THE WRONSKI MAP FOR FLAG VARIETIES

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ABSTRACT

THE WRONSKI MAP FOR FLAG VARIETIES

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In this thesis, we studied flag varieties, the Grassmann variety $G(d, n)$ and their behavior under the Wronski map. We begin by introducing algebraic, topological and geometric tools that are required to define flag varieties as projective varieties. Schubert calculus is introduced in order to understand the cohomology of the Grassmannian and flag varieties. We described Young tableau which is a helpful tool that makes some combinatorial computations, in particular of Littlewood-Richardson coefficients, easier and studied it extensively. Finally, we studied the Wronski map which sends a set of polynomials to their Wronski determinant which is given by the polynomials and their derivatives.

Keywords: Grassmann variety, Schubert calculus, flag varieties, Wronski map

ÖZ

BAYRAK VARYETELERİ İÇİN WRONSKİ GÖNDERİMİ

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Bu tezde bayrak varyeteleri, Grassmann varyeteleri $G(d, n)$ ve Wronski gönderimi altındaki davranışları çalışılmıştır. Öncelikle, bayrak varyetelerini projektif varyete olarak tanımlanması için gerekli cebirsel, topolojik ve kombinatorik araçlar tanıtıldı. Grassmann varyetesinin ve bayrak varyetelerinin kohomolojisini anlamak için Schubert kalkülüs tanımlandı. Bazı kombinatorik hesaplamaları, özellikle Littlewood-Richardson katsayılarını hesaplamayı kolaylaştıran Young tablosu tarif edildi ve üzerinde derinlemesine çalışıldı. Son olarak ise bir polinom setini, polinomların ve onların türevlerinin determinantı olarak tanımlanan Wronski determinantına gönderen Wronski gönderimini inceledik.

Anahtar Kelimeler: Grasman varyetesi, Schubert kalkülüs, bayrak varyeteleri, Wronski gönderimi

To my family and friends...

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TABLE OF CONTENTS

ABSTRACT v

ÖZ vi

ACKNOWLEDGMENTS viii

TABLE OF CONTENTS ix

CHAPTERS

1 INTRODUCTION 1

2 GRASSMANN VARIETY 3

2.1 Definition and basic notation 3

2.2 A Brief Review of Exterior Algebra 3

2.3 Plücker Embedding 8

2.4 Plücker Relations 12

2.5 Affine Charts of the Grassmannian 14

3 SCHUBERT CALCULUS ON GRASSMANNIANS 17

3.1 Schubert cells and Schubert Varieties 17

3.2 Pieri's Rule and Giambelli's Rule 20

3.3 Young Tableaux and Littlewood Richarson Rule 22

4 FLAG VARIETIES 33

4.1 Basics 33

4.2 Cohomology of Flag Varieties 37

5 WRONSKI MAP 41

5.1 Definition and history 41

5.2 Relation with Schubert Calculus 44

5.3 A Problem 47

REFERENCES 49

CHAPTER 1

INTRODUCTION

The Grassmannian or the Grassmann Variety $G(d, n)$ and flag varieties $F(d_1, ..., d_r; n)$ are fundamental objects in algebraic geometry. Grassmannians are very helpful tools to parametrize linear subspaces of an n-dimensional vector space V whereas flag varieties parametrize sequences of subspaces. In particular, Grassmannians are special types of flag varieties. Other than algebraic geometry, flag varieties are fundamental objects in combinatorics and representation theory. They also have a very rich topological structure that can be explored in many ways.

Another way that the Grassmannians can be studied is by using so called the Wronski map. It sends a set of polynomials into their Wronskian, i.e., the determinant obtained by taking derivatives of these polynomials. The Wronski map gives us a well-defined map from the Grassmannian to a projective space. Fibers of the Wronski map have been of interest for many years, and there are many works around the topic. We will explore the Wronski map in depth.

In Chapter 2, we will start with basic notations for the Grassmannian $G(d, n)$ which is nothing but the set of all d-dimensional subspaces of an n-dimensional vector space V. We will develop the necessary algebraic background including exterior algebra, in order to give the Grassmannian a variety structure via Plücker embedding. After establishing the Grassmannian as a projective variety, we will explore its defining equations. More precisely, Plücker relations and the homogeneous ideals are computed. Later in this chapter, we will develop a cover the Grassmannian with affine charts which helps us to understand some further questions about the Wronski map.

In Chapter 3, we dive into Schubert calculus. We give affine stratification for the

Grassmannian, namely we will define Schubert cells and Schubert varieties. We give a thorough example to compute Schubert varieties and their corresponding matrices. They are also important tools for exploring the topological structure of the Grassmannian. Schubert varieties define the cohomology classes in the cohomology ring $H^*(G(d,n),\mathbb{Z})$. We will give statements of Pieri's Rule and Giambelli's Rule that are helpful to compute the product of any two Schubert cycles.

Other than topological aspects, Schubert classes are important for computing Littlewood-Richardson coefficients. To compute these coefficients, we will introduce a useful tool that is called Young tableaux. It has many applications in combinatorics, and representation theory. We explore the properties and importance of Young tableaux in depth. At the end of the chapter, we will state the Littlewood-Richardson Rule involving Young tableaux.

In Chapter 4, we will extend the structure that we established in Chapter 2 for Grassmannians in order to define flag varieties. We will explore over an example how we can obtain the homogeneous ideal and defining quadratic equations of flag varieties. We will also give the cohomology structure of flag varieties.

In Chapter 5, we will define the Wronski map and its properties. Its relation with Schubert calculus will be explored.

CHAPTER 2

GRASSMANN VARIETY

2.1 Definition and basic notation

We define Grassmannian or Grassmann variety over a vector space V as follows:

Definition 2.1.1. *Let V be an n-dimensional vector space over a field. Let* d*,* $0 < d < n$*, be an integer.* ***The Grassmannian*** $G(d, n)$ *is the set of all d-dimensional subspaces of V, i.e.,*

$$G(d, n) = \{W \mid W\ is\ a\ subspace\ of\ V\ and\ dim\ W = d\}.$$

We can also consider the Grassmannian as the set of all $(d-1)$-dimensional linear subspaces of an $(n-1)$-dimensional projective space $\mathbb{P}^{n-1}$. When we consider the Grassmannian is this way, we will denote it as $\mathbb{G}(d-1, n-1)$. Thus, for instance, when $d = 2$ and $n = 4$, we have $G(2, 4) = \mathbb{G}(1, 3)$.

Example 2.1.2. *G(1,n) is the set of all* 1*-dimensional subspaces of an* n*-dimensional vector space V, which is* $\mathbb{P}^{n-1}$ *itself.*

2.2 A Brief Review of Exterior Algebra

In order to understand Grassmannians, it is crucial to understand the underlying algebra called ***exterior algebra*** or ***Grassmann algebra***. Exterior algebra has a product called the ***wedge product*** (or ***exterior product***). For the vectors u and v, the wedge product of u and v is denoted by $u \wedge v$ and called a ***bivector***. Geometrically, the magnitude of the wedge product can be interpreted as the area of the paralellogram having the sides u and v.

Definition 2.2.1. *Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let*

$$T^0(V) = \mathbb{F},\ T^1(V) = V,\ \textit{and}\ T^d(V) = V \otimes V \otimes ... \otimes V$$

be tensor products of d-many copies of the vector space V. Let $A^d(V)$ be the subspace of $T^d(V)$ spanned by all vectors of the form $v_1 \otimes v_2 \otimes ... \otimes v_d$ with $v_1, v_2, ..., v_d \in V$ and $v_i = v_j$ for some $i \neq j$.The ***d-th exterior power of the vector space V*** *is defined as*

$$\bigwedge^d V = T^d(V)/A^d(V).$$

Conventionally, an element $v_1 \otimes v_2 \otimes ... \otimes v_d + A^d(V) \in \bigwedge^d V$ is denoted by

$$v_1 \wedge v_2 \wedge ... \wedge v_d$$

If $w \in \bigwedge^{d_1} V$ and $w' \in \bigwedge^{d_2} V$ with $w = v + A^{d_1}(V)$ and $w' = v' + A^{d_2}(V)$, respectively, we define

$$w \wedge w' = v \otimes v' + A^{d_1+d_2}(V)$$

as an element of $\bigwedge^{d_1+d_2} V$. Direct sum of all exterior powers of V is defined as

$$\bigwedge V = \bigwedge^0 V \oplus \bigwedge^1 V \oplus ... \oplus \bigwedge^n V.$$

Basically, if there is no superscript in the notation, we understand it as direct sum of all exterior powers of V.

Definition 2.2.2. *Let u, v and w be elements of $\bigwedge V$ and $\alpha, \beta \in \mathbb{F}$. The* ***wedge product*** *is a multiplication operation*

$$\begin{aligned} \wedge : \bigwedge V \times \bigwedge V &\longrightarrow \bigwedge V \\ (u, v) &\longmapsto u \wedge v \end{aligned}$$

satisfying the following properties:

1. $(u \wedge v) \wedge w = u \wedge (v \wedge w)$ ***(associativity)***
2. $u \wedge (\alpha v + \beta w) = \alpha u \wedge v + \beta u \wedge w$ ***(multilinearity)***
3. $u \wedge v = -v \wedge u$ ***(anticommutativity)***

Remark 2.2.3. *If $\{u_i\}$ is a basis for V, then one can easily see that $u_i \wedge u_j = 0$ whenever $i = j$.*

Proposition 2.2.4. *Let V be an n-dimensional vector space over $\mathbb{F}$ with basis $\{v_1, v_2, ..., v_n\}$. Then, for every $d \in \{1, ..., n\}$, **d-th exterior power of a vector space V** is a vector space with basis $\{v_{i_1} \wedge ... \wedge v_{i_d}\}$ with $1 \leq i_1 \leq i_2 \leq ... \leq i_d \leq n$, i.e.,*

$$\bigwedge^d V = \{ \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_d \leq n} c_{i_1, i_2, ..., i_d} v_{i_1} \wedge ... \wedge v_{i_d} \}$$

where $c_{i_1, i_2, ..., i_d} \in \mathbb{F}$.

Definition 2.2.5. *$v \in \bigwedge^d V$ is said to be **totally decomposable**, if v can be written as*

$$u_1 \wedge ... \wedge u_d.$$

where $u_1, ..., u_d \in V$ are linearly independent.

Proposition 2.2.6. *If $v \in \bigwedge^d V$ is totally decomposable, then $v \wedge v = 0$.*

Proof. Let $v \in \bigwedge^d V$ be totally decomposable. Then we can write

$$v = u_1 \wedge ... \wedge u_d$$

for some $u_1, ..., u_d \in V$. Then

$$v \wedge v = (u_1 \wedge ... \wedge u_d) \wedge (u_1 \wedge ... \wedge u_d).$$

By Definition 2.2.2., we can move each u_i with respect to anticommutativity. So for instance once we move u_1 to the left, next to first u_1, we will get

$$u_1 \wedge (-1)^{d-1} u_1 \wedge ... \wedge u_d \wedge u_2 \wedge ... \wedge u_d)$$

Thus by Remark 2.2.3. we get $u_1 \wedge (-1)^{d-1} u_1 = 0$. So $v \wedge v = 0$. ■

Example 2.2.7. *For $d = 4$, assume $v_1, ..., v_4 \in V$ are linearly independent and $char(\mathbb{F}) \neq 2$. Then $v_1 \wedge v_2 + v_3 \wedge v_4$ is not totally decomposable, since*

$$0 = (v_1 \wedge v_2 + v_3 \wedge v_4) \wedge (v_1 \wedge v_2 + v_3 \wedge v_4)$$

$$= v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_3 \wedge v_4 \wedge v_1 \wedge v_2$$

$$= 2 v_1 \wedge v_2 \wedge v_3 \wedge v_4$$

which is a contradiction since $v_1 \wedge v_2 \wedge v_3 \wedge v_4$ is nonzero.

Remark 2.2.8. *Every nonzero element of $\bigwedge^1 V$ is trivially totally decomposable.*

Definition 2.2.9. *Let V be a finite dimensional vector space and let $v \in V$, $w \in \bigwedge^d V$. Then we say v divides w, if there exists $x \in \bigwedge^{d-1} V$ such that $w = v \wedge x$.*

Lemma 2.2.10. *Let V be a finite dimensional vector space and $v \in V$, $w \in \bigwedge^d V$. Then, v divides w if and only if $v \wedge w = 0$.*

Proof. ($\Rightarrow$) Since v divides w, there exists $\varphi \in \bigwedge^{d-1} V$ such that $w = v \wedge \varphi$. Multiply both sides by v, we get

$$v \wedge w = v \wedge v \wedge \varphi = 0 \wedge \varphi = 0$$

since $v \wedge v = 0$.

