INVARIANT METRICS AND SQUEEZING FUNCTIONS ON BOUNDED DOMAINS

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ABSTRACT

INVARIANT METRICS AND SQUEEZING FUNCTIONS ON BOUNDED DOMAINS

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In this thesis we will study the biholomorphically invariant objects called squeezing functions. They are closely releated to invariant metrics on bounded domains and describe how much a domain looks like the unit ball looking on a fixed point. In the main part of this thesis, we will give our results on squeezing functions on planar domains. In particular, our main result provides an alternative proof for the explicit formulas of squeezing functions on annuli. Also, we survey results on boundary behaviour of squeezing functions.

Keywords: Squeezing Function, Invariant Metrics, Plurisubharmonic Function

ÖZ

SINIRLI KÜMELERDE DEĞİŞMEZ METRİKLER VE SIKIŞTIRMA FONKSİYONLARI

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Bu tezde sıkıştırma fonksiyonu adlı biholomorfik değişmez objeleri çalışacağız. Sıkıştırma fonksiyonları kümelerdeki değişmez metrikler ile yakın alakalıdırlar ve bir kümenin önceden belirlenen ve içinde bulunan bir noktadan bakıldığında birim küreye ne kadar benzediğini anlatırlar. Bu tezin ana kısmında düzlemsel kümelerde sıkıştırma fonksiyonlarıyla alakalı sonuçlarımızı vereceğiz. Bilhassa, ana sonucumuz daire halkalarındaki sıkıştırma fonksiyonlarının açık formüllerine alternatif bir ispat veriyor. Aynı zamanda, yakın zamanda sıkıştırma fonksiyonlarının sınıra yakın davranışlarıyla alakalı sonuçları gözden geçiriyoruz.

Anahtar Kelimeler: Sıkıştırma Fonksiyonu, Değişmez Metrikler, Plurisubharmonik Fonksiyon

To those whom I have ever learned from

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The main part of this thesis also appears on our preprint [35].
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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Geometric function theory is a fruitful subject in mathematics which relates holomorphic mappings to their differential geometric properties. One of its most fundemental results is the Riemann mapping theorem. Poincaré's discovery of the failiure of the Riemann mapping theorem in higher dimensions led to the development of the theory of objects that remain invariant under biholomorphic transformations.

It was also Poincaré who introduced a biholomorphically invariant hyperbolic metric on the unit disc. The study of the Poincaré led the relation between differential geometry and complex analysis to become more and more appearent. After the groundbreaking works of Ahlfors and others geometric function theory in $\mathbb{C}$ has been greatly developed.

In 1927, based on the space of holomorphic functions on a domain to the unit disc, Constantin Carathéodory was the first one who introduced a generalization of the Poincaré metric to higher dimensions in the paper "Uber eine spezielle Metrik, die in der Theorie der analytischen Funktionen auftritt"[6]. Later, another generalization was given by Stefan Bergman, whose construction deeply relies on functional analytic reasoning and holomorphic function spaces. In 1967, Shoshichi Kobayashi provided another generalization of the Poincaré metric in [25]. The Kobayashi metric is related to the size of the analytic discs in a domain, and interestingly it can be considered as a dual of the Carathéodory metric. Kobayashi himself gave detailed discussion on the Kobayashi metric in the book [24]. Later, many other generalizations followed. The books [18],[20] provide a short but dense introduction to the theory of the invariant metrics.

In this thesis, we will mainly study the biholomorphic invariant called squeezing function. In the last decade, squeezing functions were constructed to study the equivalence of invariant Carathéodory and Kobayashi metrics on bounded domains. Informally, they show how well a domain can be approximated by the unit ball. This simple biholomorphic invariant led to many new results about geometric and analytic properties of certain classes of domains, thus, we consider it to be very worthy to study.

This thesis is organized as follows. In the second chapter we will give preliminaries on some concepts in complex analysis of one and several variables. We will especially discuss and give proofs of some well-known results about the invariant Carathéodory and Kobayashi metrics.

In the third chapter, we will first study the construction of squeezing functions. Afterwards, we will survey some results on boundary behaviour of squeezing functions on complex domains in both $\mathbb{C}$ and $\mathbb{C}^n$ for $n \geq 2$. We will also observe two applications of squeezing functions to the problem of determining biholomorphic equivalences of bounded domains.

In the fourth and the main chapter, we will focus on squeezing functions on planar domains. We will first discuss some generalizations of the Riemann mapping theorem which provide canonical conformal maps for finitely connected domains planar. Using some properties of these canonical conformal maps, we will give an alternate and simple proof of the explicit formulas of squeezing functions on annuli and relate the squeezing function problem on planar domains to the famous Schwarz lemma. We will also provide upper and lower bounds to squeezing functions of finitely connected planar domains which can be obtained by a minor modification of our proof of explicit formulas of squeezing functions on annuli.

CHAPTER 2

PRELIMINARIES

The following notation will be common through this theses.

We set Δ to be the unit disc in $\mathbb{C}$. For $r \in \mathbb{R}^+$ and $z \in \mathbb{C}$ denote

$$\Delta_r(a) := \{z \in \mathbb{C} : |z - a| < r\} \text{ and } \Delta_r := \Delta_r(0).$$

Similarly, we will denote the unit ball in $\mathbb{C}^n$ as $\mathbb{B}^n$, for $a \in \mathbb{C}^n$ and $r \in \mathbb{R}$ we set

$$\mathbb{B}^n_r(a) := \{z \in \mathbb{C}^n : \|z - a\| < r\} \text{ and } \mathbb{B}^n_r := \mathbb{B}^n_r(0, ..., 0).$$

Moreover, we denote the polydiscs

$$\mathbb{P}^n := \overbrace{\Delta \times \Delta \times ... \times \Delta}^{n \text{ times}} \text{ and } \mathbb{P}^n_r := \overbrace{\Delta_r \times \Delta_r \times ... \times \Delta_r}^{n \text{ times}}.$$

Further, for domains $G \subset \mathbb{C}^k$ and $D \subset \mathbb{C}^l$ let $\mathscr{O}(G, D)$ denote the set holomorphic mappings on G into D and $\mathscr{E}(G, D)$ denote the set of injective holomorphic maps (or holomorphic embeddings) of G into D.

For $r \in \mathbb{C}$, we will commonly exploit notation by setting $r := (r, 0, ..., 0) \in \mathbb{C}^n$, in particular we will denote the origin by $0 := (0, ..., 0)$.

2.1 Useful Notions in the Study of Domains

We start our thesis by defining some anaytical and geometrical notions used in the complex domains. Through our discussion in this chapter, we will observe two sharp distinctions between the complex function theory in $\mathbb{C}$ and in $\mathbb{C}^n$.

Definition 2.1.1 *Let $G \subset \mathbb{C}^n$ be a domain and let ∂G denote its boundary. ∂G is said to be C^k-smooth (with the possibility of $k = \infty$) at $p \in \partial G$ if there exists* *a neighbourhood of p, N_p and a C^k-smooth function $\rho : N_p \longrightarrow \mathbb{R}$ satisfying the properties given below.*

1. $N_p \cap G = \{x \in N_p : \rho(x) < 0\}$
2. $d\rho_x \neq 0$ *for all* $x \in N_p \cap G$

In this case, we say that ρ is a (local) defining function of G at the point $p \in \partial G$. Further, we say that G has C^k-smooth boundary if ∂G is C^k-smooth at all $p \in \partial G$.

Definition 2.1.2 *Let $G \subset \mathbb{C}^n$ be a domain. A non-constant function $f : G \longrightarrow \mathbb{R}$ is said to be an exhaustion function(or defining function) of G if for all $c < \sup_{z\in G} f(z)$ the set $G_c(f) := \{z \in G : f(z) < c\}$ is relatively compact in G.*

Convexity has been useful in determining analytical properties of domains. See for instance ([15]). Let us recall some definitions.

Definition 2.1.3 *Let G be a bounded domain in $\mathbb{C}^n$. We say that G is geometrically convex if for any $p, q \in G$, the line segment in $\mathbb{C}^n$ containing p and q as endpoints is contained in G. We say that $p \in \partial G$ is strictly geometrically convex boundary point if for any $q \in \partial G$, the line segment in $\mathbb{C}^n$ containing p,q as endpoints are contained in G. We say that G is strictly geometrically convex if all $p \in \partial G$ is a strictly geometrically convex boundary point.*

Definition 2.1.4 *Let G be a bounded domain in $\mathbb{C}^n$ with C^2-smooth boundary. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by setting $z := (z_1, ..., z_n)$, $x_{2k-1} = \Re z_k$, $x_{2k} = \Im z_k$. We say that G is analytically convex at $p \in \partial G$ (or p is an analytically convex boundary point) if there exists a C^2-smooth defining function ρ of G at p satisfying*

$$\sum_{i,j=1}^{2n} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(p) w_i w_j \geq 0 \tag{1}$$

for all real vectors $w \in \mathbb{R}^n$ satisfying

$$\sum_{i=1}^{2n} \frac{\partial \rho}{\partial x_i}(p) w_i = 0.$$

We say that G is strongly analytically convex at p (or p is a strongly analytically convex boundary point) if the inequality in (1) *is strict.*

Theorem 2.1.1 *([2]) Let G be a bounded domain in $\mathbb{C}^n$ with C^2-smooth boundary. Then G is geometrically convex if and only if it is analytically convex. $p \in \partial G$ is a strictly geometrically convex boundary point if and only if it is a strongly analytically convex boundary point. In particular G is geometrically strictly convex if and only if it is analytically strongly convex.*

Definition 2.1.5 *A map $f : \overline{\Delta} \longrightarrow \overline{G}$ is said to be a closed analytic disc in G if f is holomorphic on Δ and continuous on $\overline{\Delta}$. A domain G is said to satisfy continuity principle (Kontinuitatssatz) if for any family of closed analytic discs $\{f_t : \overline{\Delta} \longrightarrow \overline{G} : t \in [0,1]\}$ changing continuously with respect to parameter t we have that $f_0(\Delta) \subset G$ and $\forall t \in [0,1]$ $f_t(\partial\Delta) \subset G$ implies $\forall t \in [0,1]$ $f_t(\Delta) \subset G$.*

One can see that the contunuity principle is an holomorphic analogue of the condition for convexity, replacing line segments with holomorphic images of the unit disc.

