PROBLEMS OF MASSIVE GRAVITY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

AUGUST 2021

Approval of the thesis:

PROBLEMS OF MASSIVE GRAVITY

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ABSTRACT

PROBLEMS OF MASSIVE GRAVITY

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August 2021, 60 pages

In this thesis, the general properties of massive gravitation theories are outlined. Particular attention is given to the van Dam-Veltman-Zakharov discontinuity which arises in the massless limit of massive theories. Also, the observational, holographic, and thermodynamic properties of these theories are mentioned.

Keywords: Fierz-Pauli Massive Gravity, van Dam-Veltman-Zakharov Discontinuity, dRGT Massive Gravity, Gravitational Waves, Black hole Thermodynamics

ÖZ

KÜTLELİ GRAVİTASYONUN PROBLEMLERİ

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Ağustos 2021 , 60 sayfa

Bu tez de, kütleli gravitasyon teorilerinin genel özellikleri özetlenmiştir. van Dam-Veltman-Zakharov süreksizliği üzerinde özellikle durulmuştur. Ayrıca, bu teorilerin gözlemlenebilir, holografik ve termodinamik özellikleri bahsedilmiştir.

Anahtar Kelimeler: Fierz-Pauli Kütleli Gravitasyon, van Dam-Veltman-Zakharov Süreksizliği, dRGT Kütleli Gravitasyon, Gravitasyonel Dalgalar, Kara Delik Termodinamiği

To my family

ACKNOWLEDGMENTS

I want to thank Prof. Bayram Tekin for his dedication and inspiration. I have always learned something new in his classes. I want to thank Prof. Atalay Karasu for his help and Prof. Tahsin Çağrı Şişman for his suggestions.

A special thank goes to my parents and my sister for their continuous support throughout my life.

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LIST OF CONVENTIONS AND ABBREVIATIONS

We are using MTW [2] conventions. Also $c = 1$ unless indicated explicitly.

EOM: Equations of motion

BL: Boyer-Linquist Coordinates

GR: General Relativity

FP: Massless Fierz-Pauli Theory

mFP: Massive Fierz-Pauli Theory

vDVZ: van Dam-Veltman-Zakharov

PPN: Parametrized Post Newtonian

CHAPTER 1

INTRODUCTION

History of massive gravitation theories is full of failures. Giving mass to the graviton takes all the beauty and simplicity of GR (General Relativity) so one can rightfully question the necessity for such an attempt. The answer is not really the observations. Gravity is known as the longest ranged force. Although there is a need to modify gravity at the large scale to explain the accelerated expansion of the universe and the smallness of the cosmological constant, there are many modifications one can make to the GR before massive gravitation theories. The real struggle here is to understand why giving even very small amount of mass to the graviton is so disastrous. The kind of disaster we are talking about includes,

- Discontinuity in the predictions of the theory compared to GR [3],
- Acausality, superluminality, closed timelike curves [4],
- Ghost degrees of freedoms, Higuchi bound, vacuum instability [5], [6].

However, it is so absurd that something exactly zero or ridiculously small makes a discrete difference. We expect physical theories to be continuous on it's parameters.

It turns out that even the very definition of mass within the GR is not obvious. The problem is rooted in the fact that there is no local definition of energy for the gravity itself because of the diffeomorphism invariance, one can always gauge it out at that location. One solution is to introduce some sort of background or auxiliary structure. Therefore, one is led to treat GR as a field theory residing on this background which is very against Einstein's spirit.

1.1 Gravity as a Field Theory

The idea of gravity as a field theory on a background is older than GR (this is what Newton implicitly considered for example). The idea is tempting for the following reason. If we can split the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and write the Einstein-Hilbert action in terms of $h_{\mu\nu}$ on a Minkowski background, we would be able to solve the energy localization problem. This is because every Lorentz invariant *field theory* admits the canonical stress energy momentum tensor by the Noether's theorem. However, this naive splitting brings infinitely many terms because of the necessity of the inverse metric to write the Einstein-Hilbert action.

$$
\begin{aligned}
S_{EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R = \int d^4x \Bigg[& \kappa^1[\partial_\mu\partial_\nu h^{\mu\nu} - \Box h] \\
+ \kappa^2[-\frac{1}{2}(\partial_\nu h)(\partial^\nu h) - 2(\partial_\mu h^{\mu\nu})(\partial^\rho h_{\nu\rho}) + 2(\partial^\nu h)(\partial^\rho h_{\nu\rho}) - (\partial_\nu h_{\mu\rho})(\partial^\rho h^{\mu\nu}) \\
+ \frac{3}{2}(\partial_\rho h^{\mu\nu})(\partial^\rho h_{\mu\nu}) + 2h^{\mu\nu}\Big(\partial_\mu\partial_\nu h + \Box h_{\mu\nu} - 2\partial^\rho\partial_\nu h_{\mu\rho}\Big) + h(\partial_\nu\partial_\mu h^{\mu\nu} - \Box h)] + \dots \Bigg].
\end{aligned} \tag{1.1}
$$

Do we really need the inverse metric to formulate the General Relativity? This is an interesting question, one might think that Palatini formalism can solve this issue where metric and the connection are varied independently.

$$
S = \int d^4x \mathrm{g}^{\mu\nu}\Big(\partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\nu\mu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}\Big), \tag{1.2}
$$

where $\mathrm{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ is a tensor density. Variation of $\Gamma^\alpha_{\mu\nu}$ and $\mathrm{g}^{\mu\nu}$ gives roughly,

$$
\partial g + g\Gamma \sim 0 \qquad R_{\mu\nu}(\Gamma) = 0. \tag{1.3}
$$

Normally solving the first equation gives $\Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}$ the metric compatibility and the second is the vacuum Einstein equations. However, it is clear that one can not solve for Γ(g) without introducing the inverse metric.

Another possibility one can try is to retain with the quadratic part ($S_{EH}(h^2)$) of (1.1). However, this brings it's own problems when we consider the matter coupling. The free theory has equation of motions of the $\mathbf{G}_{\mu\nu}(h) = 0$ form where $\mathbf{G}_{\mu\nu}$ is the linearization of the Einstein tensor around the flat background. Since one has the identity $\partial^\mu\mathbf{G}_{\mu\nu} = 0$ we have to couple the theory into a conserved source. Once we do that, the energy momentum tensor of the matter field will not be conserved due to the coupling with gravity. To fix this, one either has to project out the non-conserved parts of the matter field energy tensor or add the energy momentum tensor of the free gravity theory to the right hand side as well. The former will lead to a consistent linear theory. The latter, $\mathbf{G}_{\mu\nu} = T^{(m)}_{\mu\nu} + t^{(g)}_{\mu\nu}$ however is tricky. Once we add this additional piece to the equation, we will need a cubic action to derive that because $t^{(g)}_{\mu\nu}$ is quadratic in h but then we can construct new $\tilde{t}^{(g)}_{\mu\nu}$ which is cubic in h out of our new cubic action and consider instead $\mathbf{G}_{\mu\nu} = T^{(m)}_{\mu\nu} + \tilde{t}^{(g)}_{\mu\nu}$. Clearly, this process goes on indefinitely. The hope is by adding this *self-interactions* one can recover the GR. There are many issues involving with this procedure though. For example, which $t^{(g)}_{\mu\nu}$ we should use, because there are infinitely many of them differing by a super-potential and the canonical one is not even symmetric!

At this point, an experienced reader can notice that our inability to formulate GR as a field theory and the defining mass (or energy) come from the same root. This is the conflict of the symmetry groups. In the field theory formulation one has the usual Poincaré invariance, but GR has a larger symmetry group which is the general diffeomorphism (gauge) invariance. Unfortunately, Noether's theorem is not fully compatible with the latter. Then what is the solution? There are a couple of strategies to overcome these difficulties:

- Try to find a positive definite tensor that is conserved [7],
- Try to formulate a quasi-local definition [8],
- Continue life with global definitions calculated at infinity [9],

all of which were studied in detail in the literature.

1.2 (In)Adequacy of the Linear Theory

As mentioned in the previous section, it is possible to construct a consistent linear spin-2 field theory out of GR. As a surprise, for many of the solar system tests this theory is perfectly adequate. However, we have to be cautious by what we mean by a *linear* theory. To agree with the usual solar system tests, like perihelion precession of planets, light bending etc, one can retain a metric which is linear[1] in the source parameters like mass but should not linearize the geodesic equation. Once the geodesic equation is also linearized, the usual solar system predictions of the linear theory will no longer agree with the GR.

The form of the solutions, that the linear theory gives, are pretty much given as, setting $8\pi G = 1$ for now,

$$\Box \tilde{h}_{\mu\nu} = T_{\mu\nu}. \tag{1.4}$$

Of course, the linear theory might give a more complicated second order differential operator, but that can be recast into a usual d'Alembertian by a gauge fixing and change of variables, at least for the linearized GR. The most general solution can be written in terms of the retarded Green's function of the d'Alembertian [2] as

$$\tilde{h}_{\mu\nu} = \int d^4x' \frac{\theta(t')\delta(t'-r')}{4\pi r'} T_{\mu\nu}(x'). \tag{1.5}$$

For a static point source one can get a solution of the form,

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 + \frac{2M}{r}\Big)dr^2 + r^2 d\Omega, \tag{1.6}$$

with a comparison with the Schwarzschild solution in GR:

$$ds^2_{Sc} = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 d\Omega. \tag{1.7}$$

The first is really the linearization of the Schwarzschild in M so there is no surprise there. Both have the coordinate singularity at $r = 2M$. However, there a couple of crucial points. Is there any event horizon in the linearized theory? Consider a radially in-falling particle in the linearized theory. Writing the conserved energy and the norm of the four velocity

$$\begin{aligned} E &= \Big(1 - \frac{2M}{r}\Big)\dot{t}, \\ -1 &= -\Big(1 - \frac{2M}{r}\Big)\dot{t}^2 + \Big(1 + \frac{2M}{r}\Big)\dot{r}^2, \end{aligned} \tag{1.8}$$

[1] Linearity of a metric is a coordinate dependent statement, the case we are referring here is the Schwarzschild coordinates.

[2] In a sense all the solutions to the linear theory can be expressed as an integral in contrast to GR.

where $\dot{t} \equiv dt/d\tau$ and τ is the proper time. From these two equations, one can write

$$\begin{aligned} \Big(\frac{dr}{dt}\Big)^2 &= \frac{1-2M/r}{1+2M/r}\Big(1-\frac{1-2M/r}{E^2}\Big), \\ \frac{d^2r}{dt^2} &= \frac{M}{r^2E^2}\Big(1+\frac{2E^2-4}{(1+2M/r)^2}\Big). \end{aligned} \tag{1.9}$$

From which we see that $dr/dt = 0$ and $d^2r/dt^2 > 0$ as $r \to 2M$. Therefore, an in-falling particle actually bounces back from $r = 2M$! Therefore, the situation of $r = 2M$ in the linear theory is very different than the Schwarzschild solution and there is no event horizon in the linearized theory.

1.3 Bi-metric, dRGT and mFP Theories

One of the recent most studied theories of massive gravity is the Bi-metric theory or Bi-gravity given by the following action [10]

$$S = M_p^2\int d^4x\sqrt{-g}\Big[R[g]+2m^2\sum_{k=0}^{4}\beta_k e_k(\gamma)\Big] + M_l^2\int d^4x\sqrt{-f}R[f], \tag{1.10}$$

where $\gamma = \sqrt{\boldsymbol{g}^{-1}\boldsymbol{f}}$, β_i's are constants and

$$\begin{aligned} &e_0 = 1 \qquad e_1 = [X] \qquad e_2 = \frac{1}{2}([X]^2-[X^2]), \\ &e_3 = \frac{1}{6}([X]^3-3[X^2][X]+2[X^3]), \\ &e_4 = \frac{1}{24}([X]^4-6[X^2][X]^2+3[X^2]^2+8[X][X^3]-6[X^4]). \end{aligned} \tag{1.11}$$

This theory has a second dynamical metric $f_{\mu\nu}$ which has it's own Planck mass. Suppose we take $g_{\mu\nu} \propto f_{\mu\nu}$, then $\gamma \propto \mathbf{1}$ is identity and $e_k(\gamma) = 0$ for $k \geq 2$. Therefore, the two dynamical metric theories decouple and one gets effectively two Einstein-Hilbert actions with cosmological constants. Hence, bi-metric theory includes all the solutions of the GR trivially.

Furthermore, setting $f_{\mu\nu}$ constant gives the dRGT massive gravity which is another massive extension of the GR.

The main subject of this thesis is a subclass of the above theories. Namely, the dRGT theory and it's weak field expansion at the second order. Setting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and

$\beta_1 = \beta_3 = \beta_4 = 0$ and $m \neq 0$ gives the so called massive Fierz-Pauli (mFP) action

$$S = M_p^2 \int d^4x \Big[-(h_{\mu\nu,\rho})^2 + 2h_{\mu\nu,\rho}h^{\mu\rho,\nu} - 2h_{,\mu}h^{\mu\nu}_{,\nu} + (h_{,\mu})^2 \\ + m^2\big[h^2 - (h_{\mu\nu})^2\big] + h_{\mu\nu}T^{\mu\nu}\Big], \tag{1.12}$$

where for completeness we added the matter coupling. This theory has been studied long ago as the simplest ghost free massive spin-2 field theory on a flat background [11]. As a reminder $M_p^2 = 2\kappa$ where κ is the Newton's constant which we restored for now. As argued before, to define a consistent linear field theory we have to assume $\partial^\mu T_{\mu\nu} = 0$. This conservation property puts constraints on the spin-2 field $h_{\mu\nu}$ as follows, writing the EOM's and taking the derivative of both sides + using the linearized Bianchi identity and the conservation of the matter stress energy tensor gives

$$G^{(1)}_{\mu\nu} + 2m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = \kappa T_{\mu\nu} \qquad \Longrightarrow \qquad \partial^\mu h_{\mu\nu} = -\partial_\nu h. \tag{1.13}$$

These are the four constraints which reduce the 10 DOF of the $h_{\mu\nu}$ to the 6. Further, taking the trace of EOM,

$$2\partial_\mu\partial_\nu h^{\mu\nu} - 2\Box h - 3m^2 h = \kappa T \qquad \text{Trace of the EOM,} \\ \Longrightarrow \qquad -\frac{3m^2h}{2} = \frac{\kappa}{2}T, \tag{1.14}$$

where in the second line we have used the above-mentioned constraints. Therefore, the trace of $h_{\mu\nu}$ is not dynamical which further reduces the number of DOF to 5. One can verify this by performing a Hamiltonian analysis but it is not necessary at this moment. We can summarize the EOM's using the constraints

$$(\Box - m^2)h_{\mu\nu} = -\frac{\kappa}{2}(T_{\mu\nu} - \frac{\eta_{\mu\nu}}{3}T + \frac{\partial_\mu\partial_\nu T}{3m^2}). \tag{1.15}$$