($\Leftarrow$) Assume $v \wedge w = 0$. Start with fixing a basis for V, $B = \{v_1, ..., v_n\}$ with $v_1 = v$. Then $v_{i_1} \wedge ... \wedge v_{i_d}$ is a basis for $\bigwedge^d V$. Since $w \in \bigwedge^d V$, we write $w = \sum v_{i_1} \wedge ... \wedge v_{i_d}$ for $1 \leq i_1 < i_2 < ... < i_d \leq n$. By the assumption we have

$$v \wedge (\sum c_{i_1,i_2,...,i_d}(v_{i_1} \wedge ... \wedge v_{i_d})) = \sum c_{i_1,i_2,...,i_d}(v \wedge v_{i_1} \wedge ... \wedge v_{i_d}) = 0.$$

Since all the multivectors $v \wedge v_{i_1} \wedge ... \wedge v_{i_d}$ are linearly independent and nonzero for $i_1 > 1$, we get $c_{i_1,i_2,...,i_d} = 0$ for $i_1 > 1$. So w is a linear combination of the basis vectors of the form $v_{i_1} \wedge ... \wedge v_{i_d} = v \wedge w_{i'}$ for $i_1 = 1$ where $w_i^{'} = v_{i_2} \wedge ... \wedge v_{i_d}$. It follows that $w = \sum c_{1,i_2,...,i_d}(v \wedge w_{i'}) = v \wedge \sum(c_{1,i_2,...,i_d} w_{i'})$. ∎

We denote the set of all $v \in V$ dividing $w \in \bigwedge^d V$ by D_w, i.e.

$$D_w = \{v \in V \mid w = v \wedge x \; for \; some \; x \in \bigwedge^{d-1} V\}.$$

Proposition 2.2.11. *D_w is a subspace of V.*

Proof. Let v_1, v_2 be vectors in D_w. Then $v_1 \wedge w = 0$ and $v_2 \wedge w = 0$ by Lemma 2.2.10. So we have

$$(v_1 + v_2) \wedge w = (v_1 \wedge w) + (v_2 \wedge w) = 0.$$

It is clear that for any $\alpha \in \mathbb{F}$, if $v \wedge w = 0$, then we have $\alpha v \wedge w = 0$. ∎

Proposition 2.2.12. *An element $w \in \bigwedge^d V$ is totally decomposable if and only if $dim(D_w) = d$.*

Proof. Suppose $w \in \bigwedge^d V$ is totally decomposable, say $w = v_1 \wedge ... \wedge v_d$. Clearly, $v_1, v_2, ..., v_d$ are linearly independent. Now, extend the set $\{v_1, v_2, ..., v_d\}$ to a basis $\{v_1, v_2, ..., v_n\}$. Then we can write for any $v \in V$, $v = a_1v_1 + a_2v_2 + ... + a_nv_n$ for some $a_i \in \mathbb{F}$. So we get,

$$0 = v \wedge w = (\sum_{i=1}^{n} a_iv_i) \wedge (v_1 \wedge ... \wedge v_d).$$

Note that, all the terms with $i \leq d$ will vanish since $v_i \wedge v_j = 0$ for $i = j$. So we get all $a_i = 0$ for $d < i \leq n$. Hence v is a linear combination of $v_1, v_2, ..., v_d$. Therefore $\{v_1, v_2, ..., v_d\}$ spans D_w and also forms a basis for D_w since they are linearly independent.

Conversely, suppose $dim(D_w) = d$. Let $\{v_1, v_2, ..., v_d\}$ be a basis for D_w. Extend it to a basis for V, namely $\{v_1, v_2, ..., v_n\}$. Then we can write

$$w = \sum c_{i_1,i_2,...,i_n}(v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_n})$$

for some $c_{i_1,i_2,...,i_n} \in \mathbb{F}$. So we get,

$$0 = v_k \wedge w = \sum c_{i_1,i_2,...,i_n}(v_k \wedge v_{i_j})$$

for $1 \leq k \leq d$, since $v_k \wedge v_{i_j} = 0$ when $i_j \in \{1, 2, ..., d\}$. Thus we have $c_{i_1,i_2,...,i_n} = 0$ for $d < k \leq n$. So we can write

$$w = \alpha v_1 \wedge v_2 \wedge ... \wedge v_d$$

for some $\alpha \in \mathbb{F}$. Thus $w \in \bigwedge^d V$ is totally decomposable. ■

We will define a map that enables us to characterize the totally decomposable vectors: Let $w \in \bigwedge^d V$, then the map

$$\varphi_w : V \longrightarrow \bigwedge^{d+1} V$$

is defined by

$$\varphi_w(v) = v \wedge w.$$

Corollary 2.2.13. *$w \in \bigwedge^d V$ is totally decomposable if and only if φ_w has rank $n - d$.*

Proof. Let φ_w be the map above. By Proposition 2.2.12, w is totally decomposable if and only if $dim(D_w) = d$. We will show $ker(\varphi_w) = D_w$. Recall that for $w \in \bigwedge^d V$,

$$D_w = \{v \in V \mid v \wedge w = 0\}$$

by Lemma 2.2.10. In other words,

$$D_w = \{v \in V \mid \varphi_w(v) = 0\} = ker(\varphi_w).$$

Thus $dim(D_w) = dim(ker(\varphi_w)) = d$. So by Rank-Nullity Theorem φ_w has rank $n - d$. ■

Corollary 2.2.14. *$w \in \bigwedge^d V$ is totally decomposable if and only if the rank of φ_w is at most $n - d$.*

Proof. Suppose $w \in \bigwedge^d V$ is totally decomposable. We need to show that it is not the case that φ_w has rank strictly less than $n - d$, since we proved that the claim holds when $dim(\varphi_w) = n - d$. Assume φ_w has rank less than $n - d$. Then by rank-nullity theorem $dim(ker(\varphi_w)) = m > d$. Since $ker(\varphi_w) = D_w$, we also have $dim(D_w) = m$. But the proof of Proposition 2.2.12 shows that $dim(D_w)$ cannot be more than d. ■

2.3 Plücker Embedding

In order to give Grassmannian a variety structure, we need to embed it to a projective space. To achieve this, we associate to a d-dimensional subspace $W \in V$ a point of $\mathbb{P}(\bigwedge^d(V))$ which is the line spanned by $v_1 \wedge ... \wedge v_d$. In other words, we have a map, called **Plücker embedding** or **Plücker map**,

$$\Psi : G(d, n) \to \mathbb{P}(\bigwedge^{d} V) \tag{2.1}$$

given by

$$U = span(u_1, ..., u_d) \mapsto [u_1 \wedge ... \wedge u_d]$$

This map is well-defined by the following lemma:

Lemma 2.3.1. *Let W be a d-dimensional subspace of a finite dimensional vector space V over $\mathbb{F}$. Let $B_1 := \{w_1, w_2, ..., w_d\}$ and $B_2 := \{w_1', w_2', ..., w_d'\}$ be two ordered bases for W. Then*

$$w_1' \wedge ... \wedge w_d' = \lambda w_1 \wedge ... \wedge w_d \; for \; some \; \lambda \in \mathbb{F}$$

where λ is the determinant of the change of basis matrix from $\{w_1, w_2, ..., w_d\}$ to $\{w'_1, w'_2, ..., w'_d\}$.

Proof. Follows from multilinearity and anticommutativity of wedge product. See [29] Prop. 11.12. ■

Proposition 2.3.2. *The Plücker map*

$$\Psi : G(d, n) \to \mathbb{P}(\bigwedge^{d} V)$$

is injective.

Proof. Let $U, W \in G(d, n)$ be two subspaces. Let $\{u_1, u_2, ..., u_d\}$ and $\{w_1, w_2, ..., w_d\}$ be bases for U and W, respectively. Recall that we have a map

$$\varphi_u : V \to \bigwedge^{d+1} V$$

defined by

$$v \longmapsto v \wedge u.$$

Now suppose $\Psi(U) = \Psi(W)$ and $u = [u_1 \wedge u_2 \wedge ... \wedge u_d]$, $w = [w_1 \wedge w_2 \wedge ... \wedge w_d]$. Then we have

$$\begin{aligned} \Psi(U) = \Psi(W) &\Leftrightarrow \exists x \in \mathbb{F}^* \textit{ such that } u = xw \\ &\Leftrightarrow \exists x' \in \mathbb{F}^* \textit{ such that } \varphi_u = x'\varphi_w \\ &\Leftrightarrow ker(\varphi_u) = ker(\varphi_w). \end{aligned}$$

So, in order to show Ψ is injective, it is enough to show that $ker(\varphi_u) = U$. It is obvious that $U \subset ker(\varphi_u)$ since $u_i \wedge u_j = 0$ for $i = j$, we have $u \wedge u = 0$. To show $ker(\varphi_u) \subset U$, note that we can extend basis $\{u_1, u_2, ..., u_d\}$ of U to a basis $\{u_1, u_2, ..., u_d, ..., u_n\}$ of V. Now, let $v \in ker(\varphi_u)$. Write $v = a_1u_1 + a_2u_2 + ... + a_nv_n = \sum a_iv_i$. Then we have

$$\begin{aligned} 0 = v \wedge u &= (a_1v_1 + ... + a_nv_n) \wedge u \\ &= (a_1v_1 + ... + a_nv_n) \wedge (u_1 + ... + u_d). \end{aligned}$$

Expanding the wedge product, we can see that all terms of the form $a_iu_i \wedge u_j$ where $i = j$, vanish. Thus we are left with the terms $a_iu_i \wedge u_j, i \neq j, i, j \geq d + 1$ which are nonzero. So we get $a_i = 0$ for $i \geq d+1$, i.e. $v = u_1 + u_2 + ... + u_d$. Hence $v \in U$. This finishes the proof. ■

Let V be an n-dimensional vector space with basis $\{v_1, v_2, ..., v_n\}$ as usual and let $U \in G(d,n)$ with basis $\{u_1, u_2, ..., u_d\}$. By Definition 2.2.4, we have

$$\bigwedge^d V = \{ \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_d \leq n} c_{i_1,i_2,...,i_d} v_{i_1} \wedge ... \wedge v_{i_d} \}$$

where $c_{i_1,i_2,...,i_d} \in \mathbb{F}$. Thus the basis for $\bigwedge^d V$ is of the form $\{v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_d}\}$. We can express the basis elements of U with respect to basis elements of V. So for $1 \leq j \leq d$, let $u_i = \sum_{j=0}^{n} a_{ij} v_j$. Then the coordinates of $\Psi(U) = [u_1 \wedge ... \wedge u_d]$ are called the **Plücker coordinates**. Once we choose a basis for V, we will see that we can represent $U \in G(d,n)$ by a $d \times n$ matrix M_U. The Plücker coordinates are the $d \times d$ minors of the matrix M_U.

Example 2.3.3. *Consider the Grassmannian G(2,5). Let $\{v_1, v_2, v_3, v_4, v_5\}$ be a basis for V. Then the basis for $\bigwedge^2 V$ is given by*

$$\{v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_1 \wedge v_5, v_2 \wedge v_3, v_2 \wedge v_4, v_2 \wedge v_5, v_3 \wedge v_4, v_3 \wedge v_5, v_4 \wedge v_5\}.$$

Let $\{u_1, u_2\}$ be a basis for $U \in G(2,5)$. Write the basis elements as follows:

$$u_1 = a_{11}v_1 + a_{12}v_2 + a_{13}v_3 + a_{14}v_4 + a_{15}v_5$$
$$u_2 = a_{21}v_1 + a_{22}v_2 + a_{23}v_3 + a_{24}v_4 + a_{25}v_5.$$

Then,

$$\begin{aligned} u_1 \wedge u_2 = &(a_{11}a_{22} - a_{12}a_{21})v_1 \wedge v_2 + (a_{11}a_{23} - a_{13}a_{21})v_1 \wedge v_3 \\ &+ (a_{11}a_{24} - a_{14}a_{21})v_1 \wedge v_4 + (a_{11}a_{25} - a_{15}a_{21})v_1 \wedge v_5 \\ &+ (a_{12}a_{23} - a_{13}a_{22})v_2 \wedge v_3 + (a_{12}a_{24} - a_{14}a_{22})v_2 \wedge v_4 \\ &+ (a_{12}a_{25} - a_{15}a_{22})v_2 \wedge v_5 + (a_{13}a_{24} - a_{14}a_{23})v_3 \wedge v_4 \\ &+ (a_{13}a_{25} - a_{15}a_{23})v_3 \wedge v_5 + (a_{14}a_{25} - a_{15}a_{24})v_4 \wedge v_5. \end{aligned}$$

So the Plücker coordinates are

$$(a_{11}a_{22} - a_{12}a_{21}, a_{11}a_{23} - a_{13}a_{21}, a_{11}a_{24} - a_{14}a_{21}, a_{11}a_{25} - a_{15}a_{21},$$
$$a_{12}a_{23} - a_{13}a_{22}, a_{12}a_{24} - a_{14}a_{22}, a_{12}a_{25} - a_{15}a_{22}, a_{13}a_{24} - a_{14}a_{23},$$
$$a_{13}a_{25} - a_{15}a_{23}, a_{14}a_{25} - a_{15}a_{24})$$

Let us denote the Plücker coordinates by $x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}$*, respectively. We can observe that the Plücker coordinates are the* 2×2 *minors of the following matrix* M_U*:*

$$M_U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix}.$$

Proposition 2.3.2 shows that the Plücker map allows us to view the Grassmannian $G(d,n)$ as a subset of the projective space $\mathbb{P}(\bigwedge^d V)$. Naturally, the next question is whether we can embed $G(d,n)$ onto a projective variety under the Plücker map or not. In other words, can we see the Grassmannian as a projective variety?

Lemma 2.3.4. $[w] \in \mathbb{P}(\bigwedge^d V)$ *lies in the image of the Grassmannian under the Plücker embedding if and only if* $w \in \bigwedge^d V$ *is totally decomposable.*

Proof. Let $[w] \in \mathbb{P}(\bigwedge^d V)$ and $[w] = \Psi(U)$ for some $U \in G(d,n)$. Let $\{u_1, u_2, ..., u_d\}$ be a basis for U. Then $[w] = [u_1 \wedge ... \wedge u_d]$. So $w = \lambda u_1 \wedge ... \wedge u_d$ is totally decomposable for some λ.

