_1^2 - 2y_1 - 2y_2 - 6, y_2^2 - 2y_1^3 + 6y_1y_2 + 10y_2 + 18y_1 + 18).$$

Therefore the generalized Chebyshev polynomial of G_2 for $k = 2$ determined from the condition $\mathcal{G}_2(h(\mathbf{x})) = h(2\mathbf{x})$ is given by

$$\mathcal{G}_2(x,y) = (x^2 - 2x - 2y - 6, y^2 - 2x^3 + 6xy + 10y + 18x + 18).$$

Note that the first few generalized Chebyshev polynomials of G_2 are given by

$$\mathcal{G}_0(x,y) = (6,6),$$

$$\mathcal{G}_1(x,y) = (x,y),$$

$$\mathcal{G}_2(x,y) = (x^2 - 2x - 2y - 6, y^2 - 2x^3 + 6xy + 10y + 18x + 18),$$

$$\mathcal{G}_3(x,y) = (x^3 - 3xy - 9x - 6y - 12, y^3 - 3x^3y + 9xy^2 - 6x^3 + 18y^2 + 45xy + 63y + 54x + 60),$$

$$\begin{aligned}\mathcal{G}_4(x,y) = (&x^4 - 4x^2y - 10x^2 - 4xy + 2y^2 - 8x + 8y + 6,\\ &y^4 + 2x^6 - 4x^3y^2 - 12x^4y + 12x^4y + 12xy^3 + 18x^2y^2 - 28x^3y - 36x^4 + 24y^3\\ &+120xy^2 + 108x^2y - 40x^3 + 134y^2 + 372xy + 162x^2 + 280y + 360x + 198),\end{aligned}$$

$$\begin{aligned}\mathcal{G}_5(x,y) = (&x^5 - 5x^3y - 15x^3 - 5x^2y + 5xy^2 - 10x^2 + 35xy + 10y^2 + 55x + 50y + 60,\\ &y^5 + 5x^6y - 5x^3y^3 - 30x^4y^2 + 10x^6 + 15xy^4 + 45x^2y^3 - 65x^3y^2 - 150x^4y\\ &+30y^4 + 240xy^3 + 360x^2y^2 - 205x^3y - 180x^4 + 255y^3 + 1200xy^2\\ &+945x^2y - 190x^3 + 920y^2 + 2415xy + 810x^2 + 1495y + 1710x + 900).\end{aligned}$$

CHAPTER 5

COEFFICIENTS OF FOLDING POLYNOMIALS ATTACHED TO LIE ALGEBRAS OF RANK TWO

In this chapter, by using Waring's formula we will find a closed formula for generalized Chebyshev polynomials attached to Lie algebras of rank 2. We follow Cox [3] for the basic definitions related to multivariate polynomials. We recall Waring's formula from [9] which will be a key tool in finding the closed forms of these polynomials. We also explain why these polynomials satisfy the recurrence relations given in [14] and prove some remarkable properties of their coefficients.

Definition 5.1. *A monomial in $x_1, x_2, ..., x_n$ is a product of the form $x_1^{d_1} \cdot x_2^{d_2} \cdots x_n^{d_n}$ where all the exponents $d_1, d_2, ..., d_n$ are nonnegative integers. The total degree of this monomial is $d_1 + d_2 + \cdots + d_n$.*

We can simplify the notation for monomials as follows:

Let $d = (d_1, ..., d_n)$ be an n-tuple of nonnegative integers. Then we set

$$x^d = x_1^{d_1} \cdot x_2^{d_2} \cdots x_n^{d_n}.$$

We also let $|d| = d_1 + d_2 + \cdots + d_n$ denote the total degree of the monomial x^d.

Definition 5.2. *Let $\mathbf{F}$ be a field and let $f = \sum_d k_d x^d$, $k_d \in \mathbf{F}$ be a polynomial in $\mathbf{F}[x_1, ..., x_n]$ where the sum is over a finite number of n-tuples $d = (d_1, ..., d_n)$. Then*

1. *k_d is called the coefficient of the monomial x^d.*
2. *The total degree of $f \neq 0$, denoted $deg(f)$, is the maximum $|d|$ such that the coefficient k_d is nonzero.*
3. *The multidegree of f is multideg$(f) = max(d \in \mathbf{Z}^n_{\geq 0} : k_d \neq 0)$.*

Definition 5.3. *Let R be a commutative ring with identity and let $R[x_1, x_2, ..., x_n]$ be the ring of polynomials in n variables. A polynomial $f \in R[x_1, x_2, ..., x_n]$ is called symmetric if $f(x_{i_1}, x_{i_2}, ..., x_{i_n}) = f(x_1, x_2, ..., x_n)$ for any permutation $i_1, i_2, ..., i_n$ of the integers $1, 2, ..., n$.*

Example 5.4. *Let x be an indeterminate over $R[x_1, x_2, ..., x_n]$ and let F be the polynomial*

$$F(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

then

$$F(x) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \cdots + (-1)^n \sigma_n$$

where

$$\sigma_k = \sigma_k(x_1, x_2, ..., x_n) = \sum_{i_1 < i_2 < ... < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

The polynomials σ_k for $1 \leq k \leq n$ serve as the most classical examples of symmetric polynomials. They are called the elementary symmetric polynomials.

Example 5.5. *Another important example of symmetric polynomials are the k^{th} power sums defined by*

$$s_k = s_k(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i^k \quad \textit{for each} \quad k = 0, 1, 2, ..., n.$$

There are two important relations between the elementary symmetric polynomials and k^{th} power sums. The first relation will allow us to explain where the recurrence relations satisfied by generalized Chebyshev polynomials come from. On the other hand, the second relation, namely Waring's formula, will be a key tool in finding the closed forms of these polynomials.

Theorem 5.6 (Newton's Identity)**.** *Let $x_1, x_2, ..., x_n$ be variables and $\sigma_1, ..., \sigma_n$ and $s_0, ..., s_n$ be the elementary symmetric polynomials and power sum polynomials in these variables respectively. Then*

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} + \cdots + (-1)^n \sigma_n s_0 = 0.$$

Proof. First of all, we know by Example 5.4 that

$$(x - x_1)(x - x_2)\cdots(x - x_n) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \cdots + (-1)^n \sigma_n.$$

If we substitute $x = x_i$ for each $i = 1, 2, ..., n$ we obtain

$$\begin{aligned} x_1^n - \sigma_1 x_1^{n-1} + \sigma_2 x_1^{n-2} + \cdots + (-1)^n \sigma_n = 0 \\ x_2^n - \sigma_1 x_2^{n-1} + \sigma_2 x_2^{n-2} + \cdots + (-1)^n \sigma_n = 0 \\ \vdots \\ x_n^n - \sigma_1 x_n^{n-1} + \sigma_2 x_n^{n-2} + \cdots + (-1)^n \sigma_n = 0. \end{aligned}$$

If we add these equalities we get

$$(x_1^n + \cdots + x_n^n) - \sigma_1(x_1^{n-1} + \cdots + x_n^{n-1}) + \cdots + (-1)^n \sigma_n n = 0.$$

This can be written as

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} + \cdots + (-1)^n \sigma_n s_0 = 0.$$

□

We will use Theorem 5.6 to explain the recurrence relations satisfied by the generalized Chebyshev polynomials.

