GRAVITY THEORIES AT LARGE NUMBER OF DIMENSIONS

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ABSTRACT

GRAVITY THEORIES AT LARGE NUMBER OF DIMENSIONS

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General Relativity is succesful in understanding the phenomena such as light bending by the Sun and the perihelion precession of Mercury that could not be understood in Newton's gravity. Within solar system scales, General Relativity is a very powerful theory but for very small or very large distances, the theory has non-renormalization issues and lack of explanation of the accelerated expansion of the universe and the galaxy rotation curves which give a hint at the need for modifications. In this thesis, Born-Infeld type modifications of General Relativity are considered. In 1998 Deser and Gibbons proposed Born-Infeld gravity theory that has some common features with the Eddington's gravitational action and the Born-Infeld electrodynamics. The Born-Infeld gravity theory, like the other two, has a determinantal action but the free variable of the theory is the metric not the connection as in the Eddington's gravity theory. In this thesis we have calculated the Wald entropy of the Born-Infeld gravity theories and showed that this dynamical entropy reduces to the geometric Bekenstein-Hawking entropy with the appropriate choice of effective gravitational constant. We also discuss black hole entropy in generic dimensions for the Born-Infeld theories.

Keywords: Black holes, thermodynamics of black holes, nonlinear electrodynamics, Born-Infeld gravity, Wald entropy, entropy in Born-Infeld gravity

ÖZ

YÜKSEK SAYIDA BOYUTLARA SAHİP KÜTLE ÇEKİM TEORİSİ

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Genel Görelilik, Newton'un kütleçekiminde anlaşılamayan Güneş'in ışığı bükmesi ve Merkür'ün günberi devinimi gibi olguları anlamakta başarılıdır. Güneş sistemi ölçeklerinde Genel Görelilik çok güçlü bir teoridir, ancak çok küçük veya çok büyük mesafeler için teorinin re-normalizasyon sorunları, evrenin ivmelenerek genişlemesi ve galaksi dönme eğrilerini açıklamasındaki eksiklikleri teorinin modifikasyon ihtiyacına dair bir ipucu vermektedir. Bu tezde, Genel Görelilik'in Born-Infeld tipi modifikasyonları ele alınmıştır. 1998'de Deser ve Gibbons, Eddington'ın kütleçekim eylemi ve Born-Infeld elektrodinamiği ile bazı ortak özelliklere sahip olan Born-Infeld kütleçekim teorisini önerdi. Born-Infeld kütleçekim teorisi, diğer ikisi gibi, determinant ile tanımlanmaktadır, ancak teorinin serbest değişkeni Eddington'ın kütleçekim teorisindeki gibi bağlantı değil, metriktir. Bu tezde, Born-Infeld kütleçekim teorilerinin Wald entropisini hesapladık ve bu dinamik entropinin, uygun etkin kütleçekim sabiti seçimi ile geometrik Bekenstein-Hawking entropisine indirgendiğini gösterdik. Ayrıca Born-Infeld teorileri için kara delik entropisini genel boyutlarda tartıştık.

Anahtar Kelimeler: Karadelikler, karadeliklerin termodinamiği, doğrusal olmayan elektrodinamik teorileri, Born-Infeld kütleçekimi, Wald entropi, Born-Infeld kütleçekim entropisi

To my parents

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I am thankful to American Physics Society (APS) for giving me the right to use our article "Entropy in Born-Infeld Gravity"[58] in this thesis without requesting permission from them.

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LIST OF ABBREVIATIONS

GR	General Relativity
EH	Einstein-Hilbert
BI	Born-Infeld
RN	Reissner-Nordström
$\eta_{\mu\nu}$	Mostly plus Minkowski metric $(-+++\ldots)$
$g_{\mu\nu}$	Generic metric
$d\Omega_2^2$	Line element on unit sphere
g	Metric determinant
∂_μ	Partial derivative
∇_μ	Metric compatible covariant derivative
$F_{\mu\nu}$	Electromagnetic field tensor
$\mathcal{L}$	Lagrangian density

NOTATION: Latin indices ijk run from 1 to $(D-1)$ and Greek indices $\mu\nu\rho$ run from 0 to D. We follow Einstein's summation convention, *i.e.* $u_\mu v^\mu = \sum_{\mu=0}^{D} u_\mu v^\mu$ and use units $k_B = c = 1$ throughout the text.

CHAPTER 1

INTRODUCTION

The life we experience may have surprises for us but to see an apple falling from a tree is not one of them thanks to Isaac Newton. He thought that there was a force which was responsible for this attraction between the apple and the Earth. In 1687 he wrote his gravity theory that would last for centuries and formulated this force as [1]

$$F = -G\frac{m_1 m_2}{r^2}, \tag{1.1}$$

where the minus sign shows that the force is attractive, G is the Newton's constant and r is the distance between the two masses m_1 and m_2. Applying this equation to the Newton's laws of motion determined the orbits of the heavenly objects around the Sun, the tidal forces of the Moon and even the mass of the Sun and the Earth [2]. The discovery of the planet Neptune whose location was predicted by the unexpected perturbations of the orbit of Uranus was another success of the Newton's theory of gravity [3, 4]. In the beginning of the 20th century, the same approach was intended to explain the perihelion precession of Mercury that was larger than the expectations of the Newton's gravity. As in the case of the discovery of Neptune, people searched for a new planet named Vulcan which was never observed. Fortunately, this problem was cured by a remedy found by Einstein. In 1915 he proposed the theory of General Relativity [5] and showed that the discreapency in Mercury's orbit could be explained not by looking for a new planet but looking for a new theory.

General Relativity is a theory of space, time and gravitation. As John Wheeler said, "Space-time tells matter how to move; matter tells space-time how to curve" [6]. The spacetime and the matter are related via Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.2}$$

where κ is some constant and $T_{\mu\nu}$ is the all the source that can be in the form of energy, momentum, pressure and stress of the matter. The left hand side contains the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the metric $g_{\mu\nu}$ which determine the geometry of the spacetime.

After successfully explained the perihelion precession of Mercury, the theory of General Relativity immediately was tested for the phenomena such as the light bending and the gravitational redshift that could not be understood by Newton's gravity. Passing these two tests proved that the General Relativity was successful to explain the phenomena within the solar system [7, 8] but out of this scale the theory is inadequate to explain the "cosmological constant problem" that is still unsolved. Einstein was the first person who speculated the existence of this constant. In 1917, to have a static universe he included the cosmological constant term Λ to his equations (1.2)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.3}$$

that can be considered as the simplest modification of the General Relativity [9, 10]. In 1922, Friedmann showed that there are two fates of the universe; either contracting or expanding but not static [11]. In 1929, Hubble showed that our universe is not contracting [12]. In fact in 1998 it was proved that our universe is expanding with an acceleration. As we know from the Newton's laws of motion, if there is an acceleration then there should be some mass-energy in the environment. Today, we named this energy as dark energy that is considered to be the 70% of the universe's energy. We started with the cosmological constant problem and ended with the dark energy. The relation between them is that the cosmological constant Λ is considered as the simplest form of the dark energy [10].

One can study the cosmological constant problem starting from its quantum field theory origin. From the quantum field theory perspective, the vacuum energy density is not zero. It follows from the fact that the processes of annihilation and creation occur with pairs and occur at any time. Therefore the universe will have an energy density even if there was nothing in it. If we apply this situation to General Relativity, we reach the idea that it does not matter whether there is an energy distribution in the universe, the existence of cosmological constant will lead to a curvature. The idea of these two theories relate the cosmological constant and the vacuum energy density of the universe. Although this relation warms our hearts, at the end of the day it turns to a nightmare. Since we expect to obtain rather close values for the cosmological constant, it should not be really important which theory we use. However this is not true, in fact the values coming from the observation and the quantum field theory are not close but differs with 10^{-122} (in Planck units) and ($\sim 10^{-2}$) respectively. The predictions of the theory and the observation do not match. This inconsistency is known as cosmological constant problem. In the other extreme, to understand the gravity in very small scales, one should consider the quantum effects. Unfortunately, no consistent theory of quantum gravity has been written so far [10].

General Relativity is mandatory to understand the physics of the black holes including their thermodynamics. In 1969, Penrose showed that it was possible to extract the rotational energy of the Kerr black hole that had an angular momentum. This process, known as Penrose process, depends on the existence of the ergosphere that is the region between the stationary limit and the outer event horizon of the Kerr black hole. In this process, he considered a particle with energy E_1 that was sent from infinity. It entered the ergoregion and then splitted into two pieces with energies E_2 and E_3 respectively such that the energy of the second particle, as measured by the observer at infinity, is negative and absorbed by the black hole. However, the third particle with the energy E_3 escapes to infinity. By using the conservation of the energy and the fact that the energy of the second particle is negative, one can conclude that $E_3 > E_1$ that means the energy of the third particle is bigger than the energy of the original particle. This extra energy is extracted from the black hole's rotational energy [13]. In 1971, Penrose and Floyd suggested that the surface area of the event horizon may naturally increase in time [14]. In 1970 independently, Christodoulou derived the mass-energy of a rotating black hole with angular momentum such that

$$m^2 = m_{\text{ir}}^2 + \frac{L^2}{4m_{\text{ir}}^2}, \tag{1.4}$$

where m is the mass-energy, m_{ir} is the irreducible mass and L is the angular momentum of a black hole. Although it was possible to increase or decrease the mass of a black hole or its angular momentum as shown by Penrose, it was not possible to decrease m_{ir} by any kind of processes. It can be only increased by irreversible transformations or does not change by reversible transformations [15]. In 1971, Hawking proved that the area of the event horizon never decreases which is given by

$$A_H = 8\pi m\big[m + (m^2 - a^2)^{1/2}\big], \tag{1.5}$$

where A_H is the area of the event horizon, m is the mass and a is the angular momentum per unit mass. If one considers the merging of two black holes with parameters m_1, a_1 and m_2, a_2 that formes a single black hole with parameters m_3, a_3, the surface area after merging should satisfy the inequality [16]

$$m_3\big[m_3 + (m_3^2 - a_3^2)^{1/2}\big] > m_1\big[m_1 + (m_1^2 - a_1^2)^{1/2}\big] + m_2\big[m_2 + (m_2^2 - a_2^2)^{1/2}\big]. \tag{1.6}$$

In 1971, Christodoulou and Ruffini generalized their ideas for charged black holes and obtained the result

$$m^2 = \big(m_{\rm ir}^2 + \frac{e^2}{4m_{\rm ir}^2}\big)^2 + \frac{L^2}{4m_{\rm ir}^2}, \tag{1.7}$$

where m is the mass-energy, $m_{\rm ir}$ is the irreducible mass and L is the angular momentum and e is the electric charge of the black hole. They also pointed out the one-to-one connection between the $m_{\rm ir}$ and the proper surface area of the event horizon which was a nondecreasing quantitiy shown by Hawking [16] such as

$$A = 16\pi m_{\rm ir}^2, \tag{1.8}$$

where A is the surface area of the event horizon [17].

In 1972, Bekenstein proposed that a black hole must have entropy, otherwise it would violate the second law of thermodynamics. One can understand this violation by the following example. Consider an observer throwing a particle into a black hole. When the particle crosses the event horizon, since it never comes back and carries entropy, the entropy of the exterior universe decreases hence the violation of the second law. To prevent this, Bekenstein considered that black holes must have a finite entropy and stated the second law as " Common entropy plus the black hole entropy never decreses". Common entropy stands for the entropy of the exterior universe and the black hole entropy is given by

$$S_{BH} = \eta\, L_P^{-2}\, A, \tag{1.9}$$

L_P is the Planck's length that are for dimensional considerations, A is the surface area of a black hole and η is a constant of order unity [18]. The relation between the black hole entropy S_{BH} and the area of a black hole A follows from the conclusions of Christodoulou [15, 17] and the Hawking [16, 19].

In 1973 Bardeen, Carter and Hawking wrote the thermodynamic-like expressions for the stationary axisymmetric solution of the Einstein equations. By using the analogy between the surface area A and the surface gravity κ of the event horizon with the entropy and temperature respectively, they postulated the four laws of black hole mechanics as follows [20]:

The Zeroth Law: For a stationary black hole, the surface gravity κ is constant on the event horizon.

The First Law: $\delta M = \frac{\kappa}{8\pi}\delta A + \Omega \delta J$,

where M is the mass, A is the surface area, Ω is the angular velocity and the J is the angular momentum of a rotating black hole. They emphasized that the relation between the surface gravity and the temperature is nothing but an analogy. If a black hole has a temperature, it must also emit rediation, which is classically impossible [20]. However as we are going to see, a black hole has a temperatue $T = \dfrac{\hbar\kappa}{2\pi}$ and it emits Hawking radiation [21].

The Second Law: The surface area of an event horizon is a non decreasing quantity, *i.e* $\delta A \geq 0$.

This states that if two black holes with surface areas A_1 and A_2 merge, then the final black hole with surface area A_3 satisfies the relation

$$A_3 > A_1 + A_2. \tag{1.10}$$

We should note that this law is a bit stronger than the second law of the thermodynamics because one can transfer entropy between two systems that cannot be done by the areas.

The Third Law: It is not possible to reduce the surface gravity κ to zero by finite number of processes.

If it was possible, then creating a naked singularity by carrying these processes would be also possible [20].

In 1975, Hawking showed that a black hole could emit radiation due to the vacuum quantum fluctuations near the event horizon and defined the entropy of a black hole as

$$S_{BH} = \frac{A}{4G\hbar}, \tag{1.11}$$

where A is the area of the event horizon, $\hbar$ is the Planck's constant and G is the Newton's constant [21].

We should emphasized that all the four thermodynamic-like laws for black holes are derived from Einstein's equations and it was thought that the relation between the laws of thermodynamics and the laws of black hole mechanics are nothing but an analogy. However, the Hawking radiation showed that this was not an analogy but an identity [20, 21].

Although General Relativity has built a kingdom within the solar system scale, one should understand the phenomena in very small or large distances; therefore, GR needs some modifications. One should keep in mind that the modified theory should reduce to the Einstein's theory when it is treated in solar system scales. The General Relativity is a theory with a massless spin-2 excitation and there are different ways to modify it. We already discussed the simplest modified gravity theory (1.3). Another way is to give mass to the graviton that is known as massive gravity. The basic idea is to change the degrees of the freedom of the theory [10].

The electron's discovery in 1897 by Thompson and the inconsistency of the Maxwell theory at the atomic scale led people to seek a new theory of electrodynamics in which the self energy of charged particles was finite at short distances. One of the attempts was to generalize the Maxwell theory nonlinearly. The idea was to introduce a maximal field to remove the inconsistency issue [22].

Mie was the first person to develope such a model by assuming that there was a maximum electric field E_0 such that the force was proportional to

$$F \sim \frac{E}{\sqrt{1 - \frac{E^2}{E_0^2}}}. \tag{1.12}$$

Although this model was promising to solve the issue of infinite energy, it was not acceptable because it was not covariant under Lorentz transformations [22, 23]. In

1932 Born, then in 1934 Born and Infeld proposed a new theory of electrodynamics with the following Lagrangian

$$\mathcal{L}_{BI} := -b^2\left(1 - \sqrt{-\det(\eta_{\mu\nu} + \frac{1}{b}F_{\mu\nu})}\right), \tag{1.13}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $F_{\mu\nu}$ is the electromagnetic field tensor. The dimensionful constant b accounts for the maximal electric field E_0 as in Mie's theory of electrodynamics [22]. Also, b is called as an absolute field by the originators of the BI theory. As we are going to study later, b is a measure of nonlinearity and when it goes to infinity, $\mathcal{L}_{BI}$ reduces to linear Maxwell theory with $\mathcal{L}_M$ [24, 25].

Because of the nonlinear feature of the theory, it is very difficult to analyze and solve the fields equations. Some attempts concerning this issue were made by Pryce [24, 26]. However, the progress in Quantum Mechanics, Dirac's contributions on classical electron and the studies on quantum field theory caused BI theory to be forgotten for a long time [22, 24, 26, 27].

Since 90's, there has been a renewed interest in BI nonlinear electrodynamics due to the research on String Theory. It was understood that D-branes with low energy can be described from a BI type action. The advent in string dualities conducted a detailed investigation on duality of BI theory [24, 28]. Also, nonlinear electrodynamics have been studied in gravity to study and understand the feature of regular black hole solutions [29].

Let us explain what we mean by the inconsistency in Maxwell electrodynamics. As we are going to discuss in detail, the energy of a charged point like particle is infinite for large fields or at small distances [30]. It was Born's motivation to generate a theory of electromagnetism such that the self energy of a point charge at origin is finite [24]. We will show that, this is not an issue in BI electrodynamics.

Although BI theory is a nonlinear theory, it has special properties that not all the nonlinear theories have. To understand what makes the BI theory special, let us first discuss the feature of nonlinear theories of electrodynamics. As we are going to study, the dynamics of this kind of theories arise from an action in the form:

$$S = \int d^D x \, \mathcal{L}(\mathcal{F}_{\mu\nu}) + \int d^D x \, A_\mu j^\mu, \tag{1.14}$$

which is valid in any dimensions. We are going to assume that the Lagrangian density $\mathcal{L}(\mathcal{F}_{\mu\nu})$ is both Lorentz and gauge invariant. However, in BI theory, there are more constraints on $\mathcal{L}(\mathcal{F}_{\mu\nu})$ such that the maximal of $\vec{E}$- field is required when $\vec{B} = 0$ and $\mathcal{L}_{BI}$ should reduce to $\mathcal{L}_M$ for small electromagnetic fields [30].

As in 1970 Boillat showed that [31] the BI theory of electrodynamics is special not only due to the constraints on the Lagrangian but also for the following reasons:

1. Maxwell and BI electrodynamics are the only causal theories with spin-1. The causal propagation is related with dominat energy condition which is [24, 25].

$$T_{00} \geq T_{\mu\nu} \quad \text{for all} \quad \mu, \nu \tag{1.15}$$

2. Like Maxwell theory, BI theory is self dual that follows from [25].

$$\vec{D} \cdot \vec{H} = \vec{B} \cdot \vec{E}. \tag{1.16}$$

3. There are no shock waves in the BI theory. It means that the phase speed does not depend on the phase which implies that BI theory is non perturbative [25].

4. There is no birefringence phenomenon in BI theory. It means that the phase velocity of light is independent from the polarizations of the external electromagnetic fields [28].

5. The BI theory obeys the correspondence principle, in other words nonlinear feature of the BI theory disappears for small electromagnetic fields [28].

We should emphasize that in BI and related nonlinear theories, vacuum is considered as a material. It means that the electric permittivity ϵ and the magnetic permeability μ of the vacuum is different than unity which implies that vacuum is not empty [28].

As we are going to study in detail, due to the Lorentz invariance of the $\mathcal{L}_{BI}$, one can infer that the field equations of BI theory have the same form as Maxwell equations. The nonlinearity of these equations come from the dependence of ϵ and μ on $\vec{E}$ and $\vec{B}$ fields. One can easily remember that in Maxwell electrodynamics, we have constant ϵ and μ such that $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$. In other words the relation between $\vec{D}$ and $\vec{E}$ and the relation between $\vec{B}$ and $\vec{H}$ is linear [24, 28, 30].

Our main interest in this thesis is Born-Infeld gravity that was proposed by Deser and Gibbons in 1998. It followed the ideas of BI electrodynamics and Eddington's gravitational action [32, 33]. In 1924, Eddington proposed a gravity action in the form

$$I_{EDD} = \int d^4 x \sqrt{\det \left[R_{(\mu\nu)}(\Gamma) \right]}, \tag{1.17}$$

where the symmetric Ricci tensor is a function of the connection Γ only [34].

In 1934, with different motivations, Born and Infeld proposed a nonlinear theory that cured the divergence of the self energy problem in Maxwell theory. BI electrodynamics was inspired from a point particle action that put a constraint on particle's velocity. The BI electrodynamics can be defined by the action

$$I_{BI} = -b^2 \int d^4 x \sqrt{-\det \left(g_{\mu\nu} + \frac{1}{b} F_{\mu\nu} \right)}, \tag{1.18}$$

where $g_{\mu\nu}$ is the metric, $F_{\mu\nu}$ is the electromagnetic field tensor and b is the maximal electromagnetic field [35].

By following these ideas, Deser and Gibbons introduced a Born-Infeld gravity action that was the gravitational analogue of (1.18) with the form

$$I_{DG} = \int d^4 x \sqrt{\det \left(a g_{\mu\nu} + b R_{\mu\nu} + c \chi_{\mu\nu} \right)}, \tag{1.19}$$

where a, b, c are the parameters of the theory and $\chi_{\mu\nu}$ is an unknown tensor that needs to be determined such that the theory will be unitary around flat and (A)dS spacetimes that are maximally symmetric with constant curvature backgrounds. The independent variable of the theory is the metric, contrary to Eddington's theory [32, 33].

We already mentioned that some of the problems arose from the quantum field theory perspective. Quantization of GR, in other words to write its corresponding quantum theory, is still an unsolved problem and the non-renormalizability of GR does not simplify this current situation. In different quantum gravity theories, GR emerges as a low energy field theory whose nonrenormalizable feature may be cured by adding terms with higher derivatives. Indeed, as shown in [36], at quadratic order in curvature, GR is renormalizable but its corresponding quantum theory is not unitary [36, 37].