Conversely, assume that $w \in \bigwedge^d V$ is totally decomposable, say $w = w_1 \wedge ... \wedge w_d$. Then the subspace W spanned by $\{w_1, w_2, ..., w_d\}$ is d-dimensional and so $W \in G(d,n)$. Observing that $\Psi(W) = [w]$, finishes the proof. ■

Before moving to the main result, we simply state a fact from linear algebra.

Remark 2.3.5. *The rank of a matrix* $M \in M_{m\times n}$ *is the largest integer* k *such that some* $k \times k$ *minor does not vanish.*

Theorem 2.3.6. $G(d,n) \subset \mathbb{P}(\bigwedge^d V)$ *is a projective variety.*

Proof. By Lemma 2.3.3 and Corollary 2.2.10, $[w]$ lies in the image of the Grassmannian if and only if the rank of the map $\varphi_w : V \longrightarrow \bigwedge^{d+1}(V)$ given by $\varphi_w(v) = v \wedge w$ is $n-d$. But in fact, by Corollary 2.2.11, we can say that

$$[w] \in G(d,n) \Leftrightarrow rank(\varphi_w) \le n-d.$$

Now, the map $\bigwedge^d V \to Hom(V, \bigwedge^{d+1} V)$ given by $w \mapsto \varphi_w$ is linear. So we can view $\varphi_w \in Hom(V, \bigwedge^{d+1} V)$ as a matrix whose entries are functions of w. Since $\varphi_w(\lambda v) = \lambda \varphi_w(v)$, these functions are homogeneous of degree 1.

By Remark 2.3.4, φ_w has rank at most $n-d$ if and only if all $(n-d+1)\times(n-d+1)$ minors vanish. Thus $[w]$ lies in the image of the Grassmannian if and only if all $(n-d+1)\times(n-d+1)$ minors vanish. In other words, one can say that $G(d,n)\subset\mathbb{P}(\bigwedge^d V)$ is a projective variety defined by the zero locus of the $(n-d+1)\times(n-d+1)$ minors of the matrix M_U of φ_w. ■

Remark 2.3.7. *M_U is the same matrix that is mentioned before when we talked about the Plücker coordinates after the proof of Proposition 2.3.2.*

2.4 Plücker Relations

The equations above give defining equations of the Grassmannian as a variety, however they do not generate whole homogeneous ideal of the Grassmannian. In order to find the homogeneous ideal of the Grassmannian, we need more tools.

Let V be an n-dimensional vector space over $\mathbb{F}$. Let V^* be the dual space of V. The pairing $\langle\cdot,\cdot\rangle : V^*\otimes V\to\mathbb{F}$ defined by $\langle w^*,v\rangle = w^*(v)$ for all $w^*\in V^*$ and $v\in V$ is a nondegenerate bilinear form.

Proposition 2.4.1. *Let V and W be a finite dimensional vector spaces over a field $\mathbb{F}$ and let $f:V\to W$ and $g:V\to W^*$ be linear maps. Then the pairing*

$$\langle g(v), f(v)\rangle : V\to\mathbb{F}$$

is a quadratic form.

Proposition 2.4.2. *Let V be an n-dimensional vector space over $\mathbb{F}$. Let V^* be the dual space of V. Let $0<d<n$ be an integer. Then we have a nondegenerate pairing*

$$\bigwedge^d V\times\bigwedge^{n-d} V\longrightarrow\bigwedge^n V\cong\mathbb{F},$$

that induces an isomorphism

$$\bigwedge^{n-d} V\cong(\bigwedge^d V)^* = \bigwedge^d V^*.$$

So, up to a scalar, we can identify $\bigwedge^d V$ with $\bigwedge^{n-d} V^*$. Let $w\in\bigwedge^d V$ and let $w^*\in\bigwedge^{n-d} V^*$ be its dual. As with the vector space V, we have a linear map

$$\varphi_{w^*}: V^*\longrightarrow\bigwedge^{n-d+1} V^*$$

defined by

$$\varphi_{w^*}(v^*) = v^* \wedge w^*.$$

Proposition 2.4.3. *An element $w \in \bigwedge^d V$ is totally decomposable if and only if φ_{w^*} has rank at most d.*

The proof of this proposition is similar to the Corollary 2.2.13. We can observe that, assuming w to be totally decomposable, the kernel of the map φ_w is the annihilator of the kernel of the map φ_{w^*}. In other words, for the tranpose maps

$$\varphi_w^t : \bigwedge^{d+1} V^* \longrightarrow V^* \ , \ \varphi_{w^*}^t : \bigwedge^{n-d+1} V \longrightarrow V$$

their images annihilate each other.

Theorem 2.4.4. *$[w]$ lies in the Grassmannian $G(d,n)$ if and only if for every pair $\alpha \in \bigwedge^{d+1} V^*$ and $\beta \in \bigwedge^{n-d+1} V$, we have*

$$\Xi_{\alpha,\beta}(w) = \langle \varphi_w^t(\alpha), \varphi_{w^*}^t(\beta) \rangle = 0$$

$\Xi_{\alpha,\beta}(w)$ are quadratic polynomials by Proposition 2.3.6. They are called the **Plücker relations** and they generate the homogeneous ideal of $G(d,n)$ which is called the **Plücker ideal**. See [23] Chapter 9.

Let us look at an example in order to understand the structure better.

Example 2.4.5. *We will explore the defining equations and homogeneous ideal of $G(2,4)$, the Grassmannian of 2-planes. Let V a be 4-dimensional vector space. Let $B = \{v_1, v_2, v_3, v_4\}$ be a basis for V. Then the canonical bases B_2, B_3 for $\bigwedge^2 V$ and $\bigwedge^3 V$, respectively, are*

$$B_2 = \{v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_2 \wedge v_3, v_2 \wedge v_4, v_3 \wedge v_4\},$$

$$B_3 = \{v_1 \wedge v_2 \wedge v_3, v_1 \wedge v_2 \wedge v_4, v_1 \wedge v_3 \wedge v_4, v_2 \wedge v_3 \wedge v_4\}.$$

Let $w \in \bigwedge^2 V$. since $w = \sum a_{ij} v_i \wedge v_j$, the map $\varphi_w : V \to \bigwedge^3 V$ given by $v \mapsto v \wedge w$ is represented by the following matrix

$$\begin{bmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{bmatrix}$$

So the Grassmann variety $G(2,4)$ is defined by the zero locus of 3×3 minors of the above matrix. If we calculate the minors for each entry, we get 16 equations. however, there is a simple way to determine homogeneous ideal for the case $G(2,4)$. Since $w \in \bigwedge^2 V$ is totally decomposable, we have $w \wedge w = 0$. If we expand $w = \sum a_{ij} v_i \wedge v_j$,

$$w = a_{12} v_1 \wedge v_2 + a_{13} v_1 \wedge v_3 + a_{14} v_1 \wedge v_4 + a_{23} v_2 \wedge v_3 + a_{24} v_2 \wedge v_4 + a_{34} v_3 \wedge v_4,$$

then

$$w \wedge w = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) v_1 \wedge v_2 \wedge v_3 \wedge v_4.$$

The condition $w \wedge w = 0$ gives us that $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$. Thus, the homogeneous ideal of the Grassmannian $G(2,4)$ is generated by the polynomial

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$

In general, the ideal of the Grassmannian $G(2,n)$ for any n, is given by similar quadratic polynomials. Let $\{v_1, v_2, ..., v_n\}$ be a basis for V, and $w = \sum x_{ij} v_i \wedge v_j \in \bigwedge^2 V$, then the homogeneous ideal of the Grassmannian $G(2,n)$ is generated by the polynomials

$$\{f_{ijkl} \coloneqq x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} = 0 \mid 0 \leq 1 < i < j < k < l \leq n\}.$$

2.5 Affine Charts of the Grassmannian

Remember that we can cover the projective space with affine charts. Similarly, the Grassmannian can also be covered with Zariski open subsets that are isomorphic to affine spaces.

Let V be an n-dimensional vector space and let P and Q be complementary subspaces of V of dimension d and $n-d$, respectively, so that $V = P \oplus Q$. Let U_Q be the subset of d-dimensional subspaces that intersect trivially with Q, i.e.,

$$U_Q = \{W \in G(d,n) \mid W \cap Q = 0\}.$$

We can observe that U_Q is Zariski open in $G(d,n)$ since if $w_1, ...w_{n-d}$ is any basis for Q and $\gamma = w_1 \wedge ... \wedge w_{n-d}$, we have

$$U_Q = \{[w] \in G(d,n) \subset \mathbb{P}(\bigwedge^d V) \mid w \wedge \gamma \neq 0\}$$

Thus U_Q corresponds to the complement of zero set of polynomials that represent a hyperplane section of $G(d, n)$.

The graph of any linear map $l\colon P \to Q$ is a d-dimensional subspace $\Gamma(l)$ defined by

$$\Gamma(l) = \{x + l(x) \mid x \in P\}.$$

Notice that for any linear map l, we have $\Gamma(l) \cap Q = 0$. Otherwise, if for $x \in P$, $x + l(x) \in \Gamma(l) \cap Q$, then it implies that $x + l(x) \in Q$ and since $l(x) \in Q$, we have $x \in Q$. But $P \cap Q = 0$. So we reach a contradiction. The following lemma is needed to establish an affine covering for the Grassmannian, though we will not prove it.

Lemma 2.5.1. *Any d-dimensional linear subspace $W \subset V$ with $W \cap Q = 0$, i.e $W \subset U_Q$, is the graph of a unique linear map*

$$l\colon P \xrightarrow{\pi_P^{-1}} W \subset V = P \oplus Q \xrightarrow{\pi_Q} Q.$$

where π_P and π_Q are natural projections onto P and Q, respectively.

Observe that π_P is also an isomorphism between P and W. Now, let $Hom(P, Q)$ be the vector space of linear maps from P to Q and U_Q as above. Define a map

$$\Theta\colon Hom(P, Q) \longrightarrow U_Q$$
$$\Theta(l) \longmapsto \Gamma(l).$$

By above discussion, Θ is a bijection with the inverse map $\Theta^{-1}\colon U_Q \longrightarrow Hom(P, Q)$. So we have the following isomorphisms

$$U_Q \cong Hom(P, Q) \cong \mathbb{A}^{d(n-d)}.$$

Choose a basis $\{v_1, v_2, ...v_n\}$ for V such that $\{v_1, v_2, ...v_d\}$ is a basis for P and $\{v_{d+1}, v_{d+2}, ...v_n\}$ is a basis for Q. So for $W \in U_Q$, $\{\pi_P^{-1}v_1, \pi_P^{-1}v_2, ..., \pi_P^{-1}v_k\}$ forms a basis for W. Thus we can represent W with the matrix

$$\begin{bmatrix} 1 & 0 & ... & 0 & a_{11} & a_{12} & \cdots & a_{1n-d} \\ 0 & 1 & ... & 0 & a_{21} & a_{22} & \cdots & a_{2n-d} \\ & & & & \cdots & & & \\ 0 & 0 & ... & 1 & a_{d1} & a_{d2} & \cdots & a_{dn-d} \end{bmatrix}$$

where the matrix $[a_{ij}]$ is the transpose of the matrix of the linear map $l\colon P \to Q$.

CHAPTER 3

SCHUBERT CALCULUS ON GRASSMANNIANS

3.1 Schubert cells and Schubert Varieties

Let $G(d, n)$ be the Grassmannian of d-dimensional subspaces of an n-dimensional vector space V. Let

$$\mathcal{F} : (0) = V_0 \subset V_1 \subset ... \subset V_n = V$$

be a full flag such that $dim(V_i) = d_i$ with basis $\{e_1, e_2, ..., e_i\}$ for V_i and let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$ be an integer partition satisfying the condition $n - d \geq \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d \geq 0$. Partitions that satisfy the condition are called **admissible partitions**. For every admissible partion $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$, the **Schubert variety** or **Schubert cycle** $\Sigma_\lambda(\mathcal{F})$ is defined by

$$\Sigma_\lambda(\mathcal{F}) = \{U \in G(d, n) \mid dim(V_{n-d+i-\lambda_i} \cap U) \geq i, \ \forall\, 1 \leq i \leq d\}.$$

The codimension of the Schubert variety is $|\lambda| = \Sigma\lambda_i$, which is also the **weight** of λ. Schubert varieties $\Sigma_\lambda(\mathcal{F})$ define the cohomology classes σ_λ in the cohomology ring $H^*(G(d, n), \mathbb{Z})$. The cohomology class σ_λ is called **Schubert class** and it is the Poincare dual of the homology class of Σ_λ. Note that σ_λ is independent of the choice of the fixed flag defining it. When the choice of the flag is obvious, we will denote $\Sigma_\lambda(\mathcal{F})$ as Σ_λ.

In order to define the cohomology ring of the Grassmannian $G(d, n)$, we need to introduce a stratification of the Grassmannian. For a given flag $\mathcal{F}$ and an admissible partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$ as above, the **Schubert cell** $\Sigma^\circ_\lambda(\mathcal{F})$ is a set of subspaces

$U \in G(d, n)$ such that

$$dim(U \cap V_j) = \begin{cases} 0, & j < n - d + 1 - \lambda_1 \\ i, & n - d + i - \lambda_i \leq j < n - d + i + 1 - \lambda_{i+1} \\ d, & n - \lambda_d \leq j \end{cases}.$$

To move forward, we will state a fundamental observation about Schubert cells.