The second relation between the k^{th} power sums and the elementary symmetric polynomials arises from the following theorem:

Theorem 5.7 (Fundamental Theorem of Symmetric Polynomials). *Every symmetric polynomial in $R[x_1, x_2, ..., x_n]$ can be written uniquely as a polynomial in the elementary symmetric polynomials $\sigma_1, \sigma_2, ..., \sigma_n$.*

As an immediate corollary of this theorem, we see that the k^{th} power sums can be written as a polynomial in $\sigma_1, \sigma_2, ..., \sigma_n$. Moreover we have the following explicit formula:

Theorem 5.8 (Waring's Formula [9, p.30]).

$$s_k = \sum (-1)^{(i_2+i_4+\cdots)} \frac{(i_1 + i_2 + \cdots + i_n - 1)! \cdot k}{i_1!\, i_2! \cdots i_n!} \sigma_1^{i_1} \cdot \sigma_2^{i_2} \cdots \sigma_n^{i_n}$$

for $k \geq 1$, where the summation is extended over all n-tuples $(i_1, i_2, ..., i_n)$ of non-negative integers with $i_1 + 2i_2 + 3i_3 + \cdots + ni_n = k$.

Before we pass to rank 2 examples, we will try to understand how to use Waring's formula with the unique rank 1 example. We will find a general formula for generalized Chebyshev polynomials associated with the Lie algebra A_1, and with the help of this formula, we will give a relation between these polynomials and the Chebyshev polynomials T_k.

5.1 Generalized Chebyshev Polynomials of A_1

Recall that in Example 4.16 we found that the generalized cosine function associated to A_1 is given by

$$h(\mathbf{x}) = e^{2\pi i u_1} + e^{-2\pi i u_1} = 2\cos 2\pi u_1$$

where $\mathbf{x} = u_1\alpha_1^\vee$.

We are looking for the polynomials $P^k_{A_1}$ determined from the following condition:

$$P^k_{A_1}(h(\mathbf{x})) = h(k\mathbf{x}). \tag{5.1}$$

If we compute the first few of these polynomials, we see that

$$\begin{aligned}
&h(1\mathbf{x}) = h(\mathbf{x}) \Longrightarrow P^1_{A_1}(x) = x\\
&h(2\mathbf{x}) = h^2(\mathbf{x}) - 2 \Longrightarrow P^2_{A_1}(x) = x^2 - 2\\
&h(3\mathbf{x}) = h^3(\mathbf{x}) - 3h(\mathbf{x}) \Longrightarrow P^3_{A_1}(x) = x^3 - 3x\\
&h(4\mathbf{x}) = h^4(\mathbf{x}) - 4h^2(\mathbf{x}) + 2 \Longrightarrow P^4_{A_1}(x) = x^4 - 4x^2 + 2\\
&h(5\mathbf{x}) = h^5(\mathbf{x}) - 5h^3(\mathbf{x}) + 5h(\mathbf{x}) \Longrightarrow P^5_{A_1}(x) = x^5 - 5x^3 + 5x.
\end{aligned}$$

In these calculations, we are writing $s_k = x_1^k + x_2^k$ in terms of $\sigma_1 = x_1 + x_2$ and $\sigma_2 = x_1x_2$ where $x_1 = e^{2\pi i u_1}$ and $x_2 = e^{-2\pi i u_1}$. Note that $\sigma_2 = 1$. Therefore we can use Waring's formula to obtain $P^k_{A_1}$ for an arbitrary integer $k \geq 1$.

Observe that when there are two indeterminates, Waring's formula gives us that

$$x_1^k + x_2^k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{(k-i-1)!\,k}{(k-2i)!\,i!}(x_1 + x_2)^{k-2i} \cdot (x_1x_2)^i.$$

Substituting $x_1 = e^{2\pi i u_1}$ and $x_2 = e^{-2\pi i u_1}$ in this equation gives that

$$e^{2\pi i k u_1} + e^{-2\pi i k u_1} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{(k-i-1)!\,k}{(k-2i)!\,i!} (e^{2\pi i u_1} + e^{-2\pi i u_1})^{k-2i}.$$

Rewriting this in terms of $h(\mathbf{x})$ and $h(k\mathbf{x})$ gives us

$$h(k\mathbf{x}) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{(k-i-1)!\,k}{(k-2i)!\,i!} (h(\mathbf{x}))^{k-2i}.$$

Using the condition 5.1 we see that

$$P_{A_1}^k(x) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{(k-i-1)!\,k}{(k-2i)!\,i!} x^{k-2i}.$$

Therefore we obtained a general formula for $P_{A_1}^k$. We can also relate these polynomials with the Chebyshev Polynomials T_k. Recall that the Chebyshev polynomials $T_k(x)$ of the first kind are defined to be $T_k(x) = \cos k\theta$ when $x = \cos\theta$.

When we turn back to Equation 5.1, we see that

$$P_{A_1}^k(h(\mathbf{x})) = h(k\mathbf{x}).$$

Substituting $h(\mathbf{x}) = 2\cos 2\pi u_1$ in this equation, we get

$$P_{A_1}^k(2\cos 2\pi u_1) = 2\cos 2\pi k u_1.$$

Writing this last equality in terms of T_k with $x = \cos 2\pi u_1$, we obtain

$$P_{A_1}^k(2x) = 2T_k(x).$$

Finding this relation allows us to find a recurrence relation satisfied by $P_{A_1}^k$. We had

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

for $n = 2, 3, ...$ with $T_0(x) = 1$ and $T_1(x) = x$. Substituting $T_n(x) = 1/2P_{A_1}^n(2x)$ in this recurrence relation we eventually obtain

$$P_{A_1}^n(x) = xP_{A_1}^n(x) - P_{A_1}^{n-2}(x)$$

for $n = 2, 3, ...$ with $P_{A_1}^0(x) = 2$ and $P_{A_1}^1(x) = x$.