From the field theory perspective, one can write the generic gravitational action in the form

$$I = \int d^4\, x\, \frac{1}{\kappa} \{(R - 2\Lambda_0) + \sum_2^{\infty} a_n \{\text{Riem, Ric, R, } \nabla\text{Riem...}\}^n\}. \tag{1.20}$$

As seen from the (1.20), the first term leads us to Einstein's gravity, the second term is its extension with cosmological constant and the rest stands for the higher order curvature terms. The main interest is to determine which curvature terms appear at which order and with which couplings [33].

BI type gravity theory is a higher curvature theory at infinite order with fixed couplings and it is a modification of General Relativity. The aim is to determine the order of n and the corresponding coefficients a_n in general. Also, by putting the constraints on (1.20) we will have a consistent theory whose spectrum is ghost and tachyon free. When we increase the number of these constraints, we can have a specific theory. The scope of the thesis [33] we have been using as a guide was to obtain a unitary theory around flat and (A)dS backgrounds that are maximally symmetric spacetimes with constant curvature.

In three dimesions, there are two, worth-mentioning, examples of BI gravity. The first one is the BI extension of new massive gravity with the action [38]

$$I_{BINMG} = -\frac{4m^2}{\kappa^2} \int d^3\, x \left[\sqrt{- \det \left(g_{\mu\nu} - \frac{1}{m^2} G_{\mu\nu} \right)} - \left(1 - \frac{\lambda_0}{2} \right) \sqrt{-g} \right], \tag{1.21}$$

that is the first example of BI gravity which is unitary around (A)dS spacetime. The second example is, at quadratic order, the expansion of (1.21) in curvature that leads the new massive gravity (NMG) theory in the form [39, 40]

$$I_{NMG} = \frac{1}{\kappa^2} \int d^3\, x \sqrt{-g} \left[-R - 2\lambda_0 m^2 + \frac{1}{m^2} \left(R^\mu_\nu R^\nu_\mu - \frac{3}{8} R^2 \right) \right], \tag{1.22}$$

that is the unique gravity theory at quadratic order which is unitary around both flat and (A)dS backgrounds. Both theories have massive spin-2 particle[41, 42, 43, 44]. A detailed explanation is provided in [33].

1.0.1 Unitary Analysis of Born-Infeld Gravity

A gravity theory is physically consistent at the tree level if its spectrum does not contain any ghosts and tachyons. A ghost is a negative kinetic energy mode that is described by an action

$$I = \int dt(-K - U), \tag{1.23}$$

where K and U are the kinetic and potential energies, respectively. The corresspond-ing energy mode $H = -K + U$ is not positive definite and this leads to an unstable vacuum. Ghost instability can be considered as a plauge that is seen at all energy scales. Nothing is sunny in the corresponding quantum theory either. In the Hilbert space, a ghost mode has a negative norm that violates the conservation of probability and leads to a non-unitary theory.

Tachyons are the negative m^2 modes that are encountered at energy scales approach-ing to m. To understand tachyon instability let us consider the following action with a scalar field ϕ such that

$$I[\phi] = \int dt\big[K(\partial_\mu \phi) - U(\phi)\big]. \tag{1.24}$$

Considering the fluctuations around the vacuum $\bar{\phi}$ with $\phi - \bar{\phi} \equiv \psi$, then the perturbed action yields $\left[\frac{dU}{d\phi}\right]_{\bar{\phi}} = 0$ for vacuum and $\left[\frac{d^2U}{d\phi^2}\right]_{\bar{\phi}} = m^2$ for squared mass. Therefore if there is a tachyon, then unstable vacuum occurs at $\left[\frac{d^2U}{d\phi^2}\right]_{\bar{\phi}} < 0$ [33].

As we mentioned before, the main interest in the thesis [33] that we use as a guide was to do the analysis of the spectrum and check its unitarity in higher curvature theories. To carry the calculations further, it is necessary to specify the corresponding free theory that solves the field equations of these higher order curvature theories. Also, this free theory is given by the action that is second order in metric fluctuations around flat and (A)dS backgrounds. As we are going to see, the unitarity analysis around different backgrounds leads different solutions in principle. Let us discuss how the features of the free theory differ around flat and (A)dS backgrounds by the following observations [33]:

Observation 1: For flat spacetime, there is no contribution beyond quadratic curva-ture terms. For (A)dS spacetimes, all terms contribute in principle.

To understand it better, let us find out the contributions coming from R^3 term. The expansion of Ricci scalar in terms of metric fluctuations is given by

$$R = \bar{R} + R_{(1)} + R_{(2)} + ... \tag{1.25}$$

which are the background curvature, first order in curvature and second order in curvature respectively. Then the second order contributions of R^3 are found as $\bar{R}R^2_{(1)}$ and $\bar{R}^2 R_{(2)}$. As immediately seen, they both vanish in flat but survive in $(A)dS$ backgrounds in principle.

Observation 2: Contributions beyond the quadratic order are in the same form as the contributions from the quadratic order in $(A)dS$ background. They differ with an overall $\bar{R}$ factor. To understand it better, let us examine the contributions coming from R^2 and R^3. The contributions from former are $R^2_{(1)}$ and $\bar{R}R_{(2)}$. Comparing these by the contributions coming from R^3 reveals us that they have the same structure with the exception of $\bar{R}$ factor.

Following these two observations, we can conclude that the structure of the most general quadratic action can be determined by the scalars $R^2, R^\mu_\nu R^\nu_\mu$ and $R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu}$. Instead of the last term, we can use the Gauss-Bonnet combination that is valid for $D > 4$.

Following these ideas, one can write the most generic theory at quadratic curvature in the form of

$$I = \int d^D x \sqrt{-g} \big[\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R^\mu_\nu R^\nu_\mu + \gamma R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} \big]. \tag{1.26}$$

The unitary analysis of (1.26) gives us a hint about the unitarity of higher order curvature theories. They will be unitary if and only if its propagator can be reduced to a propagator of a unitary theory that has already been determined. Therefore, the unitary analysis of (1.26) is rather essential [33].

Let us finalize this section by discussing the unitary analysis of Born-Infeld type gravity theories that can be obtained by analyzing tree level amplitude.

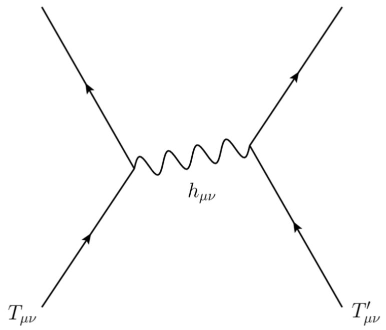

Figure 1.1: Tree-level scattering amplitude between two sources

The tree level amplitude given by Feynmann diagram (1.1) is defined as

$$A = \int d^D x \sqrt{-\bar{g}}\; T'_{\mu\nu}(x)\, h^{\mu\nu}(x), \tag{1.27}$$

where the metric fluctuaton $h_{\mu\nu}$ is given by $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ and created by the matter $T_{\mu\nu}$. To determine the unitarity of (1.26), we should reduce it to a propagator whose unitarity is known. But this is not trivial since the differential operator $\mathcal{O}_{\mu\nu}^{\rho\sigma}$ that represents the matter $T_{\mu\nu}$ in such a way that $\mathcal{O}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} = T_{\mu\nu}$ involves fourth and second derivatives therefore $\mathcal{O}_{\mu\nu}^{\rho\sigma}$ has a complex form. Also, it is not possible to invert $\mathcal{O}_{\mu\nu}^{\rho\sigma}$ directly. Following the steps studied carefully in [33], one can obtain the amplitude in the desired form of

$$\begin{aligned}\frac{A}{\kappa_e} &= 2T'_{\mu\nu}\big[(\kappa_e\beta\bar{\Box}+1)(\Delta_L^{(2)} - \frac{4\Lambda}{D-2})\big]^{-1}T^{\mu\nu} \\ &+ \frac{2}{D-2}T'\big[(\kappa_e\beta\bar{\Box}+1)(\bar{\Box}+\frac{4\Lambda}{D-2})\big]^{-1}T \\ &- \frac{2(\beta+c)}{c(D-1)(D-2)}T'\big[(\kappa_e\beta\bar{\Box}+1)(\bar{\Box}-m_s^2)\big]^{-1}T \\ &+ \frac{8\Lambda D\beta}{c(D-1)^2(D-2)^2} \\ &xT'\big[(\kappa_e\beta\bar{\Box}+1)(\bar{\Box}-m_s^2)(\bar{\Box}+\frac{2\Lambda D}{(D-1)(D-2)})\big]^{-1}T,\end{aligned} \tag{1.28}$$

where $\Delta_L^{(2)}$ is the symmetric rank-2 Lichnerowicz operator; $c \equiv \frac{4(D-1)\alpha+D\beta}{D-2}$ and the scalar excitation mass will be

$$m_s^2 = \frac{1}{c\kappa_e} - \frac{2\Lambda D}{(D-1)(D-2)}\bigg(1-\frac{\beta}{c}\bigg). \tag{1.29}$$

Some properties of the ampilute (1.28) can be discussed as follows [33]:

- When $\alpha = \beta = \gamma = 0$ one part of (1.28) produces the amplitude of GR that is unitary.
- Pole $(\kappa_e \beta \bar{\Box} + 1)^{-1}$ stands for the massive spin-2 excitation which couples to the sources in tensorial form.
- Pole $(\bar{\Box} - m_s^2)^{-1}$ stands for the massive spin-0 excitation that couples only to the trace of a source.
- These poles are multiplied with each other in the amplitude. This feature of the amplitude implies the excistence of ghosts that leads to a non-unitary theory. In fact, the unitarity of massive spin-2 conflicts with the unitarities of massive spin-0 and massless spin-2 excitations. Therefore removing the massive spin-2 mode from the spectrum by taking $\beta = 0$ gives us a unitary theory.

If we carry these procedures carefully, we will obtain the unitarity analysis of (1.26) as follows [37, 39, 43]:

- For $D = 4$, GR is a unitary theory around flat and the (A)dS backgrounds. It has a massless spin-2 excitation. Its R^2 extension is not renormalizable as GR but unitary with another spin-0 excitation. To cure non-renormalizability, one can add $R^\mu_\nu R^\nu_\mu$ that costs to lose unitarity due to new massive spin-2 ghost mode in the spectrum.
- For $D = 3$, GR has no propagating excitations. Its R^2 extension is still unitary with a massive spin-0 mode. But a generic extension of $\alpha R^2 + \beta R^\mu_\nu R^\nu_\mu$ costs an additional massive spin-2 mode to spectrum that spoils the unitarity due the the conflict between these two modes. This conflict can be solved by the proper choice of the parameters such that $8\alpha + 3\beta = 0$ which gives the NMG case [39]. This choice of parameters removes the massive spin-0 mode from the spectrum hence the resulting theory with massive spin-2 excitation will be unitary around flat and (A)dS backgrounds.
- For $D > 4$ Gauss-Bonnet (GB) term is valid. Both GR and its GB extensions are unitariy around the flat and (A)dS backgrounds with massless spin-2 exci-

tations. The R^2 extension of both theories do not spoil their unitarities and add massive spin-0 mode to spectrum. However the $R^{\mu}_{\nu}R^{\nu}_{\mu}$ extension of the theory results a ghost term to the spectrum that is a massive spin-2 excitation therefore leads a non-unitary theory.

The layout of this thesis is as follows: In the next chapter, we discuss the black hole thermodynamics in detail and calculate the corresponding thermodynamic quantities for a generic metric. The chapter three that is the bulk of this thesis is devoted to show how we calculated the Wald entropy for Born-Infeld type gravity theories. We discuss the Wald entropy of both n-dimensional and three-dimensional Born-Infeld theories and show that this dynamical entropy reduces to the geometric Bekenstein-Hawking entropy with the appropriate choice of effective gravitational constant. A conclusion chapter is followed by appendices that give related calculations with the chapters. We should note that the figure (1.1) in this chapter is taken from [33].

CHAPTER 2

BLACK HOLE THERMODYNAMICS

[1] Classical mechanics perfectly describes the motion of a particle or the orbits of heavenly objects around the Sun. But when the system we are interested in involves heat exchange also, Newton mechanics will be insufficient because it does not know how to read the coordinates of the thermal systems. Therefore to have a complete description, we need to define thermal properties of the system besides the mechanical ones; which is the scope of Thermodynamics. Although these two theories figure different parts of the same system, they have common features also. First of all, mechanical and thermal descriptions are nothing but idealizations. In classical mechanics, we use point particles which are the mechanical counterparts of closed systems in Thermodynamics. Such systems are separated from surroundings by adiabatic walls which do not allow heat exchange. Secondly, in classical mechanics the state of a particle is given by its momentum and position whereas in Thermodynamics, we describe the macroscopic systems by state functions that can be mechanical and thermal. As we know, pressure and volume are mechanical state functions for a gas; and for the thermal systems we have thermodynamic state functions like temperature for example. These state functions are well defined only in equilibrium when the properties of the system do not change anymore. Final similarity between Newtonian mechanics and Thermodynamics is that both theories depend on basic observations and laws; then a mathematical structure is constructed by using them.

One can easily conclude that Thermodynamics is valid only in equilibrium and description of the macroscopic system relies on the observations. This means that we obtain thermodynamic laws from observations.

[1] This section is based on [45].

The zeroth law shows the transitivity property of equilibrium and defines the thermal state function- the emprical temperature Θ such that

$$\Theta_A(A_1, A_2, ...) = \Theta_B(B_1, B_2, ...) = \Theta, \tag{2.1}$$

where A and B are two systems in equilibrium with each other.

The first law shows what kind of transformations we can have. In adiabatically isolated systems, the work done to transform a state depends only on initial and final points. When we remove this constraint and consider the transformations that also involves heat, then the previous result will no longer be true. Hence in general, for infinitesimal changes, one can write the first law as

$$dE = đW + đQ, \tag{2.2}$$

where $đW$ and $đQ$ are the increment work done on the system and heat put into the system respectively. Like the zeroth law, the first law generates another state function E that is known as internal energy. As it is understood from the equation (2.2), to write down the internal energy, we should find out the mechanical and thermal work properly. When things are quasistatic in other words slow enough to maintain the equilibrium, we can describe mechanical coordinates as (J_i, x_i) which are generalized forces and generalized displacements respectively. So one can write

$$đW = J_i.dx_i. \tag{2.3}$$

Therefore to get the internal energy, one should find out the form of $đQ$. One can determine it by using the analogies between Newtonian mechanics and Thermodynamics. If we use the mechanism of mechanical and thermal equilibrium, one can infer that generalized forces and the temperature have similar characters. Therefore, to determine $đQ$, we only need to find the generalized displacement corresponding to temperature.

The analog of the generalized displacements come from the second law which indicates the time ordering between transformations. As the Clausius's theorem states, one can write the heat as

$$đQ = TdS, \tag{2.4}$$

for adiabatic transformations and S stands for the another state function called the entropy.

Third law that is independent from others and concerns the behaviour of the entropy at low temperature. It reads as

$$\lim_{T \to 0} S(X,T) = 0, \tag{2.5}$$

where X stands for all choice of coordinates.

As this brief discussion shows, the way we approach mechanical and thermal systems share similarities. Keeping in mind that the mechanical forces are the indicators of the equilibrium, we determine the place of the temperature T by comparing the form of $đQ$ and $đW$. Our main interest in this chapter is the laws of black hole thermodynamics and as we will see, we are going to construct some analogies between thermodynamical systems and the black holes. In order to discuss that black holes have their own thermodynamic laws, first of all we are going to construct analogies of macroscopic functions and the notion of equilibrium for the black holes. Then we also see the relation between thermodynamic entropy and the area of the horizon. We carry our discussion for generic metrics, then these results for the Schwarzschild, the charged and the rotating black holes. The detailed discussion in classical thermodynamics can be found in the Appendix A.

2.1 Thermodynamics of the generic metric

As we studied in the first section of this chapter, we can describe thermodynamic systems in terms of a few state functions like T, P, V etc. When a system is in thermal equilibrium with the surroundings, its temperature will remain constant. We can reach another equilibrium state through transformations and final state will not depend on the intermediate stages [45, 46].

We have a similar approach in black holes. Black holes can be parametrized through few parameters like M, Q and J which stand for mass, charge and angular momentum respectively. This means that two black holes with same parameters are indistinguishable which is known as no-hair theorem. Black holes do not depend on the structure or composition of the initial star. If we consider the merging of two black holes, the event horizon of the final black hole will be spherically symmetric even though it is asymmetric through merging. Having a spherically symmetric horizon is an equilibrium state for a black hole and it implies that the force per unit mass over the horizon is constant [46].

We can reach an immediate conclusion that there is a similarity between black holes and thermodynamic systems. It is clear that the thermal equilibrium and the constant temperature are analogies of spherical symmetry and constant force over horizon which is known as surface gravity. Therefore one can expect to write the zeroth law for black holes. The main question is whether we can find similar analogies and write all thermodynamic-like laws for black holes [46].

Let us consider this example. For Schwarzchild black holes, one reads the horizon area as,

$$A_H = 4\pi r_H^2 = 16\pi G^2 M^2, \tag{2.6}$$

where G is the Newton's constant and M is the mass of the black hole. It is easy to see that horizon area is proportional to mass square of the black hole. Classically, nothing can escape from a black hole therefore mass is a nondecreasing quantity which leads us an important result

$$\triangle A \geq 0, \tag{2.7}$$

that has the similar property with the thermodynamic entropy S. Hence one can expect that black holes have entropy.

Actually black holes obey all thermodynamic-like laws and we are going to obtain these for generic black holes, then we are going to check them for two black holes Schwarzschild and Reissner-Nordström that are written in the same form of the metric [46].

2.1.1 The Zeroth Law

As the zeroth law of thermodynamics suggests, a thermal equailibrium can be given a constant temperature. In the context of black holes no matter how the inital collapse, the black hole will have a horizon at $r =$ constant also suggest the gravitational force on the horizon is constant. Therefore if we consider $r_H =$ constant as the equilibrium state, one can consider the gravitational force on the horizon as the temperature. If we define the gravitaional force or the gravitational acceleration on the horizon as the surface gravity, we can state the zeroth law for black holes as [20, 46]:

Zeroth Law: For a stationary black hole, the surface gravity κ is constant on the event horizon.

By definition, the surface gravity is the gravitational acceleration that is necessary to stay at the horizon. In Newtonian mechanics, the gravitational acceleration is

$$a = \frac{GM}{r^2}. \tag{2.8}$$

At the Shwarzschild horizon, we obtain the surface gravity (2.8) as

$$\kappa = a(r = r_H) = \frac{1}{4GM}. \tag{2.9}$$

Although it seems that one can calculate the above result by using the Newtonian theory, it is not the case actually. In General Relativity, we have to specify who measures the quantity. For an infalling observer, surface gravity diverges because irrespective of the strength of the force, his/ her fate is to pass the horizon and reach the singularity. But if the asymptotic observer measures it, it will not diverge and gives the result (2.9). The disagreement in the accelerations occurs due to the gravitational redshift [46].

Surface Gravity

Let us give the proper definition. The surface gravity is the gravitational acceleration that is necessary to hold the particle at the horizon by an observer at infinity. One can write the static metric as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2. \tag{2.10}$$

The particle's motion is discussed in detail in Appendix C, so following these steps one can have

$$\Big(\frac{dr}{d\tau}\Big)^2 = E^2 - f(r) + L^2\frac{f}{r^2}. \tag{2.11}$$

We carry our calculations without angular momentum since our particle falls radially. If we take the derivative of the equation (2.11) with respect to τ then we get

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{\partial f}{\partial r} = -\frac{GM}{r^2} = a^r, \tag{2.12}$$

that is the familiar Newtonian acceleration. However we only have the r-component of the acceleration, it is not in the covariant form a^μ. One can read its magnitude as

$$a^2 = g_{rr}a^r a^r = \frac{f'^2}{4f} \implies a = \frac{f'}{2\sqrt{f}}, \tag{2.13}$$

where $f' = \partial_r$. It can be easily captured that the above result diverges for an infalling observer as promised.

To calculate surface gravity as measured by the asymptotic observer, consider she/ he is connected to a particle at r by a massless string. If she/ he pulls the string by a proper distance δs [46], the work done by observer and work done on the particle are given by

$$W_\infty = a_\infty \delta s \quad \text{and} \quad W_r = a\delta s. \tag{2.14}$$

One can convert W_r to radiation energy E_r and it can be received by the asymptotic observer. If we use the gravitational redshift formula

$$E_\infty = \sqrt{\frac{f(r)}{f(\infty)}} E_r. \tag{2.15}$$

Since the metric is asyptotically flat, $f(\infty) = 1$. If we apply the conservation of energy to the equation (2.15) and use the equation (2.13) later, then we obtain the result

$$\kappa = a_\infty(r_H) = \frac{f'(r_H)}{2}, \tag{2.16}$$

which coincides with (2.9) for the Schwarzschild black hole [46].