On the other hand, Levi pseudoconvexity is a holomorphic analogue of analytical convexity.

Definition 2.1.6 *Let G be a bounded domain in $\mathbb{C}^n$ with C^2-smooth boundary. We say that G is Levi pseudoconvex at $p \in \partial G$ if there exists a $\mathscr{C}^2$-smooth defining function ρ of G at p satisfying*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z_j}}(p) w_i \overline{w_j} \geq 0 \tag{2}$$

for all vectors $w \in \mathbb{C}^n$ satisfying

$$\sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}(p) w_i = 0.$$

We say that G is strongly Levi pseudoconvex at p if the inequality in (2) is strict.
We say that G is Levi pseudoconvex (resp. strongly Levi pseudoconvex) if every boundary point of G is Levi pseudoconvex (resp. strongly Levi pseudoconvex).

For convenience, we say G is strongly convex if it is strongly analytically convex and G is strongly pseudoconvex if it is strongly Levi pseudoconvex.

Modern complex analysis heavily relies on pluripotential theory. The following definition of pseudoconvexity is commonly used.

Definition 2.1.7 *Let G be a bounded domain and let $\delta_G(z) = dist(z, \mathbb{C}^n \setminus G)$ where $dist$ denotes Euclidean distance in $\mathbb{C}^n$. We say that G is pseudoconvex if $-\log(\delta_G)$ is a plurisubharmonic exhaustion function of G.*

Equivalences of several notions of pseudoconvexity has been central in the study of the several complex variables in the last century. We give the following theorem below. Surely, this subjects needs a lot more discussion and we refer you to the following books [15], [23], [26].

Theorem 2.1.2 *Let $G \subset \mathbb{C}^n$ be a domain. Then G is pseudoconvex if and only if G satisfies continuity principle. Further, if G has C^2-smooth boundary, then G is pseudoconvex if and only if G is Levi pseudoconvex.*

The following is worth mentioning.

Theorem 2.1.3 *([23],[26]) Let $G \subset \mathbb{C}^n$ be a pseudoconvex domain. Then it can be exhausted by strongly pseudoconvex domains. that is for each $n \in \mathbb{N}$ there exists strongly pseudoconvex domains $G_n \subset G$ satisfying the following conditions.*

1. *For each $n \in \mathbb{N}$, G_n is relatively compact in G.*

2. *For each $n \in \mathbb{N}$, G_n is relatively compact in G_{n+1}.*

3. *$\cup_{n\in\mathbb{N}} G_n = G$.*

Let is recall the notion of domain of holomorphy.

Definition 2.1.8 *Let $G \subset \mathbb{C}^n$. We say that G is a domain of holomorphy if there exists a holomorphic function $f : G \longrightarrow \mathbb{C}$ such that $\forall z \in \partial G$ we have a neighbourhood of z, N_z such that there does not exist a holomorphic function $\tilde{f}$ on N_z such that $\tilde{f}|_{G\cap N_z} = f$.*

Informally, this means that G is a domain of holomorphy if there exists an holomorphic function on G that cannot be extended holomorphically to any larger domain.

The following result gives a concrete meaning to notion of pseudoconvexity.

Theorem 2.1.4 *([15],[23],[26]) Let $G \subset \mathbb{C}^n$ be a bounded domain. Then it is a domain of holomorphy if and only if it is pseudoconvex.*

Let us give some examples of domain of holomorphy. The unit disc Δ is a domain of holomorphy as $f(z) := \sum_{n=1}^{\infty} \frac{z^{2^n}}{2^n}$ is an analytic function on the unit disc that doesn't extend to any larger domain. Similarly, the punctured disc $\Delta^* := \Delta \setminus \{0\}$ is a domain of holomorphy, as $g(z) := \frac{1}{z} + f(z)$ is a holomorphic function on Δ^* that doesn't extend to any larger domain.

It is very well-known that if G is a planar domain with non-empty boundary, then $-\log(\delta_G)$ is plurisubharmonic ([26]). Thus, in general we have the following general result.

Theorem 2.1.5 *Let $G \subsetneq \mathbb{C}$ be a domain. Then, G is pseudoconvex. In particular, it is a domain of holomorphy.*

In higher dimensions, same assertion doesn't hold. Observe the following example.

Let $G := \mathbb{P}^2 - \frac{1}{2}\mathbb{P}^2$. We will show that G is not a domain of holomorphy. Let $f(z_1, z_2)$ be a holomorphic function on G. Then for (z_1, z_2) in G satisfying $\frac{1}{2} < |z_1| < \frac{2}{3}$ and $\frac{1}{2} < |z_2| < \frac{2}{3}$, f is given by the Cauchy integral formula

$$f(z_1, z_2) = \frac{1}{2}(2\pi i)^2 \int_{|w_1|=\frac{2}{3}} \int_{|w_2|=\frac{2}{3}} \frac{f(w_1, w_2)}{(z_1 - w_1)(z_2 - w_2)} dw_2 dw_1.$$

By Morera's theorem ([16], page 119), we see that the function

$$\tilde{f}(z_1, z_2) := \frac{1}{2}(2\pi i)^2 \int_{|w_1|=\frac{2}{3}} \int_{|w_2|=\frac{2}{3}} \frac{f(w_1, w_2)}{(z_1 - w_1)(z_2 - w_2)} dw_2 dw_1$$

is holomorphic on $\frac{2}{3}\mathbb{P}^2$. Thus by identity principle the function

$$F(z_1, z_2) := \begin{cases} f(z_1, z_2) & \text{if } (z_1, z_2) \in \mathbb{P}^2 - \overline{\frac{7}{12}\mathbb{P}^2} \\ \tilde{f}(z_1, z_2) & \text{if } (z_1, z_2) \in \frac{7}{12}\mathbb{P}^2 \end{cases}$$

is the holomorphic extension of f to $\mathbb{P}^2$. In general, we have the following result, which is often called as the generalized Hartogs phenomenon.

Theorem 2.1.6 *([28]) Let G be a domain in $\mathbb{C}^n$, K be a compact subset of G such that $D := G \setminus K$ is connected. Then any holomorphic function on D extends holomorphically to G.*

Finally, we introduce the notion of normal families of holomorphic mappings and the notion of tautness.

Definition 2.1.9 *We say that the family $\mathscr{F} \subset \mathscr{O}(G_1, G_2)$ is a normal family if for any sequence $\{f_n\} \subset \mathscr{F}$ we have that either $\{f_n\}$ has a subsequence that converges uniformly on compact subsets of G_1 to a function $f \in \mathscr{F}$ or $\{f_n\}$ has a subsequence such that for any two compact subsets $K_1 \subset G_1$ and $K_2 \subset G_2$ we have that $f_n(K_1) \cap K_2 = \emptyset$ as n goes to ∞. In this case we say that $\{f_n\}$ has a compactly divergent subsequence.*
If the family $\mathscr{O}(\Delta, G)$ itself is a normal family then we say that G is taut.

Recall the analogue of Montel's theorem in $\mathbb{C}^n$.

Lemma 2.1.1 *([26]) Let $G \subset \mathbb{C}^n$ be a domain and $\{f_i\}_{i \in I}$ be a family of holomorphic mappings which are uniformly bounded on G, that is $\exists C \geq 0$ such that $\forall i \in I$ we have that $\sup_{z \in G} \|f_i(z)\| < C$. Then $\{f_i\}_{i \in I}$ is a normal family.*

This theorem may lead one to think that any bounded domain is taut. However, the essential thing about tautness is that the limit of the holomorphic functions should belong in the family $\mathscr{O}(\Delta, G)$. We illustrate this with the following example.

Consider the punctured ball in $\mathbb{C}^2$ given by $\mathbb{B}^n \setminus \{(0,0)\}$. Let $\{f_n\} \subset \mathscr{O}(\Delta, G)$ given by $f_n(z) := (\frac{z}{2}, \frac{1}{n+2})$. The sets $K_1 := \{\frac{1}{2}\}$ and $K_2 := \{(\frac{1}{2}, z) : |z| \leq \frac{1}{2}\}$ are clearly relatively compact in Δ and $\mathbb{B}^n \setminus \{(0,0)\}$ respectively. Moreover for any n, $f_n(\frac{1}{2}) \in K_2$. Thus this family doesn't have any compactly divergent subsequence. On the other hand this family converges to the holomorphic function $f(z) = (\frac{z}{2}, 0)$.

However, f doesn't belong to the family $\mathscr{O}(\Delta, \mathbb{B}^n \setminus \{(0,0)\})$. Thus $\mathbb{B}^n \setminus \{(0,0)\}$ is not taut. Due to the Riemann's extension theorem, any holomorphic function on $\mathbb{B}^n \setminus \{(0,0)\}$ extends holomorphically to $\mathbb{B}^n$. Thus the punctured ball is not a domain of holomorphy. In particular it is not pseudoconvex. The following result shows that the example we discussed is not a surprise.

Theorem 2.1.7 *([1])Let G be a bounded taut domain. Then it is pseudoconvex.*

2.2 Automorphism Groups

Definition 2.2.1 *Let G be a bounded domain in $\mathbb{C}^n$. Let $\mathtt{aut}(G) \subset \mathscr{O}(G, G)$ be the set of biholomorphic mappings taking G to itself. The set $\mathtt{aut}(G)$ forms a group under composition of functions. We call $\mathtt{aut}(G)$ the automorphism group of G.*

Theorem 2.2.1 *([18]) Let $f \in \mathscr{E}(G_1, G_2)$ be a biholomorphism. Then f induces an isomorphism between $\mathtt{aut}(G_1)$ and $\mathtt{aut}(G_2)$ by the map $\psi \mapsto f \circ \psi \circ f^{-1}$.*

Following results are from Chapter 1 of [18]. The first one is called Cartan's uniqueness theorem. By using boundedness of G, it is proven via applying Cauchy estimates to higher derivatives of $f^n := \overbrace{f \circ \cdots \circ f}^{n \text{ times}}$.