5.2 Generalized Chebyshev Polynomials of A_2

We will find a general formula for $\mathcal{A}_k(x, y)$ for an arbitrary k. We know that these polynomials are determined from the equation

$$\mathcal{A}_k(h(\mathbf{x})) = h(k\mathbf{x})$$

where h is the generalized cosine function of A_2 found in Example 4.17. Recall that it is given by

$$h(\mathbf{x}) = \left(e^{-2\pi i u_1} + e^{2\pi i u_2} + e^{2\pi i(u_1-u_2)}, e^{-2\pi i u_2} + e^{2\pi i u_1} + e^{-2\pi i(u_1-u_2)}\right) \tag{5.2}$$

where $\mathbf{x} = u_1\alpha_1^\vee + u_2\alpha_2^\vee$. If we consider $h(k\mathbf{x})$, we will have

$$h(k\mathbf{x}) = \left(e^{-2k\pi i u_1} + e^{2k\pi i u_2} + e^{2k\pi i(u_1-u_2)}, e^{-2k\pi i u_2} + e^{2k\pi i u_1} + e^{-2k\pi i(u_1-u_2)}\right)$$

We observe that finding $\mathcal{A}_k(x, y)$ is equivalent to writing $y_1(k\mathbf{x})$ and $y_2(k\mathbf{x})$ in terms of $y_1(\mathbf{x})$ and $y_2(\mathbf{x})$ as we did in Example 4.24 for $k = 2$. By using the first component of Equation 5.2, if we let $x_1 = e^{-2\pi i u_1}$, $x_2 = e^{2\pi i u_2}$ and $x_3 = e^{2\pi i(u_1-u_2)}$, then $y_1(k\mathbf{x})$ becomes $x_1^k + x_2^k + x_3^k$. This means that we can use Theorem 5.8 to write $y_1(k\mathbf{x})$ in terms of $y_1(\mathbf{x})$ and $y_2(\mathbf{x})$. If we compute the elementary symmetric polynomials in x_1, x_2, x_3, we see that

$$\begin{aligned}
\sigma_1 &= \sum_i x_i = y_1(\mathbf{x}),\\
\sigma_2 &= \sum_{i<j} x_i x_j = y_2(\mathbf{x}),\\
\sigma_3 &= x_1 x_2 x_3 = 1.
\end{aligned}$$

To use Waring's formula, we need to find the nonnegative integer solutions of $i_1 + 2i_2 + 3i_3 = k$. If $i_3 = i$ and $i_2 = j$, then the triples $(i_1, i_2, i_3) = (k-2j-3i, j, i) \in \mathbf{N}^3$ provide solutions for this equation. Using Waring's formula we establish that

$$y_1(k\mathbf{x}) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{3} \rfloor} (-1)^j \frac{(k-j-2i-1)! \cdot k}{(k-2j-3i)!\, j!\, i!} \cdot (y_1(\mathbf{x}))^{k-2j-3i} \cdot (y_2(\mathbf{x}))^j.$$

Similarly, if we use the second component of Equation 5.2 and let $x_1 = e^{-2\pi i u_2}$, $x_2 = e^{2\pi i u_1}$ and $x_3 = e^{-2\pi i(u_1-u_2)}$ we see that $y_2(k\mathbf{x}) = x_1^k + x_2^k + x_3^k$. Computing the elementary symmetric polynomials in x_1, x_2, x_3 gives us

$$\sigma_1 = \sum_i x_i = y_2(\mathbf{x})$$
$$\sigma_2 = \sum_{i<j} x_i x_j = y_1(\mathbf{x})$$
$$\sigma_3 = x_1 x_2 x_3 = 1.$$

Therefore we have

$$y_2(k\mathbf{x}) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{3} \rfloor} (-1)^j \frac{(k-j-2i-1)! \cdot k}{(k-2j-3i)!\, j!\, i!} \cdot (y_2(\mathbf{x}))^{k-2j-3i} \cdot (y_1(\mathbf{x}))^j.$$

Hence the polynomials $\mathcal{A}_k(x,y)$, defined by $\mathcal{A}_k(h(\mathbf{x})) = h(k\mathbf{x})$, are given by

$$\mathcal{A}_k(x,y) = (f_k(x,y), g_k(x,y))$$

where

$$f_k(x,y) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{3} \rfloor} A(i,j,k) \cdot x^{k-2j-3i} \cdot y^j, \tag{5.3}$$
$$g_k(x,y) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{3} \rfloor} A(i,j,k) \cdot y^{k-2j-3i} \cdot x^j. \tag{5.4}$$

Here the coefficients $A(i,j,k)$ are given by

$$A(i,j,k) = (-1)^j \frac{(k-j-2i-1)! \cdot k}{(k-2j-3i)!\, j!\, i!} \tag{5.5}$$

where only those terms occur for which $k \geq 3i + 2j$. These polynomials have some remarkable properties. For instance, they satisfy certain recurrence relation, their coefficients are integers and they reduce to Frobenius map over a finite field with $q = p^r$ elements for prime p. We start with explaining the recurrence relation satisfied by these polynomials.

Theorem 5.9. *If*

$$\mathcal{A}_n(x,y) = (f_n(x,y), g_n(x,y)),$$

then

$$f_n = xf_{n-1} - yf_{n-2} + f_{n-3}$$
$$g_n = yg_{n-1} - xg_{n-2} + g_{n-3}.$$

Proof. We know that

$$\mathcal{A}_n(y_1(\mathbf{x}), y_2(\mathbf{x})) = (y_1(n\mathbf{x}), y_2(n\mathbf{x}))$$

where

$$y_1(\mathbf{x}) = x_1 + x_2 + x_3$$
$$y_1(n\mathbf{x}) = x_1^n + x_2^n + x_3^n.$$

Here the variables x_i for $i = 1, 2, 3$ are obtained from the first component of the Equation 5.2. We also found that the elementary symmetric polynomials in x_1, x_2, x_3 are given by

$$\begin{aligned} \sigma_1 &= y_1(\mathbf{x}) \\ \sigma_2 &= y_2(\mathbf{x}) \\ \sigma_3 &= 1. \end{aligned} \tag{5.6}$$

We have $y_1((n-j)\mathbf{x}) = s_{n-j}$ for $j = 0, 1, 2, 3$ where s_{n-j} is the $(n-j)^{th}$ power sum polynomial in variables x_1, x_2, x_3. Recall that by Equation 5.6 we have

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} + \cdots + (-1)^n \sigma_n s_0 = 0.$$

If we substitute each σ_i for $i = 1, 2, 3$ from Equation 5.6 and $s_{n-j} = y_1((n-j)\mathbf{x})$ for $j = 0, 1, 2, 3$ in this equation, we obtain

$$y_1(n\mathbf{x}) - y_1(\mathbf{x}) \cdot y_1((n-1)\mathbf{x}) + y_2(\mathbf{x}) \cdot y_1((n-2)\mathbf{x}) - y_1((n-3)\mathbf{x}) = 0.$$

If we use the same idea for the second component $y_2(n\mathbf{x})$, we get

$$y_2(n\mathbf{x}) - y_2(\mathbf{x}) \cdot y_2((n-1)\mathbf{x}) + y_1(\mathbf{x}) \cdot y_2((n-2)\mathbf{x}) - y_2((n-3)\mathbf{x}) = 0.$$

Then we obtain the recursive relations given in the theorem. □

The second property is about the multidegree of these polynomials.