Theorem 3.1.1. *The Schubert cell $\Sigma_\lambda^o(\mathcal{F})$ is isomorphic to $\mathbb{A}^{d(n-d)-|\lambda|}$.*

Proof. See [18] Theorem 4.1. ■

For each $U \in \Sigma_\lambda^o(\mathcal{F})$, we can choose a unique basis so that U is spanned by the rows of a row-reduced $d \times (n-d)$ matrix. The only nonzero entries are 1 in the i-th row and $n - d + i - \lambda_i$-th column from the left. All the entries after 1 are 0 in a row; all the entries in the column that has a leading 1's are also 0. So the matrix is of the form

$$\begin{bmatrix} * & ... & * & 1 & 0 & ... & 0 & 0 & ... & 0 \\ * & ... & * & 0 & * & ... & * & 1 & 0 & 0 \\ & & & & & ... & & & & \\ * & ... & * & 0 & * & ... & * & 0 & 1 & 0 \end{bmatrix}$$

where $*$'s are arbitrary and there are $n - \lambda_i$ many of $*$'s in i-th row. Following this construction, we can write the Grassmannian as a disjoint union of Schubert cells:

$$G(d, n) = \bigsqcup_{\lambda \ admissible} \Sigma_\lambda^o(\mathcal{F})$$

We also define the **Schubert variety** $\Sigma_\lambda(\mathcal{F})$ to be the closure of $\Sigma_\lambda^o(\mathcal{F})$. Since union of Schubert cells of real dimension not larger than i, $i = 0, 1, 2, ..., 2d(n-d)$, form a cellular CW structure of Grassmannian $G(d, n)$, we reach the following theorem, see [23, 34]:

Theorem 3.1.2. *The Grassmannian $G(d, n)$ has cohomology only in even degrees, and Schubert classes form an additive basis for the cohomology ring $H^*(G(d, n), \mathbb{Z})$. In other words, $H^{2\lambda}(G(d, n), \mathbb{Z}) = Z^{m_\lambda}$ where m_λ is the number of partitions of the integer $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$ where $n - d \geq \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d \geq 0$ and*

$$H^*(G(d, n), \mathbb{Z}) = \bigoplus_\lambda \mathbb{Z} \cdot \sigma_\lambda.$$

Example 3.1.3. *We will explore the structure of the rather famous and popular Grassmannian* $G(2,4) = \mathbb{G}(1,3)$ *that lives in* $\mathbb{P}^5$*. We will use projective perspective in order to understand the structure better. Let's take the standard full flag in* $\mathbb{P}^3$*,* $p \subset L \subset H$*, that consists of a point* p*, contained in a line* L*, contained in a plane* H *contained in* $\mathbb{P}^3$*. For* $d = 2$*,* $n = 4$ *we have dim(G(2,4))=4, and the partitions that satisfy the definition are (2,2), (2,1), (2,0), (1,1), (1,0), and (0,0).*

- *For λ=(2,2), the Schubert variety is* $\Sigma_{2,2} = \{U \mid U = L\}$*, i.e., given line L, and the Schubert cycle* $\sigma_{2,2} \in H^8(G(2,4),\mathbb{Z})$ *corresponds to the matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.
- *For λ=(2,1), the Schubert variety is* $\Sigma_{2,1} = \{U \mid p \in U \subset H\}$*, i.e., lines in a plane H that pass through p, and the Schubert cycle* $\sigma_{2,1} \in H^6(G(2,4),\mathbb{Z})$ *corresponds to the matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}$.
- *For λ=(2,0), the Schubert variety is* $\Sigma_{2,0} = \{U \mid p \in U\}$*, i.e., lines through the point p, and the Schubert cycle* $\sigma_{2,0} \in H^4(G(2,4),\mathbb{Z})$ *corresponds to the matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}$.
- *For λ=(1,1), the Schubert variety is* $\Sigma_{1,1} = \{U \mid U \subset H\}$*, i.e., lines in the plane* H*, and the Schubert cycle* $\sigma_{1,1} \in H^4(G(2,4),\mathbb{Z})$ *corresponds to the matrix* $\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix}$.
- *For λ=(1,0), the Schubert variety is* $\Sigma_{1,0} = \{U \mid U \cap L = \emptyset\}$*, i.e., lines incident to L, and the Schubert cycle* $\sigma_{1,0} \in H^2(G(2,4),\mathbb{Z})$ *corresponds to the matrix* $\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}$.
- *For λ=(0,0) the Schubert variety is* $\Sigma_0 = \mathbb{G}(1,3)$*, i.e., lines in* $\mathbb{P}^3$*, and the Schubert cycle* $\sigma_{0,0} \in H^0(G(2,4),\mathbb{Z})$ *corresponds to the matrix* $\begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}$.

By Theorem 3.1.2., we can see, for instance, that the Schubert cycles $\sigma_{2,0}$ *and* $\sigma_{1,1}$

corresponding to the codimension 2 Schubert varieties $\Sigma_{2,0}$ and $\Sigma_{1,1}$ generate the cohomology group $H^4(G(2,4),\mathbb{Z})$.

3.2 Pieri's Rule and Giambelli's Rule

In the last section, we have shown that for a given admissible partition, the cohomology ring of the Grassmannian $G(d,n)$ is additively generated by Schubert cycles. So we can express any product of two given Schubert cycles in the cohomology ring $H^*(G(d,n),\mathbb{Z})$ as a linear combination of these generators. A nice way to formulate this relation is the following theorem that arises from the well known Littlewood-Richardson rule (LR):

Theorem 3.2.1. *Let σ_λ, σ_μ be Schubert cyles with respect to the partitions λ and μ and let ν be an arbitrary partion. Then*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu}\, \sigma_\nu .$$

The coeffients $c^\nu_{\lambda,\mu}$ are called Littlewood-Richardson coefficients.

The coefficients $c^\nu_{\lambda,\mu}$ represent the degree of intersection of two Schubert cycles. Originally, Littlewood-Richardson coefficients were defined in order to multiply Schur functions, as they are a basis of symmetric functions, but in a similar context there are multiple useful computations for the Littlewood-Richardson coefficients. There are many works about Young diagrams ([23]) and Littlewood-Richardson rule ([44] [6]). Although LR is studied from the combinatorial perspective, recently ([9], vakil) gave geometric proofs. In the next section, though we will not give too much detail, using Young tableaux is one of many ways to calculate these coefficients. In this section we will give examples of special cases of Littlewood-Richardson rule.

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$ be an admissible partition. λ is called a **special partition** if $\lambda_2 = \lambda_3 = ... = \lambda_d = 0$). The Schubert class defined with respect to such a special partition is called the **special Schubert class**. For ease of notation, we will denote such a special Schubert class as σ_{λ_1} instead of $\sigma_{\lambda_1,0,0,..,0}$

Theorem 3.2.2. ***Pieri's Rule**. Let σ_λ be a special Schubert class, and let σ_μ be any*

Schubert class with an admissible partition μ. Then

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{|\nu|=|\lambda|+|\mu| \\ \mu_i \leq \nu_i \leq \mu_{i-1}}} \sigma_\nu$$

Proof. See [18]. ■

Let us work out Pieri's Rule with an example over the Grassmannian $G(2,5)$.

Example 3.2.3. *We will compute some products of Schubert classes of the Grasmmannian $G(2,5)$. Note that $dim(G(2,5))$=6 and partitions λ should satisfy admissibility criteria. Schubert classes of $G(2,5)$ are $\sigma_{3,3}$, $\sigma_{3,2}$, $\sigma_{3,1}$, σ_3, $\sigma_{2,2}$, $\sigma_{2,1}$, σ_2, $\sigma_{1,1}$, σ_1, and σ_0.*

- $\sigma_1 \cdot \sigma_{3,3} = 0$
- $\sigma_1 \cdot \sigma_{3,1} = \sigma_{3,2}$
- $\sigma_1 \cdot \sigma_{2,1} = \sigma_{3,2} + \sigma_{3,1}$
- $\sigma_2 \cdot \sigma_{2,1} = \sigma_{3,2}$
- $\sigma_2 \cdot \sigma_2 = \sigma_{3,1} + \sigma_{2,2}$
- $\sigma_2 \cdot \sigma_{3,1} = \sigma_{3,3}$
- $\sigma_3 \cdot \sigma_{1,1} = 0$
- $\sigma_3 \cdot \sigma_3 = \sigma_{3,3}$
- $\sigma_3 \cdot \sigma_1 = \sigma_{3,1}$

The next theorem describes the computation for a special Grassmannian $G(2, n+1) = \mathbb{G}(1,n)$ where the Schubert varieties with respect to the flag $\mathcal{F}$ are as follows:

$$\Sigma_{\lambda_1,\lambda_2} = \{U \in G(2, n+1) \mid U \cap V_{n-\lambda_1} \neq 0 \; and \; U \subset V_{n+1-\lambda_2}\}.$$

Theorem 3.2.4. *Suppose, for admissible partitions $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$, that $\lambda_1 - \lambda_2 \geq \mu_1 - \mu_2$. Then*

$$\begin{aligned}\sigma_{\lambda_1,\lambda_2} \cdot \sigma_{\mu_1,\mu_2} &= \sigma_{\lambda_1+\mu_1,\lambda_2+\mu_2} + \sigma_{\lambda_1+\mu_1-1,\lambda_2+\mu_2+1} + \ldots + \sigma_{\lambda_1+\mu_2,\lambda_2+\mu_1} \\ &= \sum_{\substack{|\nu|=|\lambda|+|\mu| \\ \lambda_1+\mu_1 \leq \nu_1 \leq \lambda_1+\mu_2}} \sigma_{\nu_1,\nu_2}.\end{aligned}$$

As we can see, Pieri's formula is useful for computations involving special Schubert classes. However, we need to be able to compute the product of arbitrary Schubert classes. Giambelli's Rule will show us that any Schubert class can be expressed in terms of special Schubert classes. Using both Pieri's and Giambelli's Rules, we can compute the intersection of two arbitrary Schubert classes.

Theorem 3.2.5. ***Giambelli's Rule.*** *Any Schubert class can be expressed as a linear combination of products of special Schubert classes:*

$$\sigma_{\lambda_1,\dots,\lambda_d} = \begin{vmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \sigma_{\lambda_1+2} & \cdots & \sigma_{\lambda_1+d-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \sigma_{\lambda_2+d-2} \\ & & \cdots & \cdots & \\ \sigma_{\lambda_d-d+1} & \sigma_{\lambda_d-d+2} & \sigma_{\lambda_d-d+3} & \cdots & \sigma_{\lambda_d} \end{vmatrix}$$

Following the Example 3.2.3., we can compute $\sigma_{2,2}$ as follows:

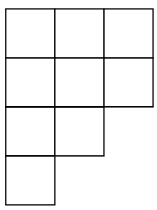
$$\sigma_{2,2} = \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 \end{vmatrix} = \sigma_2^2 + \sigma_3\sigma_1.$$

3.3 Young Tableaux and Littlewood Richarson Rule

Young tableaux or **Young diagram** is an alternative and practical way to represent Schubert classes. As we mentioned before, there is an alternative formulation for Littlewood-Richardson rule involving Young diagrams. In this section, we will introduce the basic concepts of Young diagrams and develop necessary tools in order to understand the theorem. For a detailed discussion and more information about this section, [23] is a great source.

Definition 3.3.1. *A **Young diagram** is a collection of left-justified boxes with a weakly decreasing number of boxes in each row.*

The number of boxes in each row forms a partition of an integer $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$ with the total number of boxes being $|\lambda| = \sum |\lambda_i|$. Conversely, every partition corresponds to a Young diagram. For example, partition $9 = 3 + 3 + 2 + 1$ corresponds to the Young diagram



If $|\lambda| = n$, we say that λ is a partition of n and we denote $\lambda \vdash n$. We define Λ to be the set of partitions whose diagram fit inside a $d \times (n-d)$ rectangle. The largest partition in Λ, $n-d \geq n-d \geq ... \geq n-d$ is denoted by $\square^{d}$ .We can put numbers inside the boxes. Putting positive integers in each box is called a **filling**.

Definition 3.3.2. *A **Young tableau** is a filling that is weakly increasing along each row and strictly increasing down in each column. A **standard tableau** is a Young tableau consisting of entries from 1 to n.*

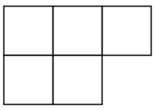
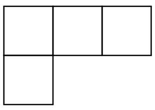
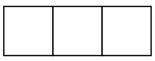
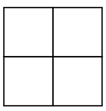
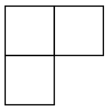
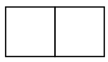
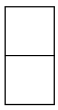
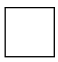
For instance, with partition $9 = 3 + 3 + 2 + 1$, Young tableau and standard Young tableau correspond to followings, though they can be filled differently in many ways, respectively:

1	2	2
2	3	5
4	5	
6		

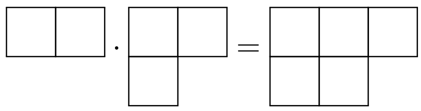
1	3	4
2	5	7
6	9	
8		

In the context of Schubert calculus, we will not deal with every partition of an integer λ. As we defined in the beginning of the chapter, we are going to look at admissible partitions $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$, $n - d \geq \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d \geq 0$. We have a correspondence between such partitions and the Schubert classes. In this way, we can represent each Schubert class σ_λ of the Grassmannian $G(d, n)$.