Theorem 2.2.2 *([18], Theorem 1.3.1) Let G be a bounded domain let $f \in \mathscr{O}(G, G)$ such that $\exists p \in G$ satisfying $f(p) = p$. Then $J_f(p) \leq 1$ where J is the Jacobian determinant. Further, if df_p is the identity matrix $diag(1, ..., 1)$, then f is the identity mapping on G.*

This theorem has two corollaries which will be of use. For the following, we recall that a circular domain is a domain that has circular symmetry in each variable. That is $G \subset \mathbb{C}^n$ is a circular domain if $\forall z := (z_1, ..., z_n) \in G$ and $\forall \{\theta_1, \theta_2, ..., \theta_n\} \subset \mathbb{R}$ we have have that $(e^{i\theta_1} z_1, e^{i\theta_2} z_2, ..., e^{\theta_n} z_n) \in G$. As an example, one can see that the unit ball, polydiscs centered at the origin and $\mathbb{C}^n$ itself are circular domains.

Corollary 2.2.1 *([18], Corollary 1.3.2.) Let $G \subset \mathbb{C}^n$ be a bounded circular domain that contains the origin. If $f \in \mathtt{aut}(G)$ fixes the origin, then it is a linear transformation.*

Corollary 2.2.2 *([18], Corollary 1.3.3.) Let G be a bounded domain, let $I_p(G) \subset \mathtt{aut}(G)$ be the isotropy subgroup at p, that is $I_p(G)$ is the subgroup of $\mathtt{aut}(G)$ consisting of elements that has p as a fixed point. Then the map $f \mapsto df_p$ is an injective homomorphism of $I_p(G)$ into $GL(n, \mathbb{C})$.*

The following result will be useful.

Theorem 2.2.3 *([18], Theorem 1.3.4.) Let G be a bounded domain and $\{f_n\} \subset \mathtt{aut}(G)$ be a sequence of holomorphic functions. Suppose that $\{f_n\}$ converges to a function f uniformly on compact subsets of G and $\exists p \in G$ such that $f(p) \in G$. Then $f \in \mathtt{aut}(G)$.*

This implies the following.

Corollary 2.2.3 *([18], Theorem 1.3.4.) Let $p \in G$, then the orbit*

$$\mathscr{O}_p(G) = \{f(p) : f \in \mathtt{aut}(G)\}$$

is closed in G.

Using this corollary, one can see that if $\{f_n\} \subset G$ tends to $f \in \mathtt{aut}(G)$ and $f_n(p)$ lie on a compact subset, then $f(p)$ also lies on that compact subset. Thus by a normal families argument we have the following corollary.

Corollary 2.2.4 *([18], Corollary 1.3.8.) Let $p \in G$ and K be a compact subset of G. Then the set $\{f \in \mathtt{aut}(G) : f(p) \in K\}$ is compact in $\mathtt{aut}(G)$.*

In particular we have the following corollaries of this result.

Corollary 2.2.5 *([18], Corollary 1.3.9.) Let $G \subset \mathbb{C}^n$ be an arbitrary domain. Then the following are equivalent.*

1. $\mathtt{aut}(G)$ *is compact.*

2. $\exists p \in G$ *such that the set* $\{f(p) : f \in \mathtt{aut}(G)\}$ *is compact in* G.

3. $\forall p \in G$ *the set* $\{f(p) : f \in \mathtt{aut}(G)\}$ *is compact in* G.

Corollary 2.2.6 *([18], Corollary 1.3.7.) Isotropy group of G at* p, $I_p(G)$ *is a compact subgroup of* $\mathtt{aut}(G)$. *Thus the image of* I_p *under the map* $f \mapsto df_p$ *is compact in* $GL(n, \mathbb{C})$.

If G is bounded, the last corollary combined with the Cauchy estimates gives that the action of $\mathtt{aut}(G)$ is proper on G. That is, the map $(\psi, z) \mapsto (\psi(z), z)$ is a proper map on $\mathtt{aut}(G) \times G$ into $G \times G$. In particular, this implies that $\mathtt{aut}(G)$ is a locally compact subgroup of the diffeomorphism group of G. The Bochner-Montgomery theorem ([18], Theorem 1.3.11) states that any locally compact subgroup of the diffeomorphism group of a smooth manifold is a lie group. Thus amazingly, we have that

Theorem 2.2.4 *([18], Corollary 1.3.13.) Let G be a bounded domain. Then, with the topology of uniform convergence on compact subset* $\mathtt{aut}(G)$ *is a Lie Group.*

Thus, for bounded domains, we have a topology on $\mathtt{aut}(G)$ that is compatible with it's group structure. In the next section with the help of squeezing functions, we will show that for certain classes of domains in $\mathbb{C}^n$, the topology of $\mathtt{aut}(G)$ can directly yield the analytical structure of G itself.

We now list some automorphisms which are going to be useful for our purposes.

Theorem 2.2.5

1. *([18], page 16) The automorphism group of the unit disc* $\Delta \subset \mathbb{C}$ *consists of Mobius transformations of the form*

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

where $\theta \in \mathbb{R}$ *and* $a \in \Delta$.

2. *([39], Theorem 2.2.5.) The automorphism group of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ is given by the maps $U \circ f_a(z)$ where U is an unitary linear transformation on $\mathbb{C}^n$, and $f_a(z)$ is given as*

$$f_a(z) := \frac{a - P_a(z) - \sqrt{1 - \|a\|^2} Q_a(z)}{1 - < z, a >}$$

Here, we have that $a \in \mathbb{B}^n$. If $a \neq 0$, $P_a(z) := \dfrac{< z, a >}{< a, a >} a$ is the orthogonal projection of z in the direction of a and $Q_a(z)$ is the orthogonal projection to the complement. If $a = 0$, $P_a(z) := 0$ and thus $Q_a(z) := z$.
In particular, for $w := (w, 0, ..., 0) \in \mathbb{B}^n$ where $w \in (0, 1)$ we have that

$$f_w(z_1, z_2, ..., z_n) = \left(\frac{z_1 - w}{1 - wz_1}, \frac{\sqrt{1 - w^2} z_2}{1 - wz_1}, ..., \frac{\sqrt{1 - w^2} z_n}{1 - wz_1} \right).$$

3. *([18], page 17) For $r \in (0, 0)$, let $\mathbb{A}_r := \Delta \setminus \overline{\Delta_r}$ be the annulus. Then, $\mathtt{aut}(\mathbb{A}_r)$ consists of the maps given by $e^{i\theta} f(z)$ where $\theta \in \mathbb{R}$ and $f(z) = z$ or the reflection $f(z) = \dfrac{r}{z}$.*
4. *([18], page 18) For $r \in (0, 1)$, let $\mathbb{G}_r := \mathbb{B}^n \setminus \overline{\mathbb{B}^n_r}$. For convenience, we call $\mathbb{G}_r$ a generalized annulus. Then the automorphism group of the generalized annulus consists of unitary transformations of $\mathbb{C}^n$.*

Let us share some remarks. One can clearly see that, for $\mathbb{B}^n$, the automorphim given by f_a takes 0 to a and a to 0. In fact, it is an involution, that is $f_a \circ f_a$ is the identity. Thus, for arbitrary $a, b \in \Delta$, $f_b \circ f_a$ takes a to b. So, $\mathtt{aut}(\mathbb{B}^n)$ is transitive. Same can be obtained in Δ, let $a, b \in \Delta$ and let $f_a(z) = \frac{z-a}{1-\bar{a}z}$ and $f_{-b}(z) = \frac{z+b}{1+\bar{b}z}$. Then $f_{-b} \circ f_a(a) = b$. This means that Δ and $\mathbb{B}^n \subset \mathbb{C}^n$ are homogenous domains, in particular for any point, the orbits of automorphisms are non-compact. Thus $\mathtt{aut}(\Delta)$ and $\mathtt{aut}(\mathbb{B}^n)$ are non-compact.

Also, the automorphism group of $\mathtt{aut}(\mathbb{A}_r)$ contains the reflection $z \mapsto \frac{z}{r}$ while $\mathtt{aut}(\mathbb{G}_r)$ doesn't. This is due to an argument involving Hartogs extension phenomenon which gives that elements of $\mathtt{aut}(\mathbb{G}_r)$ extends to elements of $\mathtt{aut}(\mathbb{B}^n)$. Thus, the origin of this distinction lies in our earlier discussions.

Let us demonstrate a simple application of automorphism groups. For $n \geq 2$, Suppose that there exists a biholomorphism $f : \mathbb{P}^n \subset \mathbb{C}^n \longrightarrow \mathbb{B}^n$. By the transitivity of the action of $\mathtt{aut}(\mathbb{B}^n)$ we can assume that $f(0) = 0$. In particular we have that the isotropy subgroups $I_0(\mathbb{P}^n)$ and $I_0(\mathbb{B}^n)$ are isomorphic. Note that since both $\mathbb{P}^n$ and $\mathbb{B}^n$ are circular, by Corollary 2.2.1 the elements of $I_0(\mathbb{P}^n)$ and $I_0(\mathbb{B}^n)$ consist of linear maps. Note that $I_0(\mathbb{B}^n)$ contains all unitary linear transformations whereas $I_0(\mathbb{P}^n$ does not. To see this we can consider the map

$$g(z_1, z_2, z_3, ..., z_n) = \left(\frac{z_1 + z_2}{\sqrt{2}}, \frac{z_1 - z_2}{\sqrt{2}}, z_3, ..., z_n \right).$$

By a direct calculation we see that $g(\mathbb{P}^n) \neq \mathbb{P}^n$. This shows that there cannot be such an $f \in I_0(\mathbb{P}^n)$. Thus, $\mathbb{P}^n$ and $\mathbb{B}^n$ are not biholomorphic. We note that both of them are simply connected proper subsets of $\mathbb{C}^n$, thus for $n \geq 2$ the Riemann Mapping Theorem fails on $\mathbb{C}^n$.