Theorem 5.10. *multideg*$(\mathcal{A}_k(x, y)) = (k, k)$.

Proof. We see that the total degree of each monomials in $f_k(x, y)$ is $k - j - 3i$ from Equation 5.3 and hence $deg(f_k(x, y)) = k$ which occurs when $i = j = 0$.

The same argument on Equation 5.4 gives that $deg(g_k(x, y)) = k$. Therefore we get multideg$(\mathcal{A}_k(x, y)) = (k, k)$. $\square$

Not only the degrees of these polynomials but also their coefficients have remarkable properties. To investigate these properties, we need the following definition.

Definition 5.11. *Let $k_1, k_2, ..., k_m$ be nonnegative integers such that $k_1 + k_2 + \cdots + k_m = n$. The multinomial coefficient is*

$$\binom{n}{k_1, k_2, ..., k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}.$$

Theorem 5.12. *The multinomial coefficient can be expressed as a product of binomial coefficients.*

Proof. We have

$$\begin{aligned}
\binom{n}{k_1, k_2, ..., k_m} &= \frac{n!}{k_1!\, k_2! \cdots k_m!} \\
&= \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!\, k_2! \cdots k_m!} \\
&= \frac{(k_1 + k_2)!}{(k_1 + k_2)!} \cdots \frac{(k_1 + k_2 + \cdots + k_{m-1})!}{(k_1 + k_2 + \cdots + k_{m-1})!} \cdot \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!\, k_2! \cdots k_m!} \\
&= \frac{(k_1 + k_2)!}{k_1!\, k_2!} \cdot \frac{(k_1 + k_2 + k_3)!}{(k_1 + k_2)!\, k_3!} \cdots \frac{(k_1 + k_2 + \cdots + k_m)!}{(k_1 + k_2 + \cdots + k_{m-1})!\, k_m!} \\
&= \binom{k_1 + k_2}{k_2} \cdot \binom{k_1 + k_2 + k_3}{k_3} \cdots \binom{k_1 + k_2 + \cdots + k_m}{k_m}.
\end{aligned}$$

$\square$

It follows from here that a multinomial coefficient must be an integer because it is the product of the binomial coefficients.

This argument will allow us to show that the coefficients of generalized Chebyshev polynomials are integers. This is previously done by Bourbaki [1]. We will give an alternative proof that gives insight into the coefficients.

Theorem 5.13. *Let k be an arbitrary nonnegative integer. Then the coefficients of $\mathcal{A}_k(x, y)$ are integers.*

Proof. From Equation 5.5, we see that the coefficients of $\mathcal{A}_k(x, y)$ are given by

$$A(i, j, k) = (-1)^j \cdot \frac{(k - j - 2i - 1)! \cdot k}{(k - 2j - 3i)!\, j!\, i!}$$

For simplicity let us put $a := k - 2i - 3j, b := j$ and $c := i$. Then we have

$$\begin{aligned}
A(i, j, k) &= (-1)^j \cdot \frac{(k - j - 2i - 1)! \cdot k}{(k - 2j - 3i)!\, j!\, i!} \\
&= (-1)^b \cdot \frac{(a + b + c - 1)! \cdot (a + 2b + 3c)}{a!\, b!\, c!} \\
&= (-1)^b \cdot \left(\frac{a \cdot (a + b + c - 1)!}{a!\, b!\, c!} + 2 \cdot \frac{b \cdot (a + b + c - 1)!}{a!\, b!\, c!} + 3 \cdot \frac{c \cdot (a + b + c - 1)!}{a!\, b!\, c!} \right) \\
&= (-1)^b \cdot \left(\frac{(a + b + c - 1)!}{(a - 1)!\, b!\, c!} + 2 \cdot \frac{(a + b + c - 1)!}{a!\, (b - 1)!\, c!} + 3 \cdot \frac{(a + b + c - 1)!}{a!\, b!\, (c - 1)!} \right) \\
&= (-1)^b \cdot (m_1 + 2m_2 + 3m_3)
\end{aligned}$$

where each of m_1, m_2, m_3 are multinomial coefficients given by

$$\begin{aligned}
m_1 &= \frac{(a + b + c - 1)!}{(a - 1)!\, b!\, c!}, \\
m_2 &= \frac{(a + b + c - 1)!}{a!\, (b - 1)!\, c!}, \\
m_3 &= \frac{(a + b + c - 1)!}{a!\, b!\, (c - 1)!}.
\end{aligned}$$

Therefore each of the terms on the right hand side of this equation is an integer, thus all of the coefficients of $\mathcal{A}_k(x, y)$ must be integers. □

The next important property of these polynomials is their behaviours on a finite field of characteristic p. Note that Bourbaki's integer coefficient conclusion is not enough to obtain the following result:

Theorem 5.14. *If q is a power of a prime p then $\mathcal{A}_q(x,y) \equiv (x^q, y^q) \pmod{p}$.*

Proof. Let p be a prime then $\mathcal{A}_p(x,y) = (f_p(x,y), g_p(x,y))$ is given as

$$f_p(x,y) = x^p + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} A(i,j,p) \cdot x^{p-2j-3i} \cdot y^j$$

$$g_p(x,y) = y^p + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} A(i,j,p) \cdot y^{p-2j-3i} \cdot x^j.$$

where the coefficients $A(i,j,p)$ are found when k is replaced by p in Equation 5.5. We know from Theorem 5.13 that the coefficients $A(i,j,p)$ are integers. We will also show that the expression $A(i,j,p)/p$ is also an integer.