2.1.2 The First Law

We already disccussed that the horizon area and the black hole's mass-square are proportial. Since the change in mass can only be positive, any change in black hole's mass increases the horizon area. Therefore we can start with the equation

$$dM \approx dA, \tag{2.17}$$

and by using dimensional analysis we can obtain the first law. We should add the Newton's constant G to the left hand side of the equation since we have G in a combination with mass in GR. Then we should add a quantity with dimension of acceleration to the right hind side of the equation to make them dimensionally equivalent. We can use surface gravity κ as an acceleration. If we differentiate the equation (2.6) and also apply the surface gravity equation (2.9), we can read the first law of black hole thermodynamics as

$$\frac{\kappa}{8\pi}dA = GdM, \tag{2.18}$$

for the Schwarzschild black hole. In the general form, it can be written as

$$dM = \frac{\kappa}{8\pi}dA + \Omega dJ + \Phi dQ, \tag{2.19}$$

where E is the mass/energy, Ω is the angular velocity, J is the angular momentum, Φ is the electrostatic potential and Q is the electric charge [20, 46].

2.1.3 The Second Law

We should stress that we haven't derived these black hole laws.We constructed analogies between classical systems and thermodynamic ones and now we are trying to extend these thermodynamic laws to black holes. Actually, if we take a look on these black hole laws, we can see that there are some problems. First of all, when we write thermodynamic laws for black holes that means we consider them as thermal systems. But a thermal system has temperature and emits radiation. However nothing can escape from a black hole, so how can this happen? Secondly, the units of area and the entropy do not match. The entropy is dimensionless but we still have a dimensionful area. Let us explain these issues carefully and discuss the solutions that leads us to the second law of black holes known as the area theorem [46]. We know that the black holes are created from a gravitational collapse that includes huge amount of matter. We also know that the matter obeys quantum mechanics. If we consider black holes quantum mechanically, we can see that the black holes emit radiation known as Hawking radiation with the temperature [21, 49]

$$T = \frac{\hbar\kappa}{2\pi}. \tag{2.20}$$

We should emphasize that this is not a classical result which is clear due to the $\hbar$. The Hawking temperature is related with the surface gravity so using this relation in the equation (2.18), we can rewrite the first law of the Schwarzschild black holes as

$$dM = \frac{\kappa}{8\pi G} dA = T \frac{1}{4G\hbar} dA. \tag{2.21}$$

Using the fact that the surface gravity is the analog of the temperature, if we compare the first law of thermodynamics $dE = TdS$ with the above equation, we obtain

$$S_{BH} = \frac{A_H}{4G} \frac{1}{\hbar} = \frac{A_H}{4L_p^2}, \tag{2.22}$$

which is the Bekenstein-Hawking entropy [21, 46]. One should notice that we have given solutions to the issues we pointed out initially. We have temperature for the Schwarz-schild black holes now. We also clarify the point that the entropy is dimensionless whereas the area is not. We divided the area by Plank length square L_p^2 hence the equation (2.22) is dimensionless now [46]. After all these calculations, we still did not derive the entropy. The basic reason is that microscopically, entropy is the measure of the degrees of the freedom of the system but we have not counted any states. But still the laws we obtain are considered as black hole laws. Another important thing is as the equation (2.22) shows, the black hole entropy depends on the area while the statistical entropy depends on the volume. This reveals that a black hole cannot have a statistical interparation at the same dimensions. However, since area in five dimensions is the volume in four dimensions, the five dimensional black hole can have a statistical interpretation in the four dimensional spacetime. That is the scope of the theory known as AdS/CFT. It basically states that the black holes can have a statistical description in the spacetime that has one dimension lower than the spacetime the gravitational theory lived. Before the derivation of the Hawking temparature, we should note that although we carried our calculations for the Schwarzschild black hole, the area theorem (2.22) is valid for all black holes as long as they are described by Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} R, \tag{2.23}$$

where G_D is the Newton's constant in D-dimensional spacetime, g is the determinant of the metric and R is the Ricci scalar. For a generic metric, one can find the entropy by using the Wald entropy formula that we are going to discuss in detail in the next chapter [46].

Hawking Temperature We follow the simplest way to derive the Hawking temperature. The main idea is that assigning a smooth Eucliean spacetime with a periodic imaginary time t_E and assuming that the temperature is the inverse of this periodicity β. Letting $t_E = it$, then the metric (2.10) takes the form

$$ds_E^2 = f(r)dt_E^2 + \frac{dr^2}{f(r)} + \dots . \tag{2.24}$$

Near the horizon $r \approxeq r_H$, one can expand the above equation as

$$ds_E^2 = f'(r_H)(r - r_H)dt_E^2 + \frac{dr^2}{f'(r_H)(r - r_H)} + \dots . \tag{2.25}$$

Let us introduce a new coordinate $\rho = 2\sqrt{\frac{(r-r_H)}{f'(r_H)}}$ that puts the above equation in a simpler form

$$ds_E^2 = d\rho^2 + \rho^2 d\left(\frac{f'(r_H)}{2}t_E\right)^2 . \tag{2.26}$$

It is clear that it looks like the plane polar coordinates if we set the periodicity 2π that is $\frac{f'(r_H)}{2}t_E = 2\pi$ that leads us

$$\beta = \frac{4\pi}{f'(r_H)} \quad \text{or} \quad T = \frac{1}{\beta} = \frac{f'(r_H)}{4\pi}. \tag{2.27}$$

If we apply the equation (2.16), we obtain the the Hawking temperature as

$$T = \frac{\hbar\kappa}{2\pi}, \tag{2.28}$$

where $T = 1/8\pi GM$ is the Hawking temperature for the Schwarzchild black hole [46].

2.1.4 The Third Law

It states that as temperature T approaches to zero, the entropy S aprroaches to some constant. It is known as Nernst theorem and also states that one can not approach zero temperature by finite number of transformations [46]. It also states that It is not possible to reduce the surface gravity κ to zero by finite number of processess [20].

2.1.5 The Black hole Thermodynamics for Charged Black Holes

We already showed that for the generic metric in the form (2.10), the thermodynamics quantities are given as

$$\kappa = \frac{f'(r_H)}{2} \quad \text{and} \quad T = \frac{\hbar\kappa}{2\pi}. \tag{2.29}$$

The Schwarzschild the Reissner-Nordström metrics have the same form as in (2.10) but differ with the function $f(r)$. For the former where $f(r) = 1 - \frac{2GM}{r}$, the above quantities are obtained as

$$\kappa = \frac{1}{4GM} \quad \text{and} \quad T = \frac{1}{8\pi GM}, \tag{2.30}$$

and the first and second laws of thermodynamics are

$$dM = \frac{\kappa}{8\pi G}dA, \quad S = \frac{A_H}{4G}, \quad A_H = 4\pi r_s^2, \tag{2.31}$$

where the r_s represents the Schwarzschild radius.

The Reissner-Nordström metric describes the spacetime outside the spherical symmetric gravitating object that couples to electric field. This means the Einstein equations are coupled to Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0, \tag{2.32}$$

$$\nabla_\mu F_{\nu\rho} + \nabla_\rho F_{\mu\nu} \nabla_\nu F_{\rho\mu} = 0, \tag{2.33}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor and the second equation is the usual Bianchi identity. If one carries a similar approach to determine Schwarzschild metric carrefully, the Reissner-Nordström solution will be obtained in the form (2.10) with the function $f(r)$ as

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}. \tag{2.34}$$

It is clear that the solution reduces to Schwarzschild solution in the absence of the electric charge Q. As it is seen, this $f(r)$ has two roots that are

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}. \tag{2.35}$$

The inside of the square root leads us to three different cases. The only physical result is given fot the case $G^2M^2 > GQ^2$ where we have the both of $r_\pm$ that are the outer and the inner horizons. Since the asymptotic observer determines the thermodynamic quantities, one should evaluate the calcuations at $r = r_+$. If we rewrite the function as $f(r) = \dfrac{(r-r_+)(r-r_-)}{r^2}$, one can read the thermodynamic quantities of Reissner-Nordström black hole as

$$\kappa = \frac{f'(r_+)}{2} = \frac{r_+ - r_-}{2r_+^2} \qquad \text{and} \quad T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{4\pi}\frac{r_+ - r_-}{r_+^2}, \tag{2.36}$$

with the thermodynamics laws

$$dM = \frac{\kappa}{8\pi G}dA + \Phi\, dQ, \quad S = \frac{A_H}{4G}, \quad A_H = 4\pi r_+^2\,. \tag{2.37}$$

Detailed calculations can be found in [7].

2.1.6 The Black hole Thermodynamics for Kerr Black holes

[2]We have discussed the static solutions of Einstein's equations that are the Schwarzschild and Reissner-Nordström black holes. Let us finalize the discussion of the black holes by introducing the Kerr black holes. Since they are rotating, the Kerr metric is not static but stationary. In the Boyer-Lindquist coordinates (t, r, θ, ϕ) the Kerr metric takes the form

$$\begin{aligned} ds^2 = &-\big(1 - \frac{2GMr}{\rho^2}\big)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\triangle}dr^2 + \rho^2 d\theta^2 \\ &+ \frac{\sin^2\theta}{\rho^2}\big[(r^2+a^2)^2 - a^2\triangle\sin^2\theta\big]d\phi^2, \end{aligned} \tag{2.38}$$

where M is the energy, J is the angular momentum and $a = J/M$ is the angular momentum per unit mass. The quantities $\triangle$ and ρ^2 are defined as

$$\triangle(r) = r^2 - 2GMr + a^2, \tag{2.39}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{2.40}$$

It is straightforward to see that metric (2.38) reduces to the Schwarzschild metric as $a \to 0$.

The metric (2.38) does not depend on the coordinates t and ϕ which leads the existence of the Killing vectors $K = \partial_t$ and $R = \partial_\phi$. As we are going to see $K = \partial_t$ is not null on the event horizon but it is spacelike.

[2] This subsection is based on [7]

The event horizon occurs where $g^{rr} = \triangle/\rho^2$ vanishes. We should note that as in the Reissner-Nordström metric, there are three possible solutions for $\triangle = r^2 - 2GMr + a^2$. The case $GM > a$ gives physical solution which gives the event horizons as

$$r_{\pm} = GM \pm \sqrt{G^2M^2 - a^2}, \tag{2.41}$$

where r_+ and r_- are the outer and the inner event horizons respectively.

One can easily capture that both horizons are null hypersurfaces. To see this, all required is to evaluate the normal vector of the hypersurface $\Sigma \equiv r_{\pm} - a = 0$ which is $n_\mu = (0, 1, 0, 0)$ at the event horizons $r_{\pm}$. Since the norm is g^{rr} and it vanishes on event horizons, we obtain the result that event horizons $r_{\pm}$ are null hypersurfaces.

We should emphasize that for the other two black holes *i.e* the Schwarzschild and the Reissner-Nordström black holes, the Killing vector $K = \partial_t$ is null on the event horizon hence event horizon is also the Killing horizon for this Killing vector. However, this situation is quite different for the Kerr black holes where $K = \partial_t$ is not null on the outer event horizon. Instead it is spacelike there. There is a surface where $K = \partial_t$ is null and this surface is known as the stationary limit surface. Inside this surface, $K = \partial_t$ is spacelike so it is not possible to be static there. A region called the ergoregion is between the stationary limit surface and the outer event horizon such that inside this region, it is not possible even for light to move in the opposite direction of the black hole. We can see this by looking at the trajectory of the light that travels in the direction ϕ for the fixed coordinates r and θ and let the motion takes place on the equatorial plane. If we apply these constraints to the metric (2.38), we have,

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\Big(\frac{g_{t\phi}}{g_{\phi\phi}}\Big)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \tag{2.42}$$

On the stationary surface where g_{tt} vanishes, we have two solutions as

$$\frac{d\phi}{dt} = 0 \quad \frac{d\phi}{dt} = -2\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a}{2G^2M^2 + a^2}, \tag{2.43}$$

which shows that the photon is rotating with the black hole inside the ergoregion. Massive particles are dragged till they reach the outer event horizon. By using equation (2.43), one can find the angular velocity of the outer event horizon as

$$\Omega_H = \frac{a}{r_+^2 + a^2}, \tag{2.44}$$

which is also the particle's minimum angular velocity there. As seen, it is slower than the photon.

We should note that in the ergoregion, it is also possible to extract the rotational energy of the Kerr black hole by Penrose process [13]. We should emphasize that although $K = \partial_t$ is not zero on the outer event horizon, the linear combination $K^\mu + \Omega_H R^\mu$ is zero on the outer horizon and the quantity Ω_H is the angular velocity of the Kerr black hole. We use this combination to calculate the surface gravity κ.

Similarly, one can calculate the angular velocity of the massive particle. One can determine the stationary limit surface by evaluating $K^\mu K_\mu = 0$ and it is given by

$$(r - GM)^2 = G^2 M^2 - a^2 \cos^2 \theta. \tag{2.45}$$

Also, we can rewrite the outer event horizon in terms of

$$(r_+ - GM)^2 = G^2 M^2 - a^2. \tag{2.46}$$

The singularity of the Kerr black hole is not $r = 0$ but it happens where $\rho = 0$. As it is seen from the equation (2.40), ρ^2 can be zero either when the all terms vanish or when $r = 0$ and $\theta = \pi/2$. Actually this reveals that the singularity is not a point but a ring [7].

The Killing vectors $K^\mu = (1, 0, 0, 0)$ and $R^\mu = (0, 0, 0, 1)$ lead us the following conserved quantities

$$E = -p^\mu K_\mu = m\left(1 - \frac{2GMr}{\rho^2}\right)\frac{dt}{d\tau} + \frac{2GMmar}{\rho^2}\sin^2\theta\frac{d\phi}{d\tau}, \tag{2.47}$$

$$L = -\frac{2GMmar}{\rho^2}\sin^2\theta\frac{dt}{d\tau} + \frac{m(r^2 + a^2)^2 - m\triangle a^2 \sin^2\theta}{\rho^2}\sin^2\theta\frac{d\phi}{d\tau}, \tag{2.48}$$

where m is the mass of the particle.

One can realize that within the ergosphere, the particle can have negative energy according to observer at infinity. We must emphasize that outside this region, any particle has positive energy therefore to escape this region to infinity, a particle should accelerate so has a positive energy. The process to extract energy from a rotating black hole is known as the Penrose process and basically it depends on the following idea. Consider a traveler moving along the geodesic and carrying a piece of rock. Outside the Kerr black hole, the initial momentum of the system containing traveler and rock is given by $p^{(0)\mu}$ that leads a conserved positive energy $E^{(0)} = -K_\mu p^{(0)\mu}$. When they enter the ergoregion, the rock and the traveler are separated from each other. Now they carry the four-momenta $p^{(1)\mu}$ and $p^{(2)\mu}$ that represent the four-momentum of the traveler and the rock respectively. By the conservation of four momentum, we have

$$p^{(0)\mu} = p^{(1)\mu} + p^{(2)\mu}, \tag{2.49}$$

whose contraction with K_μ yields energy conservation

$$E^{(0)} = E^{(1)} + E^{(2)}. \tag{2.50}$$

It is possible to arrange the trajectroy of the rock such that $E^{(2)} < 0$. If we apply the conservation of momentum (2.50), we see that

$$E^{(1)} > E^{(0)}, \tag{2.51}$$

therefore the energy of the traveler is bigger than the energy of the system containing the rock and the traveler. This extra energy is extracted from the rotational energy of the Kerr black hole. Let us show that there is a limit on the extacted energy. To see this we should use the combined Killing vector $\chi^\mu = K^\mu + \Omega_H R^\mu$ where K^μ and R^μ are the usual Killing vectors and Ω_H is the angular velocity of the Kerr black hole. The reason why we use χ^μ is that it is null on the outer event horizon so the surface $r = r_+$ is a Killing horizon for this Killing vector χ^μ. We set that the rock passes the outer event horizon therefore

$$p^{(2)\mu}\chi_\mu < 0. \tag{2.52}$$

If we apply the definitions of E and L, we obtain

$$-E^{(2)} + \Omega_H L^{(2)} < 0 \quad \rightarrow \quad L^{(2)} < \frac{E^{(2)}}{\Omega_H}. \tag{2.53}$$

Since the angular momentum of the black hole Ω_H is positive and we set the energy of the rock as negative, the angular momentum of the rock becomes negative as seen from the equation (2.53) which shows that it was thrown to move against the rotation of the black hole. If we set the infinitesimal changes of the black hole as

$$\delta M = E^{(2)} \quad \text{and} \quad \delta J = L^{(2)}, \tag{2.54}$$

where J and M are the angular momentum and the mass of the Kerr black hole. Then we can rewrite the equation (2.53) as

$$\delta J < \frac{\delta M}{\Omega_H}, \tag{2.55}$$

which shows that there is a limit on extracting the angular momentum of the black hole.

We should emphasize that we decrease the angular momentum and the mass of the black hole but we still obey the area law. Let us show that after the Penrose process, the change in the irreducible mass of the Kerr black hole is positive. Then by using the relation between the area and the M_{ir}, we also show that the change in area of the outer horizon is positive also.

To do this, let us start with the definition of the area A that is

$$A = \int \sqrt{\det h_{ij}} dx^i dx^j, \tag{2.56}$$

where h_{ij} is the induced metric on the sphere. One can easily capture that $\det h_{ij} = g_{\theta\theta}g_{\phi\phi} = \sin^2\theta(r_+^2 + a^2)^2$, then the integral becomes

$$A = 4\pi(r_+^2 + a^2). \tag{2.57}$$

If we use the relation between the square of the M_{ir} and the area [16], we will have

$$M_{ir}^2 = \frac{A}{16\pi G^2}, \tag{2.58}$$

$$= \frac{1}{2}\big(M^2 + \sqrt{M^4 - (J/G)^2}\big), \tag{2.59}$$

where the equation (2.59) is obtained after applying the equation (2.57), the definition of r_+ and the equation $J = Ma$ to the equation (2.58). To see that the change in the M_{ir} is positive after the Penrose process, we should differentiate the equation (2.59) that gives,

$$\delta M_{ir} = \frac{a}{4GM_{ir}\sqrt{G^2M^2 - a^2}}(\delta M/\Omega_H - \delta J), \tag{2.60}$$

where we also use the equation $J = Ma$ and the definition of Ω_H. If we use the equation (2.55), we can see that $\delta M_{ir} > 0$. Differentiating the equation (2.57) and then using the equation (2.60) gives

$$\delta A = 8\pi G \frac{a}{\Omega_H\sqrt{G^2M^2 - a^2}}(\delta M - \Omega_H\delta J), \tag{2.61}$$

after rearrenging the terms one can write the equation (2.61) in terms of

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J \tag{2.62}$$

where we write the surface gravity κ as

$$\kappa = \frac{\sqrt{G^2M^2 - a^2}}{2GM(GM + \sqrt{G^2M^2 - a^2})}. \tag{2.63}$$

It is clear that equation (2.62) is the first law of Kerr black hole. By using these quantities, we can write the thermodynamics of the Kerr black holes as

$$T = \frac{\hbar\kappa}{2\pi}, \quad S = \frac{A_H}{4G} \tag{2.64}$$

where A_H and κ are given in the equations (2.57) and (2.63) respectively.

CHAPTER 3

ENTROPY IN BORN-INFELD GRAVITY

3.1 Introduction

[1]It is perhaps not possible to write a classical theory of gravity that can supersede Einstein's gravity in elegance and simplicity, which, in its most succinct form, reads in the absence of matter as a variational statement

$$\delta_g \int_{\mathcal{M}} d^n x \sqrt{-g} R = 0. \tag{3.1}$$

In words, pure general relativity (GR) is a statement about the topological Lorentzian manifold $\mathcal{M}$: *finding the metrics that are critical points of the total scalar curvature on the manifold with prescribed boundary and/or asymptotic conditions to model various physical cases.* This statement is valid in generic $n-$dimensions but trivial for $n = 1$ and $n = 2$. For all other dimensions, pure GR boils down to searching for solutions of the non-linear partial differential equation $R_{\mu\nu} = 0$, that is finding Ricci-flat metrics.

In $n = 3$, the Riemann and the Ricci tensors have equal number of components (six) and so Ricci-flat metrics are locally Riemann-flat and hence there is no local gravity except at the location of the sources. But beyond $n > 3$, the innocent looking equation (3.1) and its equivalent partial differential equation form are highly complicated to solve and since their first appearance 100 years ago, many fascinating predictions of these equations (perhaps the most remarkable ones being the global dynamics of the Universe and the existence of black holes and gravitational waves) have been observed. What is rather fascinating about the classical equation $R_{\mu\nu} = 0$ is that it

[1] For this chapter, instead of D we use n to denote the number of dimensions.

leads to thermodynamics type equations such as [2]

$$\delta M = T\delta S + \Omega\delta J, \tag{3.2}$$

for black hole solutions where $T = \frac{\kappa}{2\pi}$ is the horizon temperature and κ is the surface gravity, Ω is the angular velocity of the horizon and J is the angular momentum of the black hole while M is the mass of it. S is the entropy given by the Bekenstein-Hawking formula [20, 47]

$$S = \frac{A_H}{4G}, \tag{3.3}$$

where A_H is the horizon area. The temperature T and the entropy S are defined up to a multiplicative constant which can be fixed by the help of a semi-classical analysis such as the Hawking radiation [21, 48, 49]. But the crucial point is that, classically $R_{\mu\nu} = 0$ equations for a black hole encode the equation (3.2) which is considered as the first law of black hole thermodynamics. In fact, all four laws of black hole mechanics (or thermodynamics) follow from Einstein's equations [20]. It was also a rather remarkable suggestion and demonstration that the arguments can be turned around and thermodynamics of Rindler horizons (with some assumptions) can lead to the Einstein's equations as the equation of state [50]. Note that Gibbons and Hawking showed that (3.3) result is valid for the de Sitter horizons [51] which will be relevant for this work.