2.3 The Poincaré Metric

Definition 2.3.1 *Let $v \in \mathbb{C}$ be a tangent vector to the unit disc at the point $p \in \Delta$. Then its norm under the Poincaré metric is defined as follows.*

$$P_\Delta(p; v) = \frac{|v|}{1 - |p|^2}$$

Let $\gamma : [0, 1] \longrightarrow \Delta$ be a piecewise C^1 curve. We define its Poincaré length, $L^P(\gamma)$ as

$$L^P(\gamma) = \int_0^1 P_\Delta(\gamma(t); \gamma'(t))dt.$$

Then for all $p, q \in \Delta$ we define their Poincaré distance as $d^P_\Delta(p, q) = \inf L^P(\gamma)$ where the infimum is taken over all piecewise C^1 curves $\gamma : [0, 1] \longrightarrow \Delta$ with $\gamma(0) = p$ and $\gamma(1) = q$.

As a consequence of the famous Schwarz-Pick Lemma ([16], page 264) we have the following interesting property.

Theorem 2.3.1 *([16]) The holomorphic mappings $f : \Delta \longrightarrow \Delta$ are distance non-increasing under the Poincaré metric. That is, if $f \in \mathscr{O}(\Delta, \Delta)$, $\forall p, q \in \Delta$ and $\forall v \in \mathbb{C}$ we have that $d^P_\Delta(p, q) \geq d^P_\Delta(f(p), f(q))$ and $P_\Delta(p; v) \geq P_\Delta(f(p); df_p v)$. Moreover, the inequalities given above are equalities if and only if f is biholomorphism. Thus, the automorphisms of the unit disc are isometries with respect to the Poincaré metric.*

Theorem 2.3.2 *([16], IX.2) For z in Δ,*

$$d^P_\Delta(0,z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right).$$

Moreover, $\forall p, q \in \Delta$

$$d^P_\Delta(p,q) = \frac{1}{2}\log\left(\frac{1+\left|\dfrac{p-q}{1-pq}\right|}{1-\left|\dfrac{p-q}{1-pq}\right|}\right).$$

Proof. By taking a rotation if necessary, assume that $z = |z|$. Now let γ be any piecewise C^1-curve on Δ. Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$, where γ_1 and γ_2 take values in $\mathbb{R}$. Elementary calculus shows that to minimize the Poincaré length of γ we must have γ_1 increasing. Further as γ_1 is increasing, by taking a reparametrization if necessary, we can assume that $\gamma_1(t) = t$. On any C^1 component of γ, we have that

$$P_\Delta(\gamma(t), \gamma'(t)) = \frac{\sqrt{1+(\gamma_2'(t))^2}}{1-\gamma(t)^2} \geq \frac{1}{1-t^2}.$$

Thus

$$L^P(\gamma) = \int_0^t \frac{\sqrt{1+(\gamma_2'(t))^2}dt}{1-\gamma(t)^2} \geq \int_0^z \frac{dt}{1-|t|^2} = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right).$$

Thus, the minimum of $L^P(\gamma)$ is achieved for the curve $\gamma : [0,z] \longrightarrow \Delta$ and

$$d^P_\Delta(0,z) = \frac{1}{2}\log\left(\frac{1+|z|^2}{1-|z|^2}\right).$$

Now, we consider the automorphism of the unit disc given by $f(z) = \dfrac{z-p}{1-\bar{p}z}$. By biholomorphic invariance of Poincaré metric, a direct calculation establishes the the second part of the theorem. □

Let us mention two observations from the proof of last theorem. The first one is that if $z \to \partial\Delta$ ie. $|z| \to 1$ we have that $d^P_\Delta(0,z) \to \infty$. So, with respect to the Poincaré metric, the origin (and by biholomorphic invariance any $p \in \Delta$) is infinitely far away from the boundary of the unit disc. Roughly speaking, this shows that the Poincaré metric is a complete metric.

The second observation is that, by conformal invariance under holomorphic maps and the geometry of Mobius transformations, the geodesics of the Poincaré metric are arcs of circles which are perpendicular to the unit circle. Similary, due the geometry of Mobius transformations, Poincaré ball of radius r and center $z \in \Delta$ is the Euclidean ball of radius $\frac{1-\tanh^2 r}{1-(\tanh^2 r)|z|}$ and center $\frac{1-\tanh^2 r}{1-(\tanh^2 r)|z|}z$.

Theorem 2.3.3 *([16],IX.2) The Poincaré metric induces the Euclidean topology of the unit disc.*

Proof. Note that $d^P_\Delta(0,z) = \tanh^{-1}(|z|)$. Thus about the origin the Poincaré ball of radius $\tanh^{-1}(r)$ is the Euclidean disc of radius r. Then about the origin, for a small enough $\varepsilon > 0$ so that the Poincare ball of radius $\tanh^{-1}(r)$ centered at the origin contains the euclidean disc of radius $r - \varepsilon$ centered at the origin and is contained in the euclidean disc of $r + \varepsilon$ centered at the origin. This shows that the two topologies are equivalent about the origin. Noting the transitivity of $\mathtt{aut}(\Delta)$ and openness of automorphisms of Δ, we conclude that the two topologies are equivalent on Δ. $\square$

We can naturally push the Poincaré metric to simply connected domains thanks to the Riemann Mapping Theorem. Further, by covering maps, the Poincaré metric can be pushed down to hyperbolic Riemann surfaces.

On the next chapter, we will discuss two invariant pseudometrics which are considered as generalizations of the Poincaré metric on arbitrary domains in $\mathbb{C}^n$. Their construction deeply relies on the space of holomorphic functions to and from the unit disc.

2.4 The Invariant Metrics

Definition 2.4.1 *Let $p \in G$ and $v \in \mathbb{C}^n \cong T_pG$. The norm of v at p under the infinitesimal Carathéodory metric of G is defined to be*

$$C_G(p;v) = \sup\{|df_p(v)| : f \in \mathscr{O}(G,\Delta), f(p) = 0\}.$$

Following properties of the infinitesimal Carathéodory metric follows from the definition.

Theorem 2.4.1 *([20]) Infinitesimal Carathéodory metric satisfies the following properties. Hence* $\forall p \in G$*,* $C_G(p;.)$ *defines a seminorm on* $\mathbb{C}^n \cong T_pG$*.*

1. $C_G(p;v) \geq 0$
2. $C_G(p;\lambda v) = |\lambda| C_G(p;v)$ *where* λ *is any complex number.*
3. $C_G(p;v+w) \leq C_G(p;v) + C_G(p;w)$ *for* $v, w \in \mathbb{C}^n$*.*

Theorem 2.4.2 *([20]) Holomorphic mappings are non-increasing on the Carathéodory norm of tangent vectors, that is if* $f : \Omega \longrightarrow \Omega'$ *is a holomorphic map between two domains* $G_1 \subset \mathbb{C}^{n_1}$ *and* $G_2 \subset \mathbb{C}^{n_2}$

$$C_{G_1}(p;v) \geq C_{G_2}(f(p);df_p(v)).$$

In particular, if $f : G_1 \longrightarrow G_2$ *is a biholomorphism,*

$$C_{G_1}(p;v) = C_{G_2}(f(p);df_p(v)).$$

Hence bihomorphic maps are isometries with respect to Carathéodory metric.

Proof. By composition of holomorphic functions, for any $g \in \mathscr{O}(G_2, \Delta)$ taking $f(p)$ to 0 and $df_p(v)$ to $\eta \in T_0\Delta$, the map $g \circ f \in \mathscr{O}(G_1, \Delta)$ takes $p \in G_1$ to 0 and v to η. Thus by definition, $C_{G_1}(p;v) \geq C_{G_2}(f(p);df_p(v))$.
Biholomorphic invariance follows from applying the same argument for f^{-1}. $\square$

The infinitesimal Kobayashi-Royden metric can be considered as a dual of the infinitesimal Carathéodory metric. We give its definition below.

Definition 2.4.2 *For* $p \in G$ *and* $v \in \mathbb{C}^n \cong T_pG$ *the norm of* v *under the infinitesimal Kobayashi-Royden metric is given as follows.*

$$K_G(p;v) = \inf\{|w| : f \in \mathscr{O}(\Delta, G), f(0) = p, df_0(w) = v\}$$

It also enjoys many properties of the infinitesimal Carathéodory metric.

Theorem 2.4.3 *[20] The infinitesimal Kobayashi metric satisfies the properties given below.*

1. *For any domain $G \subset \mathbb{C}^n$, $p \in G$, $v, w \in \mathbb{C}^n \cong T_pG$ and $\forall \lambda \in \mathbb{C}$ we have that*

$$K_G(p; v) \geq 0,$$

$$K_G(p; \lambda v) = |\lambda| K_G(p; v),$$

$$K_G(p; v + w) \leq K_G(p; v) + K_\Omega(p; w).$$

   Thus $K_G(p, .)$ is a seminorm in $\mathbb{C}^n$.

2. *If $f : G_1 \longrightarrow G_2$ is a holomorphic map between two bounded domains $K_{G_1}(p; v) \geq K_{G_2}(f(p); df_p(v))$. Hence, if $f : G_1 \longrightarrow G_2$ is a biholomorphism we have that*

$$K_{G_1}(p; v) \geq K_{G_2}(f(p); df_p(v)).$$

If we take $G = \mathbb{C}^n$ by Louville's theorem every holomorphic function $f : G \longrightarrow \Delta$ is constant. This shows that unlike the Poincaré metric, the Carathéodory metric is sometimes degenerate so it is indeed a pseudometric.

Similarly for the Kobayashi-Royden metric we let $G = \mathbb{C}^n$. Without loss of generality assume that v is a tangent vector of G at $p = (0, 0...0)$. Considering the maps $f_n : \Delta \longrightarrow G$ where $f_n(z) = nzv$ shows that $K_G(p; v) \leq P_\Delta(0; \frac{v}{n}) = \frac{1}{n}$. Letting $n \to \infty$ gives that $K_G(p; v) = 0$.