If $A(i,j,p)/p \in \mathbf{Q} \setminus \mathbf{Z}$ then p would divide the denominator of the expression which is equal to $A(i,j,p)/p$ but this is impossible since i and j can take the integer values only in the intervals $\left[1, \lfloor \frac{p}{3} \rfloor\right]$ and $\left[1, \lfloor \frac{p}{2} \rfloor\right]$ respectively. Hence each coefficient of $\mathcal{A}_p(x,y)$, except for the first one, is an integer multiple of p. Therefore

$$\mathcal{A}_p(x,y) \equiv (x^p, y^p) \pmod{p}.$$

Now let $q = p^r$ for some positive integer r. We have

$$\begin{aligned}
\mathcal{A}_q(x,y) &= \mathcal{A}_{p^r}(x,y) \\
&= \underbrace{\mathcal{A}_p(x,y) \circ \cdots \circ \mathcal{A}_p(x,y)}_{r \text{ times}} \\
&\equiv \underbrace{(x^p, y^p) \circ \cdots \circ (x^p, y^p)}_{r \text{ times}} \pmod{p} \\
&\equiv \left(x^{(p^r)}, y^{(p^r)}\right) \pmod{p} \\
&\equiv (x^q, y^q) \pmod{p}.
\end{aligned}$$

Here the second equality follows from the property that $\mathcal{A}_{mn} = \mathcal{A}_m \circ \mathcal{A}_n$. □

5.3 Generalized Chebyshev Polynomials of B_2

In Example 4.18, we found that the generalized cosine function of B_2 is given by

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x})) \text{ where}$$

$$y_1(\mathbf{x}) = e^{-2\pi i u_1} + e^{2\pi i u_1} + e^{-2\pi i(u_1 - 2u_2)} + e^{2\pi i(u_1 - 2u_2)}, \tag{5.7}$$

$$y_2(\mathbf{x}) = e^{-2\pi i u_2} + e^{2\pi i u_2} + e^{-2\pi i(u_1 - u_2)} + e^{2\pi i(u_1 - u_2)}. \tag{5.8}$$

If we consider Equation 5.7 and let $x_1 = e^{-2\pi i u_1}, x_2 = e^{2\pi i u_1}, x_3 = e^{-2\pi i(u_1-2u_2)}, x_4 = e^{2\pi i(u_1-2u_2)}$, then it turns out that $y_1(k\mathbf{x}) = x_1^k + x_2^k + x_3^k + x_4^k$. This means that by Theorem 5.7, $y_1(k\mathbf{x})$ can be written in terms of the following elementary symmetric polynomials in the variables x_1, x_2, x_3, x_4.

$$\begin{aligned}
\sigma_1 &= \sum_i x_i = y_1(\mathbf{x}), \\
\sigma_2 &= \sum_{i<j} x_i x_j = y_2(\mathbf{x})^2 - 2y_1(\mathbf{x}) - 2, \\
\sigma_3 &= \sum_{i<j<k} x_i x_j x_k = y_1(\mathbf{x}), \\
\sigma_4 &= x_1 x_2 x_3 x_4 = 1.
\end{aligned}$$

To use Waring's formula, we need to find the nonnegative integer solutions of $i_1 + 2i_2 + 3i_3 + 4i_4 = k$. We see that the four tuples $(i_1, i_2, i_3, i_4) = (k-4i-3j-2l, l, j, i) \in \mathbf{N}^4$ provide solutions for this equation. Using Waring's formula we have

$$y_1(k\mathbf{x}) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} (-1)^{l+i} \frac{(k-3i-2j-l-1)! \cdot k}{(k-4i-3j-2l)!\, l!\, j!\, i!} \cdot y_1^{k-4i-2j-2l} \cdot (y_2^2 - 2y_1 - 2)^l.$$

If we let $x_1 = e^{-2\pi i u_2}, x_2 = e^{2\pi i u_2}, x_3 = e^{-2\pi i(u_1-u_2)}$, and $x_4 = e^{2\pi i(u_1-u_2)}$ by Equation 5.8 then the corresponding elementary symmetric polynomials are given by

$$\sigma_1 = \sum_i x_i = y_2(\mathbf{x})$$
$$\sigma_2 = \sum_{i<j} x_i x_j = y_1(\mathbf{x}) + 2$$
$$\sigma_3 = \sum_{i<j<l} x_i x_j x_l = y_2(\mathbf{x})$$
$$\sigma_4 = x_1 x_2 x_3 x_4 = 1.$$

Applying Waring's formula to $y_2(k\mathbf{x}) = x_1^k + x_2^k + x_3^k + x_4^k$ yields

$$y_2(k\mathbf{x}) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} (-1)^{l+i} \frac{(k-3i-2j-l-1)! \cdot k}{(k-4i-3j-2l)!\, l!\, j!\, i!} \cdot y_2^{k-4i-2j-2l} \cdot (2+y_1)^l.$$

Therefore the polynomials $\mathcal{B}_k(x,y)$, defined by $\mathcal{B}_k(h(\mathbf{x})) = h(k\mathbf{x})$, are given by

$$\mathcal{B}_k(x,y) = (f_k(x,y), g_k(x,y)) \text{ where}$$

$$f_k(x,y) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} B(i,j,k,l) \cdot x^{k-4i-2j-2l} \cdot (y^2 - 2x - 2)^l \tag{5.9}$$

$$g_k(x,y) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} B(i,j,k,l) \cdot y^{k-4i-2j-2l} \cdot (2+x)^l. \tag{5.10}$$

Here the coefficients are given by

$$B(i,j,k,l) = (-1)^{l+i} \frac{(k-3i-2j-l-1)! \cdot k}{(k-4i-3j-2l)!\, l!\, j!\, i!}. \tag{5.11}$$

where only those terms occur for which $k \geq 4i + 3j + 2l$.

Theorem 5.15. *If*

$$\mathcal{B}_n(x,y) = (f_n(x,y), g_n(x,y)),$$

then

$$f_n = x(f_{n-1} + f_{n-3}) - (y^2 - 2x - 2)f_{n-2} - f_{n-4}$$
$$g_n = y(g_{n-1} + g_{n-3}) - (x+2)g_{n-2} - g_{n-4}$$

for all $n \geq 4$.

Proof. We know that

$$\mathcal{B}_n\left(y_1(\mathbf{x}), y_2(\mathbf{x})\right) = \left(y_1(n\mathbf{x}), y_2(n\mathbf{x})\right)$$

where

$$\begin{aligned} y_1(\mathbf{x}) &= x_1 + x_2 + x_3 + x_4 \\ y_1(n\mathbf{x}) &= x_1^n + x_2^n + x_3^n + x_4^n \end{aligned}$$

where the variables x_i for $i = 1, 2, 3, 4$ are coming from Equation 5.7. We also found that the elementary symmetric polynomials in x_1, x_2, x_3, x_4 are given by

$$\begin{aligned} \sigma_1 &= y_1(\mathbf{x}) \\ \sigma_2 &= y_2^2(\mathbf{x}) - 2y_1(\mathbf{x}) - 2 \\ \sigma_3 &= y_1(\mathbf{x}) \\ \sigma_4 &= 1 \end{aligned}$$