As a comparison in the linearized GR,

$$\Box h_{\mu\nu} = -\frac{\kappa}{2}(T_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}T). \tag{1.16}$$

At first it seems that $m \to 0$ limit of the mFP seems problematic but the last term $\frac{\partial_\mu\partial_\nu T}{3m^2}$ is actually a pure gauge term which can be shown by calculating that the effective action coming to the $h_{\mu\nu}$ from that part is zero.

$$\Delta S = \int d^4x T^{\mu\nu}(x)\tilde{h}_{\mu\nu}(x) \tag{1.17}$$

where

$$\tilde{h}_{\mu\nu}(x) = \kappa \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2+m^2} \frac{p_\mu p_\nu}{m^2} T(p), \tag{1.18}$$

So one has

$$\begin{aligned}
\Delta S &= \int d^4x T^{\mu\nu}(x) \tilde{h}_{\mu\nu}(x) \\
&= \kappa \int d^4x T^{\mu\nu}(x) \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2+m^2} \frac{p_\mu p_\nu}{m^2} T(p) \\
&= \kappa \int d^4x T^{\mu\nu}(x) \partial_\mu \partial_\nu \Big(\int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2+m^2} \frac{1}{m^2} T(p) \Big) \\
&= - \int d^4x \partial_\mu \big(T^{\mu\nu}(x) \big) \partial_\nu \Big(\int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2+m^2} \frac{1}{m^2} T(p) \Big) \\
&= 0.
\end{aligned} \tag{1.19}$$

One can calculate the propagator of the theory by putting the free action into the form

$$S = \int d^4x \Big[h_{\mu\nu} O^{\mu\nu\alpha\beta}(x) h_{\alpha\beta} \Big], \tag{1.20}$$

then the sourced solution would be

$$h_{\mu\nu}(x) = \int d^4x^{'} D_{\mu\nu\alpha\beta}(x - x^{'}) T^{\alpha\beta}(x^{'}). \tag{1.21}$$

It turns out that the momentum space propagators are

$$\begin{aligned}
D_{\mu\nu\rho\sigma}(p) &= \Big[\frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \Big] \frac{-i}{p^2+m^2-i\epsilon}, \\
&\text{(mFP)} \\
D_{\mu\nu\rho\sigma}(p) &= \Big[\frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma} \Big] \frac{-i}{p^2-i\epsilon}, \\
&\text{(Lin. GR)}
\end{aligned} \tag{1.22}$$

where $\theta_{\mu\nu} = \eta_{\mu\nu} + p_\mu p_\nu / m^2$ for the mFP and $\theta_{\mu\nu} = \eta_{\mu\nu} + p_\mu p_\nu / p^2$ for the linearized GR in an arbitrary frame. The zero mass divergent part as expected does not contribute to the propagator and the discrete difference between the mFP and the linearized GR comes from the last term in parenthesis, namely $1/3$ instead of $1/2$. This difference forms the experimental basis of the failure in the predictions of the mFP known as the vDVZ discontinuity. One can also write the propagator of the mFP in the $m \to 0$ limit in the co-moving frame as,

$$D_{\mu\nu\rho\sigma}(p) = \Big[P^{(2)}_{\mu\nu\rho\sigma} + \frac{P^{(0)}_{\mu\nu\rho\sigma}}{6} \Big] \frac{-i}{p^2 - i\epsilon}, \qquad \text{(mFP)} \tag{1.23}$$

where $P^{(2)}$ and $P^{(0)}$ are the spin-2 and spin-0 projection operators. It is the second term that does not decouple from the matter in the zero mass limit.

1.4 Solution to the Rotating Point Particle

Using equation (1.15) without the pure gauge part and the following conserved stress energy tensor,

$$\begin{aligned}
T_{00}(x) &= M\delta^3(\boldsymbol{x}),\\
T_{30}(x) = T_{03}(x) &= \frac{J}{2}\Big(\partial_y\delta^3(\boldsymbol{x}) - \partial_x\delta^3(\boldsymbol{x})\Big),
\end{aligned}$$

one can find the line element as

$$\begin{aligned}
(h_{\mu\nu}+\eta_{\mu\nu})dx^\mu dx^\nu = &\Big(-1+\frac{\kappa M}{6\pi}\frac{e^{-m\rho}}{\rho}\Big)dt^2 - \frac{\kappa J}{4\pi}\frac{e^{-m\rho}}{\rho}(1+m\rho)dtd\phi\\
&+\Big(1+\frac{\kappa M}{12\pi}\frac{e^{-m\rho}}{\rho}\Big)(d\rho^2+\rho^2 d\Omega).
\end{aligned} \tag{1.24}$$

One thing to note is that this solution is in the isotropic coordinates. We can re-write this metric in the familiar Boyer-Lindquist (BL) coordinates. Let,

$$r^2 = \rho^2\Big(1+\frac{\kappa M}{12\pi}\frac{e^{-m\rho}}{\rho}\Big). \tag{1.25}$$

Unfortunately ρ cannot be solved analytically in term of r. The best we can do is an approximate analytical solution. In the $m=0$ case,

$$r^2 = \rho^2 + \rho\frac{\kappa M}{12\pi} \qquad \Longrightarrow \qquad \rho = r\Bigg[\sqrt{1+\Big(\frac{\kappa M}{24\pi r}\Big)^2} - \frac{\kappa M}{24\pi r}\Bigg]. \tag{1.26}$$

Using the above form we can rewrite the metric in BL coordinates for $m=0$,

$$\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu = &\Big(-1+\frac{4\alpha}{r}+\frac{4\alpha^2}{r^2}+\frac{2\alpha^3}{r^3}-\frac{\alpha^5}{2r^5}+\dots\Big)dt^2\\
&-\frac{2J}{3M}\Big(\frac{\alpha}{r}+\frac{\alpha^2}{r^2}+\frac{2\alpha^3}{r^3}-\frac{\alpha^5}{8r^5}+\dots\Big)dtd\phi\\
&+\Big(1+\frac{2\alpha}{r}+\frac{2\alpha^2}{r^2}+\frac{\alpha^3}{r^3}-\frac{\alpha^5}{4r^5}+\dots\Big)dr^2\\
&+r^2 d\Omega,
\end{aligned} \tag{1.27}$$

where we defined $\alpha = \frac{\kappa M}{24\pi}$. We can also consider (1.25) up to the first order in m which gives

$$r^2 = \rho^2\Big(1 + \frac{\kappa M}{12\pi}\Big[\frac{1-m\rho}{\rho}\Big]\Big) + O(m^2) \qquad \Longrightarrow \qquad \rho = r\Big[\sqrt{\frac{1}{\beta} + \frac{\beta^2}{r^2}} - \frac{\beta}{r}\Big] \tag{1.28}$$

where $\beta = \frac{\alpha}{1-2\alpha m}$. Thus up to $O(m^2)$,

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = &\Big(-1 + \frac{4\alpha(1-mr)}{r} + \frac{4\alpha^2(1-mr)}{r^2} + \frac{2\alpha^3}{r^3} + \frac{2\alpha^4 m}{r^3} - \frac{\alpha^5}{2r^5} + \dots\Big)dt^2 \\ &- \frac{2J}{3M}\Big(\frac{\alpha}{r} + \frac{\alpha^2(1-mr)}{r^2} + \frac{2\alpha^3}{r^3} + \frac{2\alpha^4 m}{r^3} - \frac{\alpha^5}{8r^5} + \dots\Big)dtd\phi \\ &+ \Big(1 + \frac{2\alpha(1-mr)}{r} + \frac{2\alpha^2(1-mr)}{r^2} + \frac{\alpha^3}{r^3} + \frac{\alpha^4 m}{r^3} - \frac{\alpha^5}{4\tilde{\rho}^5} + \dots\Big)dr^2 \\ &+ r^2 d\Omega. \end{aligned} \tag{1.29}$$

These (1.27) and (1.29) forms are useful when we want to make a direct comparison with the Kerr metric in BL coordinates.

1.5 Experimental Bounds of the Graviton Mass

Observations constrain the maximum allowed mass of the graviton. However, these claimed bounds generally depend on the massive theory in consideration. Below, we have expressed some of the recent experiments and the corresponding bounds on the graviton mass in terms of the Compton wavelength.

Table 1.1: Graviton Mass Bounds

Planetary Ephersis [12]	$1.8 \times 10^{13} km$
Shapiro Time Delay [13]	$1.18 \times 10^{14} km$
GW150914 [14]	$10^{12} km$
GW151226 [14]	$3 \times 10^{12} km$
GW170104 [15]	$1.6 \times 10^{13} km$

One thing to note is, the most restrictive bounds still come from the solar system tests namely from the Shapiro time delay. There are also other estimated bounds from indirect considerations which further restricts the graviton wavelength to the size of the universe, see [16] for an extensive study.

CHAPTER 2

SOME REMARKS

2.1 Is there any Jebsen-Birkhoff's Theorem in mFP?

Once one has a theory of gravity, the first thing comes to mind is investigating possible solutions. The easiest is the spherically symmetric ones. In the case of GR, there exists the famous Jebsen-Birkhoff's Theorem which dictates exterior (vacuum) staticity once spherical symmetry is assumed. This in turn gives the unique Schwarzschild [1] solution.

If there exists a similar theorem in massive gravity, the solution space will be highly restricted. For the purposes of this thesis, we will narrow our focus into the mFP. For the case of the bi-metric massive gravity the problem seems to be complicated and may depend on the structure of the auxiliary metric [10].

Vacuum mFP equations can be written as,

$$(\Box - m^2)h_{\mu\nu} = 0, \tag{2.1}$$

with the constraints

$$\partial_\mu h^{\mu\nu} = 0, \qquad h = 0. \tag{2.2}$$

It can be seen that the most general spherically symmetric ansatz for the vacuum mFP which obeys the tracelessness condition is,

$$h_{\mu\nu}dx^\mu dx^\nu = -f(t,r)dt^2 + g(t,r)dr^2 + \Big[\frac{f(r,t)+g(r,t)}{2}\Big]r^2 d\Omega. \tag{2.3}$$

As a reminder, we do not have the luxury of fixing $\frac{f(r,t)+g(r,t)}{2} = 1$ as in GR since the action of the theory does not have the $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ gauge symmetry.

[1] Excluding the cosmological constant

Then, $\partial_\mu h^{\mu\nu} = 0$ constraint gives,

$$f(t,r) = f(r), \tag{2.4}$$

$$g(t,r) = g(t), \tag{2.5}$$

and the EOM further gives

$$\begin{aligned} \nabla^2 f &= m^2 f \\ \frac{d^2 g}{dt^2} &= -m^2 g. \end{aligned} \tag{2.6}$$

If we further assume asymptotic flatness this would require $g = 0$ and to have a non-singular metric we have to force $f = 0$ as well. Therefore, there is really no non-trivial vacuum solution to the mFP. The problem does not come from the missing gauge transformations of the theory. The situation is similar in the linearized GR as well. There is no non-trivial everywhere vacuum solution to the linearized GR! For example to get the linearized Schwarzschild solution, one has to assume a point source at the origin in the linearized GR. Therefore, the existence of Jebsen-Birkhoff theorem is a meaningful question for a non-linear theory which mFP cannot answer.

What about the Jebsen-Birkhoff theorem in a possible non-linearization of the mFP? The answer is that there is possibly no such Jebsen-Birkhoff theorem in a non-linear massive gravity. There is a conjecture in [17] claiming that "Any field theory lacking linearized spin-0 modes possesses a Birkhoff theorem.". Since massive gravity theories like mFP, have a spin-0 mode coming from graviton mass, one does not expect the Jebsen-Birkhoff theorem to be valid. Also see [18] for a direct investigation of Jebsen-Birkhoff theorem in some alternative theories like scalar-tensor theories and the results verify the previously mentioned conjecture.

2.2 What happens to the No Hair Theorem?

At first, it seems that we cannot comment about the no hair theorem in linear theory like mFP which does not have a black hole solution. However, there are some hints in the mFP which may persist in possible non-linear completions.

Once writing the effective one graviton exchange action between the two sources $t_{\mu\nu}$

$T_{\mu\nu}$ in mFP, one has

$$S = \int d^4x \Big[t^{\mu\nu} D_{\mu\nu\alpha\beta} T^{\alpha\beta}\Big],$$
$$= \int d^4x \Big[t^{\mu\nu} \frac{1}{\Box - m^2} T_{\mu\nu} - \frac{1}{3} t \frac{1}{\Box - m^2} T\Big] + \frac{1}{3m^2} \int d^4x \Big[tT\Big]. \tag{2.7}$$

The last term is a contact interaction amplified by the graviton mass which hints that non-linear completion of mFP can admit a scalar hair. There are actually known hairy black holes in the non-linear massive gravity although they are not stable [10].

2.3 ADM Energy

The usual ADM energy for the asymptotically flat metric

$$E_{ADM} = \frac{1}{\kappa} \lim_{r\to\infty} \int dS^i \Big[\partial_j h_{ij} - \partial_i h^j_j\Big], \tag{2.8}$$

will give zero for the point particle solution metric we have found because of the exponential decay terms. Instead this expression has to be complemented by

$$E_{ADM} = \frac{m^2}{\kappa} \int d^3x h^j_j + \frac{1}{\kappa} \lim_{r\to\infty} \int dS^i \Big[\partial_j h_{ij} - \partial_i h^j_j\Big], \tag{2.9}$$

which gives M for the point particle solution we presented earlier. This just comes from the integration of the (00) component of the EOM's.

2.4 Vainshtein Mechanism

As we have stated previously, the predictions of the linear theory does not agree with GR. However, it was argued by Vainshtein [19] in a seminal paper that one can recover the GR results after non-linearly completing the theory. He used the mass term of the mFP lagrangian added to the Einstein-Hilbert action. It was then possible to express the field equations perturbatively inversely in a parameter called the Vainshtein radius, $r_V = (GM/m^4)^{1/5}$. Then, as one approaches to the source, the non-linearities are amplified by r/r_V. Therefore, the predictions of the linear theory like mFP can be trusted in a distance larger than the Vainshtein radius. For the current graviton mass bounds, Vainshtein radius for a star with one solar mass is about the size of the Milky Way.