One of the current problems in gravity (or quantum gravity) research is to find a possible *microscopic* explanation of the entropy S in (3.2). Let us expound on this: from pure statistical mechanical point of view, S must be given as $S = \ln N$ where N refers to the microstates that have the same M, T, Ω and J values in (3.2). All this says that a black hole has much more internal states than meets the eye. But classical considerations suggest that black holes have no classical hair or microscopic degrees of freedom to account for the large entropy given in (3.3). Therefore even though Einstein's theory yields a macroscopic thermodynamical equation such as (3.3), it does not seem to *explain* it. Of course if the classical picture were to be correct, the four laws of black hole mechanics are not true thermodynamical equations but just a mere analogy without much deep implication. However, there is mounting evidence from various considerations (such as the microscopic degree of freedom counting for

[2] If Maxwell field is coupled to gravity one has a more general thermodynamical equation including the charge variation $\phi\delta Q$ on the right-hand side where ϕ is the electric potential on the horizon.

extremal black holes in string theory in terms of the D-brane charges [52, 53] and in 2+1 dimensional gravity in terms of the asymptotic symmetries *etc.*) that this is not the case and black holes do have temperature and entropy with perhaps important implications for quantum gravity.

In trying to understand the actual content of equations (3.2) & (3.3), it is extremely important to figure out how universal they are: namely, if the underlying theory is modified from Einstein's gravity to something else, what remains of these thermodynamical equations. Besides, we also know that both at large and short distance scales (low and high energies) GR, in its simplest form, is not sufficient to explain the observable Universe. These deficiencies are well-known: for example the accelerated expansion of the Universe and the observed age (or homogeneity and isotropy) cannot be explained by pure gravity alone, hence there are various attempts to modify it. Of course, ideally one would like to have a quantum version of gravity in order to explain the beginning of the Universe or the space and time and the singularity issues of black holes. But at the moment we do not know exactly what the symmetries, principles *etc.,* of such a theory are. In any case, the idea is that Einstein's theory captures a great deal about the *gravitating* Universe or subsystems (such as the solar system) but it is not yet the final word on it, hence the modifications.

In trying to go beyond Einstein's theory there are many routes to follow. One such route that we have advocated recently in several works [54, 55] is the following: let us write down all the easily detectable low energy properties of Einstein's theory (3.1) and try to implement them in the new modified gravity theory that we are searching for. It is quite possible that at very small distances gravity (or spacetime) will be nothing like the one proposed by Einstein's theory, so any of its low energy virtues may not survive in that regime. For example superstring theory is such an example where the spacetime or the metric of the gravity is an emergent quantity. But this does not deter us from our discussion because we shall be interested in the intermediate energy scales in string theory where Einstein's gravity appears with modified curvature corrections and the pseudo-Riemannian description of spacetime is quite accurate.

To accomplish our goal of writing a new modified theory we may list the virtues of Einstein's theory as follows:

(1) uniqueness of the vacuum: source-free GR, $R_{\mu\nu} = 0$, has a maximally symmetric solution which is the Minkowski spacetime. Classically it is the unique vacuum and other considerations, such as the positive energy theorem, suggest that this is the lowest energy state with zero energy (under the proper decay assumptions and several other physically motivated assumptions such as the cosmic censorship, dominant energy condition and the stability of the Minkowski spacetime *etc.*)

(2) masslessness of the spin-2 graviton: about its unique vacuum if one expands the metric as $g = \eta + h$, one observes that h obeys a massless Klein-Gordon type equation, with only transverse-traceless polarization.

(3) diffeomorphism invariance of the theory: the spacetime is a topological Lorentzian manifold without a prior geometry before gravity is switched on and there are no preferred coordinates even with gravity switched on.

Certainly one could add further, harder-to-see properties of the theory to this list. For example, the theory has the local causality property which can be shown in various ways such as proving that there is always a Shapiro time-delay as opposed to a time advance. It has a $3 + 1$ dimensional splitting property and a well-defined initial value (Cauchy) formulation where the full equation $R_{\mu\nu} = 0$ can be split into constraints on a spacelike Cauchy surface and evolutions off the surface in the *time* direction. But these require further assumptions on the topology of $\mathcal{M}$ and we shall not be interested in them here. Our basic working principle will be to write down a modified version of Einstein's gravity that has the above three properties. The third item on the list is easy to satisfy but the first two are quite hard.

One of the earliest modifications satisfying the above requirements was famously introduced by Einstein when he added a cosmological constant to his theory:

$$\delta_g \int_{\mathcal{M}} d^n x \sqrt{-g}(R - 2\Lambda) = 0. \tag{3.4}$$

What is interesting about this theory is that, in contrast to (3.1), this one has a dimensionful parameter Λ which will set the *sizes* of a gravitating system.[3] Now the field

[3] Note that, Newton's constant is also a dimensionful parameter that arises in the coupling of matter to gravity. Since at this stage we consider pure and classical gravity for the sake of keeping the discussion simple, we shall restore the Newton's constant below.

equations are summarized as

$$R_{\mu\nu} = \frac{2\Lambda}{n-2} g_{\mu\nu}, \tag{3.5}$$

where the number of dimensions n also makes an explicit appearance. All the desired properties are also valid for this theory: for $\Lambda > 0$ de Sitter (dS) and for $\Lambda < 0$ anti-de Sitter (AdS) is the unique vacuum, about which there is a single massless spin-2 excitation. Many solutions (such as black holes) of the $R_{\mu\nu} = 0$ equations also solve (in modified forms) the cosmological Einstein's theory. Of course currently due to the accelerated expansion of the Universe, $\Lambda > 0$ is preferred. For dS, since there is a cosmological horizon, similar to the black holes, one can define a temperature and an entropy for this horizon.

Other earlier modifications of Einstein's theory are of the following form

$$\delta_g \int_{\mathcal{M}} d^n x \sqrt{-g}\, (R - 2\Lambda_0 + \alpha R^2 + \beta R^2_{\mu\nu} + \gamma R^2_{\mu\nu\alpha\beta} + \dots), \tag{3.6}$$

which generically do not have the first two properties in our list: they have many different vacua, having different values of Λ, which cannot be compared with each other and there are many massive modes besides the massless spin-2 graviton. Typically some of these modes are ghosts and so the theory is hard to make sense both at the classical and the quantum levels.

In our search for a higher order gravity theory that has the same features as Einstein's theory, we used the ideas laid out by Deser and Gibbons [32] who were inspired by Eddingston's proposal of using generalized volumes in the action as

$$\delta_{g,\Gamma} \int_{\mathcal{M}} d^n x \Big(\sqrt{-\det\left(g_{\mu\nu} + \gamma R_{\mu\nu}(\Gamma)\right)} - \lambda_0 \sqrt{-g} \Big), \tag{3.7}$$

where one assumes that the metric g and the connection Γ are independent in this variation. This type of theories gained some interest recently (under the rubric *Eddington inspired Born-Infeld gravity* even though strictly speaking Born-Infeld's work in electrodynamics came much later than Eddington's work in gravity). For this path of the developments, we refer the reader to [33] and a recent review [56]. Instead, Deser-Gibbons's suggestion was to consider actions of the type

$$\delta_g \int_{\mathcal{M}} d^n x \sqrt{-\det(g_{\mu\nu} + \gamma R_{\mu\nu}(g) + \cdots)}, \tag{3.8}$$

where the connection is the unique Levi-Civita connection, not to be varied independently. Of course the resulting theories are quite different: the action (3.8) generically is not unitary nor does it have a unique vacuum. Only recently, the most general theory satisfying these requirements was found in [38, 57] for $3-$dimensions and in [54, 55] for four and higher dimensions. The $3-$dimensional theory is particularly elegant as it is formed by taking the determinant of the Einstein tensor plus the metric

$$I = -\frac{m^2}{4\pi G_3}\int_{\mathcal{M}} d^3x \left[\sqrt{-\det\left(\mathsf{g} + \frac{\sigma}{m^2}\mathsf{G}\right)} - \left(1 - \frac{\lambda_0}{2}\right)\sqrt{-\mathsf{g}}\right], \tag{3.9}$$

with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Here $\sigma^2 = 1$ and the theory has a unique maximally symmetric vacuum with an effective cosmological constant given as

$$\Lambda = \sigma m^2 \lambda_0 \Big(1 - \frac{\lambda_0}{2}\Big), \quad \lambda_0 \neq 2 \tag{3.10}$$

and a massive spin-2 excitation about this vacuum with a mass $M_g^2 = m^2 + \Lambda$. The Wald entropy, the $c-$function and the $c-$charge of this theory were studied in [55], but to set the stage for the generic Born-Infeld (BI) gravity in higher dimensions, we reproduce the results and give some more details on the computations and also compute the second Born-Infeld type extension of new massive gravity in $3-$dimensions, which was not computed before.

For higher dimensions, the simple form (3.9) does not lead to a unitary theory, therefore one needs to add at least quadratic terms in curvature inside the determinant. After a rather tedious computation laid out in detail in [54, 55] the following theory was found:

$$I = \frac{1}{8\pi\gamma G}\int_{\mathcal{M}} d^n x \sqrt{-g}\left(\sqrt{\det(\delta^\rho_\nu + \gamma A^\rho_\nu)} - \lambda_0 - 1\right), \tag{3.11}$$

where the $\mathsf{A}-$tensor reads

$$A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} + \gamma\left(a_1 C_{\mu\rho\sigma\lambda}C_\nu{}^{\rho\sigma\lambda} + a_3 R_{\mu\rho}R^\rho_\nu\right) + \frac{\gamma}{n}g_{\mu\nu}\left(b_1 C^2_{\alpha\beta\rho\sigma} + b_2 R^2_{\rho\sigma}\right). \tag{3.12}$$

Here, $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{n}g_{\mu\nu}R$ is the traceless-Ricci tensor and $C_{\mu\rho\sigma\lambda}$ is the Weyl tensor given as

$$C_{\mu\rho\sigma\lambda} = R_{\mu\rho\sigma\lambda} + \frac{2}{n-2}\left(g_{\rho[\sigma}R_{\lambda]\mu} - g_{\mu[\sigma}R_{\lambda]\rho}\right) + \frac{2R}{(n-1)(n-2)}g_{\mu[\sigma}g_{\lambda]\rho}. \tag{3.13}$$

Observe that there are three other possible quadratic corrections $C_{\mu\rho\nu\sigma}R^{\rho\sigma}$, $S_{\mu\rho}S_{\nu}^{\rho}$ and $S_{\rho\sigma}^2$ which do not survive since they lead to massive particles and/or destroy the uniqueness of the vacuum.

The theory has five dimensionless parameters β, a_i, b_i and a dimensionful one which is the Born-Infeld parameter γ with dimensions of L^2. Unitarity of the excitations of the theory and the requirement that the theory has a unique viable vacuum with only one massless spin-2 graviton about this vacuum leads to a further elimination of three of these parameters. Namely, the following constraints must be satisfied

$$a_1 + b_1 = \frac{(n-1)^2}{4(n-2)(n-3)}, \qquad a_3 = \frac{\beta+1}{4}, \qquad b_2 = -\frac{\beta}{4}. \tag{3.14}$$

Therefore after all the constraints are applied, there are only two free dimensionless parameters one of which can be chosen as β and the other one as a_1 or b_1. And for any values of these parameters, the theory has a massless graviton and no other particle around the unique viable vacuum given by the following equation

$$\frac{\lambda}{n-2} - 1 + (\lambda_0 + 1)\left(\frac{\lambda}{n-2} + 1\right)^{1-n} = 0, \tag{3.15}$$

where $\lambda = \gamma\Lambda$ is the *dimensionless* effective cosmological constant. Although looking quite cumbersome, remarkably for any n, the vacuum equation gives a unique viable solution consistent with the unitarity requirements. This subtle fact requires a long analysis of the root structure of this equation and we shall not repeat it here as it was done in the Appendix-C of [55].

As expected, the theory (3.11) reproduces cosmological Einstein's gravity at the first order expansion in curvature; it reproduces Einstein-Gauss-Bonnet theory at the second order expansion and a particular cubic and quartic theory in the next two order expansions. The important point here is that, all these perturbative theories are unitary on their own with the same particle content as Einstein's theory, namely they just have one massless spin-2 excitation about their unique vacuum solution. Moreover, the theory keeps all of its properties at any truncated order in an expansion in powers of the curvature. The form of the vacuum equation and the effective gravitational constant are modified up to and including the $\mathcal{O}(R^4)$ expansion but beyond that, namely at $\mathcal{O}(R^{4+k})$ with $k \geq 1$, the structure of the vacuum equation or the propagator of the theory are not affected. This is due to the fact that the action is defined with a determinant operation and only up to second order terms in curvature are kept inside the determinant.

The layout of this chapter is as follows: In the section 2, we define the Wald entropy for a generic theory and carry out the computation of the $n-$dimensional Born-Infeld gravity. In the section 3, we study a minimal version of BI gravity and consider the $n \to \infty$ limit of the theory and show that the resulting exponential gravity has the same area-law for the entropy. In the final section, we study the $3-$dimensional Born-Infeld gravity which has a massive graviton in its perturbative spectrum.

3.2 Wald Entropy in Generic BI Gravity

In the paper [58], we shall consider the locally (A)dS metrics, but for the sake of generality and possible further work, in this section we consider a general spherically symmetric and static metric and give its curvature properties which will be relevant to black holes as well as (A)dS spacetimes. Therefore we consider the following metric

$$ds^2 = -a(r)^2 dt^2 + \frac{1}{a(r)^2} dr^2 + \sum_{i,j=2}^{n-2} r^2 h_{ij}(x^k) dx^i dx^j, \tag{3.16}$$

where $h_{ij}(x^k)$ is the induced metric on $(n-2)-$dimensional sphere. The Christoffel symbols for this metric are

$$\Gamma^0{}_{10} = \frac{a'}{a}, \quad \Gamma^1{}_{00} = a^3 a', \quad \Gamma^1{}_{11} = -\frac{a'}{a}, \quad \Gamma^1{}_{ij} = -r a^2 h_{ij}, \quad \Gamma^i{}_{1j} = \frac{\delta^i_j}{r}, \tag{3.17}$$

where a is the shorthand notation for $a(r)$ and prime denotes derivative with respect to the radial coordinate r. From the definition of the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}, \tag{3.18}$$

and its contractions, the corresponding curvature terms follow

$$
\begin{aligned}
&R_{0101} = (a')^2 + aa'', \quad R_{0i0j} = ra^3a'h_{ij}, \quad R_{1i1j} = -\frac{ra'}{a}h_{ij}, \\
&R_{ijkl} = -r^2(a^2-1)(h_{ik}h_{jl} - h_{il}h_{jk}), \quad R_{00} = a^2\left((a')^2 + aa'' + \frac{n-2}{r}aa'\right), \\
&R_{11} = -\frac{1}{a^2}\left((a')^2 + aa'' + \frac{n-2}{r}aa'\right), \quad R_{ij} = -\left[(n-3)(a^2-1) + 2raa'\right]h_{ij}, \\
&R = -\left(2(a')^2 + 2aa'' + \frac{4(n-2)}{r}aa' + \frac{(n-2)(n-3)}{r^2}(a^2-1)\right). \qquad (3.19)
\end{aligned}
$$

Note that since de Sitter spacetime is maximally symmetric its curvature terms can be written collectively as

$$
R_{\mu\nu\alpha\beta} = \frac{2\Lambda}{(n-1)(n-2)}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right), \quad R_{\mu\nu} = \frac{2\Lambda}{n-2}g_{\mu\nu}, \quad R = \frac{2\Lambda n}{n-2}. \tag{3.20}
$$

In addition to that, the following exact relation will be employed for calculations:

$$
\begin{aligned}
C_{\mu\rho\sigma\lambda}C_{\nu}{}^{\rho\sigma\lambda} =\; & R_{(\mu}{}^{\rho\sigma\lambda}R_{\nu)\rho\sigma\lambda} + \frac{4}{n-2}R_{(\mu}{}^{\rho\sigma}{}_{\nu)}R_{\rho\sigma} - \frac{2n}{(n-2)^2}R^{\lambda}{}_{(\mu}R_{\nu)\lambda} \\
& + \frac{4}{(n-2)^2}RR_{\mu\nu} + \frac{2}{(n-2)^2}\left(R^2_{\rho\sigma} - \frac{1}{n-1}R^2\right)g_{\mu\nu},
\end{aligned} \tag{3.21}
$$

whose contraction yields

$$
C^2_{\alpha\beta\rho\sigma} = R^2_{\alpha\beta\rho\sigma} - \frac{4}{n-2}R^2_{\rho\sigma} + \frac{2}{(n-1)(n-2)}R^2. \tag{3.22}
$$

Now we turn to the computation of Wald entropy for the BI theory (3.11). Recall that [59, 60, 61, 62] Wald entropy arises as a Noether charge of diffeomorphism symmetry of the theory and for a generic action of the form

$$
I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^n x\sqrt{-g}\mathcal{L}\left(g_{ab}, R_{abcd}\right), \tag{3.23}
$$

it reads as [4]

$$
S_W = -2\pi\oint\left(\frac{\partial L}{\partial R_{abcd}}\varepsilon_{ab}\varepsilon_{cd}\right)dV_{n-2}, \tag{3.24}
$$

where $r = r_H$ denotes the bifurcate Killing horizon and the integral is to be evaluated on-shell. The volume element dV_{n-2} is the volume element on the bifurcation surface

[4] Note that there is an overall factor between L and $\mathcal{L}$, namely $L = \frac{1}{16\pi G}\mathcal{L}$.

Σ and ε_{ab} are the binormal vectors to Σ. For the metric (3.16) the binormal vectors can quite easily be found as

$$\varepsilon_{01} = 1, \qquad \varepsilon_{10} = -1, \tag{3.25}$$

by using the timelike Killing vector $\chi \to (-1, 0, 0, \ldots)$ in the formula

$$\varepsilon_{ab} = -\frac{1}{k}\nabla_a \chi_b, \tag{3.26}$$

where k is defined as $k = \sqrt{-\frac{1}{2}(\nabla_\alpha \chi_\beta)(\nabla^\alpha \chi^\beta)}$.

To find the Wald entropy of the de Sitter spacetime which is given by the metric

$$ds^2 = -\left(1 - \frac{2\Lambda r^2}{(n-1)(n-2)}\right)dt^2 + \left(1 - \frac{2\Lambda r^2}{(n-1)(n-2)}\right)^{-1} dr^2 + r^2 d\Omega^2_{n-2}, \tag{3.27}$$

we need to compute the derivative of the Lagrangian which can be done as follows.