If we substitute these elementary symmetric polynomials and $s_{n-j} = y_1\left((n-j)\mathbf{x}\right)$ for $j = 0, 1, 2, 3, 4$ in Equation 5.6 and assume that

$$\mathcal{B}_n(x, y) = (f_n(x, y), g_n(x, y)),$$

we obtain

$$f_n = x(f_{n-1} + f_{n-3}) - (y^2 - 2x - 2)f_{n-2} - f_{n-4}.$$

Similarly we have

$$\begin{aligned} y_2(\mathbf{x}) &= x_1 + x_2 + x_3 + x_4 \\ y_2(n\mathbf{x}) &= x_1^n + x_2^n + x_3^n + x_4^n \end{aligned}$$

where the variables x_i for $i = 1, 2, 3, 4$ are coming from Equation 5.8. We also found the corresponding elementary symmetric polynomials in these variables as

$$\begin{aligned} \sigma_1 &= y_2(\mathbf{x}) \\ \sigma_2 &= y_1(\mathbf{x}) + 2 \\ \sigma_3 &= y_2(\mathbf{x}) \\ \sigma_4 &= 1. \end{aligned}$$

Substituting these elementary symmetric polynomials and $s_{n-j} = y_2\left((n-j)\mathbf{x}\right)$ for $j = 0, 1, 2, 3, 4$ in Equation 5.6, we obtain

$$g_n = y(g_{n-1} + g_{n-3}) - (x+2)g_{n-2} - g_{n-4}.$$

$\square$

Theorem 5.16. *multideg*$(\mathcal{B}_k(x,y)) = (k,k)$.

Proof. We see that the total degree of the monomials in $f_k(x,y)$ are $k - 4i - 2j$ by Equation 5.9 and hence $deg(f_k(x,y)) = k$ which occurs when $i = j = 0$.

A similar argument on Equation 5.10 implies that the total degree of the monomials in $g_k(x,y)$ are $k - 4i - 2j - l$ and $deg(g_k(x,y)) = k$ which occurs when $i = j = l = 0$.

Therefore multideg$(\mathcal{B}_k(x,y)) = (k,k)$. $\square$

Theorem 5.17. *Let k be an arbitrary nonnegative integer. Then the coefficients of $\mathcal{B}_k(x,y)$ are integers.*

Proof. From Equation 5.11, we see that the coefficients of $\mathcal{B}_k(x,y)$ are given by

$$B(i,j,k,l) = (-1)^{l+i}\frac{(k-3i-2j-l-1)!\cdot k}{(k-4i-3j-2l)!\,l!\,j!\,i!}.$$

These coefficients can be expressed in terms of the multinomial coefficients. For simplicity let us put $a := k - 4i - 3j - 2l, b := l, c := j$ and $d := i$. Then we have

$$\begin{aligned}
B(i,j,k,l) &= (-1)^{l+i}\frac{(k-3i-2j-l-1)!\cdot k}{(k-4i-3j-2l)!\,l!\,j!\,i!} \\
&= (-1)^{b+d}\frac{(a+b+c+d-1)!\cdot(a+2b+3c+4d)}{a!\,b!\,c!\,d!} \\
&= (-1)^{b+d}\,(m_1 + 2m_2 + 3m_3 + 4m_4)
\end{aligned}$$

where each m_1, m_2, m_3 and m_4 is multinomial coefficient given by

$$m_1 = \frac{(a+b+c+d-1)!}{(a-1)!\, b!\, c!\, d!},$$

$$m_2 = \frac{(a+b+c+d-1)!}{a!\, (b-1)!\, c!\, d!},$$

$$m_3 = \frac{(a+b+c+d-1)!}{a!\, b!\, (c-1)!\, d!},$$

$$m_4 = \frac{(a+b+c+d-1)!}{a!\, b!\, c!\, (d-1)!}.$$

Since each of m_i is integer, the coefficients of $\mathcal{B}_k(x, y)$ must be integers. $\square$

Theorem 5.18. *If q is a power of a prime p then $\mathcal{B}_q(x, y) \equiv (x^q, y^q) \pmod p$.*

Proof. For a prime p, $\mathcal{B}_p(x, y) = (f_p(x, y), g_p(x, y))$ can be written as

$$f_p(x, y) = x^p + \sum_{l=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{p}{3} \rfloor} \sum_{i=1}^{\lfloor \frac{p}{4} \rfloor} B(i, j, p, l) \cdot x^{p-4i-2j-2l} \cdot (y^2 - 2x - 2)^l$$

$$g_p(x, y) = y^p + \sum_{l=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{p}{3} \rfloor} \sum_{i=1}^{\lfloor \frac{p}{4} \rfloor} B(i, j, p, l) \cdot y^{p-4i-2j-2l} \cdot (2 + x)^l,$$

where the coefficients $B(i, j, p, l)$ are found by replacing k by p in Equation 5.11. We know from Theorem 5.17 that the coefficients $B(i, j, p, l)$ are integers. Furthermore the expression $B(i, j, p, l)/p$ must also be an integer because otherwise its denominator would be divisible p which is impossible due to the possible values of i, j and l. Hence each coefficient of $\mathcal{B}_p(x, y)$, except for the first one, is an integer multiple of p. Therefore

$$\mathcal{B}_p(x, y) \equiv (x^p, y^p) \pmod p.$$

Now let $q = p^r$ for some positive integer r. Then we have

$$
\begin{aligned}
\mathcal{B}_q(x,y) &= \mathcal{B}_{p^r}(x,y) \\
&= \underbrace{\mathcal{B}_p(x,y) \circ \cdots \circ \mathcal{B}_p(x,y)}_{r \text{ times}} \\
&\equiv \underbrace{(x^p, y^p) \circ \cdots \circ (x^p, y^p)}_{r \text{ times}} \pmod{p} \\
&\equiv \left(x^{(p^r)}, y^{(p^r)}\right) \pmod{p} \\
&\equiv (x^q, y^q) \pmod{p}.
\end{aligned}
$$

Here the second equality follows from the property that $\mathcal{B}_{mn} = \mathcal{B}_m \circ \mathcal{B}_n$. □

5.4 Generalized Chebyshev Polynomials of G_2

In Example 4.19, we found that the generalized cosine function of G_2 is given by

$$h(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x})) \text{ where}$$

$$y_1(\mathbf{x}) = e^{-2\pi i u_1} + e^{2\pi i u_1} + e^{-2\pi i(u_1-u_2)} + e^{2\pi i(u_1-u_2)} + e^{-2\pi i(2u_1-u_2)} + e^{2\pi i(2u_1-u_2)},$$

$$y_2(\mathbf{x}) = e^{-2\pi i u_2} + e^{2\pi i u_2} + e^{-2\pi i(3u_1-u_2)} + e^{2\pi i(3u_1-u_2)} + e^{-2\pi i(3u_1-2u_2)} + e^{2\pi i(3u_1-2u_2)}.$$