To see the correspondence clearly on an example, we will work on $G(2, 5)$.

- $\sigma_{3,3} \longleftrightarrow$ 
- $\sigma_{3,2} \longleftrightarrow$ 
- $\sigma_{3,1} \longleftrightarrow$ 
- $\sigma_{3} \longleftrightarrow$ 
- $\sigma_{2,2} \longleftrightarrow$ 
- $\sigma_{2,1} \longleftrightarrow$ 
- $\sigma_{2} \longleftrightarrow$ 
- $\sigma_{1,1} \longleftrightarrow$ 
- $\sigma_{1,0} \longleftrightarrow$ 
- $\sigma_{0} \longleftrightarrow \emptyset$

Young diagrams make calculations easier. For instance, we saw that for a special Schubert class σ_2, we have $\sigma_2 \cdot \sigma_{2,1} = \sigma_{3,2}$. In Young diagram notation, we add corresponding boxes of σ_2 to the diagram of $\sigma_{2,1}$ one by one in each row satisfying the Young diagram conditions and the inequality in Pieri's Rule:

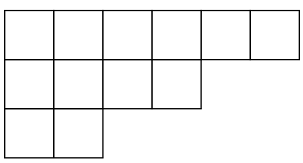
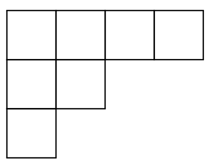


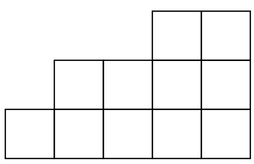
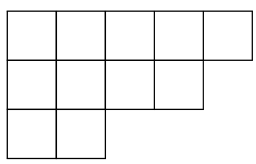
Definition 3.3.3. *A* ***skew Young diagram*** *or* ***skew shape*** λ/μ *is a difference of Young diagrams of* λ *and* μ*, where* $\lambda \geq \mu$*.*

Let $\lambda, \mu \in \Lambda$ and λ/μ be a skew shape which fits inside $d \times (n-d)$ rectangle. We denote μ^d for the skew shape $\square^d$ / μ, and let $\mu^\vee := (n-d-\mu_d, ..., n-d-\mu_1)$ denote the **dual partition** obtained by rotating μ^d by 180°. The number of boxes in λ/μ is $|\lambda/\mu| = |\lambda| - |\mu|$.

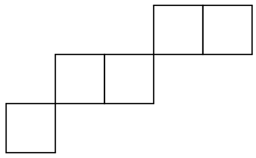
Definition 3.3.4. *A* ***standard Young tableau*** *of shape* λ/μ *is a filling of the boxes of* λ/μ *with entries* $1, 2, ..., |\lambda/\mu|$ *where the entries increase through the rows and columns. The set of all such tableaux is denoted by* ***SYT(****λ/μ****)****. When repetition is allowed in the filling , we call it simply a* ***skew Young tableau*** *and the set of all such tableaux is denoted by* $SYT(\lambda/\mu; +)$

Example 3.3.5. *Considering* $G(3,9)$*, let* $\lambda = (6,4,2)$ *and* $\mu = (4,2,1)$ *be partitions. Then we have,*

$\lambda =$  , $\mu =$ 

$\mu^d =$  , $\mu^\vee =$ 

The skew Young diagram λ/μ *is*



Definition 3.3.6. *A **row-bumping** or **row insertion** is a method of producing a new Young tableau denoted by $T \leftarrow x$ out of a given tableau T and a positive integer x. Insertion process is carried out by the following rules:*

1. *If x is greater than or equal to the all entries in the first row of T, add x in a new box on the first row.*
2. *If not, find the left-most entry of the first row of T that is strictly larger than x, and replace it with x, i.e., **bump** the entry.*
3. *Apply 1 and 2 to the entry replaced by x, starting from the second row.*
4. *Repeat the process until the bumped entry finds its position either at the bottom as a new row, or as the last box of the last row of T.*

Example 3.3.7. *Let $\lambda = (4, 3, 3, 2)$ be a partition and let T be its corresponding Young tableau with the following filling:*

1	2	2	4
3	5	7	
6	6	9	
8	9		

If we want to row-insert 4 in T, $T \leftarrow 4$, then we get

1	2	2	4	4
3	5	7		
6	6	9		
8	9			

If we want to have $T \leftarrow 3$, then we get

1	2	2	3
3	4	7	
5	6	9	
6	9		
8			

For any two tableaux T and U, row-insertion can be used to define a product operation on T and U. The product tableau $T \cdot U$ can be constructed by starting row-inserting the bottom-left entry of U into T. Then the next entry is row-inserted to the resulting tableau. By repeating this process after all entries in the bottom row is row inserted into T, and continuing the same process through the upper rows from left to right, gives us $T \cdot U$.

Example 3.3.8. *Let T and U be Young tableaux as follows:*

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&4\\ \hline 3&5&7\\ \cline{1-3} 6&6&9\\ \cline{1-3} 8&9\\ \cline{1-2}\end{array} \quad , \quad \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}$$

Then the product $T \cdot U$ is calculated as follows:

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&4\\ \hline 3&5&7\\ \cline{1-3} 6&6&9\\ \cline{1-3} 8&9\\ \cline{1-2}\end{array} \cdot \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\ \hline 3&4&7\\ \cline{1-3} 5&6&9\\ \cline{1-3} 6&9\\ \cline{1-2} 8\\ \cline{1-1}\end{array} \cdot \begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}$$

$$= \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&4&7\\ \cline{1-3} 3&6&9\\ \cline{1-3} 5&9\\ \cline{1-2} 6\\ \cline{1-1} 8\\ \cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 2\\ \hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&3&7\\ \cline{1-3} 3&4&9\\ \cline{1-3} 5&6\\ \cline{1-2} 6&9\\ \cline{1-2} 8\\ \cline{1-1}\end{array}$$

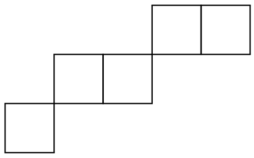
Observe that the number of the boxes in $T \cdot U$ is equal to the total number of boxes of tableaux T and U. Apart from row-insertion, there is another method called **sliding** to obtain the product of two tableaux. In order to define **sliding**, we will make some more definitions.

Definition 3.3.9. *Let T be a skew Young diagram of shape λ/μ.*

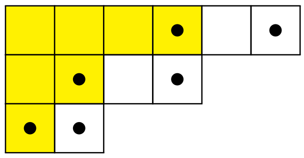
- *An **inside corner** is a box in μ where the box on the right side and below does not belong to μ.*

- *An **outside corner** is a box in λ where the box on the right side and below does not belong to λ.*

Example 3.3.10. *Following the Example 3.3.5, let T be the skew Young diagram of shape λ/μ as follows:*



In order to understand the operation and what the inside and outside corners are, we will specify the diagram of μ:

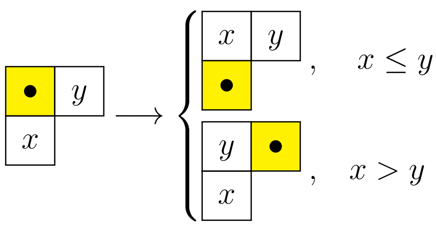


Inside corners are the yellow boxes with dots, and the outside corners are the white boxes with dots.

Definition 3.3.11. *Given a skew Young Tableau T and an inside corner, the sliding operation is carried out with the following rules:*

1. *Compare the entries of the right side and below of the inside corner. Swap boxes with the smaller entry.*
2. *If there is no box below, swap with box on the right side.*
3. *Continue this process until the inside corner becomes an outside corner.*
4. *Remove the outside corner from the tableau.*

Explicitly, in a single step of a sliding operation, one of the following happens:

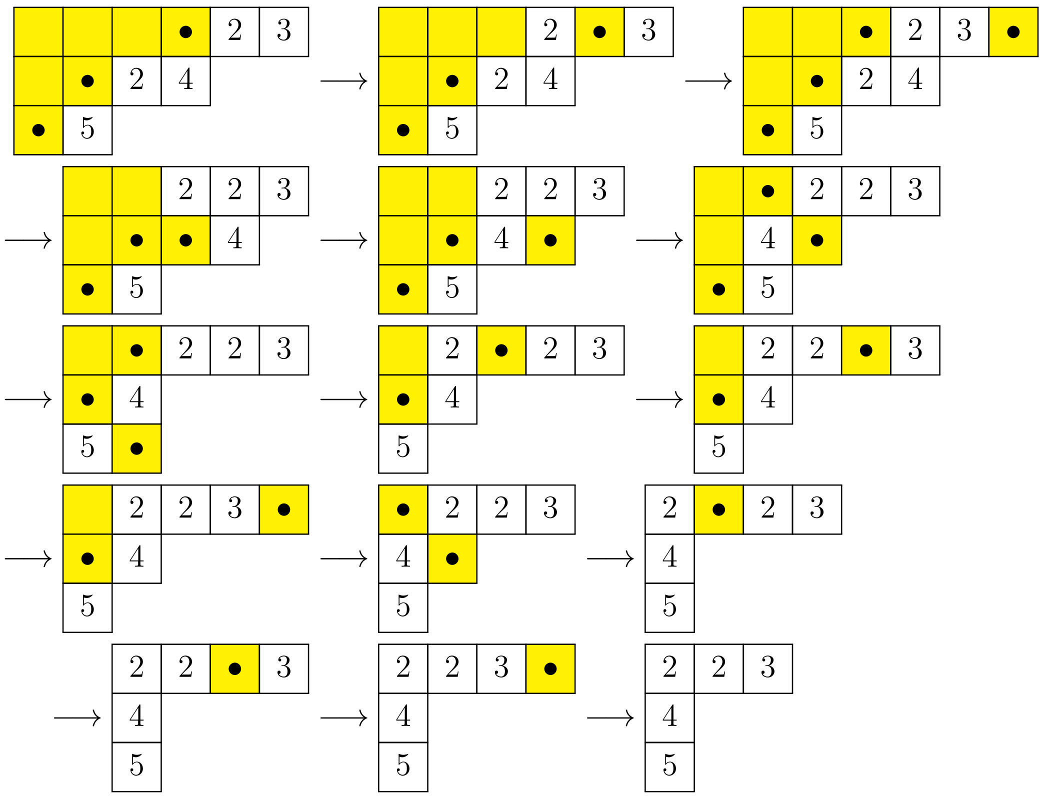


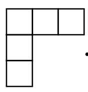
For a skew Young tableau T, this process can be done with each inside corner. If we continue this process until there is no inside corner left, the result will again be a tableau. The resulting tableau is called the **rectification** of T, denoted **rect(T)**.

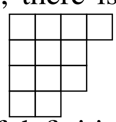
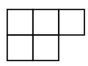
The **rectification shape** of T is the shape of $rect(T)$. When the partitions are in consideration, i.e., if $T \in SYT(\lambda/\mu)$, for partitions λ and μ, then rectification of T is also denoted as $slide_U(T)$ where $U \in SYT(\mu)$.

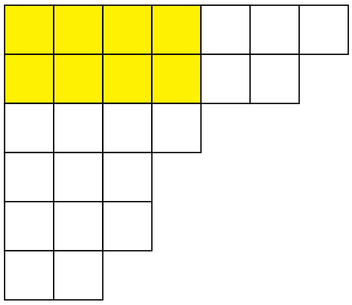
This operation is independent of the choice of an inside corner. If we have started with a different inside corner, the resulting tableau would be the same. The sliding operation that was introduced by Schützenberger and together with the rectification process, it is also referred to as **jeu de taquin**.

Example 3.3.12. *Continuing the above Example 3.3.10., let S be the skew Young tableau with some arbitrary filling. Then the rectification is carried out as follows:*



So the rectification shape of T is the resulting diagram  .