2.5 A Recent Proposal

It has long been known that that mFP Lagrangian is the only ghost free massive extension of the linearized GR. Starting from a mass term of the form

$$\Delta L = m^2(h_{\mu\nu}h^{\mu\nu} - \alpha h^2), \tag{2.10}$$

where $\alpha = 1$ corresponds to the Fierz-Pauli choice, one can work out the propagating degrees of the freedom of the resulting theory. It can be shown [5] that $\alpha \neq 1$ introduces a scalar ghost. Choosing $\alpha = 1/2$ on the other hand does not lead to a vDVZ discontinuity in the $m \to 0$ limit. Therefore, there is a trade off between introducing ghost and avoiding vDVZ discontinuity. The explanation is as follows, any mass term introduces a scalar mode and scalar modes are always attractive which causes the predictions of the massive theory to disagree with linearized GR. To cancel-out this attractive interaction one needs a ghost which has a negative energy. [2]

There is also another aspect of the mass term. It breaks the diffeomorphism invariance of the linearized GR. Therefore, mass term has "dual" purpose. A recent proposal [20] is that it is this "dual" nature of the mass term that causes the problem. If one can introduce a gauge breaking term to the linearized GR *before* adding mass term, one can get a ghost free theory without a vDVZ discontinuity.

They consider the following action,

$$S = S_{Lin.GR} + S_{gf} + S_m, \tag{2.11}$$

where $S_{Lin.GR}$ is the linearization of the GR action and

$$S_{gf} = -\frac{1}{2\beta}\int d^4x \Big[\partial_\mu h^{\mu\nu} + \alpha \partial_\mu h\Big]^2, \tag{2.12}$$

and

$$S_m = \frac{1}{2}\int d^4x (m_1^2 h_{\mu\nu}h^{\mu\nu} + m_2^2 h^2). \tag{2.13}$$

They tune the parameters α and β such that the resulting theory has a propagator free of vDVZ discontinuity and only one massive scalar mode which is not a ghost although there are two mass parameters.

[2] Remarkably this does not happen in massive spin-1 theory, as the mass goes to zero, the scalar mode decouples from the rest.

2.6 Schwarzschild Meets Yukawa

Consider the metric

$$ds^2 = (-1 + \frac{2Me^{-mr}}{r})dt^2 + \frac{dr^2}{1 - \frac{2Me^{-mr}}{r}} + r^2 d\Omega^2, \tag{2.14}$$

where we set $8\pi G = 1$. This metric has a nice a limit as $m \to 0$. Is there any such solution in GR? Of course, given the metric we can always find the corresponding matter distribution. We can calculate the corresponding stress energy tensor using $G_{\mu\nu} = T_{\mu\nu}$. The result is,

$$\begin{aligned} G_{tt} &= -\frac{2mMe^{-2mr}(e^{mr}r - 2M)}{r^3}, \\ G_{rr} &= \frac{2mM}{e^{mr}r^2 - 2Mr}, \\ G_{\theta\theta} &= -e^{-mr}m^2 Mr, \\ G_{\phi\phi} &= \sin(\theta)^2 G_{\theta\theta}. \end{aligned} \tag{2.15}$$

Then take the trace which gives

$$G = T = \frac{2mMe^{-mr}}{r^2}(2 - mr). \tag{2.16}$$

Now, we can impose different restrictions depending on the choice of the energy condition. For now, assume that $G_{tt} \geq 0$ which gives $r < 2M$. If we further have $m < 1/M$ then $T > 0$ for $r < 2M$. Hence it is possible that the given metric can be an *interior* solution to an ordinary matter distribution. But since we are forced to $r < 2M$, the metric does not have an event horizon so there is really no Yukawa like extension of the Schwarzschild with an event horizon in GR.

CHAPTER 3

MORE ON THE vDVZ DISCONTINUITY

We compile and extend most of the the weak field computations of the mFP. We have to state that some of the these calculations should be taken with a grain of salt because of the Vainshtein mechanism. However, they can be relevant for some other modified theories or astrophysical models.

Another point we want to make is on the method of the calculations. Certainly, some of these calculations can be easily seen using the PPN formalism or the effective field theory approach. We shall try to outline both approaches whenever possible.

For the calculations we will mainly use (1.24). To make a direct comparison with the GR in BL coordinates we will also employ (1.25), (1.26), (1.27) and (1.29). To prevent any confusion we will not fix the coupling constant κ and comment on the results for different values of κ. The natural choices for the κ are 12π and 16π. The former is required for the mFP to give the correct Newtonian potential in the $m \to 0$ limit. The latter is the same coupling constant in GR.

3.1 Deflection of Light

Consider a light ray passing through a non-rotating object, $J = 0$. One has for the metric (1.24) with $m = 0$ and $\kappa = 12\pi$ in the PPN notation of [21],

$$\gamma = 1/2, \tag{3.1}$$

and small angle light deflection is sensitive only to this parameter[1],

$$\delta\theta = \frac{1+\gamma}{2}\frac{\kappa}{4\pi}\frac{M}{b}, \tag{3.2}$$

since $\gamma = 1$ in GR, for $\kappa = 12\pi$ the light deflection is $3/4$ of the GR.

For the $\kappa = 16\pi$ case, all of light-matter or light-light interaction of the linearized GR and $m = 0$ mFP agree [2]. This can be seen by writing,

$$S = \kappa \int d^4x \Big[t^{\mu\nu} D_{\mu\nu\alpha\beta} T^{\alpha\beta} \Big], \tag{3.3}$$

and

$$\begin{aligned} D_{\mu\nu\rho\sigma}(p) &= \Big[\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - \frac{1}{3}\eta_{\mu\nu}\eta_{\rho\sigma}\Big]\frac{-i}{p^2 - i\epsilon}, \\ &\quad (m = 0 \text{ mFP}) \\ D_{\mu\nu\rho\sigma}(p) &= \Big[\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\Big]\frac{-i}{p^2 - i\epsilon}. \\ &\quad (\text{Lin. GR}) \end{aligned} \tag{3.4}$$

However, the the second term in these propagators do not contribute to the action if at least one of the sources is null, say $t = 0$. Therefore, there is no difference in both of the cases.

There are two more generalizations of the above result. We can consider the effect of non-zero graviton mass and rotation of the central source. We shall study both at the same time and try to solve the null geodesic equation with a source given by the metric (1.24) for a photon on the $\theta = \pi/2$ plane. Parametrizing the photon 4-velocity by $\frac{dx^\mu}{d\lambda} = (\dot{t}, \dot{\rho}, 0, \dot{\phi})$ and writing the conserved quantities

$$-\epsilon = g_{00}\dot{t} + g_{03}\dot{\phi}, \tag{3.5}$$

$$\epsilon b = g_{03}\dot{t} + g_{33}\dot{\phi}. \tag{3.6}$$

One also has,

$$0 = g_{00}\dot{t}^2 + 2g_{03}\dot{t}\dot{\phi} + g_{33}\dot{\phi}^2 + g_{11}\dot{\rho}^2. \tag{3.7}$$

[1] See [22] for a derivation.

[2] Here we are not considering the polarization of light or the angular momentum of the matter. This will be discussed in the spin interactions section.

From these equations,

$$\left(\frac{d\rho}{d\phi}\right)^2 = \frac{g_{00}g_{33} - g_{03}^2}{g_{11}} \frac{b^2 g_{00} + g_{33} + 2bg_{03}}{(bg_{00} + g_{03})^2}, \tag{3.8}$$

and evaluating the rhs for the (1.24) gives

$$\begin{aligned}\left(\frac{d\rho}{d\phi}\right)^2 =& \frac{12e^{m\rho}\pi\rho(b^2-\rho^2) - \kappa\big[(2b^2+\rho^2)M - 3bJ(1+m\rho)\big]}{\big[12e^{m\rho}\pi\rho + \kappa M\big]\big[3J(1+m\rho)\kappa - 4Mb\kappa + 24be^{m\rho}\pi\rho\big]^2} \\ &\times \big[576e^{2m\rho}\pi^2\rho^4 - 48e^{m\rho}M\pi\rho^3\kappa - \kappa^2\big(8M^2\rho^2 - 9J^2(1+m\rho)^2\big)\big].\end{aligned} \tag{3.9}$$

Although we have a complicated expression, we can consider some limiting cases. For example, to the first order in κ one gets,

$$\left(\frac{d\rho}{d\phi}\right)^2 = \rho^2 - \frac{\rho^4}{b^2} + \frac{e^{-m\rho}(J(1+m\rho) - Mb)\rho^3\kappa}{4b^3\pi} + O(\kappa^2). \qquad \text{(mFP)} \tag{3.10}$$

The effect of the source is in the last term. As a comparison, in the Schwarzschild metric expressed in isotropic coordinates,

$$\left(\frac{d\rho}{d\phi}\right)^2 = \rho^2 - \frac{\rho^4}{b^2} - \frac{4M\rho^3}{b^2} + O(M^2), \qquad \text{(Isotropic Schwarzschild)} \tag{3.11}$$

and the same equation in the spherical coordinates,

$$\left(\frac{dr}{d\phi}\right)^2 = r^2 - \frac{r^4}{b^2} - 2Mr + O(M^2), \qquad \text{(Schwarzschild)} \tag{3.12}$$

finally in the BL coordinates of the Kerr metric,

$$\left(\frac{dr}{d\phi}\right)^2 = r^2 - \frac{r^4}{b^2} - 2Mr + \frac{4r^3JM^2}{b^3} + O(M^3). \qquad \text{(Kerr)} \tag{3.13}$$

Things to note are,

- Light deflection in the $\kappa = 16\pi$ and $m = 0$ case, mFP agrees with the Schwarzschild solution as expected.
- In mFP the effect of rotation, J, in mFP comes at the same order of M in contrast to GR where there is no first order J effect on the deflection of light. This is a surprise but it is an artifact of the linear theory.

Finally, we turn into the effect of non-zero m which we ignored so far. Our aim is to solve (3.10) when $J = 0$. Defining $u = 1/\rho$,

$$\left(\frac{du}{d\phi}\right)^2 = u^2 - \frac{1}{b^2} - \frac{ue^{-\frac{m}{u}}M\kappa}{4b^2\pi} + O(\kappa^2), \tag{3.14}$$

and the deflection angle can be approximated as,

$$\delta\phi \sim \pi - 2\int_0^{x^*} \frac{dx}{\sqrt{1 - x^2 + \frac{x e^{-\frac{mb}{x}} M\kappa}{4b\pi}}}, \tag{3.15}$$

and where $x = ub$ and x^* is the biggest positive root of the denominator. Since an analytical expression is not possible, we can resort to a numerical integration which is given in Figure 3.1.

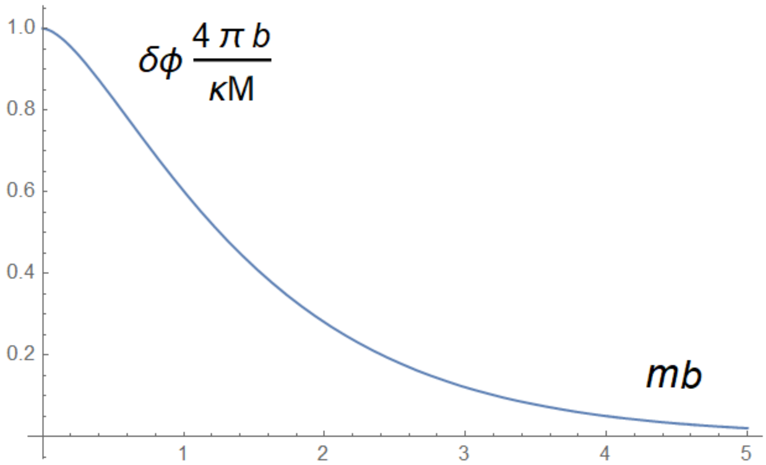


Figure 3.1: Weak deflection angle for a photon in mFP with graviton mass m and impact parameter b

3.2 Shapiro Time Delay

Shapiro time delay is the time difference measured by an observer as light signal bounces back and forth from a distant object in the absence and presence of an intermediate third mass [23]. Coordinate travel time for a photon that is projected spherically outward at a coordinate distance ρ_0 reaches ρ_1 and bounces back in the presence of a non-rotating point source in GR and mFP can be calculated. Here, we outline the results in the first order of M and subtract the flat space travel time,

$$\Delta t = 2\int_{\rho_0}^{\rho_1} d\rho \sqrt{\frac{-g_{11}}{g_{00}}} - 2(\rho_1 - \rho_0), \tag{3.16}$$

$$\Delta t = 2\frac{\kappa M}{8\pi}(\mathrm{Ei}(-m\rho_1) - \mathrm{Ei}(-m\rho_0)), \qquad \text{(mFP)} \tag{3.17}$$

$$\Delta t = 4M(\ln(\rho_1) - \ln(\rho_0)). \qquad \text{(Isotropic Schwarzschild)} \tag{3.18}$$

Here, the results are in the isotropic coordinates but conversion to the spherical coordinates is possible using (1.26) for example. As stated before, for $m = 0$ and $\kappa = 16\pi$ there is no difference in both cases.

3.3 Equatorial Circular Photon Orbits

Equatorial photon geodesics play an important role in some astrophysical calculations or black hole observations. Here, our aim is to show the discrete discontinuity in these orbits within the mFP in contrast to GR. However, we again emphasize that these calculations will not represent the most relevant nature of massive gravitation theories because of the Vainshtein's mechanism. As one goes near the source, the non-linearities become import and nonlinear massive theories are known to agree with GR.

We will consider two cases

- A non-rotating source with a varying graviton mass,
- A rotating source with a small graviton mass.

To find circular photon geodesics in the equatorial plane of the source, we have to set the rhs of (3.9) and it's derivative with respect to ρ to zero.

For the first case, a non-rotating source, one gets the following equation from this procedure

$$\rho\Big(6e^{m\rho}\pi\rho - M\kappa\Big)\Big(72e^{2m\rho}\pi^2\rho^2 - 3e^{m\rho}\pi\rho(5+3m\rho)M\kappa - M^2\kappa^2\Big) = 0. \tag{3.19}$$

This equation can be analytically solved for small m. For example, when $m = 0$, one gets a fourth order polynomial [3] equation and the positive solutions are

$$\begin{aligned}\rho_{ph1} &= \kappa M\frac{5+\sqrt{57}}{48\pi},\\ \rho_{ph2} &= \frac{M\kappa}{6\pi},\end{aligned} \tag{3.20}$$

[3] Although fourth order, one root is zero and the other root is already separated out.

and which can be expressed in terms of the spherical coordinates using (1.26),

$$
\begin{aligned}
r_{ph1} &= \frac{M\kappa}{24\pi}\sqrt{(51+7\sqrt{57})/2},\\
r_{ph2} &= \frac{M\kappa}{2\sqrt{6}\pi}.
\end{aligned} \tag{3.21}
$$

For an arbitrary m, one can resort to the numerical methods and it can be observed that the root coming from separated out first order polynomial is always less than the positive root coming from the second order part. Therefore, one might be tempted to think that the innermost circular photon geodesic is given by the simpler part $6e^{m\rho_+}\pi\rho_+ - M\kappa = 0$. However, we see later that the effect of rotation of the source does not effect this root. Therefore, this root is a mathematical artifact of the equations and does not represent something physical. Hence, we consider the positive solutions of the second order equation

$$
72e^{2m\rho_+}\pi^2\rho_+^2 - 3e^{m\rho_+}\pi\rho_+(5+3m\rho_+)M\kappa - M^2\kappa^2 = 0, \tag{3.22}
$$

whose solutions in spherical coordinates are plotted in Figure 3.2.