Similar to the works done in the previous two sections we have

$$
\begin{aligned}
x_1 &= e^{-2\pi i u_1} & & & \sigma_1 &= y_1 \\
x_2 &= e^{2\pi i u_1} & & & \sigma_2 &= 3 + y_1 + y_2 \\
x_3 &= e^{-2\pi i(u_1-u_2)} & &\Longrightarrow & \sigma_3 &= y_1^2 - 2y_2 - 4 \\
x_4 &= e^{2\pi i(u_1-u_2)} & & & \sigma_4 &= 3 + y_1 + y_2 \\
x_5 &= e^{-2\pi i(2u_1-u_2)} & & & \sigma_5 &= y_1 \\
x_6 &= e^{2\pi i(2u_1-u_2)} & & & \sigma_6 &= 1.
\end{aligned}
\tag{5.12}
$$

This gives us that

$$y_1(k\mathbf{x}) = \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{l=0}^{\lfloor \frac{k}{4} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{5} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{6} \rfloor} C(i,j,k,l,m,n) \cdot \sigma_1^{k-2n-3m-4l-4j-6i} \cdot \sigma_2^{n+l} \cdot \sigma_3^{m}.$$

From the second component $y_2(\mathbf{x})$, we make the following change of variables and obtain the elementary symmetric polynomials in terms of y_1 and y_2 as follows:

$$
\begin{aligned}
x_1 &= e^{-2\pi i u_2} & \sigma_1 &= y_2 \\
x_2 &= e^{2\pi i u_2} & \sigma_2 &= y_1^3 - 3y_1y_2 - 9y_1 - 5y_2 - 9 \\
x_3 &= e^{-2\pi i(3u_1 - u_2)} \quad \Longrightarrow & \sigma_3 &= -2y_1^3 + y_2^2 + 6y_1y_2 + 18y_1 + 12y_2 + 20 \\
x_4 &= e^{2\pi i(3u_1 - u_2)} & \sigma_4 &= y_1^3 - 3y_1y_2 - 9y_1 - 5y_2 - 9 \\
x_5 &= e^{-2\pi i(3u_1 - 2u_2)} & \sigma_5 &= y_2 \\
x_6 &= e^{2\pi i(3u_1 - 2u_2)} & \sigma_6 &= 1.
\end{aligned}
\tag{5.13}
$$

Therefore we have

$$
y_2(k\mathbf{x}) = \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{l=0}^{\lfloor \frac{k}{4} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{5} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{6} \rfloor} C(i,j,k,l,m,n) \cdot \sigma_1^{k-2n-3m-4l-4j-6i} \cdot \sigma_2^{n+l} \cdot \sigma_3^m
$$

where the coefficients $C(i,j,k,l,m,n)$ are given by

$$
C(i,j,k,l,m,n) = (-1)^{n+l+i} \frac{(k-n-2m-3l-4j-5i-1)! \cdot k}{(k-2n-3m-4l-5j-6i)!\, n!\, m!\, l!\, j!\, i!}. \tag{5.14}
$$

Therefore the polynomials $\mathcal{G}_k(x,y)$, defined by $\mathcal{G}_k(h(\mathbf{x})) = h(k\mathbf{x})$, are given by

$$
\mathcal{G}_k(x,y) = (f_k(x,y), g_k(x,y)) \;\text{ where}
$$

$$
\begin{aligned}
f_k(x,y) = \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{l=0}^{\lfloor \frac{k}{4} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{5} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{6} \rfloor} & C(i,j,k,l,m,n) \cdot x^{k-2n-3m-4l-4j-6i} \\
& \cdot (3+x+y)^{n+l} \cdot (x^2 - 2y - 4)^m
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
g_k(x,y) = \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{l=0}^{\lfloor \frac{k}{4} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{5} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{6} \rfloor} & C(i,j,k,l,m,n) \cdot y^{k-2n-3m-4l-4j-6i} \\
& \cdot (x^3 - 3xy - 9x - 5y - 9)^{n+l} \\
& \cdot (-2x^3 + y^2 + 6xy + 18x + 12y + 20)^m
\end{aligned}
\tag{5.16}
$$

where the coefficients $C(i,j,k,l,m,n)$ are as in the Equation 5.14 above.

Theorem 5.19. *If*

$$\mathcal{G}_n(x,y) = (f_n(x,y), g_n(x,y)),$$

then

$$\begin{aligned} f_n &= x(f_{n-1} + f_{n-5}) - (x+y+3)(f_{n-2} + f_{n-4}) \\ &\quad + (x^2 - 2y - 4)f_{n-3} - f_{n-6} \\ g_n &= y(g_{n-1} + g_{n-5}) - (x^3 - 3xy - 9x - 5y - 9)(g_{n-2} + g_{n-4}) \\ &\quad + (y^2 - 2x^3 + 6xy + 18x + 12y + 20)g_{n-3} - g_{n-6} \end{aligned}$$

for all $n \geq 6$.

Proof. We know that

$$\mathcal{G}_n(y_1(\mathbf{x}), y_2(\mathbf{x})) = (y_1(n\mathbf{x}), y_2(n\mathbf{x}))$$

where $y_1(n\mathbf{x})$ and $y_2(n\mathbf{x})$ are the n^{th} power sums for the variables given in Equation 5.12 and 5.13 respectively. Therefore substituting the elementary symmetric polynomials in these equations accordingly and $s_{n-j} = y_1((n-j)\mathbf{x})$ for the first component and $s_{n-j} = y_2((n-j)\mathbf{x})$ for the second component where $j = 0,1,2,3,4,5,6$ in Equation 5.6, we obtain the expressions for f_n and g_n given in the theorem. □

Theorem 5.20. *multideg*$(\mathcal{G}_k(x,y)) = \left(k, k + \left\lfloor \frac{k}{2} \right\rfloor\right)$.

Proof. If we consider Equation 5.15, we see that the total degree of the monomials in $f_k(x,y)$ is $k - n - m - 3l - 4j - 6i$ and hence $deg(f_k(x,y)) = k$ which occurs when $n = m = l = j = i = 0$.