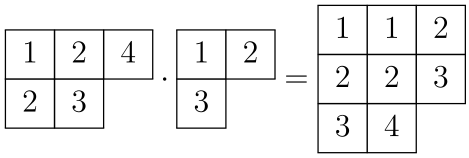
For given Young tableaux T and U, there is a way to express their product using sliding and rectification. Let $T =$  $U =$  be two Young tableaux, omitting the filling for the purpose of definition. Then let $T * U$ denote the following tableau:

$T * U =$ 

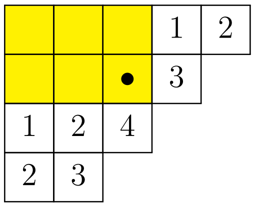
Then we have

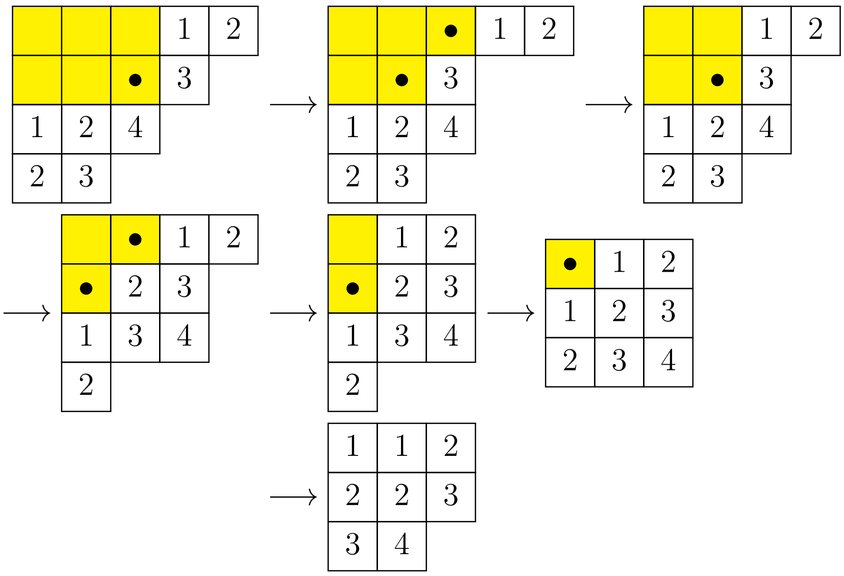
$$T \cdot U = rect(T * U).$$

Example 3.3.13. *For the purpose of this example, we are going to choose smaller tableau with standard filling in order to make calculations easier. So let* $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ *and* $U = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$. *Then* $T \cdot U$ *using row-insertion is*

$T \cdot U =$ 

Let us calculate the rectification of $T * U$*:*

$T * U =$ 



Definition 3.3.14. *If for* $T \in SYT(\lambda/\mu; +)$ *and* $T' \in SYT(\lambda'/\mu'; +)$*, we have* $rect(T) = rect(T')$*, then we say that* T *and* T' *are* ***equivalent*** *and denote* $T \sim T'$*.*

Definition 3.3.15. *For $T, T' \in SYT(\lambda/\mu; +)$, if $slide_T$ and $slide'_T$ are identical operations, then we say that T and T' are **dual equivalent**, and write $T \sim^* T'$.*

Definition 3.3.16. *For any tableaux $U_\circ$ of shape μ and $V_\circ$ of shape ν, we define the following sets:*

$$\mathcal{S}(\mu/\lambda, U_\circ) = \{S \; of \; shape \; \mu/\lambda \mid rect(S) = U_\circ\}$$
$$\mathcal{T}(\lambda, \mu, V_\circ) = \{[T, U] \mid T \; of \; shape \; \lambda, U \; of \; shape \; \mu, \; rect(T * U) = V_\circ\}$$

Proposition 3.3.17. *For any tableaux $U_\circ$ of shape μ and $V_\circ$ of shape ν, there is a canonical one-to-one correspondence*

$$\mathcal{S}(\mu/\lambda, U_\circ) \leftrightarrow \mathcal{T}(\lambda, \mu, V_\circ).$$

Corollary 3.3.18. *Cardinalities of the sets $\mathcal{S}(\mu/\lambda, U_\circ)$ and $\mathcal{T}(\lambda, \mu, V_\circ)$ are independent of choice of $U_\circ$ or $V_\circ$, and depend only on the shapes λ, μ and ν. The cardinalities of these sets are denoted by $c^{\nu}_{\lambda,\mu}$ and are called **Littlewood-Richardson coefficients**.*

Proof. See [23] ∎

Definition 3.3.19.
- *Given a tableau or skew tableau T, a **word** of T, denoted $w(T)$, is defined by reading the entries of T from left to right and bottom to top.*
- *A lattice word $w = x_1 x_2 ... x_r$ is called a **reverse lattice word**, if when it is read backwards from the end to any letter, the sequence $x_r x_{r-1} ... x_s$ contains as many 1's as it does 2's, at least as many 2's as 3's, and so on for all positive integers.*
- *A skew tableau T is called a **Littlewood-Richardson skew tableau** if its word $w(T)$ is a reverse lattice word.*

Example 3.3.20. *Let* $T = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 2 & 2 & \\ \hline 3 & & & \\ \hline \end{array}$ *and* $T' = \begin{array}{|c|c|c|c|} \hline & & 2 & 3 \\ \hline & 1 & 3 & \\ \hline 2 & & & \\ \hline \end{array}$. *Then $w(T) = 32211$ and $w(T') = 21323$. Observe that $w(T)$ is a reverse lattice word whereas $w(T')$ is not. Thus, T is a Littlewood-Richardson skew tableau.*

Definition 3.3.21. *A skew tableau is said to have **content** $\mu = (\mu_1, , , \mu_l)$ if its entries consist of μ_1 1's, μ_2 2's, an so on up to μ_l l's.*

Now we are finally ready to state the Littlewood-Richardson rule:

Theorem 3.3.22. *Let σ_λ, σ_μ be Schubert cycles with respect to the partitions λ and μ and let ν be an arbitrary partition. Then*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu}\, \sigma_\nu .$$

where the coefficients $c^\nu_{\lambda,\mu}$ are the number of Littlewood-Richardson skew tableau of shape ν/λ of content μ.

Alternatively, $c^\nu_{\lambda,\mu}$ counts the different ways of a tableaux V of shape ν that can be written as a product of tableau T of shape λ and a tableau U of shape μ.

Theorem 3.3.23. *The Littlewood-Richardson coefficient*

$$c^\nu_{\lambda,\mu} = \sum_\sigma \sigma_{\lambda^\vee}\sigma_\mu\sigma_\nu$$

is the number of dual equivalence classes in SYT(λ / μ;+) with rectification shape ν.

Proof. See [42]. ■

Example 3.3.24. *The followings are all of the Littlewood-Richardson skew tableau on the skew shape $(4,3,2,1)/(2,1,1)$:*

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & & \\ 1 & & & \end{array}, \quad \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & & \\ 2 & & & \end{array}, \quad \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & & \\ 3 & & & \end{array}, \quad \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 3 & & \\ 1 & & & \end{array}$$

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 3 & & \\ 2 & & & \end{array}, \quad \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 3 & & \\ 4 & & & \end{array}, \quad \begin{array}{cccc} & & 1 & 1 \\ & 2 & 2 & \\ & 3 & & \\ 1 & & & \end{array}$$

$$\begin{array}{cccc} & & 1 & 1 \\ & 2 & 2 & \\ & 3 & & \\ 3 & & & \end{array}, \quad \begin{array}{cccc} & & 1 & 1 \\ & 2 & 2 & \\ & 3 & & \\ 4 & & & \end{array}.$$

As $\nu = (4,3,2,1)$ and $\lambda = (2,1,1)$ fixed, the Littlewood-Richardson numbers $c^{\nu}_{\lambda,\mu}$, as μ changes, are as follows:

$$c^{\nu}_{\lambda,\mu} = 1 \; for \; \mu = (4,2),(3,3),(4,1,1),(3,1,1,1),(2,2,2) \; and \; (2,2,1,1)$$

$$c^{\nu}_{\lambda,\mu} = 3 \; for \; \mu = (3,2,1)$$

$$c^{\nu}_{\lambda,\mu} = 0 \; for \; all \; other \; \mu.$$

CHAPTER 4

FLAG VARIETIES

4.1 Basics

We can generalize the results for Grassmannians that we established in Chapter 2. Before moving into a more formal definition, we can picture it verbally. The generalization is called **flag varieties**, and they are basically defined as homogeneous spaces whose points are **flags**.

Definition 4.1.1. *Let $V = k^n$ where k is an algebraically closed field. A **flag** $\mathcal{F}$ in V is*

$$\mathcal{F} : (0) = V_0 \subset V_1 \subset ... \subset V_n = V$$

*such that $dim(V_i) = d_i$. The **signature** of a flag is defined to be the sequence*

$$(dim(V_1), dim(V_2), ..., dim(V_n)).$$

Let $d_1, d_2, ..., d_r, n$ be integers such that $0 < d_1 < d_2 < ... < d_r < n$. We define $F(d_1, d_2, ..., d_r; n)$ to be the set of all possible flags in V with signature $(d_1, d_2, ..., d_r, n)$. $F(d_1, d_2, ..., d_r; n)$ is called an ***r-step flag variety***.

Example 4.1.2. *If $r = 1$, then we have flag $(0) \subset V_{d_1} \subset V$. But this exactly means that $F(d_1; n) = G(d_1, n)$.*

When $r = n-1$, $F(d_1, d_2, ..., d_r; n)$ is called a **full flag variety** or **complete flag variety**. Otherwise it is called a **partial flag variety**. One special case of partial flag varieties is when $r = 2$, which is called a **2-step flag variety**. Important results have been established for both partial varieties and 2-step varieties such as Coşkun's works [8, 9] on Littlewood-Richardson Rule and many more. See [1, 6, 7, 35, 51].

We can observe that $F(d_1, d_2, ..., d_r; n)$ is contained in $G(d_1, n) \times ... \times G(d_r, n)$. We have already seen in **Example 3.1.2.** that this is the case when $r = 1$, i.e., $F(d_1; n) = G(d_1, n)$. For the general case, we have a canonical inclusion

$$\Phi : F(d_1, d_2, ..., d_r; n) \hookrightarrow G(d_1, n) \times ... \times G(d_r, n)$$

sending

$$\mathcal{F} : V_0 \subset V_1 \subset ... \subset V_r \longmapsto (V_1, V_2, ..., V_r).$$

Thus we can write the following:

$$F(d_1, d_2, ..., d_r; n) = \{(V_1, V_2, ..., V_r) \in \prod_r G(d_r, n) \,|V_0 \subset V_1 \subset ... \subset V_r\}.$$

As we did in the previous chapter, we can endow a projective variety structure to $F(d_1, d_2, ..., d_r; n)$ via Plücker embedding. Since we saw the inclusion part, we need to show that it is a closed subset.

Proposition 4.1.3. *$F(d_1, d_2, ..., d_r; n)$ is a Zariski closed subset of $G(d_1, n) \times ... \times G(d_r, n)$.*

Proof. We already know that this is true when $r = 1$. Notice that

$$\Phi : F(d_1, d_2, ..., d_r; n) \to G(d_1, n) \times ... \times G(d_r, n)$$

$$\mathcal{F} : \{V_i\} \longmapsto (V_1, V_2, ..., V_r)$$

is an embedding. Consider the projection map

$$\pi_{i,j} : G(d_1, n) \times ... \times G(d_r, n) \to G(d_i, n) \times G(d_j, n)$$

for any $1 \leq i < j \leq r$. Consider the restriction to $F(d_1, d_2, ..., d_r; n)$. Then we have

$$\Phi(F(d_1, d_2, ..., d_r; n)) = \bigcap_{1 \leq i < j \leq n} \pi_{i,j}^{-1}(\pi_{i,j}(\Phi(F(d_1, d_2, ..., d_r; n)))).$$

Since $\pi_{i,j}(\Phi(F(d_1, d_2, ..., d_r; n))) = F(d_i, d_j; n)$, it is enough to show that $F(d_i, d_j; n)$ is Zariski closed in $G(d_i, n) \times G(d_j, n)$ for $i < j$.

Now, let $(U, W) \in G(d_i, n) \times G(d_j, n)$. Let $\{u_1, u_2, ..., u_i\}$ be a basis for U and $\{w_1, w_2, ..., w_j\}$ be a basis for W. Let $u = u_1 \wedge u_2 \wedge ... \wedge u_i$ and $w = w_1 \wedge w_2 \wedge ... \wedge w_j$. As in the Grassmannian case, we have maps

$$\varphi_u : V \to \bigwedge^{i+1} V$$

and

$$\varphi_w : V \to \bigwedge^{j+1} V.$$

Direct sum of these maps will give us

$$\varphi_u \oplus \varphi_w : V \to \bigwedge^{i+1} V \oplus \bigwedge^{j+1} V$$

and ker$(\varphi_u \oplus \varphi_w) = U \cap W$. Furthermore, by rank-nullity we have

$$rank(\varphi_u \oplus \varphi_w) = n - ker(\varphi_u \oplus \varphi_w) = n - dim(U \cap W) \geq n - i.$$

Thus, by Remark 2.3.4, $\varphi_u \oplus \varphi_w$ has rank at most $n - i$ if and only if all $(n - i + 1) \times (n - i + 1)$ minors vanish. So $F(d_i, d_j; n)$ can be defined by the zero locus of the $(n - i + 1) \times (n - i + 1)$ minors of the matrix $\varphi_u \oplus \varphi_w$. ■

As in the case of Grassmannians, those equations are not as simple as possible. We can find finer polynomials such that they generate the homogeneous ideal defining the flag varieties. To see this over 2-step flag varieties, let V be an n-dimensional vector space with basis $\{v_1, v_2, ..., v_n\}$ and let $F(d_1, d_2; n)$ be a 2-step flag variety:

$$F(d_1, d_2; n) = \{(U, W) \in G(d_1, n) \times G(d_2, n) \mid U \subset W\}$$

Let $\{u_1, u_2, ..., u_{d_1}\}$ and $\{w_1, w_2, ..., w_{d_2}\}$ be bases for U and W, respectively. Since we can see each Grassmannian $G(d_1, n)$, $G(d_2, n)$ in $\bigwedge^{d_1} V$ and $\bigwedge^{d_2} V$, we can write

$$u_1 \wedge u_2 \wedge ... \wedge u_{d_1} = \sum a_{\mathbf{i}} v_{i_1} \wedge ... \wedge v_{id_1}$$

$$w_1 \wedge w_2 \wedge ... \wedge w_{d_2} = \sum a_{\mathbf{j}} v_{j_1} \wedge ... \wedge v_{j_{d_1}}$$

where

$$\mathbf{i} = \{(i_1, i_2, ..., i_{d_1}) \mid 1 \leq i_1 < i_2 < ... < i_{d_1} \leq n\}$$

$$\mathbf{j} = \{(j_1, j_2, ..., j_{d_2}) \mid 1 \leq j_1 < j_2 < ... < j_{d_2} \leq n\}$$

are index sets. So the defining equations for $F(d_1, d_2; n)$ are given by

$$(u_1 \wedge u_2 \wedge ... \wedge u_{d_1}) \cdot (w_1 \wedge w_2 \wedge ... \wedge w_{d_2}) -$$
$$\sum (u_1 \wedge ... \wedge w_1 \wedge ... \wedge w_k \wedge ... \wedge u_{d_1}) \cdot (u_{i_1} \wedge ... \wedge v_{i_k} \wedge w_{k+1} \wedge ... \wedge w_{d_2}) = 0$$

for $1 \leq k \leq d_2$.