We will mainly focus on bounded domains. In that case, we have the following result.

Theorem 2.4.4 *([20]) Let $G \subset \mathbb{C}^n$ be a bounded domain. Then, it's Carathéodory and Kobayshi-Royden metrics are non-degenerate, that is $\forall p \in G$ and $\forall v \in \mathbb{C}^n \setminus \{0\}$ we have that $C_G(p; v) \neq 0$ and $K_G(p; v) \neq 0$.*

Proof. Let T_G denote either the Carathéodory or Kobayashi metric of G, let $p \in G$, $v \in T_pG$. By the biholomorphic invariance of the Poincaré metric we take a unitary transformation of $\mathbb{C}^n$ if necessary and assume that $v \in \{(s, 0, ..., 0) : s \in \mathbb{R}^+\}$. As G is bounded $\exists r > 0$ such that $G \subset \mathbb{B}^n_r$. Now consider the map $P : G \to C$ given by $P(z_1, ..., z_n) = z_1$. It is clear that $P(G) \subset \Delta_r$. Now, consider the inclusion map

$i : P(G) \hookrightarrow \Delta_r$ and $f : \Delta_r \longrightarrow \Delta$ given by $f(z) = \frac{z}{r}$. The distance decreasing property of the invariant metrics gives

$$T_G(p; v) \geq T_{P(G)}(P(p), |v|) \geq T_{\Delta_r}(P(p), |v|) = T_\Delta\left(\frac{P(p)}{r}, \frac{|v|}{r}\right) = \left(\frac{r|v|}{r^2 - |P(p)|^2}\right) > 0.$$

Thus the invariant metrics on G are non-degenerate. $\square$

Continuity and semi-continuity is required to define distances on domains based on these metrics, and we have the following results about regularity of the invariant metrics.

Theorem 2.4.5 *([20])*

1. *For an arbitrary domain G, if we consider the infinitesimal Carathéodory metric as a function on the tangent bundle TG, it is continuous.*
2. *For an arbitrary domain G, if we consider the infinitesimal Kobayashi metric as a function on the tangent bundle TG, it is upper-semicontinuous.*
3. *If $G \subset \mathbb{C}^n$ is a bounded domain, it's Kobayashi metric is continuous on the tangent bundle TG.*

We note that the holomorphic maps from G into Δ form a normal family. This fact is used in the proof of first statement in the last theorem. Similarly, boundedness of G allows us to use a similar normal families argument to get continuity of the Kobayashi metric on TG.

Another application of this fact shows that for any domain G there exists a function $f \in \mathscr{O}(G, \Delta)$ such that $f(p) = 0$ and $P_\Delta(0; df_p v) = C_G(p; v)$. So the supremum in the definition of the infinitesimal Carathéodory metric can be replaced with a maximum. We will introduce the squeezing functions in the next chapter and give an analogous proof of existence of extremal functions with respect to squeezing functions.

Let us define the integrated forms of Carathéodory and Kobayashi metrics.

Definition 2.4.3 *Let $\gamma : [0, 1] \longrightarrow G$ be a piecewise C^1 curve. We define its Carathéodory* *length (resp. Kobayashi-Royden length) as*

$$L_G^C(\gamma) = \int_0^1 C_G(\gamma(t);\gamma'(t))dt \text{ (resp. } L_G^K(\gamma) = \int_0^1 K_G(\gamma(t);\gamma'(t))dt).$$

Then the integrated Carathéodory (resp. integrated Kobayashi-Royden) distance between two points $p, q \in \Omega$ is defined as

$$d_G^C(p,q) = \inf L_G^C(\gamma) \text{ (resp. } d_G^K(p,q) = \inf L_G^K(\gamma))$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0,1] \longrightarrow G$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Following properties of the integrated Carathéodory and Kobayashi-Royden distances trivially follow from the properties of their infinitesimal forms.

Corollary 2.4.1 *Let $d_G^T(.,.)$ denote either the integrated Carathéodory or integrated Kobayashi-Royden distance. Then the following hold.*

1. *d_G^T is a pseudo-distance on G. If G is bounded then it is a distance.*
2. *Holomorphic maps are distance decreasing with respect to d_G^T that is $\forall f \in \mathscr{O}(G_1, G_2)$ and $\forall p, q \in G_1$ we have that*

$$d_{G_1}^T(p,q) \geq d_{G_2}^T(f(p), f(q)).$$

Therefore d^T is invariant under biholomorphic maps.

Introducing the integrated distances allows us to give a metric space structure to domains in $\mathbb{C}^n$. We will make use of the following topological result.

Theorem 2.4.6 *([3],[4]) Let $G \subset \mathbb{C}^n$ be a domain and suppose that C_G (resp. K_G) is non-degenerate, then the topology on G induced from d_G^C (resp. d_G^K) coincides with its Euclidean topology.*

We note that using continuity (or semi-continuity in the case of the Kobayashi metric) one can see that the previous result follows with the arguments as in the proof of Theorem 2.3.3.

Due to the Theorem 2.4.4, if G is bounded we have that K_G, C_G are non-degenerate. So in particular, the previous result implies the following.

Corollary 2.4.2 *Let $G \subset \mathbb{C}^n$ be a bounded domain then the topology on G induced from d_G^C and d_G^K are equivalent and they coincide with the Euclidean topology of G.*

There are other notions of Carathéodory distance and Kobayashi distance. Before we define them, we would like to note that they were constructed earlier than their infinitesimal counterparts.

Definition 2.4.4 *([18]) The Carathéodory distance for the points $p, q \in G$ is defined as*

$$d_G^c(p, q) = \sup\{f(p), f(q) : f \in \mathscr{O}(G, \Delta)\}.$$

Definition 2.4.5 *([18]) The set of holomorphic maps $\tau = \{f_i : \Delta \longrightarrow G\}_{i=1}^N$ is said to be a holomorphic chain connecting the points $p, q \in G$ if there are points $\{p_1, p_2 ... p_{N+1}\}$ such that $f_1(p_1) = p$, $f_N(p_{N+1}) = q$ and $\forall i \in \{1, 2..., N-1\}$ we have that $f_i(p_{i+1}) = f_{i+1}(p_{i+1})$. The length of the chain τ on G is defined as*

$$L_G(\tau) = \sum_{i=1}^{N-1} P_\Delta(p_i, p_{i+1}).$$

The Kobayashi distance of $p, q \in G$ is defined to be

$$d_G^k(p, q) = \inf L_G(\tau)$$

where the infimum is taken over all chains in G connecting p and q.

Theorem 2.4.7 *([18]) Let d_G^t denote either d_G^k or d_G^c. Then the following hold.*

1. *d_G^t is a pseudometric on G.*
2. *$\forall f : G_1 \in \mathscr{O}(G_1, F_2)$ and $\forall p, q \in G$ $d_{G_1}^t(p, q) \geq d_{G_2}^t(f(p), f(q))$. Thus biholomorphic mappings are isometries with respect to d^t.*

As in our previous discussions, one can see that the Carathéodory and Kobayashi distances are degenerate for $G = \mathbb{C}^n$ and they are non-degenerate for bounded domains.

Theorem 2.4.8 *For the invariant distances the following hold.*

1. *([38]) For any domain G and any $p, q \in G$ we have that $d_G^k(p,q) = d_G^K(p,q)$.*
2. *([4]) In general we only have that $d_G^c(p,q) \leq d_G^C(p,q)$.*

Using the definition of the Riemann integral, one sees that

$$d_G^C(p,q) = \sup_k \sum_{i=1}^{k-1} d_G^c(p_i, p_{i+1})$$

where the supremum is taken over any finite family of points $P = \{p_1, p_2, ..., p_k\}$ satisfying $p_1 = p$, $p_k = q$. So the fact that $d_G^C(p,.) \geq d_G^c(p,.)$ should not be a surprise.

For a detailed discussion on the Carathéodory and Kobayashi distances see [20]. We won't further deal with them in this theses. After this point when we say Carathéodory metric (resp. Kobayashi metric) we mean either the infinitesimal or integrated form of Carathéodory(resp. Kobayashi-Royden) metric.

The following gives the comparasion of these two metrics.

Theorem 2.4.9 *([20]) For any domain G and any $p, q \in G$ and $v \in T_pG$ the following holds.*

1. *$K_G(p;v) \geq C_G(p;v)$.*
2. *$d_G^K(p,q) \geq d_G^C(p,q)$.*

Proof. We will prove the first statement as the second follows from the first.
Let $p \in G$, $v \in \mathbb{C}^n$ be arbitrary. As we noted earlier there exists an $f \in \mathscr{O}(G, \Delta)$ such that $|df_p(v)| = C_G(p;v)$.
Let $g \in \mathscr{O}(\Delta, G)$ be any function satisfying $g(0) = p$ and $dg_0\nu = v$. We have that $f \circ g \in \mathscr{O}(\Delta, \Delta)$, $f \circ g(0) = 0$ and $d(f \circ g)_0\nu = df_p(v)$, then by Schwarz lemma we have

$$|\nu| \geq |df_pv| = C_G(p;v).$$

As g was arbitrary

$$K_G(p;v) \geq C_G(p;v). \square$$

Definition 2.4.6 *A domain G is said to be complete in the sense of Carathéodory (resp. Kobayashi) if (G, d_G^C) (resp. (G, d_G^K)) is complete as a metric space.*

If G is complete in the sense of Carathéodory (resp. Kobayashi) we also say that the Carathéodory (resp. Kobayashi) metric of G is complete.