An interesting observation is about the polynomial equation (3.19) for $m = 0$. This is a fourth order equation. For the Schwarzschild solution in GR one has the analogous equations,

$$
r - 3M\kappa/16\pi = 0, \qquad \text{(Spherical coordinates)} \tag{3.23}
$$

$$
\rho\Big(4\rho^2 - 8\rho M\kappa/(16\pi) + \big[\kappa M/(16\pi)\big]^2\Big) = 0. \qquad \text{(Isotropic coordinates)} \tag{3.24}
$$

Therefore, the order of the polynomial equation clearly depends on the choice of the coordinates but still mFP equation is one degree higher than the GR equation in the $m = 0$ limit; although we mentioned that the extra root is not physical.

For the second case, a rotating source, one gets the following equation

$$\begin{aligned}
&- 3456\pi^3 r^4 e^{3mr} + 432\pi^2 r^3 \tilde{M} e^{2mr}(mr+3) \\
&+ \tilde{M}^2 \Big(\tilde{M} \left(9a^2 m(mr+1) - 8r\right) \\
&- 3am\sqrt{\tilde{M}^2 \left(9(amr+a)^2 - 8r^2\right) - 48\pi r^3 \tilde{M} e^{mr} + 576\pi^2 r^4 e^{2mr}}\Big) \\
&- 18\pi \tilde{M} e^{mr} \Big(a(mr(mr+1)+1) \\
&\times \sqrt{\tilde{M}^2 \left(9(amr+a)^2 - 8r^2\right) - 48\pi r^3 \tilde{M} e^{mr} + 576\pi^2 r^4 e^{2mr}} \\
&- \tilde{M}(mr+1)\left(3a^2(mr(mr+1)+1) - 4r^2\right) \Big) = 0,
\end{aligned} \tag{3.25}$$

where we have $a = J/M$ and $\tilde{M} = \kappa M$. See Figure 3.3 for the numerical results.

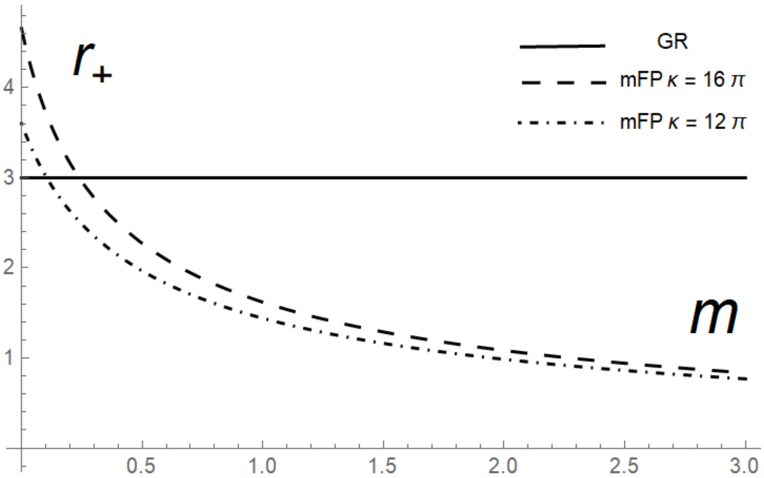


Figure 3.2: Equatorial circular photon geodesic radius in mFP for a non-rotating source in spherical coordinates *vs* m in units of $M = 1$ and $\kappa = 12\pi$. The GR result is $r_{ph} = 3$.

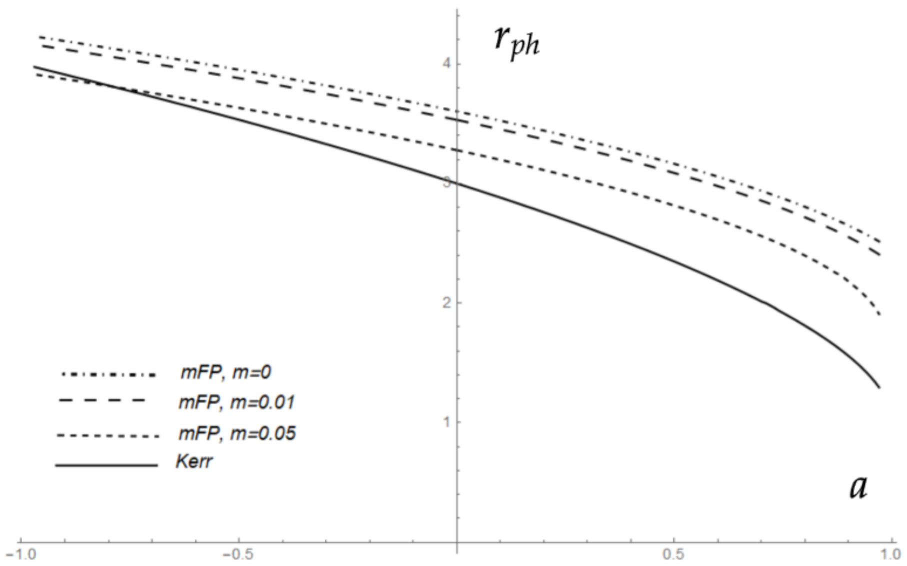


Figure 3.3: Equatorial circular photon geodesic radius in mFP for a source in spherical coordinates *vs* a in units of $M = 1$ and $\kappa = 12\pi$. The GR result (Kerr) is also given for $\kappa = 16\pi$.

3.4 Equatorial Circular Timelike Orbits

The analysis of the time-like orbits is similar to the one in the first section of this chapter. We have

$$-\epsilon = g_{00}\dot{t} + g_{03}\dot{\phi}, \tag{3.26}$$

$$\epsilon b = g_{03}\dot{t} + g_{33}\dot{\phi}. \tag{3.27}$$

One also has,

$$-1 = g_{00}\dot{t}^2 + 2g_{03}\dot{t}\dot{\phi} + g_{33}\dot{\phi}^2 + g_{11}\dot{\rho}^2. \tag{3.28}$$

From these equations,

$$g_{11}\frac{d^2\rho}{d\tau^2} = V_{eff}(\rho), \tag{3.29}$$

and

$$V_{eff}(\rho) = -1 + \epsilon^2 \frac{g_{33} - 2bg_{03} + b^2 g_{00}}{g_{03}^2 - g_{00}g_{33}}. \tag{3.30}$$

For circular orbits one should have $V_{eff} = 0$ and $dV_{eff}/d\rho = 0$. Using these conditions one arrives at

$$\epsilon = \frac{g_{00} + \Omega g_{03}}{\sqrt{-g_{00} - 2\Omega g_{03} - g_{33}\Omega^2}}, \tag{3.31}$$

$$\epsilon b = \frac{g_{03} + \Omega g_{33}}{\sqrt{-g_{00} - 2\Omega g_{03} - g_{33}\Omega^2}}, \tag{3.32}$$

$$\Omega = \frac{-g_{03,\rho} + \sqrt{g_{03,\rho}^2 - g_{00,\rho} g_{33,\rho}}}{g_{33,\rho}}. \tag{3.33}$$

therefore timelike circular orbits in the equatorial plane are always possible unless $-g_{00} - 2\Omega g_{03} - g_{33}\Omega^2 \leq 0$ for some parameters ϵ and b. To determine the inner edge one can impose $V_{eff,\rho\rho} = 0$ for stability. This further gives the following equation

$$\epsilon^2 g_{33,\rho\rho} - 2\epsilon^2 b g_{03,\rho\rho} + b^2 g_{00,\rho\rho} - (g_{03}^2 - g_{00} g_{33})_{,\rho\rho} = 0. \tag{3.34}$$

This equation for inner-most stable circular orbit (ISCO), evaluated for the mFP ($M = 1\ J = 0$) metric gives

$$\begin{aligned} &864\pi^3\rho^3 e^{3m\rho}(m^2\rho^2 - m\rho - 1) + 432\pi^2\kappa\rho^2 e^{2m\rho}(m\rho + 1)^2 \\ &- 18\pi\kappa^2\rho e^{m\rho}(m^3\rho^3 + 2m^2\rho^2 - 2m\rho - 2) - \kappa^3(m^2\rho^2 - m\rho - 1) = 0, \end{aligned} \tag{3.35}$$

as a comparison for the Schwarzschild solution ($M = 1$) in isotropic coordinates,

$$(\rho - 1/2)^2(4\rho^2 - 20\rho + 1) = 0. \tag{3.36}$$

Numerical results for the ISCO radius is given below. It turns out ISCO radius after a critical value of m in mFP diverges and the radius has a non-monotonic dependence on m.

3.5 Perihelion Precession

To see a possible vDVZ discontinuity in the perihelion problem, the easiest choice is to employ the PPN formalism. Let us take the isotropic form of a metric,

$$ds^2 = -Adt^2 + B(dr^2 + r^2 d\Omega). \tag{3.37}$$

Perihelion precession up to M/r order is influenced by the expansion of the product term $B(A - 1)$ up to M^2/r^2, see [22]. Therefore, we need to expand B to M/r and

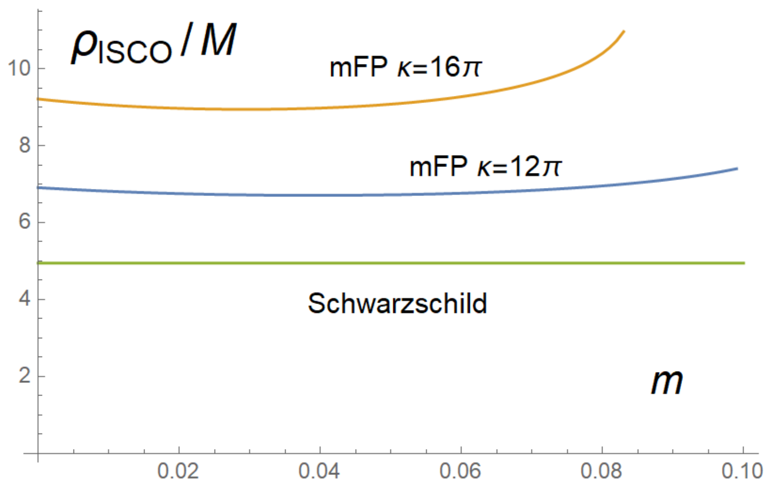


Figure 3.4: Isotropic ISCO radius in unit of $G = 1$.

A to M^2/r^2. In Schwarzschild coordinates one has

$$A = (1 - M/2\rho)^2(1 + M/2\rho)^{-2} = 1 - 2M/\rho + 2M^2/\rho^2 + \dots \quad , \tag{3.38}$$

$$B = (1 + M/2\rho)^4 = 1 + 2M/\rho + \dots \quad . \tag{3.39}$$

Thus in GR

$$B(A - 1) = (1 + 2M/\rho)(-2M/\rho + 2M^2/\rho^2) = -2M/\rho - 2M^2/\rho^2. \tag{3.40}$$

In mFP with $m = 0$ and $\kappa = 12\pi$,

$$A = 1 - 2M/\rho, \tag{3.41}$$

$$B = 1 + M/\rho. \tag{3.42}$$

Thus in mFP

$$B(A - 1) = (1 + M/\rho)(-2M/\rho) = -2M/\rho - 2M^2/\rho^2. \tag{3.43}$$

Therefore, there is no vDVZ discontinuity for the perihelion precession problem between Schwarzschild and FP with $m = 0$ and $\kappa = 12\pi$. There was a claim in the original [3] article about $2/3$ discontinuity in the perihelion precession which turns out to be incorrect and unnoticed so far.

3.6 Gravitational Waves

Theory of gravitational waves plays a central role in current observations. Therefore, it is one of the most important topics in a modified gravity to discuss.

We will discuss gravitational waves in the linearized regime [4]. Therefore, Vainshtein mechanism does not curse the result of this section.

There are a couple of ways to treat the gravitational waves. One can take a quantum mechanical viewpoint and consider tree level graviton processes (although there is no quantization here) or treat the gravitational waves within the Einstein's equations. We will comment on both approaches.

3.6.1 Introduction

The equation for a gravitational wave in a generic theory is given by

$$\mathcal{D}_2 \phi_{\mu\nu} = -\kappa t_{\mu\nu}, \tag{3.44}$$

where $\mathcal{D}_2$ is some hyperbolic second order differential operator and $\phi_{\mu\nu}$ and $t_{\mu\nu}$ are some linear functions of the metric perturbation and the stress tensor. In GR,

$$\mathcal{D}_2 = \Box, \qquad \phi_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu} h/2, \qquad t_{\mu\nu} = T_{\mu\nu}, \qquad \kappa = 16\pi G, \tag{3.45}$$

in mFP,

$$\mathcal{D}_2 = \Box - m^2, \qquad \phi_{\mu\nu} = h_{\mu\nu}, \qquad t_{\mu\nu} = T_{\mu\nu} - \eta_{\mu\nu} T/3, \qquad \kappa = 16\pi G \text{ or } 12\pi G. \tag{3.46}$$

Solution for a given theory can be expressed by the Green's function of the corresponding differential operator

$$\begin{aligned}
\phi^{GR}_{\mu\nu}(x) &= \kappa \int d^4y \frac{\delta(|\vec{x}-\vec{y}| - (x^0 - y^0))}{4\pi|\vec{x}-\vec{y}|} \theta(x^0 - y^0) t_{\mu\nu}(y), \\
\phi^{mFP}_{\mu\nu}(x) &= \kappa \int d^4y \frac{e^{-m|\vec{x}-\vec{y}|}\delta(|\vec{x}-\vec{y}| - (x^0 - y^0))}{4\pi|\vec{x}-\vec{y}|} \theta(x^0 - y^0) t_{\mu\nu}(y).
\end{aligned} \tag{3.47}$$

[4] In the case of mFP, the linear theory is all we have.

Also, when the sources are irrelevant one can write gravitational waves as a superposition of different polarizations

$$\phi_{\mu\nu}(x) = \int d\omega d^3k \epsilon_{\mu\nu}(k) e^{ikx}, \tag{3.48}$$

where $k^\mu = (\omega, \vec{k})$ and $k^2 = 0$ for GR and $k^2 = -m^2$ for mFP. However, in GR one has diffeomorphism invariance. Therefore, a gauge has to be imposed. This restricts the allowed basis for the polarization tensor by imposing conditions like $k^\mu \epsilon_{\mu\nu} = 0$, $\epsilon^\mu_\mu = 0$. Because of these constraints GR has only two tensor modes in the polarization tensor. In mFP, however, there is no such condition and all allowed six modes (four tensor + two scalar) modes are allowed.