By defining $f(\mathsf{A}) = \sqrt{-\det(g_{\mu\nu} + \gamma A_{\mu\nu})}$ and using $\sqrt{\det M_{\mu\nu}} = \exp\left[\frac{1}{2}\mathrm{Tr}\ln M_{\mu\nu}\right]$ we obtain

$$\frac{\partial \mathcal{L}}{\partial R_{abcd}} = f(\mathsf{A})\left((\mathsf{g} + \gamma\mathsf{A})^{-1}\right)^{\mu\nu}\frac{\partial A_{\mu\nu}}{\partial R_{abcd}}. \tag{3.28}$$

Here, when calculating the derivative one needs to use

$$\frac{\partial R}{\partial R_{abcd}} = g^{c[a}g^{b]d}, \quad \frac{\partial R^2_{\mu\nu}}{\partial R_{abcd}} = g^{a[c}R^{d]b} - g^{b[c}R^{d]a}, \quad \frac{\partial R^2_{\mu\nu\alpha\beta}}{\partial R_{abcd}} = 2R^{abcd}, \tag{3.29}$$

and it is a lot easier if $A_{\mu\nu}$ is written in the Riem-Ric-R (RRR) basis as it is done in [54]. By denoting a bar on top of the background elements we can evaluate all the terms in de Sitter background with the help of (3.20). The derivative of the $\mathsf{A}-$tensor reads

$$\begin{aligned}\left(\frac{\partial A_{\mu\nu}}{\partial R_{abcd}}\right)_{\bar{R}} = &\left(\frac{1+\beta}{4} + \frac{\lambda}{n-2}a_3\right)\left(\delta^b_{(\mu}\delta^d_{\nu)}\bar{g}^{ac} - \delta^b_{(\mu}\delta^c_{\nu)}\bar{g}^{ad} - \delta^a_{(\mu}\delta^d_{\nu)}\bar{g}^{bc} + \delta^a_{(\mu}\delta^c_{\nu)}\bar{g}^{bd}\right) \\ &+ \left(-\frac{\beta}{2n} + \frac{2\lambda}{n(n-2)}b_2\right)\left(\bar{g}^{ac}\bar{g}^{bd} - \bar{g}^{ad}\bar{g}^{bc}\right)\bar{g}_{\mu\nu},\end{aligned} \tag{3.30}$$

where $\lambda = \gamma\Lambda > 0$. The value of the other tensors can be computed as

$$\bar{g}_{\mu\nu} + \gamma\bar{A}_{\mu\nu} \;=\; \bar{g}_{\mu\nu}\left(1 + \chi\right), \tag{3.31}$$

where $\chi = \frac{2\lambda}{n-2} + \frac{4\lambda^2}{(n-2)^2}(a_3 + b_2)$ and finally one has

$$f(\mathsf{A})_{\bar{R}} \;=\; (1+\chi)^{n/2}. \tag{3.32}$$

Using these with the implementation of the binormal vectors one obtains the Wald entropy for the $n-$dimensional de Sitter spacetime as

$$S_W = \frac{A_H}{4G}\left(1+\frac{4\lambda}{n-2}(a_3+b_2)\right)\left(1+\frac{2\lambda}{n-2}+\frac{4\lambda^2}{(n-2)^2}(a_3+b_2)\right)^{\frac{n-2}{2}}. \quad (3.33)$$

After inserting the constraints (3.14) on a_3 and b_2 for the unitarity of the theory, Wald entropy takes the form

$$S_W = \frac{A_H}{4G}\left(1+\frac{\lambda}{n-2}\right)^{n-1}, \quad (3.34)$$

over which two important limits of small and large γ can be found as

$$\lim_{\gamma\to 0} S_W = \frac{A_H}{4G} = S_{BH}, \quad (3.35)$$

$$\lim_{\gamma\to\infty} S_W = \frac{A_H}{4G}\left(\frac{\Lambda}{n-2}\right)^{n-1}\gamma^{n-1}. \quad (3.36)$$

As expected when the Born-Infeld parameter γ is taken to be very small the theory gives the results of Einstein's theory and the Bekenstein-Hawking and Wald results match. In the large $\gamma-$limit the crucial result to note is the increase in entropy when γ increases, i.e. when we add more curvature to the spacetime, entropy increases dramatically with the cubic order in four dimensions.

The *effective gravitational constant* of the theory again might be calculated from the derivative of the Lagrangian with respect to Riemann tensor: [5]

$$\frac{1}{\kappa_{\text{eff}}} = \frac{1}{G}\frac{\bar{R}_{\mu\nu\rho\sigma}}{\bar{R}}\left(\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\right)_{\bar{R}}, \quad (3.37)$$

which in our case gives

$$\frac{1}{\kappa_{\text{eff}}} = \frac{1}{G}\left(1+\frac{\lambda}{n-2}\right)^{n-1}. \quad (3.38)$$

This result has some important implications since it allows us to rewrite the Wald entropy in a nicer form

$$S_W = \frac{A_H}{4\kappa_{\text{eff}}}, \quad (3.39)$$

which is consistent with the results of [64]. Namely, as far as the dynamical entropy is concerned, BI gravity that has the same particle content of Einstein's theory has the Bekenstein-Hawking form. The only change is that instead of G, κ_{eff} must be used.

[5] For more details on this see [63].

And since $\kappa_{\text{eff}} < G$ for dS, as can be seen from (3.38), the entropy is increased due to the high curvature terms.

Until one carries out the computations leading to our equations (3.33) and (3.34) for Born-Infeld gravity theories that was defined in the previous papers and the ones in 3 dimensions and infinite number of dimensions, it is absolutely not clear at all that the entropy of the de Sitter Space in these theories obey the area law. This is actually quite a surprise since almost any other theory with some powers of curvature will not have this property even for the de Sitter case. Here the main idea is to see a possible connection between the particle content, vacuum and entropy properties of a theory at hand. The Born-Infeld type theories that were constructed in [33] are in some sense very natural generalizations of Einstein's gravity as far as their particle content and vacua are concerned. In this work we have shown another shared property of these models: that is both Einstein's theory and the specific BI gravity theories that we discussed obey the area law for entropy. This statement is important as it shows the equivalence of the dynamical and geometric entropies in these models.

3.3 Two Special BI gravities

Observe that the Wald entropy (3.34) does not depend on the parameters β, a_3 and b_2. Namely, these parameters are not constrained further by the positivity of the Wald entropy. Obeying the constraints they must satisfy, one can write down a *minimal* theory without any parameters which reads as [54]

$$\mathcal{L} = \frac{2}{\gamma}\left[\left(1 + \frac{\gamma}{n}R + \frac{\gamma^2(n-1)^2}{4n(n-2)(n-3)}\chi_{GB} - \frac{\gamma^2(n-2)}{4n^2}R^2\right)^{n/2} - \lambda_0 - 1\right], \tag{3.40}$$

where $\chi_{GB} = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the Gauss-Bonnet term and the Wald entropy for the dS spacetime is still given by (3.34). This theory has further studied in [65] where it was shown that the Schwarzschild-Tangherlini black hole is only an approximate solution.

General relativity in the large number of dimensions has some interesting properties (see [66]). Here we can also consider the $n \to \infty$ limit of the BI gravity given by

(3.40) to get [54]

$$\mathcal{L} = \frac{2}{\gamma}\left(\exp\left[\frac{\gamma}{2}R + \frac{\gamma^2}{8}\left(R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu}\right)\right] - \lambda_0 - 1\right), \tag{3.41}$$

where the Wald entropy reads as

$$S_W = \frac{A_H}{4G}\left(1 - \frac{2\lambda(2n-3)}{(n-1)(n-2)}\right)\exp\left[\frac{\lambda n}{n-2}\left(1 - \frac{\lambda(2n-3)}{(n-1)(n-2)}\right)\right]. \tag{3.42}$$

Here, in addition to the small$-\gamma$ limit

$$\lim_{\gamma\to 0} S_W = \frac{A_H}{4G} = S_{BH}, \tag{3.43}$$

which gives the expected result and reduces to the Bekenstein-Hawking entropy, one can also check the large$-n$ limit

$$\lim_{n\to\infty} S_W = \frac{A_H}{4G}e^{\lambda}, \tag{3.44}$$

that satisfies the relation $S_W = \frac{A_H}{4\kappa_{\text{eff}}}$ where κ_{eff} can be found as (or can be obtained from (3.38) by taking the $n \to \infty$ limit) $\frac{1}{\kappa_{\text{eff}}} = \frac{e^{\lambda}}{G}$.

3.4 BI extension of New Massive Gravity

As we noted in the Introduction, the Born-Infeld gravity in $2+1$ dimensions has a particularly simple form which was obtained as an infinite order expansion of the quadratic new massive gravity [39, 40, 44]. BI theory has the nice property that it coincides at any order in curvature expansion with the theory obtained from the requirements of AdS/CFT [67, 68] and moreover it appears as a counter term in AdS$_4$ [69]. Both BI and new massive gravity describe a massive spin-2 particle with 2 degrees of freedom in $2+1$ dimensions and have very rich solution structures, including the BTZ and various other black holes. For further work on this theory see [70, 71].

For the theory given in (3.9), we can define the Wald entropy of the horizon of the BTZ black hole in terms of the derivative of the Lagrangian with respect to the Ricci tensor as

$$S_W = -2\pi \oint \left(\frac{\partial L}{\partial R_{\rho\sigma}} g^{\mu\nu}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}\right) d^3x. \tag{3.45}$$

To carry out the computation, let us define

$$f(R_{\mu\nu}) = \sqrt{-\det\left(\mathsf{g} + \frac{\sigma}{m^2}\mathsf{G}\right)}, \tag{3.46}$$

which yields

$$\frac{\partial f}{\partial R_{\rho\sigma}}=\frac{\sigma}{4m^2}f\left(\left(\mathsf{g}+\frac{\sigma}{m^2}\mathsf{G}\right)^{-1}\right)^{\mu\nu}\left(\delta^\rho_\mu\delta^\sigma_\nu+\delta^\rho_\nu\delta^\sigma_\mu-g_{\mu\nu}g^{\rho\sigma}\right), \tag{3.47}$$

where the derivative of the Ricci tensor is taken by respecting its symmetry, $\frac{\partial R_{\mu\nu}}{\partial R_{\rho\sigma}}=\frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu+\delta^\rho_\nu\delta^\sigma_\mu\right)$.

Locally the BTZ solution is AdS with the curvature values

$$R_{\mu\rho\nu\sigma}=\Lambda\left(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\rho\nu}\right),\qquad R_{\rho\sigma}=2\Lambda g_{\rho\sigma},\qquad R=6\Lambda, \tag{3.48}$$

and from here one can calculate the following relevant quantities

$$\bar{\mathsf{g}}+\frac{1}{m^2}\bar{\mathsf{G}}\ =\ \left(1-\frac{\sigma\Lambda}{m^2}\right)\bar{\mathsf{g}}, \tag{3.49}$$

$$f(R_{\mu\nu})_{\bar{R}}\ =\ \sqrt{-\bar{g}}(1-\frac{\sigma\Lambda}{m^2})^{3/2}, \tag{3.50}$$

which leads to [6]

$$\left(\frac{\partial L}{\partial R_{\rho\sigma}}\right)_{\bar{R}}=\frac{\sigma}{16\pi G_3}\sqrt{1-\frac{\sigma\Lambda}{m^2}}\,\bar{g}^{\rho\sigma}. \tag{3.51}$$

Since the binormal vectors are the same as before, one arrives at

$$S_W=\frac{A_H}{4}\frac{\sigma}{G_3}\sqrt{1-\frac{\sigma\Lambda}{m^2}}, \tag{3.52}$$

which is positive for $\sigma=+1$. This result matches [72]. To put this result into the Bekenstein-Hawking form, let us compute the effective gravitational constant which reads

$$\frac{1}{\kappa_{\text{eff}}}=\frac{1}{G_3}\frac{\bar{R}_{\mu\nu}}{\bar{R}}\left(\frac{\partial\mathcal{L}}{\partial R_{\mu\nu}}\right)_{\bar{R}}=\frac{\sigma}{G_3}\sqrt{1-\frac{\sigma\Lambda}{m^2}}, \tag{3.53}$$

leading to the relation $S_W=\frac{A_H}{4\kappa_{\text{eff}}}$. Since $\Lambda<0$, entropy is again increased due to the higher curvature terms as in the generic Born-Infeld gravity.

As it is noted in [38], there is another theory that can be considered as an extension of the new massive gravity whose action reads

$$I=-\frac{m^2}{4\pi G_3}\int d^3x\left(\sqrt{-\det\left[g_{\mu\nu}+\frac{\sigma}{m^2}\left(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\right)\right]}-\left(1-\frac{\lambda_0}{2}\right)\sqrt{-g}\right). \tag{3.54}$$

[6] Note that $\sqrt{-g}$ appearing in $f(R_{\mu\nu})$ is promoted to integral density as was done in (3.11).

By carrying out the analogous computations for this action one arrives at the Wald entropy of the BTZ solution as

$$S_W = -\frac{A_H}{4}\frac{\sigma}{G_3}\sqrt{1+\frac{\sigma\Lambda}{m^2}}, \tag{3.55}$$

where this time entropy is positive for $\sigma = -1$ and the relation $S_W = \frac{A_H}{4\kappa_{\text{eff}}}$ is again satisfied with effective gravitational constant calculated as $\frac{1}{\kappa_{\text{eff}}} = -\frac{\sigma}{G_3}\sqrt{1+\frac{\sigma\Lambda}{m^2}}$.

CHAPTER 4

CONCLUSION

We have studied the Wald entropy for Born-Infeld type gravity theories which were constructed to carry the vacuum and excitation properties of Einstein's gravity. In other words, these BI theories have a unique maximally symmetric solution (despite the many powers of curvature) and a massless spin-2 graviton about the vacuum for $n \geq 3+1$ dimensions (and a massive graviton for $n = 2+1$ dimensions). Wald entropy (dynamical entropy coming from diffeomorphism invariance) usually differs from the Bekenstein-Hawking entropy for generic gravity theories. But here we have shown that these two notions of entropy coincide for BI gravity theories in generic dimensions with the slight modification that instead of the Newton's constant of Einstein's theory, the effective gravitational coupling appears. Basically the following expression is valid:

$$S_W = S_{BH} = \frac{A_H}{4\kappa_{\text{eff}}}, \tag{4.1}$$

where κ_{eff} encodes all the information about the underlying theory as far as its de Sitter entropy is concerned. Hence, besides having similar properties at the vacuum and linearized levels, these theories share the same entropy properties for their de Sitter solutions. This is a rather interesting result and one might conjecture that it might be a property of gravity theories that have only massless gravity in their spectrum. (The $3-$dimensional case is somewhat different as there is no massless bulk graviton and one necessarily considers massive gravity). To be able to understand this potentially deep connection between various notions of entropy and the particle content of the underlying gravity theory, further solutions of BI gravity theories must be constructed.

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APPENDIX A

CLASSICAL THERMODYNAMICS

A.1 The Zeroth Law

[1] It shows that the thermal equilibrium has a transitive nature and provides a non-mechanical state function called emprical temperature. One can express it as follows:

If two systems are separately in equilibrium with another system, then these two systems are also in equilibrium with each other.

When systems are in equilibrium, we can start measuring their properties. If two systems are in equilbrium with each other, then this brings a constraint on their coordinates such that they cannot change independently. Assume a space spanned by coordinates of two systems A and C. If we put a mark on the equilibrium points, we obtain a surface

$$f_{AC}(A_1, ...; C_1, ...) = 0. \tag{A.1}$$

Similar argument can be done for systems B and C. In both equations, we can write say C_1 in terms of others which leads us to

$$F_{AC}(A_1, ...; C_2, ...) = F_{BC}(B_1, ...; C_2, ...). \tag{A.2}$$

We also have another constraint coming from the equilibrium between A and B, that is

$$f_{AB}(A_1, ...; B_1, ...) = 0. \tag{A.3}$$

Any parameters of A and B satisfying the equation (A.3) will satisfy the equation (A.2) which implies that the result is independent of the choice of coordinates C.

[1] This chapter is based on [45]

Therefore cancelling the coordinates C leads us

$$\Theta_A(A_1, A_2, ...) = \Theta_B(B_1, B_2, ...), \tag{A.4}$$

where a state function Θ stands as an emprical temperature. We should note that the zeroth law only shows the existence of isotherms that are the curves along which the temperature does not change.

We should stress that there is an analogy between the force in mechanics and temperature in Thermodynamics. They are the indicators of equilibrium and we use this analogy to construct the form of heat later.

To define a temperature scale, isotherms are not enough; we need a reference system and an ideal gas provides this. According to observations,

$$\lim_{P\to 0} PV = \text{constant}, \tag{A.5}$$

along the isotherm of any gas that is sufficently dilute. By using the triple point of water-ice-gas system, one can obtain the temperature of a system by using this ideal gas relation.

A.2 The First Law

When the system is adiabatically isolated, *i.e* the system undergoes a mechanical work only; the work required to transfrom the state of a sytem depends only on the initial and final states, not the way we perform the work or the intermediate states the system experiences. As an equation, one has

$$\Delta E = \Delta W = E(\mathbf{x}_f) - E(\mathbf{x}_i). \tag{A.6}$$

From the mechanical perspective, for a particle that moves in a potential, we can find the required work by using the change in the potential energy. Similar approach can be observed in thermodynamics in which we construct another state function the internal energy to describe transformations that the system is exposed to. If we remove the adiabatic constraint and let the system allow heat exchanges, the equation (A.6) does not hold anymore. Therefore for general transfromations, we have

$$\Delta E = \Delta W + \Delta Q, \tag{A.7}$$

where $\Delta E, \Delta Q$ and ΔW stands for the change in the internal energy, the heat supplied to the system and the work done on the system respectively. It is important to note that neither ΔW nor ΔQ are state functions. They depend on the path that means how the work is performed or how heat is supplied to the system. For an infinitesimal transformations, one can write [2]

$$dE = đQ + đW. \tag{A.8}$$

One can perform a transformation sufficiently slowly so that the system always remains in equilibrium. These transformations are known as quasistatic transformations. In such processes, one can determine the coordinates of the system at each stage and can compute them in principle. Therefore for an infinitesimal quasistatic transformation; one can define the mechanical work as

$$đW = \sum_i J_i \, dx_i. \tag{A.9}$$

The displacements are extensive quantities which are proportional to the size of the systems whereas forces are independent of the size of the system and are intensive quantities. One can define a gas through mechanical coordinates such as pressure and volume that are intensive and extensive quantities respectively.

We also mentioned that the forces are signs of the equilibrium. Consider a piston separating two systems such that it does not move anymore. This shows that the pressure of each sides are equal to each other. Therefore we can consider pressure as a generalized force. By using the same argument, we can conclude that temperature acts as a generalized force and we want to find out the corresponding displacement. As we will see, the second law defines another state function which acts as a generalized displaecment corresponds to the temperature. That state function is the entropy.

Let us give another useful property of an ideal gas before concluding the first law. As we mentioned earlier, for any sufficiently dilute gas, along an isotherm (A.5) holds. According to an experiment known as Joule's free expansion, if we change the state of a gas adiabatically (but not necessarily quasistatically) from an inital volume to a final volume, its temperature does not change. Since the process is adiabatic and no external work is done on the system $\Delta W = \Delta Q = 0$ therefore its internal energy

[2] $đQ$ is a one-form, but Q is not a function.

remains the same. From the ideal gas law, we know the product PV is constant along the isotherm and is proportional to the temperature. During the experiment both the pressure and volume change but the temperature remains the same. This leads us the result that the internal energy of an ideal gas depends only on its temperature, *i.e* $E(P, V) = E(V, T) = E(T)$.

Heat capacity is a thermodynamic quantity that is a change in the temperature due to heat change in the system that can be defined as

$$C = \frac{đQ}{dT}. \tag{A.10}$$

Since heat is not a function of state but depends on the path, neither is heat capacity a state function. We can add heat at least two ways to the system; by keeping its volume or pressure constant that leads us to C_V or C_P respectively. By using the result of an Joule's experiment, one can find out the result

$$C_P - C_V = N, \tag{A.11}$$

where N is the number of particles in the ideal gas.

A.3 The Second Law

In the 19th century during the industrial revolution, the need of the engines to do work entailed people to understand the mechanism of the conversion of heat to work. Therefore the advent of heat engines led to dramatic developments in thermodynamcis.

One can define an idealized heat engine as an engine which takes heat Q_H from a hot source, converts some of its portion to work W, then releases the waste Q_C to a sink at lower temperature. The efficiency η of the heat engine is defined as

$$\eta = \frac{W}{Q_H} \leq 1, \tag{A.12}$$

by conservation of energy.

An idealized refrigerator is an engine that runs backwards, which extracts heat Q_C from a cold reservoir by using work W and releasing the heat Q_H at a higher temperature. The performance of a refrigerator is defined as

$$\omega = \frac{Q_C}{W}. \tag{A.13}$$

The basis of the second law is that the heat flows naturally from a hotter body to colder one. There are a number of different versions of the second law, let us note the most famous versions as follows:

Kelvin's statement : It is not possible to convert heat to work without any dumping to the environment.

Clausius's statement : It is not possible to transfer heat from a colder to hotter body without any external work.

One can easily prove that these statements are equivalent and infer that there is neither an ideal refrigerator nor an ideal heat engine. This brings us another important notion in thermodynamics.

A.3.1 Carnot Engine

A Carnot engine is any reversible engine that runs in a cycle and operates between only two temperatures. Reversibility in time can be achieved by changing the direction of inputs and outputs. Frictionless motion can be considered its mechanical analog. We should note that, since time reversion implies equilibrium, a reversible process should be quasistatic also, however reverse is not true in general. By running in a cycle we mean that, at the end of the process, the system returns to its initial state.

Operating between two temperatures is the characteristic of the Carnot engine. To understand what that means, let us discuss a Carnot engine that uses monatomic ideal gas as a substance. By using the ideal gas law, we can select two isotherms that are at tempretaures T_H and T_C. We demand that our Carnot engine operates between these temperatures and we have two adiabatic paths to complete the cycle. One can easily calculate that along adiabatic curves

$$PV^{\gamma} = \text{constant}, \tag{A.14}$$

where $\gamma = 5/3$ for this case. We should note that as understood from the above equation, adiabatic curves and isotherms are two different curves.

Carnot Theorem: Within all the engines operating between two tempetarures T_H and T_C, the Carnot engine is the most efficient one.

The immediate proof comes from the fact that heat flows naturally from a hotter to a colder body and by the equation (A.12).

Corollary: All the Carnot engines operating between T_H and T_C have the same efficiency $\eta(T_H, T_C)$.