Theorem 2.4.10 *([38]) Let T_G denote C_G (resp. K_G) and d_G^T denote d_G^C (resp. d_G^K). Then the following are equivalent.*

1. *(G, d_G^T) is complete as a metric space.*
2. *$\forall r > 0$ the set $\{z \in G : d^T(p, z) < r\}$ is relatively compact in G.*
3. *For every isometry $\gamma : [0, a) \longrightarrow G$, there exists an isometry $\tilde{\gamma} : [0, a] \longrightarrow G$ such that $\tilde{\gamma}|_{[0,a)} = \gamma$.*

Theorem 2.4.11 *([1, 2]) Let G be a bounded domain. Suppose that G is complete in the sense of Carathéodory or Kobayashi then G is pseudoconvex.*

Proof. If G is complete in the sense of Carathéodory then by Theorem 2.4.9 the Kobayashi ball of radius r and center z is contained in the Carathéodory ball of radius r and center z. So, by Theorem 2.4.10 it is complete in the sense of Kobayashi as well.
Now, let G be a domain that is complete in the sense of Kobayashi. We will show that G is taut.
Firstly, by the distance decreasing property of Kobayashi metric the family $\mathscr{O}(\Delta, G)$ is equicontinuous with respect to Kobayashi distances.
Note that as G is complete in the sense of Kobayashi, the Kobayashi metric of G is non-degenerate. So the Kobayashi topology of G is equivalent to its Euclidean topology. The same assertion also holds for Δ. Thus the family $\mathscr{O}(\Delta, G)$ is equicontinuous with respect to euclidean distances as well.
Now we let $\{f_n\} \in \mathscr{O}(\Delta, G)$ be a sequence that doesn't have any compactly divergent subsequence. Thus, by passing to a subsequence if necessary we can find relatively compact subsets K and L of Δ and G respectively, satisfying $f_n(K) \cap L \neq \emptyset$. We fix

$z_0 \in K$ and $w_0 \in L$ and set $C := \sup_{w \in L} d_G^K(w, w_0)$. By the triangle inequality and the distance decreasing property of Carathéodory metric $\forall z \in \Delta$ and $\forall n$ the estimate

$$d_G^K(f_n(z), w_0) \leq d_G^K(f_n(z_0), w_0) + d_G^K(f_n(z_0), f_n(z)) \leq d_\Delta^K(z, z_0) + C$$

is satisfied. So, $f_n(z)$ lies in the Kobayashi ball of radius $d_\Delta^K(z, z_0) + C$ centered at w_0 inside G. By completeness of the Kobayashi metric this implies that for any $z \in \Delta$, $f_n(z)$ lies on a compact subset of G. This observation combined with the equicontinuity of $\mathscr{O}(\Delta, G)$ shows that the sequence $\{f_n\}$ has a subsequence $\{f_{n,k}\}$ converging uniformly on compact subsets to some continuous function f from Δ to G. As the uniform limits of holomorphic functions are holomorphic $f \in \mathscr{O}(\Delta, G)$, we see that G is taut.
Thus, by Theorem 2.1.7 G is pseudoconvex. $\square$

A partial converse of this theorem is the following.

Theorem 2.4.12 *([20]) If G is a taut domain then the Kobayashi distance of G is non-degenerate.*

We conclude this chapter by giving the explicit formulas of invariant metrics on the unit disc $\Delta \subset \mathbb{C}^n$ and the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$.

Theorem 2.4.13 *Let T denote either the Carathéodory or Kobayashi metric. Then we have that:*

1. *([18])* $T_\Delta(p; v) = P_\Delta(p; v) = \dfrac{|v|}{1 - |p|^2}$.
2. *([17])* $[T_{\mathbb{B}^n}(p; v)]^2 = \dfrac{|v|^2}{1 - |p|^2} + \dfrac{| < v, p > |^2}{(1 - |p|^2)^2}$.

Proof. By biholomorphic invariance we can assume that $p = 0$. Let $f \in \mathscr{O}(\Delta, \Delta)$ such that $f(0) = 0$. Then the Schwarz lemma asserts that $|df_0| \leq 1$ so $|df_0 v| \leq |v|$. Considering the identity function, we have that $C_\Delta(0; v) = |v|$.
Let $p \in \Delta$. Now, for the Kobayashi-Royden metric let $g \in \mathscr{O}(\Delta, \Delta)$ with $g(0) = p$ such that $dg_0 \eta = v$. As $p = 0$, the Schwarz lemma gives that $|dg_0| \leq 1$, hence $|\eta| = \frac{|v|}{|dg_0|} \geq |v|$. Again, considering the identity function gives that $C_\Delta(0; v) = |v|$.

Consider the automorphism of the unit disc $f(z) = \dfrac{z-p}{1-\overline{p}z}$ and observe that $f'(p) = \dfrac{1}{1-\overline{p}p}$. By biholomorphic invariance we obtain that

$$C_\Delta(p;v) = C_\Delta\left(0; \frac{v}{1-\overline{p}p}\right) = \frac{|v|}{1-|p|^2} = P_\Delta(p;v).$$

This proves the first part.

For the second part we will exploit the notation by denoting the origin with 0 and the tangent vector $(1, 0...0) \in T_0\mathbb{B}^n$ with 1.

Let $f \in \mathscr{O}(\Delta, \mathbb{B}^n)$ such that $f(0) = (0, ..., 0)$ and $df_0\nu = 1$. Consider the map $g(z_1, z_2...z_n) = z_1$ so that $dg_0 1 = 1$. As $g \circ f \in \mathscr{O}(\Delta, \Delta)$ is a map fixing the origin, we have that $|\nu| \geq |d(f \circ g)_0\nu| = |dg_0 1| = 1$. The map $z \mapsto (z, 0...0)$ yields $K_{\mathbb{B}^n}(0; 1) = 1$. Biholomorphic invariance and properties of Kobayashi metric shows that $K_{\mathbb{B}^n}(0; v) = |v|$.

To get the Carathéodory metric at the origin, first take the map $f : \Delta \longrightarrow \mathbb{B}^n$ where f takes z to $(z, 0...0)$. Now let $g \in \mathscr{O}(\mathbb{B}^n, \Delta)$ be arbitrary. Observe that $g \circ f(0) = 0$ and $df_0(1) = 1$. Thus, $d(g \circ f)_0 1 = dg_0 1$. By Schwarz lemma, $|d(g \circ f)_0 1| \leq 1$. Hence for any $g \in \mathscr{O}(\mathbb{B}^n, G)$ with $g(0) = 0$, $|dg_0(1)| \leq 1$. Considering the projection to first coordinate gives $C_{\mathbb{B}^n}(0; 1) = 1$ hence $C^n_{\mathbb{B}}(0; v) = 1$. Now let $p := (p, 0...0) \in \mathbb{B}^n$ and $v \in T_p\mathbb{B}^n$. The map defined by

$$(z_1, z_2..., z_n) \mapsto \left(\frac{z_1 - p}{1 - pz_1}, \frac{z_2\sqrt{1-p^2}}{1-pz_1}, ..., \frac{z_n\sqrt{1-p^2}}{1-pz_1}\right)$$

is an automorphism of the unit ball that takes the point p to 0. An easy calculation shows that the derivative of this map at p is given by the diagonal matrix

$$df_p = diag\left(\frac{1}{1-p^2}, \frac{1}{\sqrt{1-p^2}}, ..., \frac{1}{\sqrt{1-p^2}}\right).$$

By biholomorphic invariance, for the point $p = (p, 0...0)$ we have that

$$[T_{\mathbb{B}^n}(p;v)]^2 = [T_{\mathbb{B}^n}(0; df_p v)]^2 = |df_p v|^2 = \frac{|v|^2}{1-p^2} + \frac{p^2|v_1|^2}{(1-p^2)^2}$$

Applying a unitary transformation shows that $\forall p \in \mathbb{B}^n$ and $v \in T_p\mathbb{B}^n$,

$$[T_{\mathbb{B}^n}(p;v)]^2 = \frac{|v|^2}{1-|p|^2} + \frac{|<v,p>|^2}{(1-|p|^2)^2}. \square$$

A direct calculation similar to the proof of Theorem 2.3.2 leads to the next corollary.

Corollary 2.4.3

1. *Let $d^T_{\mathbb{B}^n}$ denote either the Carathéodory or the Kobayashi distance on the unit ball. Then* $\forall z \in \mathbb{B}^n$,
$$d^T_{\mathbb{B}^n}(0, z) = \frac{1}{2}\log\left(\frac{1 + \|z\|}{1 - \|z\|}\right).$$
2. *In particular, for* $r \in (0, 1)$,
$$d^T_{\mathbb{B}^n}(\partial\mathbb{B}^n_r, z) = \left|\frac{1}{2}\log\left(\frac{1 + r}{1 - r}\right) - \frac{1}{2}\log\left(\frac{1 + \|z\|}{1 - \|z\|}\right)\right|.$$

Corollary 2.4.4 *For $r \in (0, 1)$ let $T(r) := d^T_{\mathbb{B}^n}(0, r)$ denote either the Carathéodory or Kobayashi distance on the unit ball from 0 to the circle $\{z \in \mathbb{C}^n : \|z\| = r\}$. Then $T(r) = \tanh^{-1}(r)$ and for $x \in (0, \infty)$ and $T^{-1}(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. In particular, $T(r)$ is a strictly increasing function. Further $r \to 0$ if and only if $T(r)$ tends to 0 and $T(r)$ tends to ∞ if and only if r tends to 1.*

CHAPTER 3

THE SQUEEZING FUNCTION

3.1 General Properties of Squeezing Functions

Recall that $\mathscr{E}(G, \mathbb{B}^n)$ denotes the set of holomorphic embeddings of G into $\mathbb{B}^n$.

Definition 3.1.1 *Let $G \subset \mathbb{C}^n$ be a bounded domain and $f \in \mathscr{E}(G, \mathbb{B}^n)$ such that $f(z) = 0$. The squeezing number of f is given as*

$$S_G^f(z) = \sup\{r : \mathbb{B}_r^n \subset f(G)\}. \tag{3}$$

Then the squeezing function of G is defined to be

$$S_G(z) := \sup\{S_G^f(z) : f \in \mathscr{E}(G, \mathbb{B}^n), f(z) = 0\}. \tag{4}$$

A domain G is said to be holomorphically homogenous regular (HHR) if its squeezing function is uniformly bounded below by a positive constant, that is there exists a $r > 0$ such that for all $z \in G$ we have that $S_G(z) > r$.

It is clear from the definition that the squeezing function is biholomorphically invariant, that is if $f : G_1 \longrightarrow G_2$ is a biholomorphism, $S_{G_1}(z) = S_{G_2}(f(z))$.