A useful approximation is when the source is localized to a small region and we are considering the gravitational waves far away from it, $|\vec{x} - \vec{y}| \approx |\vec{x}| \equiv r$. In this case, one can approximate the integrals by

$$\begin{aligned}
\phi^{GR}_{\mu\nu}(x) &= \frac{\kappa}{4\pi r} \int d^4y t_{\mu\nu}(t - r, \vec{y}), \\
\phi^{mFP}_{\mu\nu}(x) &= \frac{\kappa e^{-mr}}{4\pi r} \int d^4y t_{\mu\nu}(t - r, \vec{y}).
\end{aligned} \tag{3.49}$$

It is also possible to express the spatial components of t_{ij} in terms of the quadruple moment tensor. Using the conservation of the source

$$\begin{aligned}
\partial_0 T_{00} + \partial_i T_{0i} &= 0, \\
\partial_0 T_{0i} + \partial_j T_{ij} &= 0,
\end{aligned} \tag{3.50}$$

and by integration by parts,

$$\begin{aligned}
\int d^3x T_{0i}(x) &= -\partial_0 \int d^3x x^i T_{00}(x), \\
\int d^3x T_{ij}(x) &= \partial_0^2 \int d^3x x^i x^j T_{00}(x) = \ddot{I}_{ij}.
\end{aligned} \tag{3.51}$$

Therefore,

$$h^{GR}_{ij}(x) = \frac{\kappa}{4\pi r} \ddot{Q}_{ij}, \tag{3.52}$$

$$h^{mFP}_{ij}(x) = \frac{\kappa e^{-mr}}{4\pi r} \Big[\ddot{Q}_{ij} - \frac{\delta_{ij}}{3} \ddot{Q}_0 \Big], \tag{3.53}$$

where $Q_{ij} = I_{ij} - \delta_{ij} I/3$ and $Q_0 = \int d^3y T_{00}(x)$.

3.6.2 Radiated Energy

Since we fixed the background, one can define a partially conserved tensor to represent the energy of the gravitational waves. The definition is however not unique. These tensors differ by super-potentials. Let us use the following definition

$$K_{\mu\nu} = \frac{1}{2}\Big[\frac{\partial L}{\partial(\partial_\mu h_{\alpha\beta})}\partial_\nu h_{\alpha\beta} - \eta_{\mu\nu}L + \mu \leftrightarrow \nu\Big]. \tag{3.54}$$

Then it turns out, one gets

$$K_{0i} = -\frac{1}{2}(\partial_i \xi_{\alpha\beta})(\partial_0 \xi^{\alpha\beta}) + O(m^2) \tag{3.55}$$

which is evaluated using the equations of motion outside the source and $\xi_{\alpha\beta} = h_{\alpha\beta} - \eta_{\alpha\beta}h/2$ for GR and $\xi_{\alpha\beta} = h_{\alpha\beta} - \eta_{\alpha\beta}h$ for mFP, we also omitted the terms multiplied by m^2 as they can be neglected. The emitted power is

$$P = \int_S dS^i \left\langle K_{0i} \right\rangle . \tag{3.56}$$

Averaging is necessary to ensure that the result is gauge independent in the case of GR. The results are then,

$$P_{GR} = -\frac{1}{5}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\right\rangle \tag{3.57}$$

$$P_{mFP} = -\frac{\kappa}{5\kappa_{GR}}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij} - \frac{\ddot{Q}\dot{Q}_0}{3} + \frac{\dot{Q}_0^2}{4}\right\rangle \qquad m \longrightarrow 0. \tag{3.58}$$

Alternatively, one can use the effective field approach which we follow from [24]. Emission rate can written by

$$d\Gamma = \frac{\kappa^2}{(2\pi)^2}\frac{1}{8}\int T_{\mu\nu}(k^{'})T^*_{\alpha\beta}(k^{'})P^{\mu\nu\alpha\beta}\frac{\delta(\omega-\omega^{'})}{\omega}d^3k, \tag{3.59}$$

where $P^{\mu\nu\alpha\beta}$ is the total spin projection operator mentioned in the Section (1.3) which for the mFP case is the sum of spin-2 and spin-0 parts. By further using the dispersion relation $k^2 = \omega^2 - m^2$ and the explicit form of the projection operator, energy emission rate can be written as

$$\frac{dE}{dt} = \frac{\kappa^2}{(2\pi)^2}\frac{1}{8}\int \Big[T_{\mu\nu}T^{\mu\nu}(k^{'}) - \frac{T^2}{3}(k^{'})\Big]\delta(\omega-\omega^{'})\omega^2\Big(1 - m^2/\omega^2\Big)d\omega d\Omega_k. \tag{3.60}$$

Finally, the term inside the square bracket can be expressed only in terms of the spatial part of the energy-momentum tensor using $k^{\mu}T_{\mu\nu} = 0$ and $\hat{k}^i = \frac{k^i}{\sqrt{\omega^2 - m^2}}$. The result is

$$\Big[T_{\mu\nu}T^{\mu\nu}(k^{'}) - \frac{T^2}{3}(k^{'})\Big] = T^{ij}\Lambda_{ijlm}T^{*lm}, \tag{3.61}$$

$$\Lambda_{ijlm} = \delta_{il}\delta_{jm} - \frac{1}{3}\delta_{ij}\delta_{lm} - 2\Big(1 - \frac{m^2}{\omega^2}\Big)\hat{k}_j\hat{k}_m\delta_{il} + \frac{2}{3}\Big(1 - \frac{m^2}{\omega^2}\Big)^2\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l \tag{3.62}$$

$$+ \frac{1}{3}\Big(\hat{k}_l\hat{k}_m\delta_{ij} + \hat{k}_i\hat{k}_j\delta_{lm}\Big). \tag{3.63}$$

Also using,

$$\int d\Omega_k\hat{k}_i\hat{k}_j = \frac{4\pi}{3}\delta_{ij}, \tag{3.64}$$

$$\int d\Omega_k\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m = \frac{4\pi}{15}(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}). \tag{3.65}$$

$$\begin{aligned} P_{mFP} =& \frac{8G}{5}\int\Bigg[\Big\{\frac{5}{2} - \frac{5}{3}\Big(1 - \frac{m^2}{\omega^2}\Big) + \frac{2}{9}\Big(1 - \frac{m^2}{\omega^2}\Big)^2\Big\}|T_{ij}|^2 \\ &+ \Big\{-\frac{5}{6} + \frac{5}{9}\Big(1 - \frac{m^2}{\omega^2}\Big) + \frac{1}{9}\Big(1 - \frac{m^2}{\omega^2}\Big)^2\Big\}|T^i_i|^2\Bigg]\delta(\omega - \omega^{'})\omega^2\Big(1 - \frac{m^2}{\omega^2}\Big)^{\frac{1}{2}}d\omega, \end{aligned} \tag{3.66}$$

where $\kappa = 12\pi$ is set. A similar analysis for GR gives

$$P_{GR} = \frac{8G}{5}\int\Bigg[|T_{ij}|^2 - \frac{1}{3}|T^i_i|^2\Bigg]\delta(\omega - \omega^{'})\omega^2 d\omega. \tag{3.67}$$

3.6.3 Gravitational Wave Memory

Gravitational wave memory is a permanent displacement effect after a gravitational passes through freely falling test masses. Because of it's observational possibility in eLISA, it caught a recent attention.

Theoretically, there are two memory effects called linear and non-linear under the same name. Here we focus on the linear memory. In GR, the linear memory associated with the DC part of the gravitational wave can be expressed in terms of the initial and final configurations of point masses in the PPN framework as

$$\Delta h^{TT}_{ab} = \frac{4}{r}\Delta\Bigg[\sum_i \frac{m_i}{\sqrt{1 - v_i^2}}\frac{(v_i)_a(v_i)_b}{1 - v_i\cos\theta_i}\Bigg], \tag{3.68}$$

here Δ signifies the subtraction between $t = \pm\infty$ limits. Separation between test masses after the memory effect can be neatly expressed in terms of an object called memory tensor

$$\Delta d^a = \frac{1}{r}\Delta^a_b \Theta(t-r) d^b. \tag{3.69}$$

Then, an interesting question is how the structure of the memory tensor is modified for the massive gravity and in particular for mFP theory? It turns out [25] memory tensor in mFP has additional time-like and off-diagonal contributions. Also, the limit $m \to 0$ exhibits vDVZ discontinuity. The most important result is, however, memory tensor is suppressed by $e^{-m_g r}$ as expected and for the current bounds on the graviton mass and binary black hole event locations, there is effectively no measurable memory effect.

3.7 Geodetic Effect (de Sitter Precession)

Due to the curvature or relativistic nature of the space-time there are interesting effects which are absent in Newtonian physics. Spinning particles are known to precess when orbiting around a massive central object.

The theory of these spin effects is rich. Although there are many ways to formulate the dynamics of spin in a curved space-time like gravitomagnetism etc, we will take a more geometric approach.

Change of a vector in the frame of an observer is described by the Fermi-Walker transport.

$$u^\mu \nabla_\nu S^\nu = -u^\mu a_\nu S^\nu, \tag{3.70}$$

where u^μ and a^μ are four velocity and acceleration of the observer. For a freely falling observer, $a^\mu = 0$ and this reduces to the usual parallel transport equation.

Geodetic effect considers the scenario in which the massive object is not rotating. Therefore, we consider (1.24) with $J = 0$.

The equations we are trying to solve are,

$$u_\mu u^\mu = -1, \qquad S_\mu S^\mu = 1, \qquad u^\mu S_\mu = 0 \qquad u^\nu \nabla_\nu S^\mu = 0. \tag{3.71}$$

We will work with a spherically symmetric metric with the form

$$g_{\mu\nu} = \text{diag}\Big[g_{00}(x^1), g_{11}(x^1), g_{22}(x^1), g_{33}(x^1, \theta)\Big], \tag{3.72}$$

through out the calculations. Let $u^\mu = u^0(1, 0, 0, \Omega)$ and confine everything to $\theta = \pi/2$ then from $u_\mu u^\mu = -1$

$$u^0 = \sqrt{\frac{-1}{g_{tt} + g_{\phi\phi}\Omega^2}}, \tag{3.73}$$

$$u^0_{GR} = \sqrt{\frac{1}{1 - \frac{2GM}{r} - \Omega^2 r^2}}, \tag{3.74}$$

$$u^0_{FP} = \sqrt{\frac{1}{1 - \frac{\kappa M e^{-mr}}{6\pi r} - \Big(r^2 + \frac{\kappa M e^{-mr}(1+mr+m^2r^2)}{12\pi m^2 r}\Big)\Omega^2}}. \tag{3.75}$$

To find Ω one can use the geodesic equation for the radial coordinates. Then for the metric form in consideration one gets

$$\Omega^2 = -\frac{\partial_1 g_{00}}{\partial_1 g_{33}}. \tag{3.76}$$

Finally, for the mFP metric,

$$\Omega^2 = \frac{\kappa M e^{-m\rho}(1 + m\rho)}{12\pi\rho^3} + O(M^2). \tag{3.77}$$

In GR,

$$\Omega^2_{GR} = \frac{GM}{r^3}. \tag{3.78}$$

Let us parametrize the trajectory of the particle with respect to ϕ. Then, $u^\mu \nabla_\mu S^\nu = 0$ gives,

$$d_\phi S^0 + \Gamma^0{}_{01}\frac{S^1}{\Omega} = 0, \tag{3.79}$$

$$d_\phi S^1 + \Gamma^1{}_{00}\frac{S^0}{\Omega} + \Gamma^1{}_{33}S^3 = 0, \tag{3.80}$$

$$d_\phi S^2 + \Gamma^2{}_{33}S^3 = 0, \tag{3.81}$$

$$d_\phi S^3 + \Gamma^3{}_{31}S^1 + \Gamma^3{}_{32}S^2 = 0, \tag{3.82}$$

with

$$\Gamma^0{}_{01} = \frac{g^{00}}{2}\partial_\rho g_{00}, \qquad \Gamma^1{}_{00} = -\frac{g^{11}}{2}\partial_\rho g_{00},$$
$$\Gamma^1{}_{33} = -\frac{g^{11}}{2}\partial_\rho g_{33}, \qquad \Gamma^2{}_{33} = -\frac{g^{22}}{2}\partial_\theta g_{33},$$
$$\Gamma^3{}_{31} = \frac{g^{33}}{2}\partial_\rho g_{33}, \qquad \Gamma^3{}_{32} = \frac{g^{33}}{2}\partial_\theta g_{33}. \tag{3.83}$$

From $u^\mu S_\mu = 0$, one also has

$$S^0 = \frac{S^3 \Omega g_{33}}{g_{00}}. \tag{3.84}$$

Inserting this into the first and last equations in (3.79) recovers (3.76). Finally, (3.79) gives for (1.24) with $J = 0$ and $\theta = \pi/2$,

$$d_\phi S^1 + \left[-\rho - \frac{e^{-m\rho}(1+m\rho)M\kappa}{24\pi} + O(M^2) \right] S^3 = 0, \tag{3.85}$$
$$d_\phi S^3 + \left[\frac{1}{\rho}, -\frac{e^{-m\rho}(1+m\rho)M\kappa}{24\pi\rho^2} + O(M^2) \right] S^1 = 0, \tag{3.86}$$

as a comparison, for the Schwarzschild metric in spherical coordinates

$$d_\phi S^1 + \left[-r + M \right] S^3 = 0, \tag{3.87}$$
$$d_\phi S^3 + \left[\frac{1}{r} \right] S^1 = 0, \tag{3.88}$$

and in isotropic coordinates

$$d_\phi S^1 + \left[-\rho + O(M^2) \right] S^3 = 0, \tag{3.89}$$
$$d_\phi S^3 + \left[\frac{1}{\rho} - \frac{M}{\rho^2} + O(M^2) \right] S^1 = 0. \tag{3.90}$$

3.8 Lense-Thrring Effect

Also known as the frame-dragging effect, this phenomenon causes spin-vector of an observer to precess solely due to the rotation of the source.