In terms of homogeneous coordinates, we have the following quadratic equation:

$$x_{\mathbf{i}} \cdot x_{\mathbf{j}} - \sum x_{\mathbf{i}'} x_{\mathbf{j}'} = 0$$

where $\mathbf{i}'$ is obtained by changing first k indices of $\mathbf{i}$ with k indices of $\mathbf{j}$ by preserving the order. Thus the homogenous ideal of $F(d_1, d_2; n)$ is generated by the above polynomials.

Let us see this over an example.

Example 4.1.4. *Let V be a 4-dimensional vector space with basis $\{v_1, v_2, v_3, v_4\}$. Let $F(1,2;4)$ be a 2-step flag variety*

$$F(1,2;4) = \{(U, W) \in G(1,4) \times G(2,4) \mid U \subset W\}$$

where $\{u_1\}$ and $\{w_1, w_2\}$ are bases for the subspaces U and W, respectively. Write $u_1 = \sum a_i v_i$ and $w_1 \wedge w_2 = \sum_{i<j} b_{ij} v_i \wedge v_j$. Then

$$\begin{aligned} u_1 \wedge w_1 \wedge w_2 &= \sum (a_1 b_{23} - a_2 b_{13} + a_3 b_{12}) v_1 \wedge v_2 \wedge v_3 \\ &= \sum (a_1 b_{24} - a_2 b_{14} + a_4 b_{12}) v_1 \wedge v_2 \wedge v_4 \\ &= \sum (a_1 b_{34} - a_3 b_{14} + a_4 b_{13}) v_1 \wedge v_3 \wedge v_4 \\ &= \sum (a_2 b_{34} - a_3 b_{24} + a_4 b_{23}) v_2 \wedge v_3 \wedge v_4 \end{aligned}$$

In order to have $U \subset W$, we need $u_1 \wedge w_1 \wedge w_2 = 0$. So in terms of homogeneous coordinates, the homogeneous ideal of $F(1,2;4)$ is generated by the quadratic equations $x_1x_{23} - x_2x_{13} + x_3x_{12}$, $x_1x_{24} - x_2x_{14} + x_4x_{12}$, $x_1x_{34} - x_3x_{14} + x_4x_{13}$, $x_2x_{34} - x_3x_{24} + x_4x_{23}$. So

$$\begin{aligned} F(1,2;4) = V(&x_1x_{23} - x_2x_{13} + x_3x_{12}, x_1x_{24} - x_2x_{14} + x_4x_{12}, \\ &x_1x_{34} - x_3x_{14} + x_4x_{13}, x_2x_{34} - x_3x_{24} + x_4x_{23}). \end{aligned}$$

In the general case, the concept is the same. For any partial flag variety $F(d_1, d_2, ..., d_r; n)$, the defining quadratic equations are

$$\begin{aligned} &(u_1 \wedge u_2 \wedge ... \wedge u_p) \cdot (w_1 \wedge w_2 \wedge ... \wedge w_q) - \\ &\sum (u_1 \wedge ... \wedge w_1 \wedge ... \wedge w_k \wedge ... \wedge u_p) \cdot (u_{i_1} \wedge ... \wedge v_{i_k} \wedge w_{k+1} \wedge ... \wedge w_q) = 0 \end{aligned}$$

for $1 \leq k \leq q$ and $p \leq q$ in $\{d_1, d_2, ..., d_r\}$. Simirlarly, in terms of homogeneous coordinates, we can write

$$x_{\mathbf{i}} \cdot x_{\mathbf{j}} - \sum x_{\mathbf{i}'} x_{\mathbf{j}'} = 0$$

where

$$\mathbf{i} = \{(i_1, i_2, ..., i_p) \mid 1 \leq i_1 < i_2 < ... < i_p \leq n\}$$

$$\mathbf{j} = \{(j_1, j_2, ..., j_q) \mid 1 \leq j_1 < j_2 < ... < j_q \leq n\}$$

with $p \leq q$ in $\{d_1, d_2, ..., d_r\}$.

4.2 Cohomology of Flag Varieties

Cohomology of flag varieties is a generalization of the concept for the Grassmannian. As we have seen above, Grassmannians are considered as a special case of flag varieties. Since the Grassmannian $G(d, n) = F(d_1, d_2, ..., d_r; n)$ with $r = 1$ is a flag variety, we can define the same structure for the flag varieties.

Let

$$\mathcal{F} : (0) = V_0 \subset V_1 \subset ... \subset V_n = V$$

be a complete flag such that $dim(V_i) = d_i$. Let $\{v_1, v_2, ..., v_n\}$ be a basis for V and a basis for each V_i is the first i elements of this basis, namely $\{v_1, v_2, ..., v_i\}$. Schubert varieties for the r-step flag varieties are parametrized by the sequences S of integers of length n with the integers $1, 2, ..., r+1$, where $d_i - d_{i-1}$ of the entries are i. Given that S_t is the t-th place in the sequence S, the Schubert variety is defined as follows

$$\Sigma_\omega(\mathcal{F}) = \{(U_1, ..., U_r) \in F(d_1, ...d_r; n) \mid \\ dim(U_i \cap V_j) \geq \#\{t \mid S_t \leq i\ ,\ t \leq j\}\}.$$

A basis for the cohomology of flag varieties is formed by the Poincare duals of the classes of all Schubert varieties. For each Schubert class σ_λ in $G(d, n)$, there is a special Schubert class $\sigma_\lambda^{(k)}$ in $F(d-k, d+k; n)$ given by

$$\Sigma_\lambda^{(k)} = \{(U_1, U_2) \mid dim(U_1 \cap V_{n-i-\lambda_{d-i}}) \geq d-k-i,\ dim(U_2 \cap V_{n-d+j-\lambda_j}) \geq j\}$$

where $1 \leq i \leq n-d$ and $1 \leq j \leq d$.

As an alternative to the sequence notation, for the cohomology classes of the flag varieties, we are going to introduce a similar notation to the one we used for Grassmannians. We can use a pair of sequences λ, δ of length d_r where the sequence $\lambda_1 \geq ... \geq \lambda_{d_r}$ has the digit in the $(n - d_r + i - \lambda_i)$-th position less than $r + 1$. Schubert classes of the r-step flag variety is denoted by $\sigma^{\delta_1,...,\delta_{d_r}}_{\lambda_1,...,\lambda_{d_r}}$ where λ_i's correspond to the partition that is defining the d_r-plane U_r considered as Schubert class in $G(d_r, n)$; whereas the upper δ_i's are recording the digits in the gth d_r where the sequence $\lambda_1 \geq ... \geq \lambda_{d_r}$ has the digit in the sequence w and δ_i are the integers between 1 and r, of which d_1 of them is 1, and $d_i - d_{i-1}$ of them is i.

Translation between the two notations is fairly easy. In order to form an integer sequence of length n, put $r+1$ to every position in the sequence except the $(n - d_r + j - \lambda_j)$-th position. For the $(n - d_r + j - \lambda_j)$-th position, place δ_j.

With this notation, we can also define the Schubert varieties as follows:

$$\begin{aligned}\Sigma^{\delta}_{\lambda}(\mathcal{F}) = \{(U_1, ..., U_r) \in F(d_1, ...d_r; n) \mid \\ dim(U_i \cap V_{n-d_r+j-\lambda_j}) \geq \#\{t \leq j \mid \delta_t \leq i\}\}.\end{aligned}$$

Example 4.2.1. *Let* $(0) = V_0 \subset V_1 \subset ... \subset V_9 = V$ *be a flag and let F(2,4;9) be a 2-step flag variety. In order to find the Schubert variety corresponding the sequence 1,3,3,2,3,3,1,3,2 in F(2,4;9), which can be expressed by* $\sigma^{1,2,1,2}_{5,3,1,0}$, *first we find the flag elements that are needed for each* U_i. *For* U_2, *considering* $n - d_r + j - \lambda_j$, *those are the flag elements* V_1, V_4, V_7, V_9. *Then the flag elements for* U_1 *is determined by the upper* δ_i*'s which assign indexes to the flag elements that are less than or equal to* 1. *Those are* V_1 *and* V_7 *Then the Schubert variety is the following:*

$$\begin{aligned}\Sigma^{1,2,1,2}_{5,3,1,0} = \{(U_1, U_2) \in F(2, 4; 9) \mid & dim(U_2 \cap V_1) \geq 1, dim(U_2 \cap V_4) \geq 2, \\ & dim(U_2 \cap V_7) \geq 3, dim(U_2 \cap V_9) \geq 4, \\ & dim(U_1 \cap V_1) \geq 1, dim(U_1 \cap V_7) \geq 2\}.\end{aligned}$$

Partial flag varieties have a long history of interest. In particular, their cohomology rings have been studied in various ways. As there are many interpretations of LR coefficients of the cohomology ring of the Grassmannians, it is quite natural to ask whether there is a way to express the Littlewood-Richarson rule for the cohomology of partial flag varieties. Although there is no LR rule for an arbitrary partial flag variety, there are results for some cases. Recently Buch, Kresch, Purbhoo and Tamvakis in [7] proved the famous puzzle conjecture of Knutson involving triple interctions for two step flag varieties. I. Coşkun gave a geometric proof for two step flag varieties and gave a geometric rule for computing the LR coefficients for partial flag varieties [8, 9]

Apart from the classical cohomology of flag varieties, their quantum cohomology is particularly a point of interest. There are many studies in this topic including [4], [5], [10].

CHAPTER 5

WRONSKI MAP

5.1 Definition and history

Historically, the Wronskian is a determinant that is used to find the dependency of the solutions of differential equations. It is named after the Polish mathematician Józef Hoene-Wroński. For each d linearly independent polynomials of degree at most n, the Wronskian assigns a non-zero polynomial of degree at most $d(n-d)$. In other words, if $(f_1, f_2, ..., f_d)$ are linearly independent polynomials of degree at most n, then their Wronski determinant is

$$Wr(f_1, f_2, ...f_d) = \begin{vmatrix} f_1 & f_2 & \cdots & f_d \\ f_1' & f_2' & \cdots & f_d' \\ & & \vdots & \\ f_1^{(d-1)} & f_2^{(d-1)} & \cdots & f_d^{(d-1)} \end{vmatrix}.$$

where f_i' denotes the usual derivative and $f_i^{(d-1)}$ is the $d-1$-st derivative of the polynomial f_i.

It is quite clear from the basic rules of determinant that $W(f_1, f_2, ...f_d) = 0$ if and only if the polynomials $(f_1, f_2, ...f_d)$ are linearly dependent. Also, again from the properties of the determinant operation, if we multiply the polynomials $(f_1, f_2, ...f_d)$ by a $d \times d$ matrix M, then the Wronskian is multiplied by $det(M)$. These give some insights why and how we can apply Wronskian to Grassmannians and projective space.

We now give the precise relation of Wronskian with our topic of interest. Let $\mathbb{F}[z]$ be the polynomial ring over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathbb{F}_n[z]$ denote the $n+1$-

dimensional vector space of polynomials of degree at most n:

$$\mathbb{F}_n[z] = \{f(z) \in \mathbb{F}[z] \mid deg(f(z)) \leq n\}.$$

Let $0 < d < n$. Let $X = G(d, \mathbb{F}_{n-1}[z])$ denote the Grassmannian of d-dimensional subspaces of an n-dimensional subspace $\mathbb{F}_{n-1}[z]$. As we proved earlier, dim$X$ = $d(n-d)$. So for $U \in X, U = span(f_1, f_2, ...f_d)$, the Wronskian is

$$Wr(f_1, f_2, ...f_d) = \begin{vmatrix} f_1 & f_2 & \cdots & f_d \\ f_1' & f_2' & \cdots & f_d' \\ & & \vdots & \\ f_1^{(d-1)} & f_2^{(d-1)} & \cdots & f_d^{(d-1)} \end{vmatrix}.$$

Calculating this determinant, we can see that the resulting is a polynomial of degree at most $d(n-d)$, let $N = d(n-d)$. By above mentioned properties of the determinant, the Wronskian gives a well-defined map

$$Wr \colon X \longrightarrow \mathbb{P}(\mathbb{F}_N[z]).$$

This map is called the **Wronski map**. For each element $x \in X$, we denote any representative of $Wr(x)$ in $\mathbb{F}_N[z]$ by $Wr(x; z)$. It is natural to say that a point $x \in X$ is real if the subspace x in $\mathbb{F}_{n-1}[z]$ has a basis $f_1, f_2, ..., f_d \in \mathbb{R}_{n-1}[z]$.