The idea of the squeezing function arose in [31, 32] where the authors wanted to compare the invariant metrics on Teichmüller spaces of Riemann Surfaces, and also on [43] where the study included the punctured ones. As a first observation, we have the following.

Theorem 3.1.1 *([31]) Let G be a HHR domain. Then the invariant metrics on G are equivalent.*

Proof. Let $r \in (0,1)$ be a constant so that $S_G(z) > r$. Let $z \in G$ be arbitrary. Now, let $f \in \mathscr{E}(G, \mathbb{B}^n)$ such that $f(z) = 0$ and $f(G) \supset \mathbb{B}^n_r$. Then we have $C_G(z; v) = C_f(G)(f(z); df_z(v))$ and $K_G(z; v) = K_f(G)(f(z); df_z(v))$.
Moreover, considering the inclusion map $i_{f(G)} : f(G) \hookrightarrow \mathbb{B}^n$ gives

$$C_{f(G)}(f(z); df_z(v)) \geq C_{\mathbb{B}^n}(f(z); df_z(v)) = K_{\mathbb{B}^n}(f(z); df_z(v)).$$

Rescaling and considering the inclusion map $i_{\mathbb{B}^n_r} : \mathbb{B}^n_r \hookrightarrow f(G)$ shows that

$$K_{\mathbb{B}^n}(f(z); df_z(v)) = rK_{\mathbb{B}^n_r}(f(z); df_z(v)) \geq rK_{f(G)}(f(z); df_z(v)).$$

Thus we have that

$$C_G(z; v) = C_{f(G)}(f(z); df_z(v)) \geq rK_{f(G)}(f(z); df_z(v)) = rK_G(z; v).$$

On the other hand, due to Theorem 2.4.9 we have that $K_G(z; v) \geq C_G(z; v)$. Thus the estimate

$$K_G(z; v) \geq C_G(z; v) \geq rK_G(z; v)$$

is satisfied for all $z \in G$ and $v \in \mathbb{C}^n$. Hence we conclude that the two invariant metrics are equivalent on G. □

Letting $r \to S_G(z)$ gives:

Corollary 3.1.1 *The invariant metrics satisfy the estimate*

$$K_G(z; v) \geq C_G(z; v) \geq S_G(z)K_G(z; v)$$

The Hurwitz theorem ([16],page 231) asserts that for planar domains G, if the sequence of injective holomorphic functions $\{f_n\}_{n \in \mathbb{N}} \subset \mathscr{O}(G, \mathbb{C})$ converge to a holomorphic function $f \in \mathscr{O}(G, \mathbb{C})$, then f is either injective or constant. The following lemma can be considered as a generalization to higher dimensions [10].

Lemma 3.1.1 *([10], Theorem 2.1.) Let $G \subset \mathbb{C}^n$ be a bounded domain and $z \in G$. Let $\{f_i\}_{i=1}^{\infty} \subset \mathscr{E}(G, \mathbb{C}^n)$ be a sequence of maps such that $\forall i$, f_i maps z to origin and $\exists \varepsilon > 0$ such that $\mathbb{B}^n_\varepsilon \subset \cap_{i \in \mathbb{N}} f_i(G)$. If f_i converge to a function f uniformly on compact subsets of G, then f is injective.*

Theorem 3.1.2 *([10], Theorem 2.2.) There exists an extremal map for the squeezing function problem, that is for any bounded domain $G \subset \mathbb{C}^n$ and any $z \in G$ there exists an $f \in \mathscr{E}(G, \mathbb{C}^n)$ such that $f(z) = 0$ and $f(G) \supset \mathbb{B}^n_{S_G(z)}$.*

Proof. Let $z \in G$ be an arbitrary point. Let $\{f_i\}_{i=1}^{\infty} \subset \mathscr{E}(G, \mathbb{B}^n)$ be a sequence such that f_i maps z to the origin and $f_i(G) \supset \mathbb{B}^n_{r_i}$ for and increasing sequence $\{r_i\}_{i=1}^{\infty}$ converging to $S_G(z)$. Since this family is bounded, by Theorem 2.1.1, by passing to a subsequence if necessary we can assume that f_i converge to some holomorphic function $f : G \longrightarrow \mathbb{C}^n$. By the previous lemma f is injective, so it is an open map. Thus, as $f_i(G)$ is contained in $\mathbb{B}^n$, we have that $f(G)$ is contained in $\mathbb{B}^n$.
To show that $f(G) \supset \mathbb{B}^n_{S_G(z)}$ we need to show that $\mathbb{B}^n_{r_i} \subset f(G)$ for all $i \in \mathbb{N}$. Let $g_i = f_i^{-1}|_{\mathbb{B}^n_{r_i}}$ so that $f_j \circ g_j$ is the identity mapping on $\mathbb{B}^n_{r_i}$ for $j \geq i$. By passing to a subsequence if necessary, we assume that g_j converges to a holomorphic function g. As $g_j(0) = z$, we have that $g(0) = z$ so there exists a neighbourhood $N_0 \subset \mathbb{B}^n_{r_i}$ of the origin so that $g(N_0) \subset G$. By construction $f \circ g|_{N_0} = id_{N_0}$, so $det(J_g(0)) \neq 0$. As each g_j is injective, $det(J_{g_j}(p)) \neq 0$ at any point, since $det(J_{g_j})$ converges to $det(J_g)$, by Hurwitz's theorem, $det(J_g)$ is either non-zero or zero everywhere. As $det(J_g(0)) \neq 0$, $det(J_g)$ is nowhere zero, thus g is an open map. This implies that $\forall i$, $g(\mathbb{B}^n_{r_i}) \subset G$. Since $f \circ g$ is identity on $\mathbb{B}^n_{r_i}$, $f(G) \supset \mathbb{B}^n_{r_i}$ and this establishes the theorem. □

The following corollary is immediate.

Corollary 3.1.2 *Let $z_0 \in G$. Then the following are equivalent.*

1. *$S_G(z_0) = 1$*
2. *G is biholomorphic to the unit ball $\mathbb{B}^n$.*
3. *$\forall z \in G$ we have that $S_G(z) = 1$.*

Theorem 3.1.3 *([10], Theorem 3.1.) Let $G \subset \mathbb{C}^n$ be a bounded domain. Then S_G is continuous.*

Proof. Let $z \in G$ be an arbitrary point and and $\{z_i\}_{i=1}^{\infty}$ be sequence in G converging to z.

Due to Theorem 3.1.2 there exists a $f \in \mathscr{E}(G, \mathbb{B}^n)$ such that $f(z) = 0$ and $f(G) \supset \mathbb{B}^n_{S_G(z)}$. As $z_i \to z$, by continuity of f we have that $|f(z_i)| \leq \varepsilon_i$ for some sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ converging to 0. Considering

$$f_i(z) = \frac{f(z) - f(z_i)}{1 + \varepsilon_i} \in \mathscr{E}(G, \mathbb{B}^n)$$

yields

$$S_G(z_i) \geq \frac{S_G(z) - \varepsilon_i}{1 + \varepsilon_i}.$$

So we have $\liminf_{i \to \infty} S_G(z_i) \geq S_G(z)$.
Now let $\{g_i\}_{i=1}^{\infty} \subset \mathscr{E}(G, \mathbb{B}^n)$ be a sequence such that $g_i(z_i) = 0$ and $g_i(G) \supset \mathbb{B}^n_{S_G(z_i)}$.
By distance decreasing property of Kobayashi metric we have that

$$d^K_{\mathbb{B}^n}(g_i(z_i), g_i(z)) \leq d^K_{g_i(G)}(g_i(z_i), g_i(z)) = d^K_G(z_i, z). \tag{5}$$

As G is bounded, by Theorem 2.4.4 its Kobayashi metric is non-degenerate. Moreover by Theorem 2.4.6, the Kobayashi metric of G is generates the Euclidean topology of G. So, as z_i goes to z, $d^K_G(z_i, z)$ tends to 0. Thus, $g_i(z)$ tends to 0 by (5). In particular, there exists a decreasing sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ converging to 0 such that $|g_i(z)| < \varepsilon_i$.
Now we define

$$h_i(x) = \frac{g_i(x) - g_i(z)}{1 + \varepsilon_i}.$$

Noting the injectivity of h_i we see that

$$S_G(z) \geq \frac{S_G(z_i) - \varepsilon_i}{1 + \varepsilon_i}.$$

Hence we get that

$$\limsup_{i \longrightarrow \infty} S_G(z_i) \leq S_G(z).$$

As $z \in G$ was arbitrary, we get that S_G is continuous. $\square$

Squeezing functions are stable with respect to exhaustion of domains, this property will be useful.

Theorem 3.1.4 *([9]) Let $G \subset \mathbb{C}^n$ be a domain and $\{G_i\}_{i=1}^{\infty}$ be a sequence of domains such that $\forall n \in \mathbb{N}$ $G_i \subset G$ and $G_i \subset G_{i+1}$. Further, suppose that $G = \cup_{i \in \mathbb{N}} G_i$. Then we have that $\forall z \in G$*

$$\lim_{n \longrightarrow \infty} S_{G_i}(z) = S_G(z)$$

Proof. Fix $z \in G$ and for G_i containing z, let $f_i \in \mathscr{E}(G_i, \mathbb{B}^n)$ be a holomorphic embedding such that $f_i(G_i) \supset \mathbb{B}^n_{S_{G_i}(z)}$. By Theorem 2.1.1, we can assume that f_i converges uniformly on compact subsets to a holomorphic map f on G taking z to the origin.

We claim that f is injective. To see this, for $j \geq i$ let g_j be the local inverse of f_j on G_i. Again by due to the normal families argument, by passing to a subsequence if necessary we assume that $\{g_j\}$ is convergent to $g \in \mathscr{O}(G_i, \mathbb{B}^n)$.