In the coordinate system of the observer, $x^0 = t$, the precession rate is [26]

$$\Omega_k = \frac{g_{00}}{2\sqrt{-g}} \epsilon_{ijk} \partial_j \left(\frac{g_{0i}}{g_{00}} \right) \left(\partial_k - \frac{g_{0k}}{g_{00}} \partial_0 \right) \tag{3.91}$$

Evaluating this expression to first order in κ for the (1.24) and to first order in G for the Kerr metric,

$$\vec{\Omega}_{FP} = \frac{\kappa}{16\pi}\frac{e^{-m\rho}}{\rho^3}\Big[(1+m\rho+m\rho^2)\vec{J} - (3+3m\rho+m^2\rho^2)(\vec{J}\cdot\hat{\rho})\hat{\rho}\Big], \tag{3.92}$$

$$\vec{\Omega}_{Kerr} = \frac{M}{r^3}\Big[\vec{a} - (3\vec{a}\cdot\hat{r})\hat{r}\Big]. \tag{3.93}$$

3.9 Einstein-Infeld-Hoffmann Potential

Einstein-Infeld-Hoffmann (EIH) potential is a first order general relativistic interaction potential between two (spinning) point sources. It's form is quite valuable in certain approximate calculations. To calculate the EIH potential between two sources, one can use the effective field theory approach

$$U = -\frac{\kappa}{4t}\int d^4x d^4x' T^{(1)}_{\mu\nu}(x') D_{\mu\nu\alpha\beta}(x,x') T^{(2)}_{\alpha\beta}(x), \tag{3.94}$$

where $D_{\mu\nu\alpha\beta}$ is the propagator of the theory given in Ch. 2. Let

$$T_{00}(x) = M_{1(2)}\Big(1 + \frac{v^2_{1(2)}}{2} - \frac{J^k_{1(2)} v^i \epsilon_{ikj}\partial_j}{2}\Big)\delta^3(\boldsymbol{x} - \boldsymbol{x}_{1(2)}), \tag{3.95}$$

$$T_{i0}(x) = \Big(-M_{1(2)}v^i + \frac{J^k_{1(2)}\epsilon_{ikj}\partial_j}{2}\Big)\delta^3(\boldsymbol{x} - \boldsymbol{x}_{1(2)}), \tag{3.96}$$

and note that stress part of the energy momentum tensor do not contribute to the EIH order. Then the result is [27]

$$\begin{aligned} U_{mFP} = &-\frac{\kappa}{12\pi}\frac{e^{-mr}}{r}\Bigg[M_1M_2\Big(\frac{4}{3} + \frac{4}{3}(v_1^2+v_2^2) - 4\vec{v}_1\cdot\vec{v}_2\Big) \\ &+ \frac{1+mr+m^2r^2}{r^2}\Big(\vec{J}_1\cdot\vec{J}_2 - 3\vec{J}_1\cdot\hat{r}\vec{J}_2\cdot\hat{r}\frac{1+mr+m^2r^2/3}{1+mr+m^2r^2}\Big) \\ &+ \frac{1+mr}{r}\Big(\frac{4M_1(\hat{r}\times\vec{v}_2)\cdot\vec{J}_2}{3} - \frac{4M_2(\hat{r}\times\vec{v}_2)\cdot\vec{J}_2}{3} \\ &- 2M_1(\hat{r}\times\vec{v}_1)\cdot\vec{J}_2 + 2M_2(\hat{r}\times\vec{v}_2)\cdot\vec{J}_1\Big)\Bigg], \end{aligned} \tag{3.97}$$

$$\begin{aligned} U_{GR} = &-\frac{G}{r}\Bigg[M_1M_2\Big(1 + \frac{3}{2}(v_1^2+v_2^2) - 4\vec{v}_1\cdot\vec{v}_2\Big) + \frac{1}{r^2}\Big(\vec{J}_1\cdot\vec{J}_2 - 3\vec{J}_1\cdot\hat{r}\vec{J}_2\cdot\hat{r}\Big) \\ &+ \frac{1}{r}\Big(3M_1(\hat{r}\times\vec{v}_2)\cdot\vec{J}_2 - 3M_2(\hat{r}\times\vec{v}_2)\cdot\vec{J}_2 \\ &- 2M_1(\hat{r}\times\vec{v}_1)\cdot\vec{J}_2 + 2M_2(\hat{r}\times\vec{v}_2)\cdot\vec{J}_1\Big)\Bigg], \end{aligned} \tag{3.98}$$

where $\vec{r} = \vec{x}_1 - \vec{x}_2$ and m is the graviton mass. Again, one can see that $m \to 0$ limit of mFP does not recover the GR.

3.10 Critical Coordinate Velocity

There is an interesting concept regarding the radial motion of the particles in the weak field limit GR [28]. When written in terms of Schwarzschild coordinates, the coordinate velocity of the particles shot from infinity satisfies

$$v^2 = v_\infty^2 + \frac{2M}{r}(1 - 3v_\infty^2), \tag{3.99}$$

for $r \gg 2M$. Therefore, to an observer at infinity a critical velocity of $v_c = \sqrt{1/3}$ appears. The value of this velocity does not change by a various isotropic, harmonic and asymptotically flat coordinate transformations. It is also not related to the dimension of the space. In the mFP framework a similar equation can be found to be,

$$v^2 = v_\infty^2 + \frac{M\kappa e^{-m\rho}}{12\pi\rho}(2 - 5v_\infty^2). \tag{3.100}$$

Therefore, although relativistic fields seen by the particles in the $v_\infty \ll 1$ limit are equal ($\frac{2M}{r}$) in GR and mFP ($\kappa = 12\pi$), the critical coordinate velocities are different.

3.11 Kretschmann scalar

We can calculate Kretschmann scalar for the (1.24),

$$R^2_{\mu\nu\alpha\beta} = K = -\kappa^2 \frac{f(M,J,m)}{96\pi^{28}} e^{-2m\rho}, \tag{3.101}$$

where

$$\begin{aligned}
f(M,J,m) &= 9J^2m^6\rho^6\sin^2(\theta) + 36J^2m^5\rho^5\sin^2(\theta) - 12M^2m^4\rho^6 \\
&+ 45J^2m^4\rho^4\sin^2(\theta) + 54J^2m^4\rho^4 - 40M^2m^3\rho^5 - 108J^2m^3\rho^3\sin^2(\theta) \\
&+ 324J^2m^3\rho^3 - 100M^2m^2\rho^4 - 486J^2m^2\rho^2\sin^2(\theta) + 810J^2m^2\rho^2 \\
&- 120M^2m\rho^3 - 648J^2m\rho\sin^2(\theta) + 972J^2m\rho - 60M^2\rho^2 \\
&- 324J^2\sin^2(\theta) + 486J^2,
\end{aligned} \tag{3.102}$$

and the limit $m \to 0$ gives,

$$K = \kappa^2 M^2 \frac{10\rho^2 - 54a^2\cos^2\theta - 27a^2}{16\pi^2\rho^8}, \qquad a \equiv J/M. \tag{3.103}$$

Whereas Kerr space-time in large r limit gives

$$K = 48G^2M^2\frac{r^2 - 15a^2\cos^2(\theta)}{r^8}. \tag{3.104}$$

CHAPTER 4

ASTROPHYSICS

4.1 Introduction

In this chapter, we will mainly discuss the observational properties of the black holes in massive gravity. This means we abandon the linear theory of mFP and consider the black hole solutions in the dRGT theory of gravity which is the non-linear generalization of mFP.

Consider the action

$$S = \int d^4x\sqrt{-g}\frac{1}{2\kappa^2}\Big[R + m_g^2 U(g,\phi^a)\Big], \tag{4.1}$$

where $\kappa^2 = 8\pi$ with $G = 1$ and

$$U = U_2 + \alpha_3 U_3 + \alpha_4 U_4, \tag{4.2}$$

$$\begin{aligned}
U_2 &= [K]^2 - [K^2],\\
U_3 &= [K]^3 - 3[K][K^2] + 2[K^3],\\
U_4 &= [K]^4 - 6[K^2][K]^2 + 8[K][K^3] + 3[K^2]^2 - 6[K^4],
\end{aligned} \tag{4.3}$$

and

$$K^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\lambda} f_{\alpha\beta}\partial_\lambda\phi^\alpha\partial_\nu\phi^\beta}. \tag{4.4}$$

For each choice of the auxiliary metric $f_{\alpha\beta}$ there is a class of solutions. One can classify solutions in which both the metric and the auxiliary metric are diagonal or not *etc*. One particular choice studied in the literature is setting,

$$f_{\alpha\beta} = \mathrm{diag}\Big[0, 0, l^2, l^2\sin^2\theta\Big],$$

and it can be shown [29] that

$$ds_g^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega, \tag{4.5}$$

is a (vacuum) solution to the equations of motions of the above action with

$$f(r) = 1 - \frac{2M}{r} + \frac{\Lambda r^2}{3} + \gamma r + \zeta, \tag{4.6}$$

where Λ, γ, ζ appear as integration constants and they are related with the parameters of the theory as

$$\Lambda = 3m_g^2(1 + \alpha + \beta), \tag{4.7}$$

$$\gamma = -lm_g^2(1 + 2\alpha + 3\beta), \tag{4.8}$$

$$\zeta = l^2 m_g^2(\alpha + 3\beta), \tag{4.9}$$

and

$$\alpha = 3\alpha_3 + 1 \qquad \beta = 4\alpha_4 + \alpha_3. \tag{4.10}$$

Therefore, massive gravity theories introduce cosmological constant naturally. However, there are also other contributions like the global monopole term ζ and a linear term γr.

4.2 Black Hole Shadow

The next question is "Can we deduce Λ, γ, ζ of the dRGT Massive Gravity Theory by observing the shadow of a black hole?".

We can review the analytical approach to the black hole shadow. Consider a spherically symmetric metric of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + D(r)d\Omega. \tag{4.11}$$

Using two constants of the motion

$$E = A(r)\dot{t}, \qquad L = D(r)\dot{\Phi}, \tag{4.12}$$

and the null geodesic equation (we set $\theta = \pi/2$)

$$-A(r)\dot{t}^2 + B(r)\dot{r}^2 + D(r)\dot{\phi}^2 = 0, \tag{4.13}$$

it can be written that

$$\Big(\frac{dr}{d\phi}\Big)^2 = \frac{D(r)}{B(r)}\Big(\frac{D(r)}{A(r)}\frac{E^2}{L^2} - 1\Big). \tag{4.14}$$

Introduce a quantity

$$h^2(r) = \frac{D(r)}{A(r)}, \tag{4.15}$$

also let r_{ph} be the radius of the photon sphere and r_O be the radius of an observer. Then, the shadow size observed by the location of this observer will be

$$\sin^2\alpha = \frac{h^2(r_{ph})}{h^2(r_0)}. \tag{4.16}$$

Finally, r_{ph} can be found via

$$\frac{dh^2(r)}{dr}\Big|_{r=r_{ph}} = 0. \tag{4.17}$$

In the case of the metric in (4.5), one can write

$$r_{ph} = \frac{-1-\zeta+\sqrt{1+6M\gamma+2\zeta+\zeta^2}}{\gamma}, \tag{4.18}$$

as a short-hand notation define

$$\xi \equiv 1+\zeta-\sqrt{6M\gamma+(1+\zeta)^2}.$$

Then,

$$\sin^2\alpha = \frac{\xi^2 f(r_O)}{r_O^2\gamma^2\Big((-\xi-1-\zeta)+\frac{2M\gamma}{\xi}+\frac{\xi^2\lambda}{3\gamma^2}\Big)}. \tag{4.19}$$

From this formula, we focus on some limiting cases. First, consider $\zeta \neq 0$ and $\Lambda = \gamma = 0$

$$\sin^2\alpha = \frac{27M^2(1-\frac{2M}{r_O}+\zeta)}{(1+\zeta)^3 r_O^2} \quad\longrightarrow\quad \frac{27M^2}{(1+\zeta)^2 r_O^2} \quad \text{for large } r_O. \tag{4.20}$$

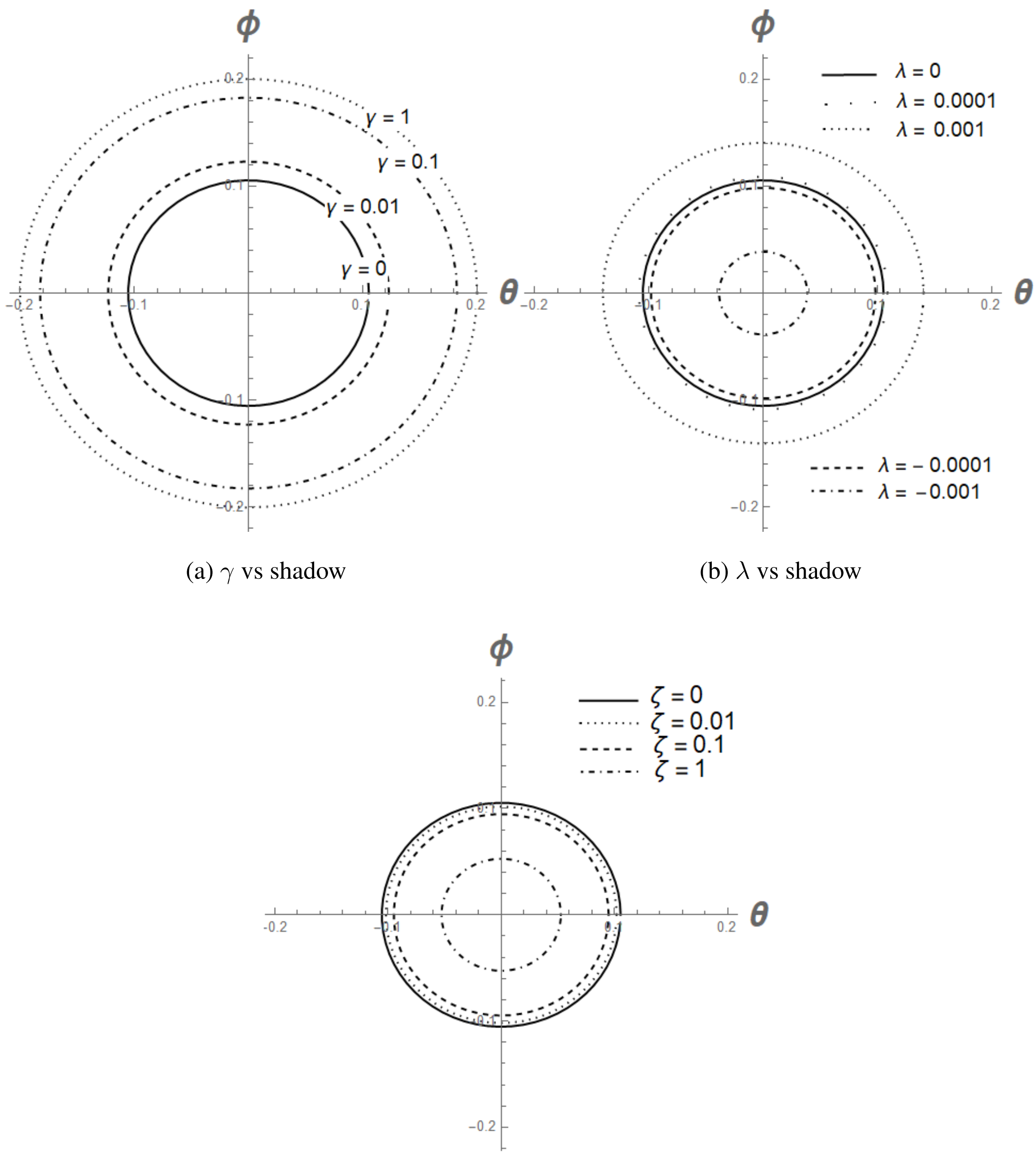
Second, consider $\Lambda \neq 0$ and $\gamma = \zeta = 0$

$$\sin^2\alpha = \frac{27M^2(1-\frac{2M}{r_O}+\frac{\Lambda r_O^2}{3})}{(1+9M^2\Lambda)r_O^2}. \tag{4.21}$$

Finally, consider $\gamma \neq 0$ and $\Lambda = \zeta = 0$

$$\sin^2\alpha = \frac{(-1+\sqrt{6M\gamma+1})^3(1-\frac{2M}{r_O}+\gamma r_O)}{\gamma^2(1+4M\gamma-\sqrt{6M\gamma+1})r_O^2}. \tag{4.22}$$

The last two cases do not have a well defined $r_O \longrightarrow \infty$ limit as the geometry is not asymptotically flat.