What we obtain is remarkable. Remember that earlier we used an ideal gas both as a substance in Carnot engine and to construct a temperature scale. But now, we find that efficiency of all Carnot engines running between the same temperatures is the same. Efficiency depends only on the temperatures, therefore we can use it to construct a temperature scale named thermodynamic temperature scale. But first, one has to determine the form of the efficieny in terms of T_H and T_C. Consider two Carnot engines one of which runs between temperatures T_1 and T_2 and another one runs between temperatures T_2 and T_3 where $T_1 > T_2 > T_3$. By using (A.12), one can calculate Q_3 in terms of the efficiencies $\eta(T_1, T_2)$ and $\eta(T_2, T_3)$. One can also write Q_3 in terms of $\eta(T_1, T_3)$ which is the efficiency of the composite function. Equating these two terms gives us

$$1 - \eta(T_1, T_2) = \frac{1 - \eta(T_1, T_3)}{1 - \eta(T_2, T_3)} = \frac{f(T_2)}{f(T_1)} \equiv \frac{T_2}{T_1}, \tag{A.15}$$

where we wrote the last term by convention. Therefore in terms of T_H and T_C we have

$$1 - \eta(T_H, T_C) = \frac{Q_C}{Q_H} = \frac{T_C}{T_H}. \tag{A.16}$$

We should note that although we have two temperatures Θ and T, by running Carnot engine between the emprical temperatures, one can easily see that they are the same quantity.

Now we are ready to define another and a very important state function which serves as a corresponding displacement for temperature. As we will see, entropy is a direct conclusion of Clausius's Theorem.

A.3.2 Entropy

Clausius's Theorem: For any cyclic process whether reversible or not, $\oint \frac{đQ}{T} \leq 0$, where $đQ$ is the heat increment delivered to the system at temperature T.

To understand this, let us divide the cycle into many transformations. We also have an external bath at temperature T_0 such that at each stage, the Carnot engine takes $đQ_R(s)$ from that bath and delivers $đQ_{(s)}$ to the system at temperature $T_{(s)}$. We should emphasize that our only requirement is that we have a cyclic process, we even do not require being in equilibrium. Then there is an ambiguity in the temperature $T_{(s)}$ we have. If we are not even in equilibrium, how can we define a state function that is valid only in equilibrium? We can consider this temperature as the temperature of different engines that deliver heat to the system. Another answer is that for each cycle, heat can be delivered through different processes. So one can consider $T_{(s)}$ as the temperature of different transformations. At each stage, the Carnot engine delivers heat $đQ_{(s)}$ to the system at temperature $T_{(s)}$ from a heat reservoir at fixed temperature T_0. Assume that the Carnot engine extracts heat $đQ^R_{(s)}$ at each stage. Since it is a Carnot engine, sometimes it delivers heat to the system or extracts heat, hence the sign of $đQ_{(s)}$ is not specified. At the end of the cycle, *i.e* when the sytem returns to its original state, the net extracted heat from the source is

$$\oint đQ^R_{(s)} = T_0 \oint \frac{đQ_{(s)}}{T_{(s)}}, \tag{A.17}$$

where we used (A.16) for the RHS of the equation. By Kelvin's statement, it is not possible to convert heat to work completely, but the reverse is true so we have

$$\oint đQ^R_{(s)} = T_0 \oint \frac{đQ_{(s)}}{T_{(s)}} \leq 0, \tag{A.18}$$

and since T_0 is positive, one can finally find

$$\oint \frac{đQ_{(s)}}{T_{(s)}} \leq 0. \tag{A.19}$$

Let us understand what we obtain by considering the following cases:

1. Consider a reversible transformation where the equation (A.19) holds for both directions which leads us to

$$\oint \frac{đQ_{(s)}}{T_{(s)}} = 0. \tag{A.20}$$

2. Consider a reversible transformation from points A to B and return back to A again. Then we can write (A.20) as

$$\int_A^B \frac{đQ_{(1)}}{T_{(1)}} + \int_B^A \frac{đQ_{(2)}}{T_{(2)}} = 0. \tag{A.21}$$

As one can immediately realize that the integral does not depend on the path but on the initial and final states only. Hence the second law of thermodynamics defines a new state function called the *entropy* denoted by S. Therefore for a reversible process

$$\int_A^B \frac{đQ_{(1)}}{T_{(1)}} = S(B) - S(A). \tag{A.22}$$

3. Consider an irreversible process from points A to B and use a reversible process form B to A to complete the cycle. Then by using (A.19) and (A.22) , one can write

$$\int_A^B \frac{đQ_{(irr)}}{T} \leq S(B) - S(A). \tag{A.23}$$

4. For any irreversible transformation, (A.23) can be writen in differential form as

$$đQ \leq TdS, \tag{A.24}$$

and equality holds for reversible processes.

To understand what the equation (A.24) tells, consider a box which is adiabatically isolated, clamped with a piston and in equilibrium initially. If we remove the clamp and let the system evolve to another equilibrium state, we observe that the adiabatic condition leads us $\delta S \geq 0$. It means that an aidabatic system has maximum entropy in equilibrium.

5. Finally, we can write the most generic formula of the thermodynamics as

$$dE = đQ + đW = TdS + \sum J_i dx_i. \tag{A.25}$$

The above equation also tells us that if there are i different ways to perform work to a system and one way to add heat, then we need $(i+1)$ coordinates to describe a thermodynamic system.

As we will discuss, entropy is not always the most appropriate function. Therefore for different situations, we can define different thermodynamic potentials.

A.4 Thermodynamic Potentials and Path to Equilibrium

Enthalpy: Consider a spring and a mass attached to it in an adiabatically isolated box. The spring will be extended due to a constant force on it. Then one can write,

$$đQ = 0, \tag{A.26}$$

$$đW \leq J_i dx_i, \tag{A.27}$$

the second line is an inequaility since there is always a loss due to friction. Remembering that the generalized forces J_i are constant, equations A.26 and A.27 lead to

$$dE \leq J_i dx_i \implies dH = d(E - J_i x_i) \leq 0, \tag{A.28}$$

where H is known as the enthalpy. This implies that the system evolves to a state minimizing the enthalpy. It is obvious that H is a function of state. In equilibrium, one can write

$$dH = dE - d(J_i x_i) = TdS - x_i dJ_i, \tag{A.29}$$

hence natural coordinates for enthalpy are $(S, \mathbf{J})$. Therefore one can calculate the equilibrium point just from

$$x_{eq} = -\left.\frac{\partial H}{\partial J_{eq}}\right|_S, \tag{A.30}$$

at constant S.

Helmholtz Free Energy: Consider an isothermal process where the temperature remains the same and there is no work done to the system, so one can write

$$đQ \leq TdS \quad \text{and} \quad đW = 0, \tag{A.31}$$

$$\implies dF = d(E - TS) \leq 0, \tag{A.32}$$

where F is known as Helmholtz free energy. It is obvious that F is a function of state hence in equilibrium, we have

$$dF = dE - d(J_i x_i) = -SdT + J_i dx_i. \tag{A.33}$$

It shows that natural coordinates for Helmholtz free energy are $(T, \mathbf{x})$. One can calculate the equilibrium forces and entropyn as

$$J_i = \left.\frac{\partial F}{\partial x_i}\right|_S \quad \text{and} \quad S = -\left.\frac{\partial F}{\partial T}\right|_{\mathbf{x}}. \tag{A.34}$$

Gibbs Free Energy: It is another potential that is useful for isothermal processes containing external constant force. Then one can immediately write

$$đQ \leq TdS \quad \text{and} \quad đW \leq J_i dx_i, \tag{A.35}$$

$$\implies dG = d(E - TS - \mathbf{J.x}) \leq 0, \tag{A.36}$$

where G is known as Gibbs free energy. It is obvious that G is a function of state hence in equilibrium, one can write

$$dG = dE - d(TS) - d(J_i x_i) = -SdT - x_i dJ_i, \tag{A.37}$$

that shows the natural coordinates for Gibbs free energy are $(T, \mathbf{J})$.

So far, we have assumed that the number of particles is constant. But in chemical reactions for example, the number of particles will change and this leads to a change in internal energy. One can define this change in terms of chemical work

$$đW_{chem} = \mu_i dN_i, \tag{A.38}$$

where N and μ indicates the different number of particles of each substances with corresponding chemical potentials respectively.

In a situaiton where the temperature of the system is fixed, if there is no external work done to the system but there is a chemical work with constant chemical potential, one can write

$$đQ \leq TdS \quad \text{and} \quad đW_{chem} \leq \mu_i dN_i, \tag{A.39}$$

$$\implies d\mathscr{G} = d(E - TS - \mu_i N_i) \leq 0, \tag{A.40}$$

where $\mathscr{G}$ is known as grand potential. It is obvious that $\mathscr{G}$ is minimized in equilibrium and its variation can be written as

$$d\mathscr{G} = -SdT - J_i dx_i - N_i d\mu_i, \tag{A.41}$$

hence natural coordinates for grand potential are $(T, \mathbf{x}, \boldsymbol{\mu})$.

Before stating the third law, let us introduce a very important mathematical result concerning the above expressions.

Extensivity: As we already stated, extensive quantities are proportional to the size of the system. Consider we write the energy as $E = E(S, \mathbf{x}, \mathbf{N})$ and increase the coordinates $(S, \mathbf{x}, \mathbf{N})$ by factor λ, we obtain

$$E(\lambda S, \lambda \mathbf{x}, \lambda \mathbf{N}) = \lambda E(S, \mathbf{x}, \mathbf{N}). \tag{A.42}$$

From the equations (A.24) and (A.38), one can also write

$$dE = TdS + \mathbf{J} \cdot d\mathbf{x} + \mu \cdot d\mathbf{N}. \tag{A.43}$$

If we take derivative of (A.42) with respect to λ at $\lambda = 1$, compare the results with the equation (A.43), one can read

$$E = TS + \mathbf{J} \cdot \mathbf{x} + \mu \cdot \mathbf{N}. \tag{A.44}$$

Its derivative with the equation (A.43), we obtain the Gibbs-Duhem relation

$$SdT + \mathbf{x} \cdot d\mathbf{J} + \mathbf{N} \cdot d\mu = 0, \tag{A.45}$$

that shows that we cannot describe the given system by using intensive quantities because they are related as seen from above.

A.5 The Third Law

In experiments that are held at low temperatures, it is observed that change in entropy approaches to zero. This result is independent from what we introduced so far. We can state the third law as follows:

At absolute zero, entropy of all systems is zero.

Mathematically, we can write it as

$$\lim_{T \to 0} S(X, T) = 0. \tag{A.46}$$

where X stands for all choice of coordinates. By using this law, one can obtain that $\lim_{T \to 0} C_X(T) = 0$. From this law, we also have that it is not possible to reach absolute zero by cooling any system in a finite number steps.

APPENDIX B

SYMMETRIES, KILLING VECTORS AND CONSERVED QUANTITIES

In General Relativity, we want to determine the geometry outside a gravitating object and to do so, we need to solve the Einstein's equations. However this is not easy because of the nonlinear behavior of the theory. By nonlinearity we mean that the theory, therefore the equations do not obey the superposition principle. If we have two masses m_1 and m_2, then the field created by m_1 and m_2 is not equal to the field created by $m_1 + m_2$. As seen, things are quite different than a linear theory say electrodynamics. Therefore to proceed in GR, we need some approximations or better the concept of symmetry. In this chapter we want to explain what we mean by symmetry [7, 73, 74].

A symmetry can be assigned to a manifold if its geometry does not change under transfromations maping the manifold to itself. We are interested in the symmetries of the metric that are known as isometries. Sometimes isometries can be detected easily as in the case of four dimensional Minkowski spacetime where the metric is invariant under translations $x^\mu \to x^\mu + a^\mu$ where a^μ is fixed and under Lorentz transformations $x^\mu \to \Lambda^\mu_\nu x^\nu$ where Λ^μ_ν describes Lorentz transformations.

Let us consider spacetimes with generic metric $g_{\mu\nu}$ and study their isometries. Consider an action as follows:

$$S = -\, m \int d\tau, \tag{B.1}$$

$$= -m \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda, \tag{B.2}$$

$$= \int L d\lambda, \tag{B.3}$$

where $L = -m\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$ and $\dot{x}^\mu := \dfrac{dx^\mu}{d\lambda}$ where λ is an affine parameter.

If we use the Euler-Lagrange equations,

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^{\mu}} - \frac{\partial L}{\partial x^{\mu}} = 0, \tag{B.4}$$

we can immediately observe that if the metric is independent of a fixed coordinate then the momentum in this direction is conserved [46]. Let us summarize what we have found so far:

$$\partial_{\gamma} g_{\mu\nu} = 0 \implies x^{\gamma} \to x^{\gamma} + a^{\gamma}\, is\, a\, symmetry, \tag{B.5}$$

$$\partial_{\gamma} g_{\mu\nu} = 0 \implies \frac{dp_{\gamma}}{d\tau} = 0 \;\textit{i.e}\; p_{\gamma}\; is\, conserved. \tag{B.6}$$

One can get these results by writing the geodesic equation in terms of the momentum as $p^{\lambda}\nabla_{\lambda}p^{\mu} = 0$. Applying the covariant derivative, after performing metric compatibility, using definition of Christoffel connection and performing related contractions lead us to the same result (B.6) [7].

The gravitating objects we are interested in have some symmetry. The Schwarzchild metric which describes the geometry outside the spherical symmetric body like stars or planets as well as the outside region of the Schwarzchild black hole or the Reissner-Nordström black hole which are charged have static metrics. For the case of the Kerr black hole that is rotating, the metric is stationary instead of being static. The solutions of vacuum Einstein's equations should obey the symmetries of the sources [7]. One can read these symmetries s follows:

1. Static metric imposes two conditions. The first one is that the metric components do not depend on the time component. The second one is that the metric is invariant under the time reversal symmetry $t \to -t$. This immediately shows that there will be no cross terms like $dtdx^{i}$ in the metric.

2. If an object is spherically symmetric, then it has the symmetries of the sphere S^{2} that is given by the group $SO(3)$. Symmetries are related with Killing vectors and for the sphere they are

$$R = \partial_{\phi}, \tag{B.7}$$

$$S = \cos\phi\, \partial_{\theta} - \cot\theta\, \sin\phi\, \partial_{\phi}, \tag{B.8}$$

$$T = -\sin\phi\, \partial_{\theta} - \cot\theta\, \cos\phi\, \partial_{\phi}. \tag{B.9}$$

One can verify that these vectors form an algebra that means we have commutation relations $[R,S]=T,[S,T]=R,[T,R]=S$. A manifold has a spherical symmetry if and only if it has the Killing vectors of the sphere R,S,T that satisfies the algebra.

3. A vacuum metric with spherical symmetry has a timelike Killing vector. As the Birkhoff theorem suggests, a spherically symmetric metric does not depend on time even if we start with a time dependent metric. This concludes that we have a timelike Killing vector.

4. If a metric has a timelike Killing vector as $r\to\infty$, then it is called stationary.

As we are going to see, these symmetries help us to determine the particle's motion around the black holes [7].

B.1 Killing Vectors

We are interested in the symmetries of the metric that are quite related with the Killing vectors. If the Lie derivative of the metric with respect to a vector K is zero, then K is called a Killing vector. Mathematically we can write it as

$$\mathcal{L}_K g_{\mu\nu} = K^\alpha \nabla_\alpha g_{\mu\nu} + g_{\nu\beta}\nabla_\mu K^\beta + g_{\mu\beta}\nabla_\nu K^\beta, \tag{B.10}$$

$$= \nabla_{(\mu} K_{\nu)} = 0, \tag{B.11}$$

where the last equation is the Killing equation. We know that the Minkowski spacetime has 10 symmetries that are four spacetime translations, three boosts and three spacetime rotations that we already discussed. Now let us study the isometries of spacetime with the general metric $g_{\mu\nu}$ [7]. If $g_{\mu\nu}$ is independent of a coordinate x^γ, then we can label the vector ∂_γ as K:

$$K = \partial_\gamma, \tag{B.12}$$

and we can write it in component form as

$$K^\mu = (\partial_\gamma)^\mu = \delta^\mu_\gamma. \tag{B.13}$$

By using this isometry in the direction of K^μ, we can write the four-momentum in direction γ in terms of

$$p_\gamma = \delta^\mu_\gamma p_\mu = K^\mu p_\mu. \tag{B.14}$$

From Euler-Lagrange equations we know that the four-momentum p_γ is conserved. Therefore along the geodesic we can write it as

$$\frac{dp_\gamma}{d\tau} = 0 \quad \Longrightarrow \quad \frac{dx^\mu}{d\tau}\nabla_\mu p_\gamma = p^\mu \nabla_\mu (p^\nu K_\nu), \tag{B.15}$$

$$\Longrightarrow \quad p^\mu \nabla_\mu (p^\nu K_\nu) = 0. \tag{B.16}$$

Let us examine the term (B.16) in detail. We can expand it as

$$p^\mu \nabla_\mu (p^\nu K_\nu) = p^\mu p^\nu \nabla_\mu K_\nu + p^\mu K^\nu \nabla_\mu p_\nu, \tag{B.17}$$

$$= p^\mu p^\nu \nabla_{(\mu} K_{\nu)}, \tag{B.18}$$

$$= 0, \tag{B.19}$$

where the second term in RHS of the equation (B.17) vanishes since it is a geodesic equation and we write the term (B.18) by contracting the symmetric part of $\nabla_\mu K_\nu$ with $p^\mu p^\nu$. As long as the K^μ is a Killing vector that satisfies the Killing equation (B.11), we have a conserved quantity $K^\mu p_\mu$ along the geodesic. What we obtain is quite important and will be very useful to study geodesics of black holes. We obtain that if metric is independent of a coordinate x^γ, then we have a Killing vector $K = \partial_\gamma$ such that we have a conserved quantity $K^\mu p_\mu$ along the geodesic. As we are going to see that a timelike Killing vector leads a conserved energy whereas a spacelike Killing vector leads a conserved angular momentum [7].

B.2 Conserved Quantities

We started with the symmetries of the solutions and defined their properties. Then we studied that symmetries are related with Killing vectors of the manifold and we obtained that if we have a Killing vector then we will have a conserved quantity along the geodesic. If we have a metric with components that are independent of time, then we have a timelike Killing vector $K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$ with conserved energy $E = -K^\mu p_\mu$ [7]. Now, consider the Killing vectors R, S, T that can be considered as the three components of the angular momentum. If the angular momentum is conserved then we can consider the motion takes place in plane or without loss of generality we can consider the plane as the equatorial plane that is $\theta = \pi/2$. Since direction is determined by two Killing vectors, we are left with $R^\mu = (0, 0, 0, 1)$ to determine the magnitude of the angular momentum.

Therefore we have one timelike Killing vector K^μ that leads us to conserved energy and one spacelike Killing vector R^μ that leads us to conserved angular momentum [7]. Let us calculate these conserved quantities that can be written in the form

$$E = -K_t \frac{dt}{d\lambda} \quad \text{and} \quad L = R_\phi \frac{d\phi}{d\lambda}, \tag{B.20}$$

where λ is an affine parameter and the quantities E and L are energy and angular momentum of the particle per unit mass respectively. Let us calculate these quantities for the generic metric (2.10). To do so, we should find K_μ and R_μ that take the form

$$K_\mu = \big(-f(r), 0, 0, 0\big) \quad \text{and} \quad R_\mu = \big(0, 0, 0, r^2\big), \tag{B.21}$$

which lead us to the conserved quantities as

$$E = f(r)\,\dot{t} \quad \text{and} \quad L = r^2\dot{\phi}, \tag{B.22}$$

where the dot represents the derivative with respect of the affine parameter λ. If we apply these results to the Schwarzschild solution where $f(r) = 1 - \frac{2GM}{r}$, we can write the result as

$$K_\mu = \bigg(-\big(1 - \frac{2GM}{r}\big), 0, 0, 0\bigg) \quad \text{and} \quad R_\mu = \bigg(0, 0, 0, r^2\bigg), \tag{B.23}$$

therefore along the geodesic we have the following conserved quantities as

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = \bigg(1 - \frac{2GM}{r}\bigg)\dot{t}, \tag{B.24}$$

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2\dot{\phi}. \tag{B.25}$$

We should mention that there is also another form of energy that can be measured by an observer as $E = -p_\mu U^\mu_{obs}$. This and the energy in (B.22) are completely different energies, they are not even proportional. This is due to the four-velocity is normalized but the Killing vector K^μ is not. We can define these two energies such that $E = -p_\mu U^\mu_{obs}$ is kinetic energy of the particle whereas $E = -K_\mu\, p^\mu$ is the total conserved energy that also contains the gravitational potential energy. Along the geodesic, the energy (B.22) is conserved. The other form of the energy is used for massless particles where we can consider it as the photon's observed frequency. One can use this to study the gravitational redshift of the photon [7].

APPENDIX C

MOTION AROUND BLACK HOLES

A black hole is a place in the spacetime with strong gravity such that nothing even light can escape from there. The event horizon is considered as a boundary which separates the spacetimes which are causally connected and which are not.

Although general relativity is necessary to understand their physcis properly, a black hole-like objects known as dark stars can be encountered and studied by using Newton's gravity [46]. As we know, a particle which is under the influence of a gravitational field can escape this field and it is an issue of velocity. One can easily determine this escape velocity by using the conservation of energy.