On the other hand, Equation 5.16 implies that the total degree of the monomials in $g_k(x,y)$ is $k+n-l-4j-6i$ and hence we have $deg(g_k(x,y)) = k + \left\lfloor \frac{k}{2} \right\rfloor$ which occurs when $n = \left\lfloor \frac{k}{2} \right\rfloor$ and $l = j = i = 0$. Therefore multideg$(\mathcal{G}_k(x,y)) = \left(k, k + \left\lfloor \frac{k}{2} \right\rfloor\right)$. □

Theorem 5.21. *Let k be an arbitrary nonnegative integer. Then the coefficients of $\mathcal{G}_k(x,y)$ are integers.*

Proof. From Equation 5.14 we see that the coefficients of $\mathcal{G}_k(x,y)$ are given by

$$C(i,j,k,l,m,n) = (-1)^{n+l+i} \frac{(k-n-2m-3l-4j-5i-1)! \cdot k}{(k-2n-3m-4l-5j-6i)!\, n!\, m!\, l!\, j!\, i!}.$$

If we let $a := k - 2n - 3m - 4l - 5j - 6i, b := n, c := m, d := l, e := j$ and $f := i$, then these coefficients can be written as

$$C(i, j, k, l, m, n) = (-1)^{b+d+f}(m_1 + 2m_2 + 3m_3 + 4m_4 + 5m_5 + 6m_6)$$

where $m_1, m_2, ..., m_6$ are the multinomial coefficients given by

$$m_1 = \frac{(a+b+c+d+e+f-1)!}{(a-1)!\, b!\, c!\, d!\, e!\, f!},$$
$$m_2 = \frac{(a+b+c+d+e+f-1)!}{a!\, (b-1)!\, c!\, d!\, e!\, f!},$$
$$m_3 = \frac{(a+b+c+d+e+f-1)!}{a!\, b!\, (c-1)!\, d!\, e!\, f!},$$
$$m_4 = \frac{(a+b+c+d+e+f-1)!}{a!\, b!\, c!\, (d-1)!\, e!\, f!},$$
$$m_5 = \frac{(a+b+c+d+e+f-1)!}{a!\, b!\, c!\, d!\, (e-1)!\, f!},$$
$$m_6 = \frac{(a+b+c+d+e+f-1)!}{a!\, b!\, c!\, d!\, e!\, (f-1)!}.$$

Since each of m_i is integer, the coefficients of $\mathcal{G}_k(x, y)$ must be integers. $\square$

Theorem 5.22. *If q is a power of a prime p then $\mathcal{G}_q(x, y) \equiv (x^q, y^q) \pmod{p}$.*

Proof. Let p be a prime, then the polynomials $\mathcal{G}_p(x, y) = (f_p(x, y), g_p(x, y))$ can be written as

$$f_p(x, y) = x^p + \sum_{n=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{m=1}^{\lfloor \frac{p}{3} \rfloor} \sum_{l=1}^{\lfloor \frac{p}{4} \rfloor} \sum_{j=1}^{\lfloor \frac{p}{5} \rfloor} \sum_{i=1}^{\lfloor \frac{p}{6} \rfloor} C(i, j, p, l, m, n) \cdot x^{p-2n-3m-4l-4j-6i}$$
$$\cdot (3 + x + y)^{n+l} \cdot (x^2 - 2y - 4)^m$$

$$g_p(x, y) = y^p + \sum_{n=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{m=1}^{\lfloor \frac{p}{3} \rfloor} \sum_{l=1}^{\lfloor \frac{p}{4} \rfloor} \sum_{j=1}^{\lfloor \frac{p}{5} \rfloor} \sum_{i=1}^{\lfloor \frac{p}{6} \rfloor} C(i, j, p, l, m, n) \cdot y^{p-2n-3m-4l-4j-6i}$$
$$\cdot (x^3 - 3xy - 9x - 5y - 9)^{n+l}$$
$$\cdot (-2x^3 + y^2 + 6xy + 18x + 12y + 20)^m$$

where the coefficients $C(i, j, p, l, m, n)$ are found when k is replaced by p in Equation 5.14. We know from Theorem 5.21 that the coefficients of $\mathcal{G}_p(x, y)$ are integers.

We can also say that each of $C(i,j,p,l,m,n)/p$ must also be an integer because otherwise its denominator would be divisible by p which is impossible due to the possible values of i,j,l,m and n. Hence each coefficient of $\mathcal{G}_p(x,y)$, except for the first one, is an integer multiple of p. Therefore

$$\mathcal{G}_p(x,y) \equiv (x^p, y^p) \pmod{p}.$$

Now let $q = p^r$ for some positive integer r. Then we have

$$\begin{aligned}
\mathcal{G}_q(x,y) &= \mathcal{G}_{p^r}(x,y) \\
&= \underbrace{\mathcal{G}_p(x,y) \circ \cdots \circ \mathcal{G}_p(x,y)}_{r \text{ times}} \\
&\equiv \underbrace{(x^p, y^p) \circ \cdots \circ (x^p, y^p)}_{r \text{ times}} \pmod{p} \\
&\equiv \left(x^{(p^r)}, y^{(p^r)}\right) \pmod{p} \\
&\equiv (x^q, y^q) \pmod{p}.
\end{aligned}$$

Here the second equality follows from the property that $\mathcal{G}_{mn} = \mathcal{G}_m \circ \mathcal{G}_n$. $\square$

REFERENCES

[1] N. Bourbaki. *Lie Groups and Lie Algebras: Chapters 4-6*. Springer Science & Business Media, 2008.

[2] L. Boza, E. M. Fedriani, J. Nunez, and A. F. Tenorio. A historical review of the classifications of lie algebras. *Rev. Un. Mat. Argentina*, 54(2):75–99, 2013.

[3] D. Cox, J. Little, and D. OShea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013.

[4] K. Erdmann and M. J. Wildon. *Introduction to Lie algebras*. Springer Science & Business Media, 2006.

[5] M. E. Hoffman and W. D. Withers. Generalized chebyshev polynomials associated with affine weyl groups. *Transactions of the American Mathematical Society*, 308(1):91–104, 1988.

[6] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[7] T. Koornwinder. Two-variable analogues of the classical orthogonal polynomials. In *Theory and application of special functions*, pages 435–495. Elsevier, 1975.

[8] Ö. Küçüksakallı. Bivariate polynomial mappings associated with simple complex lie algebras. *Journal of Number Theory*, 168:433–451, 2016.

[9] R. Lidl and H. Niederreiter. *Finite fields*. Number 20. Cambridge university press, 1997.

[10] J. C. Mason and D. C. Handscomb. *Chebyshev polynomials*. CRC press, 2002.

[11] P. Ricci. Una proprietà iterativa dei polinomi di chebyshev di prima specie in più variabili. *Rend Mat Appl*, 6:555–563, 1986.

[12] T. J. Rivlin. *Chebyshev polynomials*. Courier Dover Publications, 2020.

[13] C. Wells and R. Lidl. Chebyshev polynomials in several variables. *Journal für die reine und angewandte Mathematik*, 1972(255):104–111, 1972.

[14] W. D. Withers. Folding polynomials and their dynamics. *The American Mathematical Monthly*, 95(5):399–413, 1988.