(a) γ vs shadow

(b) λ vs shadow

(c) ζ vs shadow

Figure 4.1: Effect of γ,λ and ζ on the black hole shadow for a static observer at $r_0 = 50M$ and we set $M = G = 1$.

In a recent article, [30], black hole shadow dependence on the linear term γr has been addressed. Their result suggests that black hole shadow decreases with increasing γ. This seems to contradict with our results given on Figure 4.1. However, in [30] they consider only the radius of photon sphere instead of (4.16). In fact, expanding (4.22) for small γ gives

$$\sin^2\alpha = \frac{27M^2}{r_0^3}\Big[(r_0 - 2M) + \gamma(r_0^2 - 9Mr_0 + 18M^2)\Big] + O(\gamma^2). \tag{4.23}$$

For $r_0 \geq 6M$ the effect of small γ on the black hole shadow, $\sin^2\alpha$, is always positive.

4.3 Novikov-Thorne Model

Another observable property of the massive gravity in the strong field regime could be the accretion disk around a black hole.

Novikov-Thorne model, see [31] for full details, of accretion disks assumes a disk with negligible thickness starting from the ISCO radius and has the Keplerian orbital frequency. In the model radiation flux emitted by the surface of accretion disk is;

$$F(r) = \frac{\dot{M}_0}{4\pi M^2} f_{disk}(r), \tag{4.24}$$

where $\dot{M}_0$ is the mass accretion rate to hole and $f_{disk}(r)$ is given by

$$f(r) = -\Omega_{,r}\frac{M^2}{\sqrt{-g}(\tilde{E} - \Omega\tilde{L})^2}\int_{r_{ISCO}}^{r} dr(\tilde{E} - \Omega\tilde{L})\tilde{L}_{,r}, \tag{4.25}$$

and

$$\tilde{E} = \frac{g_{00} + \Omega g_{03}}{\sqrt{-g_{00} - 2\Omega g_{03} - g_{33}\Omega^2}}, \tag{4.26}$$

$$\tilde{L} = \frac{g_{03} + \Omega g_{33}}{\sqrt{-g_{00} - 2\Omega g_{03} - g_{33}\Omega^2}}, \tag{4.27}$$

$$\Omega = \frac{-g_{03,r} + \sqrt{g_{03,r}^2 - g_{00,r}g_{33,r}}}{g_{33,r}}. \tag{4.28}$$

Also, r_{ISCO} is given by the solution of the following equation

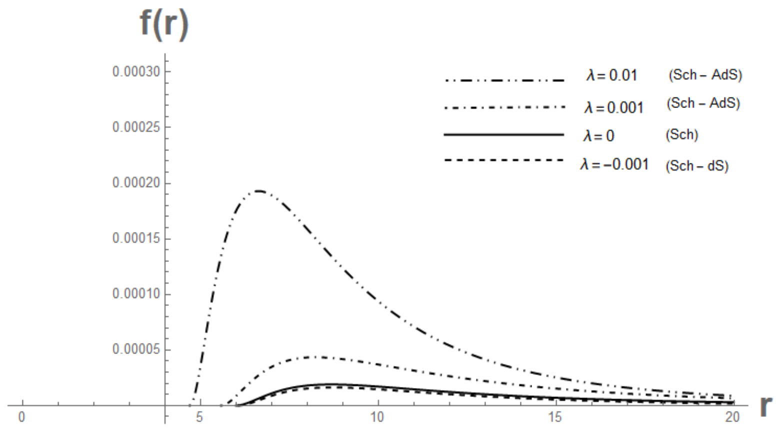
$$\tilde{E}^2 g_{33,rr} - 2\tilde{E}\tilde{L}g_{03,rr} + \tilde{L}^2 g_{00,rr} - (g_{03}^2 - g_{00}g_{33})_{,rr} = 0. \tag{4.29}$$

For the metric (4.5), the ISCO equation is

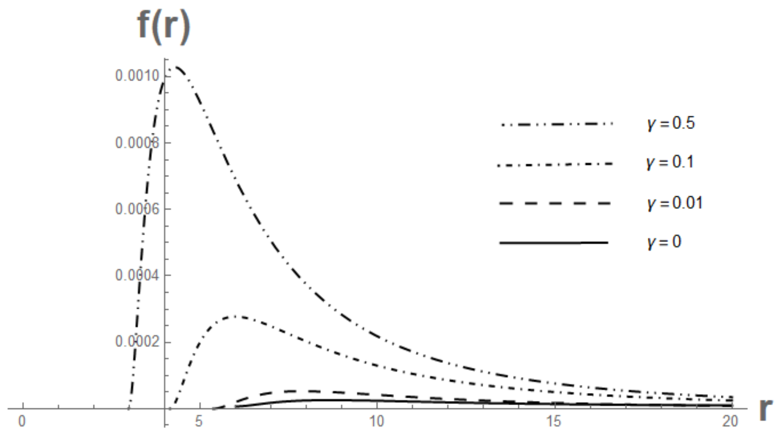
$$3\gamma\Lambda r^5 + (3\gamma^2 + 8(1+\zeta)\Lambda)r^4 \tag{4.30}$$

$$+ (9\gamma(1+\zeta) - 30M\Lambda)r^3 - 36M\gamma r^2 + 6M(1+\zeta)r - 36M^2 = 0. \tag{4.31}$$

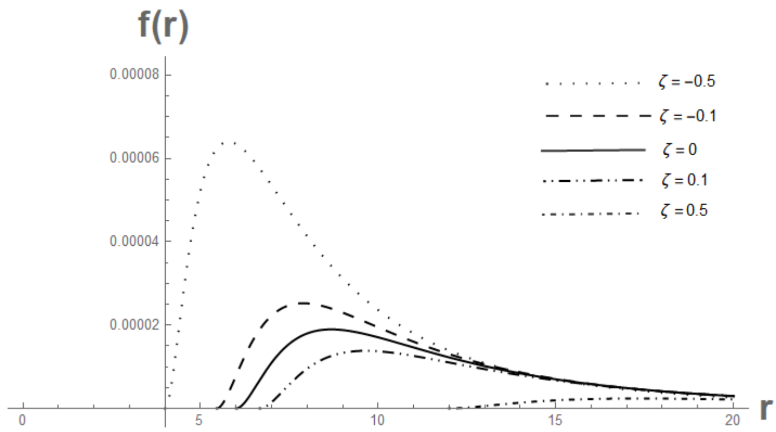
We have plotted (4.25) for different scenarios in which only one of γ, ζ, Λ is non-zero in Figure 4.2.



(a) γ *vs* $f(r)$



(b) λ *vs* $f(r)$



(c) ζ *vs* $f(r)$

Figure 4.2: Effect of γ,λ and ζ on the dimensionless accretion disk radiation flux function $f(r)$, we set $M = G = 1$.

4.4 Ray Tracing

Ray tracing is a general name under which there are various methods to integrate photon geodesics. In many areas like computer graphics *etc*, the problem is very simple since the space-time is taken to be flat. On a curved space-time one has to parallel transport the tangent vectors hence the problem reduces into solving geodesic equations. Using the symmetries of space-time can simplify the problem considerably, *e.g.* reducing the number of equations. However, here we want to stick with the direct integration method without any assumption about the symmetries of the underlying space-time. Let the Lagrangian for a photon be

$$L = g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{4.32}$$

where τ is an affine parameter. Then, the Hamiltonian is

$$H = \frac{\partial L}{\partial(\frac{dx^\mu}{d\tau})}p^\mu - L = g^{\mu\nu}p_\mu p_\nu, \tag{4.33}$$

with $p_\mu = g_{\mu\nu}(x)\frac{dx^\nu}{d\tau}$ and Hamilton's equations are

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu}p_\nu, \tag{4.34}$$

$$\frac{dp_\mu}{d\tau} = -\frac{\partial g^{\alpha\beta}}{\partial x^\mu}p_\alpha p_\beta. \tag{4.35}$$

These are nothing but the usual geodesic equations. For the metric (4.5), these equations give

$$\begin{gathered}\frac{dt}{d\tau} = -\frac{p_0}{f}, \qquad \frac{dr}{d\tau} = fp^1, \qquad \frac{d\theta}{d\tau} = \frac{p_2}{r^2}, \qquad \frac{d\phi}{d\tau} = \frac{p_3}{r^2\sin^2\theta}, \\ \frac{dp_0}{d\tau} = \frac{dp_3}{d\tau} = 0, \qquad \frac{dp_1}{d\tau} = -\frac{f^{'}}{f^2}p_0^2 + f^{'}p_1^2 - \frac{2}{r^3}(p_2^2 + \frac{p_3^2}{\sin^2\theta}), \qquad \frac{dp_2}{d\tau} = \frac{p_3^2\cos\theta}{r^2\sin^3\theta}.\end{gathered} \tag{4.36}$$

To set the initial conditions, we can use a tetrad for the metric and map the observers' local sky to the initial momentum. Let

$$e^\mu_\alpha e^\nu_\beta g_{\mu\nu} = \eta_{\alpha\beta}.$$

Then,

$$p_\mu = g_{\mu\nu}e^\nu_\alpha k^\alpha, \tag{4.37}$$

with $k^\alpha = (1, \sin\theta_0\cos\phi_0, \cos\theta_0, \sin\theta_0\sin\phi_0)$ will be null. Because,

$$\begin{aligned} p^\mu p_\mu &= (e^\mu_\beta k^\beta) g_{\mu\nu} e^\nu_\alpha k^\alpha = \eta_{\alpha\beta} k^\alpha k^\beta \\ &= -1 + (\sin\theta_0\cos\phi_0)^2 + (\sin\theta_0\sin\phi_0)^2 + \cos^2\theta_0 = 0. \end{aligned} \tag{4.38}$$

For (4.5), we have simply $e^\mu_\alpha = \text{diag}[1/\sqrt{f}, \sqrt{f}, 1/r, 1/(r\sin\theta)]$. Therefore, for an observer with local sky coordinates (θ_0, ϕ_0) we can set the momentum initial conditions for (4.36) as

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{f}} \\ \sqrt{f}\sin\theta_0\cos\phi_0 \\ \frac{\cos\theta_0}{r} \\ \frac{sin\theta_0\sin\phi_0}{r\sin\theta} \end{pmatrix}. \tag{4.39}$$

Therefore, for a given initial position $x^\mu(0)$ and a direction (θ_0, ϕ_0) one can use (4.39) and (4.36) to find the trajectory of a photon by integrating the radial part till the event horizon or other points of interest like the surface of the accretion disk *etc*. Some examples are given in Figure 4.3.

4.5 Modified Tolman-Oppenheimer-Volkoff (TOV) Equations

Effect of graviton mass also alters the mass radius relation of the stars. One can construct the modified version of the TOV equations to approach the problem, [32],[33],[34], also see [**?**].

First, model neutron star as a perfect fluid

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}. \tag{4.40}$$

Then, using the a spherical metric ansatz of the form,

$$g_{\mu\nu} = \text{diag}(-f(r), 1/g(r), r^2, r^2\sin^2\theta), \tag{4.41}$$

one can solve for the equations of motions of the dRGT theory with the singular auxiliary metric of Section (4.1),

$$\frac{dP}{dr} = (\rho + P)\frac{4\pi M(r) + 2\pi r^3(8\pi P + \frac{2\Lambda}{3}) + 2\gamma r^2}{r\Big(-4\gamma r^2 + 8\pi M(r) + 4\pi r(\zeta + \frac{\Lambda r^2}{3} - 1)\Big)}, \tag{4.42}$$

with $M(r) = \int dr 4\pi r^2 \rho(r)$. In the limit $m = 0$, $\Lambda = \gamma = \zeta = 0$ one gets

$$\frac{dP}{dr} = (\rho + P)\frac{M(r) + 4\pi P r^3}{r\Big(2M(r) - r\Big)}, \tag{4.43}$$

the ordinary TOV equation.

This time instead of fixing Λ, γ, ζ one by one, we go back to the original definition of these parameters in terms of the constants of the dRGT theory

$$\Lambda = 3m_g^2(1 + \alpha + \beta), \tag{4.44}$$

$$\gamma = -lm_g^2(1 + 2\alpha + 3\beta), \tag{4.45}$$

$$\zeta = l^2 m_g^2(\alpha + 3\beta), \tag{4.46}$$

and consider different values of (α,β).

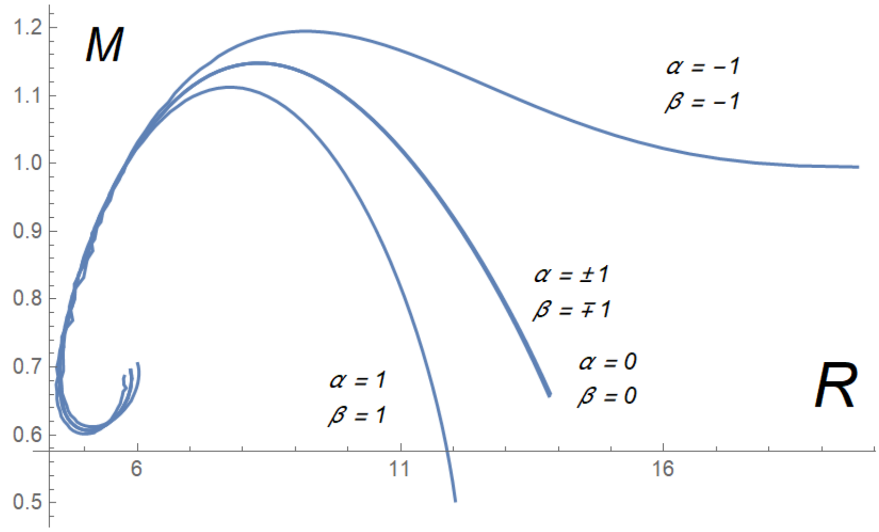


Figure 4.4: Mass and radius are given in km's, in geometrized units $G = c = 1$. We set $l = 1\ km$ and $m^2 = 10^{-4}\ km^2$. EOS is taken as $\rho = \rho_0 + \frac{KP^\Gamma}{\Gamma - 1}$ and $\Gamma = 5/3$.