Now consider a visible star whose light reaches to us by escaping its surface. At the late 1700's, Michell and Laplace independently found that a star with the density of the Sun that has $M \approxeq 500M_S$ would be a dark star. From the conservation of energy, one can easily determine the criterion for a star to be invisible as [46, 75]

$$r = \frac{2GM}{c^2}, \tag{C.1}$$

which corresponds to the radius of the Schwarzchild black hole. We should note that one cannot infer neither the dark stars and the black holes are the same nor the Newtonian gravity is sufficient to understand black holes because of the followings [46, 75] :

1. The speed of light in Newtonian gravity is arbitrary. As seen from (C.1), if the light travels away from the surface of the star, its speed decreases. But this is not the case in Special Relativity or General Relativity.

2. Michell and Laplace assumed that the star was dark for us but actually it was

shining there. The reason we could not see is that, even though the light escaped from the surface of the star, it would go back to the surface because of the strong gravity. But in General Relativity, even for a temporary escape from the surface is not possible.

Before diving to the particle's motion, let us briefly discuss the particle action in both flat and curved spacetimes. Then we will derive the particle's motion for a generic metric, and study these results for the Scwarzschild metric in detail. These will help us understand the physics of black holes and important notions we described in black hole thermodynamcis.

C.1 Particle Action

C.1.1 Flat Spacetime Case

We can describe the particle's coordinates as $x^\mu = (t, \vec{x})$. The relativistically invariant line element can be written as

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{C.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. We can classify the line element as follows:

$$ds^2 \begin{cases} < 0 \text{ timelike} \\ = 0 \text{ null} \\ > 0 \text{ spacelike} \end{cases}$$

Massive particles travel along timelike trajectories and read their proper time as

$$ds^2 = -d\tau^2. \tag{C.3}$$

We can use it to define particle's action

$$S = -m \int d\tau. \tag{C.4}$$

It is easy to see that in the non-relativistic limit, action takes the classical form

$$S = -m \int d\tau \approx -m\,(1 - \frac{v^2}{2})\,dt, \tag{C.5}$$

where the first and the second terms represent the rest mass and the usual kinetic energy respectively [46].

The trajectory of a particle in spacetime is called a world-line. For any arbitrary parameter λ along the world-line, one can determine particle's motion by $x^\mu(\lambda)$. By using this parametrization with equations (C.2) and (C.4) , one can write

$$S = -m \int d\lambda \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{C.6}$$

$$= -m \int \mathcal{L} d\lambda, \tag{C.7}$$

where $\dot{x}^\mu = \dfrac{dx^\mu}{d\lambda}$. For timelike paths λ is an affine parameter. Following the definition, one can easily obtain the canonical momentum as

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^2}}, \tag{C.8}$$

$$= m\frac{dx_\mu}{d\tau}. \tag{C.9}$$

As it is easily seen from the above result, $p^2 = -m^2$ which means that its components are not independent, hence it constraints the particle's motion.

One can define the particle's four velocity as

$$u^\mu := \frac{dx^\mu}{d\tau}. \tag{C.10}$$

It is easy to see that $u^2 = -1$. One should note that the action in (C.4) and the four velocity can be used only for massive particles. We cannot define proper time for massless particles hence cannot use this action and we exclude the spacelike trajectories since they are not physical.

As we already mentioned in the beginning of the section, particles travel along world-lines which extremizes the action. From (C.6) it can be seen that Lagrangian does not depend on x^μ but the derivatives, therefore the corresponding momentum p_μ is constant which describes the free motion [46] .

C.1.2 Curved Spacetime Case

When we move to General Relativity, we replace the Minkowski metric $\eta_{\mu\nu}$ with a curved spacetime metric $g_{\mu\nu}$. One can write the corresponding line element, action and the canonical momentum as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{C.11}$$

$$S = -m \int d\tau = -m \int d\lambda \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}, \tag{C.12}$$

$$p_\mu = m \frac{\dot{x}_\mu}{\sqrt{-\dot{x}^2}}. \tag{C.13}$$

It is easy to capture from the above equations that $p^2 = -m^2$ still remains true and if the metric does not depend on coordinates x^μ then the corresponding momentum p_μ is constant.

In the curved spacetime, particles travel along the world-lines that extremize the action and are called the geodesics. Variation of the action (C.12) with respect to x^μ determines the equation of the motion

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0, \tag{C.14}$$

which is the geodesic equation and

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} (\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}), \tag{C.15}$$

is known as Christoffel connection.

We can determine particle's motion by solving the geodesic equation. But as we are going to discuss in the following sections, instead of solving this equation we will use the conserved quantities to describe the particle's motion around a black hole [46].

C.2 Motion Around the Schwarzchild Black Holes

As we know, one can determine the metric by solving the Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{C.16}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, G and $T_{\mu\nu}$ are the Newton's constant and energy-momentum tensor for the matter fields respectively. One example of such a matter field is

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}, \tag{C.17}$$

where Λ is known as the cosmological constant. With no cosmological constant and no matter field, by applying the trace of (C.16), Einstein's equations can be written in the form

$$R_{\mu\nu} = 0, \tag{C.18}$$

whose solution is the Schwarzschild black hole that is the simplest black hole. The Schwarzschild metric in coordinates (t, r, θ, ϕ) is written as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{C.19}$$

where $d\Omega_2^2$ is the line element on a unit sphere and is written as $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ [46].

Before focusing on the geodesics and motion around Schwarzchild black hole, let us examine the metric (C.19) in detail [7, 46]:

1. The parameter M is the mass of the gravitating body or in our case the mass of the Schwarzschild black hole. One can easily capture this result by studying the metric in weak field limit and comparing the component g_{00} with the Newtonian potential.

2. The metric is asymptotically flat that means as $r \to \infty$ it becomes Minkowskian.

3. The horizon is located at $r = 2GM$ where g^{rr} vanishes.

4. As Birkhoff theorem shows the metric (C.19) is the unique static and spherically symmetric solution of the vacuum Einstein's equation.

5. Any spherically symmetric vacuum metric has a timelike Killing vector. We can understand this as follows. Our metric is independent of coordinate t then the metric is invariant under time translations hence we have a Killing vector K that is written in components $K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$. One can easily see that it is timelike for $r > 2GM$.

C.2.1 Null hypersurfaces, Event and Killing horizons

A hypersurface Σ of a D-dimensional manifold is a $D-1$ dimensional surface which can be defined as a level set of some function

$$\Sigma \equiv f(x) - a = 0, \tag{C.20}$$

where f is a function of coordinates x^μ and a is some constant.

The normal of this hypersurface is given by $n_\mu = \partial_\mu f$ where $\partial_\mu f$ is the gradient of the hypersurface Σ. For a generic hypersurface, one can introduce a normal vector such that $n^\mu = (n^0, n^1, 0, 0)$ with the norm as $n^\mu n_\mu = -(n^0)^2 + (n^1)^2$.

Now consider a vector t^μ that is tangent to the same hypersurface at a point. By using the fact that t^μ and n^μ are orthogonal to each other, we will obtain

$$t^\mu = \sigma(n^1, n^0, a, b), \tag{C.21}$$

where σ, a, b are some functions. The norm of the tangent vector $t^\mu t_\mu = \sigma^2[-n^\mu n_\mu + (a^2 + b^2)]$ with the norm of normal vector results as

1. If $n^\mu n^\mu < 0$ then Σ is a spacelike hypersurface and t^μ can only be spacelike.
2. If $n^\mu n^\mu > 0$ then Σ is a timelike hypersurface and t^μ can be spacelike, timelike or null.
3. If $n^\mu n^\mu = 0$ then Σ is a null hypersurface and t^μ can be null or spacelike.

We can apply these results to the Schwarzschild black hole for the hypersurface $\Sigma \equiv r - 2GM = 0$ with the normal vector $n_r = (0, 1, 0, 0)$. One can easily see that its norm is $n^\mu n_\mu = g^{\mu\nu} n_\mu n_\nu = 1 - \frac{2GM}{r}$.We have the following results:

$$\text{Surfaces with constant r are} \begin{cases} \text{spacelike if } r < 2GM, \\ \text{timelike if } \ r > 2GM, \\ \text{null if } \ r = 2GM. \end{cases}$$

As we have shown, the horizon is a null surface that separates timelike hypersurfaces with those that are spacelike [74, 76, 77].

As we know, $r = 2GM$ is the event horizon for the Schwarzschild black hole and it is a null surface. The event horizon is a boundary between the spacetimes that are causally connected and that are not. Event horizons occur at the point $r = r_H$ and one can find this point by requiring that the normal vector of $\Sigma \equiv r - \text{constant} = 0$ hypersurface will be null on Σ. One can have the normal vector as $n_\mu = (0, 1, 0, 0)$ with the norm

$$n^\mu n_\mu = g^{\mu\nu} n_\mu n_\nu = g^{rr} \tag{C.22}$$

that vanishes on event horizon. We can conclude that the event horizon $r = r_H$ is a hypersurface where g^{rr} changes sign from positive to negative [7].

A null hypersurface Σ is a Killing horizon for a Killing vector χ^μ if χ^μ is null on it. It is easy to capture that χ^μ is also a normal vector of Σ.

Let us examine the relation between the event and the Killing horizons as follows:

1. For static metrics, the Killing vector is $\chi^\mu = K^\mu = (\partial_t)^\mu$

2. For stationary metrics, the Killing vector χ^μ is a combination of the Killing vectors $K^\mu = (\partial_t)^\mu$ and $R^\mu = (\partial_\phi)^\mu$ such that $\chi^\mu = K^\mu + \Omega_H R^\mu$ where Ω_H is the angular velocity of the rotating black hole.

We should note that, for both cases, the event horizon is a Killing horizon of the Killing vectors χ^μ. However for the Kerr black holes, the Killing vector $K^\mu = (\partial_t)^\mu$ is not null on the event horizon but null on the surface called the stationary limit surface. There is a region between the event horizon and the stationary limit surface called the ergoregion. Inside this region since K^μ is spacelike, it is not possible for an object to be stationary even though it is located outside the event horizon. The stationary limit surface is also known as the infinite redshift surface [7].

Killing horizons are important because for every Killing horizon there is a quantity called the surface gravity κ that is the acceleration of the static observer at the horizon which is measured by the observer at infinity. The hypersurface Σ is a Killing horizon for the Killing vector χ^μ. Since it is normal to Σ, it should obey the geodesic equation along the Killing horizon, so we have

$$\chi^\mu \nabla_\mu \chi^\nu = -\kappa \chi^\nu. \tag{C.23}$$

The RHS is there because the geodesic equation may not be affinely parametrized. By using the Killing's equation and the equation $\chi_{[\mu \nabla_\nu \chi_\sigma]} = 0$, one can rewrite the equation (C.23) as

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)|_\Sigma \tag{C.24}$$

where Σ represents the horizon [7] .

We already calculated the surface gravity κ by using the particle motion. For Kerr spacetime, the above formula is useful.

C.2.2 Gravitational Redshift

The gravitational redshift is one of the classical tests of General Relativity. It shows how the photons gain or lose energy depending on whether they fall or rise in a gravitational field. Let us explain how the gravity redshifts the wavelength of the photon. Consider two static observers A and B such that A sends out light with frequency ν_A and the observer B receives it with the frequency ν_B. We demand that observers are static since we want to exclude the effects of Doppler-like shifts [7, 46, 73, 74]. We are sure that both observers agree on the number of pulses they send and receive with respect to their proper times. Mathematically this leads us to

$$\nu_A d\tau_A = \nu_B d\tau_B, \tag{C.25}$$

We know that observers are static hence $dr, d\phi, d\theta$ vanish and the equation $ds^2 = -f(r)dt^2 = -d\tau^2$ leads us to

$$\frac{\nu_A}{\nu_B} = \frac{d\tau_B}{d\tau_A} = \sqrt{\frac{|g_{00}(r_B)|}{|g_{00}(r_A)|}}. \tag{C.26}$$

Since the energy is $E = h\nu$ we can write the above equation in the form of

$$\frac{E_B}{E_A} = \sqrt{\frac{|g_{00}(r_A)|}{|g_{00}(r_B)|}}. \tag{C.27}$$

We should note that the coordinate times of both observers are the same. It can be captured by looking at the radial equation of the photon. Since $ds^2 = g_{00}dt^2 + g_{rr}dr^2 = 0$, we have

$$\int dt = \int \sqrt{\frac{g_{rr}}{-g_{00}}} dr. \tag{C.28}$$

As it is seen, the RHS depends only on coordinate r, therefore the coordinate times of the each observer do not differ [7, 73, 74].

Let us study the result (C.27) for the Schwarzschild metric. Consider the observer A is far from the horizon ($r_A >> 0$) and the observer B is the asymptotic observer (r_B is at near infinity). If we apply these conditions to the equation (C.27), we will obtain

$$E_B = \sqrt{g_{00}(r_A)} E_A, \tag{C.29}$$

since $g_{00}(r_B) = 1$ at infinity. If we expand the equation (C.29) around $r_A >> GM$, we will have $\sqrt{1 - 2GM/r_A} \equiv 1 - GM/r_A$ that is smaller than 1 that concludes the discussion by the result

$$E_B < E_A, \tag{C.30}$$

which shows that the energy of the photon decreases as it travels from a strong gravitational field to weak one. If we choose A to be located at horizon, since $E_B \to 0$ the photon gets infinite redshift [7, 46, 73, 74].

C.2.3 Particle Motion

Consider a generic spherically symmetric metric of the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{C.31}$$

If we want to understand the particle's motion, we should find the geodesics that the particle follows. To do so, we need to solve the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{C.32}$$

To proceed further, we should find the Christoffel connections that are not obvious all the time. Even though we can determine these, it will not be easy to solve the geodesic equation. If the symmetries of the spacetime exist, then we can use them to determine the particle motion instead of solving the equation C.32. In the Appendix (B), we already determined the symmetries of a spherically symmetric, static metric that (C.31) also obeys and obtained the conserved quantities E and L. By using these quantities as well as the normalization of the four-velocity vector $u^2 = -1$ for masssive and $u^2 = 0$ for massless particles, we can have the following equations:

$$\dot{r}^2 = E^2 - f(r) - \frac{L^2 f(r)}{r^2} \quad \text{for massive particles,} \tag{C.33}$$

$$\dot{r}^2 = E^2 - \frac{L^2 f(r)}{r^2} \quad \text{for massless particles,} \tag{C.34}$$

where $\dot{r} = \dfrac{d}{d\lambda}$ where λ is not an affine parameter for the massless case. We should also note that for the massive case, the quantities E and L represent the energy and angular momentum per unit mass.

Let us study this to find the geodesics of the Schwarzschild black hole. For this case, one can write the geodesic equation as

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0, \tag{C.35}$$

$$\begin{aligned}\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)(\frac{dt}{d\lambda})^2 - \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}(\frac{dr}{d\lambda})^2 \\ -(r-2GM)\big[(\frac{d\theta}{d\lambda})^2 + \sin^2\theta(\frac{d\phi}{d\lambda})^2\big] = 0,\end{aligned} \tag{C.36}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \sin\theta\cos\theta(\frac{d\phi}{d\lambda})^2 = 0, \tag{C.37}$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0, \tag{C.38}$$

that are obtained after the proper Christoffel connections [7]. Instead of dealing with these complicated equations, we can use the symmetries and the conserved quantities of the Schwarzschild solution. If one puts $f(r) = 1 - \frac{2GM}{r}$ in the equation (C.33), then the result would be

$$\dot{r}^2 = E^2 - \big(1 - \frac{2GM}{r}\big)\big(1 + \frac{L^2}{r^2}\big) = E^2 - V(r) \quad \text{for massive particles,} \tag{C.39}$$

$$\dot{r}^2 = E^2 - \frac{L^2}{r^2}\big(1 - \frac{2GM}{r}\big) = E^2 - V(r) \quad \text{for massless particles.} \tag{C.40}$$

Also, to determine the orbits of the particle, one should focus on the potential $V(r)$

[7], [74, 76]. If we rewrite the equation (C.39) in the form

$$\dot{r}^2 = E^2 - 1 + \frac{2GM}{r} - \frac{L^2}{r^2} + \frac{2GML^2}{r^2}, \tag{C.41}$$

where $E = 1$ represents the rest mass energy of the particle when it is at rest at infinity, the third term looks like the centrifugal force. Up to now, we are familiar with these terms from Newtonian mechanics, however the last term is new and it is the characteristic of the Genereal Relativity. The explanation of the light bending and the perihelion precession of Mercury follows from this term. The effect of this term becomes important at distances near the horizon where it can be comparable with the centrifugal force term [46].

C.2.4 Radial fall of a massive particle

Consider a particle falling radially into a Schwarzschild black hole. Since it is falling radially, $\dot{\phi} = 0$ therefore its angular momentum L vanishes. If we apply these results to radial equation C.41 and use the equation B.24 we have

$$\frac{dr}{d\tau} = \left(E^2 - 1 + \frac{2GM}{r}\right)^{1/2}, \tag{C.42}$$

$$\frac{dt}{d\tau} = E\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{C.43}$$

It is easy to see that the derivative of the equation C.42 with respect to τ and then take the Newtonian limit $\tau \approx t$ gives

$$\frac{d^2 r}{dt^2} \approx -\frac{GM}{r^2}, \tag{C.44}$$

which is nothing but the Newtonian gravity [46, 73, 74, 77].

Let us finalize this discussion by determining how long it takes for a particle to reach the horizon. Assume that initially the paritcle is at some point r_0 and if we integrate the radial equation (C.42), we will have

$$\int d\tau = -\int_{r_0}^{2GM} \frac{dr}{\left(E^2 - 1 + \frac{2GM}{r}\right)^{1/2}}, \tag{C.45}$$

where minus represents that the particle is infalling. Since the integral is finite, the particle reaches the horizon in a finite proper time. However for an observer at infinity, this is not the case. To find the corresponding time, we should find the expression of $\frac{dr}{dt}$ that can be captured by combining the equations (C.42) and (C.43) that gives,

$$\int dt = \int_{r_0}^{2GM} \frac{E dr}{\big(1 - \frac{2GM}{r}\big)\big(E^2 - 1 + \frac{2GM}{r}\big)^{1/2}}, \tag{C.46}$$

which diverges logarithmically. Hence for the observer at infinity, it takes infinite coordinate time for the particle to reach the horizon [46, 73, 74, 77].

APPENDIX D

LINEAR AND NONLINEAR ELECTRODYNAMICS

D.1 Classical Electrodynamics

[1] Unlike Newton's theory, classical electrodynamics is manifestly relativistic that allows us to extend the theory of electrodynamics to higher dimensions. Maxwell equations determine the dynamics of electromagnetic fields. In the Heaviside-Lorentz unit system which is much more suitable to discuss the relativity, one can write the Maxwell equations as

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{D.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{D.2}$$

$$\vec{\nabla} \cdot \vec{E} = \rho, \tag{D.3}$$

$$\vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t}. \tag{D.4}$$

The last two equations are known as non homogenous Maxwell equations because they involve the source terms where ρ and $\vec{j}$ are the charge and current densities respectively. The first two equations do not contain any sources and they are homogenous Maxwell equations.

A charged particle in any electromagnetic field experiences a force which is given by the Lorentz force law of the form

$$\vec{F} = q\big(\vec{E} + \vec{v} \times \vec{B}\big). \tag{D.5}$$

Let us focus on the homogenous Maxwell equations and see that in fact $\vec{E}$ and $\vec{B}$ fields can be written in terms of scalar and vector potentials. Following the equation

[1] This chapter is based on [30]

(D.2), one can write the magnetic field $\vec{B}$ as

$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{D.6}$$

where $\vec{A}$ is the vector potential. If we apply this result to the equation (D.1) and use the fact that the curl of a divergence is zero, we can write the electric field $\vec{E}$ as

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi, \tag{D.7}$$

where ϕ is a scalar potential.

Although the electric and magnetic fields seem more familiar to us, the potentials $(\phi, \vec{A})$ are more fundamental quantities because as we will see, the Hamiltonian is written by using these potentials, not the fields. Let us end this section by introducing gauge transformations which relate two different sets of potentials $(\phi, \vec{A})$ and $(\phi' \vec{A}')$ through equations:

$$\phi' = \phi - \frac{\partial \epsilon}{\partial t}, \tag{D.8}$$

$$\vec{A}' = \vec{A} + \nabla\epsilon, \tag{D.9}$$

where ϵ is the gauge parameter. It is straightforward to show that $\vec{E}$ and the $\vec{B}$ fields remain unchanged under these transformations which shows that the potentials corresponds to same electromagnetic fields are not unique.

If we can relate two different sets of potentials by gauge transformations and determine the corresponding gauge parameter, then they are physically equivalent and give rise to the identical electromagnetic fields. But sometimes, not in Minkowski spacetime but in spacetimes that have compact spatial dimensions, it is not possible to find such a gauge parameter that equations (D.8) and (D.9) hold. In such a situation, eventhough they have the same $\vec{E}$ and the $\vec{B}$ fields, the sets of potentials $(\phi, \vec{A})$ and $(\phi', \vec{A}')$ are not gauge equivalent therefore they are physically different.