Proposition 5.1.1. *The Wronski map is well-defined.*

Proof. Let $U \in X$ be a d-dimensional subspace of $\mathbb{F}_N[z]$. Let $\{f_1, f_2, ..., f_d\}$ and $\{g_1, g_2, ..., g_d\}$ be bases for U. We can write every g_i as follows:

$$\begin{aligned} g_1 &= a_{11}f_1 + ... + a_{1d}f_d \\ g_2 &= a_{21}f_1 + ... + a_{2d}f_d \\ &\vdots \\ g_d &= a_{d1}f_1 + ... + a_{dd}f_d \end{aligned}$$

Thus, the coefficients of f_i's form a $d \times d$ matrix M that is basically the change of basis matrix from $\{f_1, f_2, ..., f_d\}$ to $\{g_1, g_2, ..., g_d\}$. Then under the Wronski map we have

$$\begin{vmatrix} g_1 & g_2 & \cdots & g_d \\ g_1' & g_2' & \cdots & g_d' \\ & & \vdots & \\ g_1^{(d-1)} & g_2^{(d-1)} & \cdots & g_d^{(d-1)} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ & & \vdots & \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{vmatrix} \cdot \begin{vmatrix} f_1 & f_2 & \cdots & f_d \\ f_1' & f_2' & \cdots & f_d' \\ & & \vdots & \\ f_1^{(d-1)} & f_2^{(d-1)} & \cdots & f_d^{(d-1)} \end{vmatrix}.$$

So the Wronskian differs by the determinant of the change of basis matrix M. ■

Example 5.1.2. *Let us see how the Wronski map is applied over an example. Let* $\mathbb{C}_3[z]$ *be a vector space of polynomials of degree at most 3. Let* $G(2, \mathbb{C}_3[z]) = G(2, 4)$ *be the Grassmannian of* 2*-dimensional subspaces of* $\mathbb{C}_3[z]$*. Suppose a subspace* $U \in G(2, \mathbb{C}_3[z])$ *is spanned by the polynomials* $\{f_1, f_2\}$*. We can write* $\{f_1, f_2\}$ *as follows:*

$$f_1 = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$
$$f_2 = b_0 + b_1 z + b_2 z^2 + b_3 z^3$$

So the Wronskian of f_1 *and* f_2 *is*

$$\begin{aligned} \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} &= \begin{vmatrix} a_0 + a_1 z + a_2 z^2 + a_3 z^3 & b_0 + b_1 z + b_2 z^2 + b_3 z^3 \\ a_1 + 2a_2 z + 3a_3 z^2 & b_1 + 2b_2 z + 3b_3 z^2 \end{vmatrix} \\ &= (a_0 b_1 - a_1 b_0) + 2(a_0 b_2 - a_2 b_0) z \\ &+ (3(a_0 b_3 - a_3 b_0) + a_1 b_2 - a_2 b_1) z^2 \\ &+ 2(a_1 b_3 - a_3 b_1) z^3 + (a_2 b_3 - a_3 b_2) z^4 \end{aligned}$$

As we can see the resulting polynomial is of degree 4. Hence it lives in $\mathbb{C}_4[z]$*.* $\mathbb{P}(\mathbb{C}_4[z])$ *has of dimension* 4*, which is the same as* $dim(G(2, 4))$*.*

The Wronski map appears in many applications. In 1983, Eisenbud and Harris in [17] proved the following theorem:

Theorem 5.1.3. $Wr \colon X \longrightarrow \mathbb{P}(\mathbb{C}_N[z])$ *is a flat, finite morphism of schemes.*

In 1995, B. and M. Shapiro conjectured about the reality problem of the fibre of the Wronski map, later known as the Shapiro conjecture. There are many studies about the Shapiro conjecture including [48] in which the conjecture has been studied extensively and presented with computational evidence. The conjecture has been proved by Eremenko and Gabrielov for special cases in [21] and finally it was proved by Mukhin, Tarasov and Varchenko in [40, 41].

Theorem 5.1.4. *If* $f(z) \in \mathbb{R}_{n-1}[z]$ *is a polynomial with real roots, then every point in the fibre* $Wr^{-1}(f(z))$ *is real and* $Wr^{-1}(f(z))$ *is reduced.*

Remark 5.1.5. *When $\mathbb{F} = \mathbb{C}$, there is an easy formula to compute the degree of the Wronski map. In order to compute the degree, we need to change the set up a little bit. We will define the Wronski map as follows*

$$Wr\colon G(m, \mathbb{C}_{m+p-1}[z]) \longrightarrow \mathbb{P}(\mathbb{C}_{mp}[z])$$

It is clear that the dimensions of both $G(m, \mathbb{C}_{m+p-1}[z])$ and $\mathbb{P}(\mathbb{C}_{mp}[z])$ are mp. Then the degree of the Wronski map, i.e., the number of points in a fiber, is given by

$$\#^G_{m,p} := \frac{(mp)! \cdot 1! \cdot 2! \cdots (p-1)!}{m! \cdot (m+1)! \cdots (m+p-1)!}.$$

For a more detailed discussion, see [20].

Remark 5.1.6. *When $p = 2$, the number $\#^G_{m,2}$ is the m-th Catalan number. See [25].*

Example 5.1.7. *Considering the Example 5.1.2., in this setting we have $m = 2$ and $p = 2$. So the degree of the Wronski map above is:*

$$\#^G_{2,2} = \frac{4! \cdot 1!}{2! \cdot 3!} = 2.$$

It means that there are 2 points in the fiber $Wr^{-1}(g(z))$ of the Wronski map.

Remark 5.1.8. *Observe that the degree of the Wronski map is equal to the number of standard Young tableaux of maximal partition, i.e, |**SYT**($\square^d$)|. In the light of the example above, for $G(2,4)$, the possible standard Young tableaux are the followings:*

1	2
3	4

,

1	3
2	4

The Wronski map has been an interest on its own and with its relation to Grassmannians. In [44], Purbhoo studied the Wronski map, with Shapiro Conjecture, on orthogonal Grassmannians $OG(n, 2n+1) \subset G(n, 2n+1)$. Purbhoo came up with a new geometric proof for the Littlewood-Richardson rule for $OG(n, 2n+1)$. In [20] Eremenko and Gabrielov studied the computation of degrees of the real Wronski maps. In [30], they studied the congruences for the fibers of the Wronski map.

5.2 Relation with Schubert Calculus

We will recall some definitions from earlier in order to understand the relation better. Let Λ denote the set of partitions whose diagrams fit inside $d \times (n-d)$ rectangle. Let

$\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$ be an admissible partition. There is a one-to-one correspondence between the partitions whose diagram fit inside the largest rectangle $\square^d$ and subsets of $\{1, ..., n\}$ with d-elements: for each partition $\lambda \in \Lambda$, we have

$$J(\lambda) = \{j + \lambda_{d+1-j} \mid 1 \leq j \leq d\}.$$

Let $x \in X$ be a d-dimensional subspace spanned by the polynomials $f_1(z), f_2(z), ..., f_d(z)$ of $\mathbb{C}_{n-1}[z]$ whose Plücker coordinates are $[p_\lambda(x)]_{\lambda\in\Lambda}$. Plücker coordinates $p_\lambda(x)$ are defined to be the maximal minors of the matrix $A_J(\lambda)$ with the column set $J(\lambda)$ where the matrix $A_{ij} = [z^{j-1}]f_i(z)$ is a $d \times n$ matrix whose entries are the coefficients of the polynomials $f_i(z)$.

In [42], Purbhoo showed that the Wronskian can be expressed in terms of Plücker coordinates:

Proposition 5.2.1. *The Wronskian $Wr(x; z)$ is (up to a scalar multiple) given explicitly in terms of the Plücker coordinates of x by*

$$Wr(x; z) = \sum_{\lambda\in\Lambda} q_\lambda p_\lambda(x) z^{|\lambda|}.$$

where q_λ is the Vandermonde determinant

$$q_\lambda = \begin{vmatrix} 1 & ... & 1 \\ k_1 & ... & k_d \\ \vdots & ... & \vdots \\ k_1^{d-1} & ... & k_d^{d-1} \end{vmatrix} = \prod_{1\leq i<j\leq d} (k_j - k_i),$$

and $k_j = j + \lambda_{d+1-j}$.

Let us now define the Schubert cells and Schubert varieties. For each $a \in \mathbb{CP}^1$, let $\mathcal{F}$ be a flag in $\mathbb{C}_{n-1}[z]$:

$$\mathcal{F}(a)\colon \{0\} \subset F_1(a) \subset F_2(a) \subset ... \subset F_{n-1}(a) \subset \mathbb{C}_{n-1}[z].$$

If $a \in \mathbb{C}$, then

$$F_i(a) = (z + a)^{n-i}\mathbb{C}[z] \cap \mathbb{C}_{n-1}[z]$$

is the set of polynomials in $\mathbb{C}_{n-1}[z]$ that are divisible by $(z + a)^{n-i}$. Let Λ be the set of partitions whose diagrams fit inside a $d \times (n - d)$ rectangle and let $J(\lambda)$ be as above. Then for every $\lambda \in \Lambda$, the **Schubert cell** relative to the flag $\mathcal{F}(a)$ is

$$X^{\circ}_{\lambda}(a) = \{x \in X \mid dim(x \cap F_i(a)) = |J(\lambda) \cap \{n-i+1, ..., n\}|\}.$$

The **Schubert variety** is the closure of the Schubert cell, i.e., $X_{\lambda}(a) := \overline{X^{\circ}_{\lambda}(a)}$. Codimension of $X_{\lambda}(a)$ is $|\lambda|$. If the codimension is 1, i.e., $\lambda = \square$, then $X_{\square}(a)$ is called a **Schubert divisor**.

There is a strong connection between the roots of the Wronskian and the Schubert varieties. In the beginning of the chapter, we said that we will use the notation $Wr(x; z)$ for the Wronskian as the representative of $Wr(x)$ in $\mathbb{C}_N[z]$. Now we are going to elaborate why this notation makes sense. If $Wr(x; z)$ has degree strictly less than N, then we say that $N - deg(Wr(x; z))$ many roots are at infinity. If $Wr(x; z) = \prod_{i=1}^{k}(z + a_i)$, then the roots of $Wr(x; -z)$ form a multiset $\pi(x) := \{a_1, a_2, ..., a_N\}$ where $a_{k+1} = ... = a_N = \infty$ if $k < N$.

Theorem 5.2.2. *Let $x \in X$ be a closed point, $a \in \mathbb{CP}^1$, and $k \geq 0$ an integer. Then $a \in \pi(x)$ with multiplicity at least k if and only if $x \in X_{\lambda}(a)$ for some $\lambda \vdash k$.*

Proof. See [42]. ■

In this context, we denote the fibers of the Wronski map Wr at the point $\prod_{a_i \neq \infty}(z + a_i)$ as

$$X(\mathbf{a}) := \pi^{-1}(\mathbf{a}) = \{x \in X \mid \pi(x) = \mathbf{a}\}$$

where $\mathbf{a} = \{a_1, a_2, ..., a_N\} \subset \mathbb{CP}^1$ is a multiset.

Mukhin, Tarasov and Varchenko in [40, 41] proved the following theorem concerning the reality of the intersections of the Schubert varieties:

Theorem 5.2.3. *If $a_1, a_2, ..., a_k \in \mathbb{RP}^1$ are distinct real points, and $\lambda_1, \lambda_2, ..., \lambda_k$ are partitions with $|\lambda_1| + |\lambda_2| + ... + |\lambda_k| = dim(X)$, then the intersection*

$$X_{\lambda_1}(a_1) \cap X_{\lambda_2}(a_2) \cap ... \cap X_{\lambda_k}(a_k)$$

is finite, transverse and real.

5.3 A Problem

As we have seen that the fibers of the Wronski map are widely studied in many ways. A further question that may lead to an interesting results is the following: Suppose we have a base Grassmannian $G(d_2, n)$. For each point $x \in G(d_2, n)$, we can consider another Grassmannian $G(d_1, d_2)$. In the general picture, we are looking at a 2-step flag variety $F(d_1, d_2, n)$. At first, we would like to keep the base Grassmannian $G(d_2, n)$ unchanged. We know that each $x \in G(d_1, d_2)$ is sent to the projective space $\mathbb{P}(\mathbb{C}_{d_1(d_2-d_1)}[z])$ via the Wronski map. Consider the projective bundle E over the Grassmannian $G(d_2, n)$. For an open cover $\{U_i\}$ for the Grassmannian $G(d_2, n)$, E_{U_i} will look like $U_i \times \mathbb{P}^r$, for some r. We would like to define a map from the the Grassmannian $G(d_1, d_2)$ to the projective bundle over the Grassmannian $G(d_2, n)$ via the Wronski map. This bundle E can be studied further in the sense that whether it is uniform or not.

In general, vector bundles, in particular uniform vector bundles, over the Grassmannian are widely studied. It is proved by Grothendick in [27] that any vector bundle on a projective line over an algebraically closed field splits as a direct sum of lines bundles. While this is the case for 1-dimensional projective spaces, i.e. projective lines, when the dimension is 2 or higher, it is more difficult to give such classification. More details about uniform vector bundles on flag varieties can be found in [15, 46, 47, 52]

Furthermore, we know that we can cover the Grassmannian with affine charts. If we change the fixed affine chart for the base Grassmannian $G(d_2, n)$, how would it affect the Grassmannian $G(d_1, d_2)$ and how the image of the Wronski map is affected by such a change?

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