We start integrating TOV equations for a case $l = 1\ km$, $m^2 = 10^{-4}\ km^2$ (this is not the Compton wavelength, it is the the value in geometrized units and the value is taken to be very large compared to the observations) in geometrized units $G = c = 1$. Center density starts from $\rho_c = 10^{15} g/cm^3$ to $10^{18} g/cm^3$. The equation of state used is $\rho = \rho_0 + \frac{KP^\Gamma}{\Gamma - 1}$ and $K = 5.38 \times 10^9 g^{-2/3} cm^4 s^{-2}$, $\Gamma = 5/3$.

As the results suggest, the main effect for the structure of neutron stars in dRGT massive gravity is the sum of $\alpha + \beta$, the parameters separately does not effect total mass and radius much.

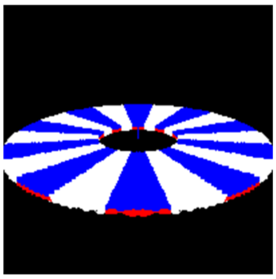
4.6 Rotating Black holes and dRGT Theory

Rotating black holes solutions are very critical for astrophysical calculations. Zero angular momentum is only theoretically possible. Existence of ergo-sphere for rotating black holes also gives rise to interesting thermodynamic considerations like Penrose process and black hole bombs.

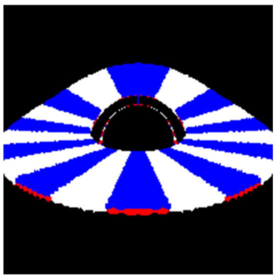
From the point of Cosmic Censorship conjecture, angular momentum over mass square ratio J/M^2 should be less than 1, otherwise a naked singularity arises. Although theoretically impossible, the physical possibility of super-extremal black holes with $J/M^2 > 1$ have been investigated in realistic accretion scenarios. Historically, first it has been found that accretion by a finite amount of matter can overspin the black hole, [35]. Then, this pathology has been solved by Thorne by considering the photon capture from the accretion disk, [1]. Since capture cross section is greater for counter-rotating photon their effect is to decrease the evolution of the spin parameter of the black hole, see Figure 4.5. Surprisingly, the maximum spin parameter J/M^2 by accretion is around 0.998, [1], see Figure 4.6.

Therefore, the most logical step is to look for rotating black hole solutions in dRGT theory and compare their properties with the GR. However, as of now no satisfactory rotating black hole solutions in dRGT theory has been found [1]. Lack of black hole solutions is one of the biggest obstacle towards testing the massive gravity theories.

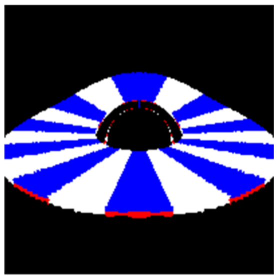
[1] There seems to be rotating black string solutions [36].



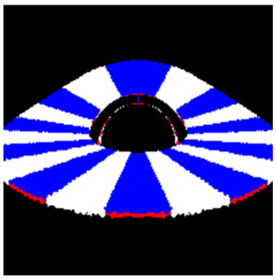
(a) $M = 0, r_0 = 50$



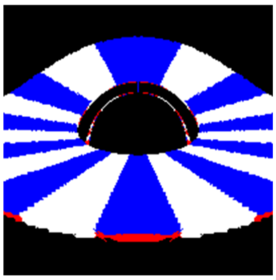
(b) $M = 1$



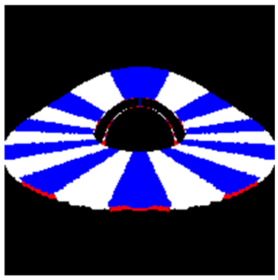
(c) $M = 1, \eta = 0.1$



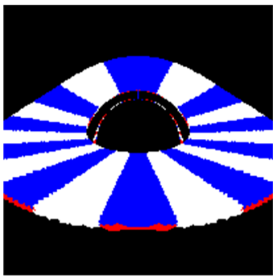
(d) $M = 1, \lambda = 10^{-4}$



(e) $M = 1, \lambda = 10^{-3}$



(f) $M = 1, \lambda = -10^{-4}$



(g) $M = 1, \gamma = 0.01$

(h) $M = 1, \gamma = 0.1$

Figure 4.3: Effect of different parameters on the observed black hole image with accretion disks. Units are chosen so that $G = 16\pi$. Inclination angle of the observer with respect to the disk is 20^o. (a) No black hole, only disk is present hence no deflection of light. (b) A black hole with $M = 1$. (c) A black hole with global monopole term (d-e) A black hole in AdS space-time. (f) A black hole in dS space-time. (g-h) A black hole with linear term.

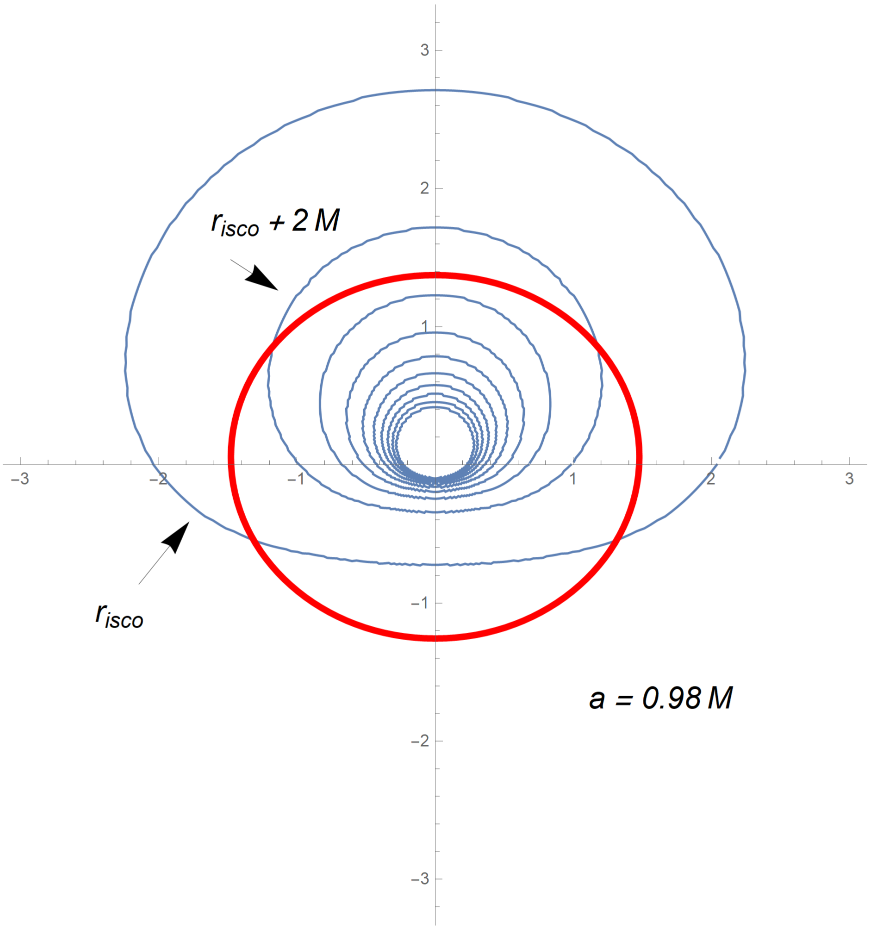


Figure 4.5: Blue curves are the boundary of the region where inside of it photons are captured by the hole for an observer at the equatorial plane for a rotating black hole with $a = 0.98$. (Same as black hole shadow) Axes are the local sky angles ϕ, θ of the observer. Below the horizontal axis are the photons co-rotating with the black hole and above the horizontal axis is the photons counter-rotating with the black hole. Biggest blue curve is for an observer located at $r = r_{ISCO}$ and the inner blue curves represent the observer at distances incremented by $2M$. The red curve is the boundary of the region where inside of it photons are captured by the hole for an observer at the equatorial plane and $r = r_{ISCO}$ for a non-rotating black hole.

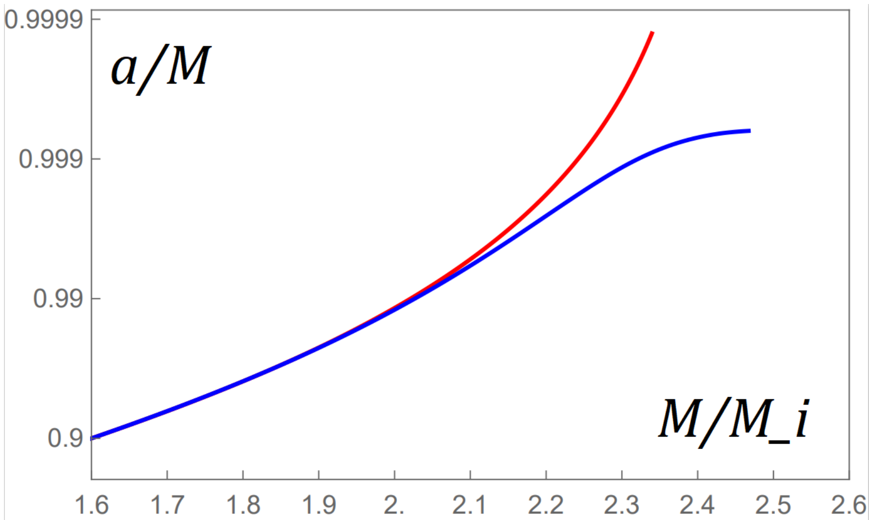


Figure 4.6: Evolution of a black hole spin parameter vs the accretion of rest mass. The red curve is accretion without the effect photon capture and it has been seen that after a finite amount of rest mass of $\sim 1.4M_i$ captured the spin parameter can exceed 1. The blue curve is with the effect of photon capture. Interestingly, photon capture starts to be significant near the extremal limit. A similar figure has been obtained in [1], our simulation results deviate from it slightly.

CHAPTER 5

OTHER ASPECTS

5.1 Instability of Black Holes in Massive Gravity

Bi-gravity Schwarzschild solution has a Gregory-Lafamme Instability similar to the black string metric [37]. The similarity stems from the idea that mass term of the massive gravity in combination with the four dimensional black hole metric behaves as a five dimensional black string with a compactified dimension which induces a Kaluza-Klein mode (graviton mass) and the fact that five dimensional black strings are unstable [38]. This instability is in the linear level.

5.2 Holography

It has been known within the AdS/CFT conjecture that gravitation theories can be used to study various properties of the condensed matter or strongly coupled systems.

It has been argued that diffeomorphism invariance of the gravitation theory corresponds to a momentum conservation at the field theory side [39], [40]. Since massive gravitons break this diffeomorphism invariance, massive gravitation theories can be used to understand systems with momentum dissipation. An example is a lattice with impurities where translation invariance is broken and no conservation of momentum exists. It is also desirable in some situations to keep the response parameters of the dual theory finite like DC conductivity or non-zero resistivity which necessitates a massive theory of gravitation.

5.3 Black Hole Thermodynamics

Consider again the black hole metric (4.5) of the dRGT theory

$$ds_g^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega, \tag{5.1}$$

with

$$f(r) = 1 - \frac{2M}{r} + \frac{\Lambda r^2}{3} + \gamma r + \zeta. \tag{5.2}$$

remembering

$$\Lambda = 3m_g^2(1 + \alpha + \beta), \tag{5.3}$$

$$\gamma = -lm_g^2(1 + 2\alpha + 3\beta), \tag{5.4}$$

$$\zeta = l^2 m_g^2(\alpha + 3\beta). \tag{5.5}$$

Mass and the Hawking temperature can be written in terms of the horizon radius given by $f(r_+) = 0$,

$$M = \frac{r_+}{2}\left(1 + \zeta + \frac{\Lambda r_+^2}{3} + \gamma r_+\right), \tag{5.6}$$

$$T = \frac{1 + \zeta \Lambda r_+^2 + 2\gamma r_+}{4\pi r_+}. \tag{5.7}$$

From these, one can verify using $dS = dM/T$ that

$$S = \pi r_+^2 = \frac{A_h}{4}, \tag{5.8}$$

holds. For the heat capacity, one has simply

$$C = T\frac{\partial S}{\partial T} = \frac{2\pi r_+^2(1 + \Lambda r_+^2 + 2\gamma r_+ + \zeta)}{\Lambda r_+^2 - (1 + \zeta)}. \tag{5.9}$$

One thing to note is that there exists a local minimum for $T(r_+)$ if α and β are suitably chosen. Because of this interesting feature, thermodynamics of black holes in dRGT theory have some unique features. Here we outline some of these features studied in the literature,

- In massive gravity $T = 0$ has a root, this means that after reaching $T = 0$ by evaporation there can still be a massive object called *remnant*.

$$r_0 = \frac{\sqrt{\gamma^2 - (1 + \zeta)\Lambda} - \gamma}{\Lambda}. \tag{5.10}$$

Substituting this into $M(r_0)$ and $S(r_0)$, one can write the *remnant* mass and entropy. (For positive $M(r_0)$, $\gamma < 0$ and $\Lambda > 0$ (AdS), see [41] for details.)

- Also, it has been shown that evaporation takes finite amount of time, see [42].

- Identifying $P = -\frac{\Lambda}{8\pi}$, one gets $V = \frac{\partial M}{\partial P} = \frac{4}{3}\pi r_+^3 \to r_+ = (\frac{3V}{4})^{1/3}$. Then one can find $P(V,T)$, see [43]. Define $p = P/P_c$ *etc*

$$p = \frac{8T}{3v^{1/3}} - \frac{2}{v^{2/3}} + \frac{1}{v^{3/4}}. \tag{5.11}$$

- Hawking-Page transition temperature is modified to

$$T = \frac{1}{4\pi r_h}\Big(3 + \frac{r_h^2}{L^2}\Big[1 - \frac{m^2 L^2}{2}\Big]\Big). \tag{5.12}$$

where L is the AdS radius, see [44].

CHAPTER 6

CONCLUSION

After a brief review of the currently studied massive gravity theories, we focused our attention on the simplest of all, massive spin-2 Fierz-Pauli theory. We briefly touched upon the pathologies of this theory mainly due to its linear nature. Afterward, we investigated the existence of van Dam-Veltman-Zakharov discontinuity for various solar system tests for the same theory. These discontinuities arise in all massive gravitation theories after linearization, therefore they are of some theoretical interest. Then, we switched to the de Rham-Gabbazade-Tolley theory which is the most straightforward non-linear extension of the massive spin-2 Fierz-Pauli theory. There, we analyzed strong field tests like black hole shadow and accretion disks on a class of static black hole solutions. Unfortunately, the absence of rotating black hole solutions in this theory limits the availability of such strong field topics. Finally, we summarized what has been known about the thermodynamics of these black holes so far.

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