D.2 Relativistic Electrodynamics

In order to construct explicitly the relativistic form of the Maxwell's theory, we define a four-vector A^μ such as

$$A^\mu = \big(\phi, \vec{A}^1, \vec{A}^2\, \vec{A}^3\big), \tag{D.10}$$

that combines the scalar and vector potentials we already discussed. By using the Minkowski metric, one can easily find $A_\mu = \big(-\phi, \vec{A}^1, \vec{A}^2\, \vec{A}^3 \big)$ and create the electromagnetic field strength tensor $F_{\mu\nu}$ as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{D.11}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. By using the equations (D.6), (D.7), (D.10) and D.11, we can obtain the matrix $F_{\mu\nu}$ as follows

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}.$$

The relativistic formulation of the gauge transformations (D.8) and (D.9) are written in terms of

$$A'_\mu = A_\mu + \partial_\mu \epsilon, \tag{D.12}$$

where A'_μ and A_μ are potentials which are related through gauge transformations, and $\epsilon(x)$ is some function of spacetime.

One can suspect that $F_{\mu\nu}$ is gauge invariant because it encodes the $\vec{E}$ and $\vec{B}$ fields which are invariant under gauge transformations. One can easily see that is true by using the equations (D.11), (D.12) and the fact partial derivatives commute.

Our final aim is to obtain the relativistic formulation of the Maxwell equations, in other words the equations written in terms of $F_{\mu\nu}$. Let us start with homogenous Maxwell equations (D.1) and (D.2). Consider an object $T_{\mu\nu\lambda}$ with the following combination:

$$T_{\mu\nu\lambda} = \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}. \tag{D.13}$$

With a little effort and using equation (D.11), it is straightforward to see that $T_{\mu\nu\lambda}$ vanishes, so

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{D.14}$$

The choice $(\mu, \nu, \lambda) = (1, 2, 3)$ produces the equation (D.2) and the rest of the choices gives the equation (D.1). Therefore we can conclude that the equation (D.14) is the relativistic formulation of the homogenous Maxwell equations. We should note that this equation is known as "Bianchi identity" and it can be also written in terms of a dual tensor $\tilde{F}^{\mu\nu}$ such that

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \text{where} \quad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{D.15}$$

The equations (D.3) and (D.4) contain sources so we must introduce a current four-tensor j^μ such that

$$j^\mu = \left(\rho, \vec{j^1}, \vec{j^2}, \vec{j^3}\right), \tag{D.16}$$

where ρ and $\vec{j}$ stands for the charge and current densities respectively. By using the Minkowski metric and the equation (D.11), it is straightforward to obtain $F^{\mu\nu}$ as a matrix

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}.$$

So we have,

$$\partial_\nu F^{\mu\nu} = j^\mu, \tag{D.17}$$

Therefore the equations (D.15) and eqD.17 are precisely the relativistic Maxwell equaitons.

We should note that these equations are valid in any dimensions. All we have to do is to define the Lorentz vector A^μ as $A^\mu = \left(\phi, \vec{A}\right)$ where $\vec{A}$ is the $(D-1)$ spatial dimensional vector potential. We should note that in generic dimensions, definition of $\tilde{F}^{\mu\nu}$ changes.

D.3 Non-linear and Born-Infeld Electrodynamıcs

D.3.1 The non-linear electrodynamics

How can we describe the electromagnetism in the presence of materials? We are sure that the original equations (D.1) -(D.4) are still valid but we are also aware that they need some modifications. These modifications are necessary due to the materials which are responsible for the contributions of polarization charges or magnetization currents to charge and current densities respectively. Mathematically, these contributions are conducted by $\vec{D}$ and $\vec{H}$ fields and the corresponding Maxwell equations can be written in the form

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{D.18}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{D.19}$$

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{D.20}$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}. \tag{D.21}$$

We should note that, ρ and $\vec{j}$ are free charges which means they are not bounded to a specific atom or molecule, or simply to a material. Phenomenologically, these $\vec{D}$ and $\vec{H}$ fields are related to $\vec{E}$ and $\vec{B}$ fields through the equations

$$\vec{D} = \vec{D}(\vec{B}, \vec{E}) \quad \text{and} \quad \vec{H} = \vec{H}(\vec{B}, \vec{E}). \tag{D.22}$$

In linear dielectrics where the electric permittivity ϵ and magnetic permeability μ are constants, one can write the equation (D.22) as $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$. However, these relations may not be linear in either nonlinear or BI electrodynamics and the quantities ϵ and μ depend on the fields $\vec{E}$ and $\vec{B}$ [28].

The equations (D.18)-(D.21) with the relations (D.22) are the equations of nonlinear electrodynamics. Let us determine the relativistic formulation of these equations. It is obvious that the Bianchi identity still holds so the equation (D.14) is the relativistic formulation of the equations (D.18) and (D.19). To write the equations (D.20) and (D.21) relativistically, we follow a different path. We already have $\partial_\nu F^{\mu\nu} = j^\mu$ so consider a quantity $G^{\mu\nu}$ such that

$$\partial_\nu G^{\mu\nu} = j^\mu, \tag{D.23}$$

where $G^{\mu\nu}$ is the analog of $F^{\mu\nu}$. The matrix form of $G^{\mu\nu}$ can be constructed from $F^{\mu\nu}$ by replacing $\vec{E}$ with $\vec{D}$ and replacing $\vec{B}$ with $\vec{H}$. One can read the matrix as

$$G^{\mu\nu} = \begin{bmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{bmatrix}.$$

We should emphasize that the equation (D.23) is valid in any dimensions and together with the equation (D.14), they form the equations of nonlinear electrodynamics in any number of dimensions.

Our final discussion is to find the relation between $G^{\mu\nu}$ and $F^{\mu\nu}$. We are going to achieve this by showing that the equation (D.23) is obtained from a general Lagrangian with specific properties we are going to mention.

Consider the following action which is valid in any dimensions:

$$S = \int d^d x\, \mathcal{L}(F_{\mu\nu}) + \int d^d x\, A_\mu j^\mu. \tag{D.24}$$

We assume $\mathcal{L}(F_{\mu\nu})$ depends only on the tensor $F_{\mu\nu}$, not on its derivatives. Following the gauge invariance of the field strength $F_{\mu\nu}$, we can see that the Lagrangian density $\mathcal{L}$ has the same feature. To obtain the field equations (D.23), we should apply the variation to the action (D.24) that gives

$$\delta S = \int d^d x\, \delta\mathcal{L}(F_{\mu\nu}) + \int d^d x\, j^\mu \delta A_\mu. \tag{D.25}$$

By using the fact that $F_{\mu\nu}$ is antisymmetric, for any function $M(F_{\mu\nu})$, one can find the relation

$$\delta M = \frac{1}{2}\frac{\partial M}{\partial F_{\mu\nu}}\delta F_{\mu\nu}. \tag{D.26}$$

If we apply this result to the equation (D.25), with integration by parts and relabelling, we can read the final result as

$$\delta S = \int d^d x\, \delta A_\mu \left[\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\right) + j^\mu\right]. \tag{D.27}$$

It is obvious that the comparison with the equation (D.23) gives the relation between $F^{\mu\nu}$ and $G^{\mu\nu}$ such that

$$G^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}. \tag{D.28}$$

It is clear that we have an antisymetric tensor $G^{\mu\nu}$ by construction. As long as the nonlinear Lagrangian is given, the equation (D.28) leads us to not only the relation between the field strength tensors $F^{\mu\nu}$ and $G^{\mu\nu}$ but also the relations in the form

$$\vec{D} = \frac{\partial\mathcal{L}}{\partial\vec{E}} \quad \text{and} \quad \vec{H} = -\frac{\partial\mathcal{L}}{\partial\vec{B}}. \tag{D.29}$$

One can obtain these equations by applying the $G^{\mu\nu}$ and the $F_{\mu\nu}$ matrices to the equation (D.28). We should note that, as it is seen from the equation (D.7), $\vec{E}$ is related with time derivative of vector potential so one can consider $\vec{E}$ as velocity. Then from the equation (D.29), $\vec{D}$ acts as the canonical momentum corresponding to velocity $\vec{E}$. Therefore one can construct the Hamiltonian density as

$$\mathcal{H} = \vec{D} \cdot \vec{E} - \mathcal{L}. \tag{D.30}$$

As a final remark, we should mention the properties of Lagrangian density in the action (D.24). We already explained that $\mathcal{L}(F_{\mu\nu})$ depends only on the field strength tensor $F_{\mu\nu}$ and not on its derivatives. We require that the Lagrangian density must be both gauge and Lorentz invariant. $\mathcal{L}$ is already gauge invariant due to the gauge invariance of $F_{\mu\nu}$. To be Lorentz invariant, we are seeking for objects without indices. In fact, there are only two independent invariants which are formed only from $F_{\mu\nu}$ but none of its derivatives. They are written in the form

$$s = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(E^2 - B^2), \tag{D.31}$$

$$p = -\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = \vec{E} \cdot \vec{B}. \tag{D.32}$$

In four dimensional spacetime, one can write the most general Lagrangian as an arbitrary function of these invariants. The invariant s is Maxwell Lagrangian density and p is built from a dual field strength $\tilde{F}^{\mu\nu}$.

D.3.2 Born-Infeld electrodynamics

As we already mentioned, Born-Infeld electrodynamics is a special kind of nonlinear theory with particular properties. In the theory, the self-energy of a charged point like particle is finite at small distances which is quite different than the Maxwell electrodynamics.

Let us begin with four dimensional spacetime. In the nonlinear theories with the action (D.24), we already determined that the Lagrangian density $\mathcal{L}$ is both Lorentz and gauge invariant as well as it is a function of $F_{\mu\nu}$ but none of its derivatives. In Born-Infeld theory, we add two more constraints on $\mathcal{L}$ as follows:

1. The vanishing magnetic field $\vec{B} = 0$ should lead to the maximal electric field $\vec{E}_{max}$

2. For small electromagnetic fields, $\mathcal{L}(F_{\mu\nu})$ should produce to Maxwell Lagrangian.

The maximal electric field is borrowed from string theory in which the critical electric field is written as $E_{crit} = 1/2\pi\alpha' \equiv b$. This condition on electric field was considered necessary by Born and Infeld to get a finite self energy of the particle. To proceed further, we should find a way to impose these conditions to the Lagrangian. In Special Relativity, the constraint on the velocity is given in the point particle Lagrangian as

$$L = -m\sqrt{1 - v^2}. \tag{D.33}$$

This gives a hint on how we can impose the condition $E_{crit} \leq b$ to the Lagrangian. For an expression with the Lorentz invariant s only, we have

$$\mathcal{L} = -b^2 \sqrt{1 - \frac{(\vec{E}^2 - \vec{B}^2)}{b^2}} + b^2. \tag{D.34}$$

It is easy to capture that this Lagrangian (D.34) satisfies the requirements. Let us introduce the Born-Infeld Lagrangian in four dimensional spacetime

$$\mathcal{L}_{BI} = -b^2 \sqrt{1 - \frac{(\vec{E}^2 - \vec{B}^2)}{b^2} - \frac{(\vec{E} \cdot \vec{B})^2}{b^4}} + b^2. \tag{D.35}$$

It is easily seen that the Lagrangian $\mathcal{L}_{BI}$ satisfies the constraints we demand. When $\vec{B} = 0$, $E \leq b$ is satisfied. Also for small fields, it reduces to $\mathcal{L}_{\mathcal{M}} = \frac{1}{2}(E^2 - B^2)$.

The crucial point about Born-Infeld Lagrangian is that it can be written in terms of a determinant which is valid in any dimensions as opposed to (D.35):

$$\mathcal{L} = -b^2 \sqrt{-\det\left(\eta_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\right)} + b^2. \tag{D.36}$$

We can easily show that the equations (D.35) and (D.36) are equal to each other in four dimensions. To do this, we should show that the Lagrangian density in (D.36) is Lorentz invariant. To make calculations easier, let us take $b = 1$ and write the equation (D.36) in terms of

$$\mathcal{L} = -\sqrt{-\det\left(\eta_{\mu\nu} + F_{\mu\nu}\right)} + 1. \tag{D.37}$$

It is straightforward to see that the determinants of two vectors M and $\bar{M}$ with components $M_{\mu\nu}$ and $M^{\mu\nu}$ respectively are equivalent. If we apply this to the inside of the equation (D.37), we get

$$\det(\eta + F) = \det(\bar{\eta} + \bar{F}), \tag{D.38}$$

where the $\bar{\eta}$ and $\bar{F}$ are matrices with components $\eta^{\mu\nu}$ and $F^{\mu\nu}$ respectively. We are done when we prove the Lorentz invariance of the $\det(\bar{\eta} + \bar{F})$. To show this, all we need to show is that $\det(\bar{\eta} + \bar{F})$ is unchanged under the Lorentz transformations which can be written as

$$\eta'^{\mu\nu} + F'^{\mu\nu} = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}(\eta^{\rho\sigma} + F^{\rho\sigma}), \tag{D.39}$$

$$\bar{\eta}' + \bar{F}' = \Lambda(\bar{\eta} + \bar{F})\Lambda^{T}, \tag{D.40}$$

$$\Rightarrow \det(\bar{\eta}' + \bar{F}') = \det(\bar{\eta} + \bar{F}), \tag{D.41}$$

which proves the Lorentz invariance of the equation (D.37). To see the equivalence of the Lagrangians (D.36) and (D.37), we can write the latter in terms of

$$\det(\eta + F) = \det(\eta(1 + \eta F)) = -\det(1 + \eta F), \tag{D.42}$$

where we used the fact that $\det \eta = -1$. By following the exact calculation, one can show that the determinat of the matrix $1 + \eta F$ in four dimensional spacetime reduces to the result (D.35).

Let us end this chapter by showing that the self energy of a point particle is finite in Born-Infeld electrodynamics as opposed to Maxwell theory. To simplify the calculations, we can use the $\mathcal{L}_{BI}$ in (D.35) with the assumption that $\vec{B} = 0$. Then we read the Lagrangian density as

$$\mathcal{L} = -b^2\sqrt{1 - \frac{\vec{E}^2}{b^2}} + b^2. \tag{D.43}$$

Since we are given the Lagrangian, we can immediately use the equation (D.29) to find the $\vec{D}$-field in terms of

$$\vec{D} = \frac{\vec{E}}{\sqrt{1 - E^2/b^2}}. \tag{D.44}$$

If we solve the above equation for $\vec{E}$ then the result

$$E^2 = b^2\left(\frac{D^2}{D^2+b^2}\right) \quad \text{and} \quad \vec{E} = \frac{\vec{D}}{\sqrt{1+D^2/b^2}}, \tag{D.45}$$

from where we see that the electric field is bounded by b as expected. By calculating its magnitude, one can see that there is no bound on D. In fact, it should be infinitely large to have $E = b$. If we apply this electric field to the equation (D.43), we can the Lagrangian density as

$$\mathcal{L} = -\frac{b^2}{\sqrt{1+D^2/b^2}} + b^2. \tag{D.46}$$

Then the Hamiltonian density can be given as

$$\mathcal{H} = b^2\sqrt{1+\frac{D^2}{b^2}} - b^2. \tag{D.47}$$

This is the Born-Infeld energy density which is valid in any number of dimensions when $\vec{B} = 0$. We are going to use it to find the self energy of a point particle in Maxwell and Born-Infeld theories.

In Maxwell theory, energy density $\mathcal{H} = \vec{D}\cdot\vec{E} - \mathcal{L}$ is proportional to $E\cdot D = E^2$ and we know that E is a function like $1/r^2$. Therefore the integral $\int d^3x\, E^2$ is proportional with $1/r^2$ which is not finite for small r. It shows that the energy of a point particle is infinite in Maxwell theory.

In Born-Infeld theory, the energy density is given by the equation (D.47) that takes the form

$$\mathcal{H} \approx bD = E_{crit}D, \tag{D.48}$$

for large D fields. Therefore the integrand $\int d^3x\, E_{crit}D$ is proportional to dr that converges.

Let us summarize what we obtained. We showed that the energy density in Maxwell theory is proportional to $E \cdot D$. Since D and E are functions of r^{-2} the term $d^3x\, E^2$ will diverge for small fields. On the other hand, the energy density in Born-Infeld theory is proportional to bD that leads a convergent integrand for large fields and small distances.

APPENDIX E

A BRIEF LOOK AT BLACK HOLE ENTROPY IN BORN-INFELD GRAVITY

In chapter 3 we studied the entropy of the de Sitter space for generic gravity and did not discuss black hole entropy (except for the $n = 2{+}1$ dimensions). The main reason for this is that the BI theories studied here are quite recent and no exact black hole type solutions have been found for these theories. The only relevant work up to now is [65] where perturbative solutions were constructed. But for the sake of completeness and as an initial estimate to what happens to the entropy of Schwarzschild-Tangherlini black holes in the BI gravity, let us carry out the following computation which should be considered as a preliminary work. To be specific, let us take the Lagrangian (3.40) which reads

$$\mathcal{L} = \frac{2}{\gamma}\left(f(R_{\mu\nu\alpha\beta})^{n/2} - \lambda_0 - 1\right), \tag{E.1}$$

where

$$f(R_{\mu\nu\alpha\beta}) = 1 + pR + uR^2_{\mu\nu\alpha\beta} - 4uR^2_{\mu\nu} + (u-v)R^2, \tag{E.2}$$

with the coefficients

$$p = \frac{\gamma}{n}, \quad u = \frac{\gamma^2(n-1)^2}{4n(n-2)(n-3)}, \quad v = \frac{\gamma^2(n-2)}{4n^2}. \tag{E.3}$$

With the help of (3.29), the derivative of the Lagrangian with respect to the Riemann tensor can be found as

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial R_{abcd}} = \frac{n}{\gamma} f(R_{\mu\nu\alpha\beta})^{\frac{n-2}{2}} \Bigg[& \Big[p + 2(u-v)R\Big] g^{c[a} g^{b]d} + 2uR^{abcd} \\ & -4u\left(g^{a[c}R^{d]b} - g^{b[c}R^{d]a}\right)\Bigg]. \end{aligned}\tag{E.4}$$

Initially, we want to keep everything general by studying the $n-$dimensional Schwarzschild-Tangherlini black hole metric

$$ds^2 = -\left(1 - \frac{c}{r^{n-3}}\right) dt^2 + \left(1 - \frac{c}{r^{n-3}}\right)^{-1} dr^2 + r^2 d\Omega_{n-2}^2, \tag{E.5}$$

where $r = r_H = c^{\frac{1}{n-3}}$ is the radius of the black hole horizon. After making a general derivation we will study the $n = 4$ case.

With some effort, one can find the relevant curvature tensor components and make the following table:

Black Hole, $a = \sqrt{1 - \frac{c}{r^{n-3}}}$		
Curvature Terms	$n-$dimensions	$n = 4, c = 2M$
$\boldsymbol{R_{0101}}$	$-\frac{(n-2)(n-3)}{2r^2}\frac{c}{r^{n-3}}$	$-\frac{2M}{r^3}$
$\boldsymbol{R_{0i0j}}$	$\frac{n-3}{2}\frac{c}{r^{n-1}}\left(1 - \frac{c}{r^{n-3}}\right) g_{ij}$	$\frac{M}{r^3}\left(1 - \frac{2M}{r}\right) g_{ij}$
$\boldsymbol{R_{1i1j}}$	$-\frac{n-3}{2}\frac{c}{r^{n-1}}\left(1 - \frac{c}{r^{n-3}}\right)^{-1} g_{ij}$	$-\frac{M}{r^3}\left(1 - \frac{2M}{r}\right)^{-1} g_{ij}$
$\boldsymbol{R_{ijkl}}$	$\frac{c}{r^{n-1}}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$	$\frac{2M}{r^3}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$
$\boldsymbol{R^2_{\mu\nu\alpha\beta}}$	$\frac{(n-1)(n-2)^2(n-3)}{r^4}\left(\frac{c}{r^{n-3}}\right)^2$	$\frac{48M^2}{r^6}$

Table E.1: List of non-zero curvature terms for the black hole solutions in both generic $n-$dimensions and also in $n = 4$. Note that all the Ricci tensor components and hence the Ricci scalar vanishes.

Having all these information one can obtain the Wald entropy in $n-$dimensions for (E.1) as

$$S_W = \frac{A_H}{4G}\left(1 + \frac{\gamma(n-1)^2}{2r_H^2}\right)\left(1 + \frac{\gamma^2(n-2)(n-1)^3}{4nr_H^4}\right)^{\frac{n-2}{2}}. \tag{E.6}$$

In four dimensions, where we have $c = 2M,\ a = \sqrt{1 - \frac{2M}{r}}$, the result reduces to

$$S_W = \frac{A_H}{4G}\left(1 + \frac{27\gamma^2}{128M^4}\right)\left(1 + \frac{9\gamma}{8M^2}\right). \tag{E.7}$$

What we learn from (E.6) and (E.7) is that in the BI gravity, the entropy of the spherically symmetric black holes increases compared to the Einstenian result. We also see that as $\gamma \to 0$, one recovers the Bekenstein-Hawking entropy. It is quite possible that one can write S_W as the geometric entropy with a modified gravitational constant. But for this to work out, we need the exact black hole solutions in the BI gravity which we currently lack.

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