Notice that the Jacobian of g_j at the origin, $\det(dg_{j_0})$ is bounded above by Cauchy estimates ([26]). Hence, $\det(df_{j_z})$ is bounded below. Then $\det(df_z) > 0$, in particular f is an open map. Thus $f(G)$ contains a neighbourhood of the origin. So by Lemma 3.1.1, $\lim f_i = f$ is injective. So it is an open map. Thus we have that $f(G) \subset \mathbb{B}^n$.

Let $\varepsilon > 0$ be arbitrary. As the sequence $\{G_i\}$ exhausts G, we have that for large i, $f(G_i) \supset \mathbb{B}^n_{S_G(z)-\varepsilon}$, so $S_{G_i}(z) \geq S_G(z) - \varepsilon$. This shows that

$$S_G(z) \leq \liminf S_{G_i}(z).$$

Let $R = \limsup_{i \to \infty} S_{G_i}(z)$. By passing to a subsequence if necessary assume that $S_{G_i}(z)$ tends to R. Further, let $\{f_i\}$ be the corresponding subsequence of $\{f_i\}$ satisfying $f_i(G_i) \supset \mathbb{B}^n_{S_{G_i}(z)}$. It is clear that for any $\varepsilon > 0$, for large i, $f_i(G_i) \supset \mathbb{B}^n_{R-\varepsilon}$. Take h_i to be the inverse of f_i on $\mathbb{B}^n_{R-\varepsilon}$. By passing to a subsequence if necessary, we assume that $\{f_i\}$ converges to $f \in \mathscr{O}(G, \mathbb{B}^n)$ and $\{h_i\}$ converges to $h \in \mathscr{O}(\mathbb{B}^n_{R-\varepsilon}, G)$. As in our earlier proofs using Lemma 3.1.1 the injectivity of h and f can be shown. Also, $f \circ h$ is identity on $\mathbb{B}^n_{R-\varepsilon}$. Hence $f(h(\mathbb{B}^n_{R-\varepsilon})) = \mathbb{B}^n_{R-\varepsilon}$ and $f(G) \supset \mathbb{B}^n_{R-\varepsilon}$. This implies that

$$S_G(z) \geq R = \limsup_{i \longrightarrow \infty} S_{G_i}(z).$$

This finishes the proof.

For a sequence of decreasing domains, we only have a weaker result.

Theorem 3.1.5 *([9]) Let $G \subset \mathbb{C}^n$ be domain and $\{G_i\}_{i=1}^{\infty}$ be sequence of domains such that $\forall i \in \mathbb{N}$, $G_i \subset G$ and $G_i \supset G_{i+1}$. Further suppose that $G = \cap_{i \in \mathbb{N}} G_i$. Then we have that $\forall z \in G$*

$$\limsup_{i \longrightarrow \infty} S_{G_i}(z) \leq S_G(z).$$

Theorem 3.1.6 *([44], [10]) Let G be a HHR domain, then the Carathéodory metric of G is complete.*

Proof. Let $S_G(z) > r$ for all $z \in G$. For any $\varepsilon > 0$, let

$$C(p) := \left\{ z \in G : d_G^C(p, z) < \frac{1}{2} \log \left(\frac{1 + \frac{r}{2}}{1 - \frac{r}{2}} \right) \right\}.$$

In other words, $C(p)$ is the Carathéodory ball in G of radius $\frac{1}{2} \log \left(\frac{1-\frac{r}{2}}{1-\frac{r}{2}} \right)$ about p. Let $f : G \longrightarrow \mathbb{B}^n$ be a holomorphic embedding such that $f(p) = 0$ and $\mathbb{B}_r^n$ is relatively compact in $f(G)$.

Now observe that

$$f(C(p)) = \left\{ y \in f(G) : d_{f(G)}^C(0, y) < \frac{1}{2} \log \left(\frac{1 + \frac{r}{2}}{1 - \frac{r}{2}} \right) \right\}.$$

By distance decreasing property of Carathéodory metric we have that

$$f(C(p)) \subset \left\{ y \in \mathbb{B}^n : d_{\mathbb{B}^n}^C(y, f(p)) < \frac{1}{2} \log \left(\frac{1 + \frac{r}{2}}{1 - \frac{r}{2}} \right) \right\} = \mathbb{B}_{\frac{r}{2}}^n.$$

Thus $f(C(p)) \subset \mathbb{B}_{\frac{r}{2}}^n \subset \mathbb{B}_r^n$. So, $f(C(p))$ is relatively compact in $f(G)$. Since f is a biholomorphism, $C(p)$ is relatively compact in G. Since G is bounded, by Theorem 2.4.6 the topology of (G, d_G^C) coincides with the Euclidean topology of G. Hence, for any $\varepsilon > 0$, the Carathéodory balls in G of radius less than ε forms a basis in the Carathéodory topology. This shows that Carathéodory balls of arbitrary radii are compact are G. So, by Theorem 2.4.10 the Carathéodory metric of G is complete. □

Corollary 3.1.3 *Let G be a HHR domain, then G is pseudoconvex.*

Proof. Let G be a HHR domain. By Theorem 3.1.6 the Carathéodory metric of G is complete. Then by Theorem 2.4.11 G is pseudoconvex.
□

3.2 Boundary Behaviour of Squeezing Functions on Planar Domains

Let us start with an easy observation.

Theorem 3.2.1 *([10]) Let $G = \Delta^* := \Delta \setminus \{0\}$. Then $S_G(z) = |z|$.*

Proof. By Riemann's removable singularity theorem, any $f \in \mathscr{E}(G, \Delta)$ extends to a $\tilde{f} \in \mathscr{O}(\Delta, \Delta)$.
Now, let $f \in \mathscr{E}(G, \Delta)$ such that $f(z) = 0$ and $f(G) \supset \Delta_{S_G(z)}$. Let $\tilde{f}$ be the holomorphic extension of f to Δ. Distance decreasing property of Poincaré metric gives

$$d^P_\Delta(0, \tilde{f}(0)) \le d^P_\Delta(0, z) = \frac{1}{2} \log\left(\frac{1 + |z|}{1 - |z|}\right).$$

Thus, $d(0, \tilde{f}(0)) \le |z|$ and hence $S_G(z) \le |z|$.
Now consider the automorphism $f(x) = \dfrac{x - z}{1 - \overline{z}x} \in \mathscr{E}(G, \Delta)$ taking z to 0. By biholomorphic invariance of the Poincaré metric

$$f(G) \supset \{x \in \Delta : d^P_\Delta(0, x) \le d^P_\Delta(0, z)\} = \Delta_{|z|}.$$

Thus we obtain $S_G(z) = |z|$. $\Box$

The following observation remarkably simplifies the squeezing function problem for planar domains. For bounded planar domains let us denote the special embeddings

$$\mathscr{S}\mathscr{E}(G, \Delta) = \{f \in \mathscr{E}(G, \Delta) : \partial\Delta \subset \partial f(G)\}. \tag{6}$$

We note that the family $\mathscr{S}\mathscr{E}(G, \Delta)$ is non-empty. To see this we let K denote the unbounded component of $\mathbb{C} \setminus G$. Then let $L := \{\mathbb{C} \setminus G\} \setminus K$. It is clear that $G \cup L$ is simply connected so we can find a conformal map that takes $G \cup L$ onto the unit disc. As $\partial(G \cup L)$ gets mapped to $\partial\Delta$ we have that $\partial f(G) \supset \partial\Delta$.

Theorem 3.2.2 *([10],[35]) For a bounded planar domain G, and $z \in G$ there doesn't exist a function f in the set $\mathscr{E}(G, \Delta) \setminus \mathscr{S}\mathscr{E}(G, \Delta)$ such that $f(z) = 0$ and $f(G) \supset \Delta_{S_G(z)}$. Therefore we have that*

$$S_G(z) = \sup\{r : \Delta_r \in f(G)\}$$

for some $f \in \mathscr{S}\mathscr{E}(G, \Delta)$ such that $f(z) = 0$.

Proof. Let $f \in \mathscr{E}(G, \Delta) \setminus \mathscr{S}\mathscr{E}(G, \Delta)$ such that $f(z) = 0$. Now consider the domain $D := f(G) \cup L$ where L is the union of bounded components of $\mathbb{C} \setminus f(G)$. Let g be the Riemann map of D taking 0 to itself.

As $D \subsetneq \Delta$, by the Schwarz lemma we have that $(g^{-1})'(0) < 1$. Thus, $g'(0) > 1$. This implies that there exists a $\delta > 0$ such that for $v \in \mathbb{C} \setminus \{0\}$ we have

$$C_D(0; v) > C_\Delta(0; v) + \delta|v|.$$

Then, by the continuity of Carathéodory metric, we can find an $\varepsilon > 0$ and a positive $\tilde{\delta} \leq \delta$ such that $\forall p \in \Delta_\varepsilon$ and for an arbitrary $v \in \mathbb{C} \setminus 0$ we have that

$$C_D(p; v) > C_\Delta(p; v) + \tilde{\delta}|v| \tag{7}$$

We claim that for any $p \in \partial L$

$$dist(0, p) < dist(0, f(p)).$$

To see this, notice that by (7), for any C^1-curve γ connecting p and 0 we have

$$L^C_\Delta(\gamma) \leq L^C_D(\gamma) - \tilde{\delta}\varepsilon. \tag{8}$$

Due to (8) and biholomorphic invariance of Carathéodory metric we have that

$$d^C_\Delta(0, p) \leq d^C_D(0, p) - \tilde{\delta}\varepsilon = d^C_\Delta(0, f(p)) - \tilde{\delta}\varepsilon.$$

Thus, on the unit disc, the Carathéodory distance of p to 0 is less than the Carathéodory distance of $f(p)$ to 0. We note that due to Corollary 2.4.4

$$d^C_\Delta(0, z) = \frac{1}{2}\log\left(\frac{1 + |z|}{1 - |z|}\right)$$

is an increasing function of $|z|$. It follows that that $\exists C > 0$ such that $\forall p \in \partial L$, $|p| + C \leq |f(p)|$. Hence, if $f(G)$ contains a disc of radius r, $g \circ f(G)$ contains a disc of radius $r + C$. Then we have that $S^f_G(z) < S^{g \circ f}_G(z)$. As $g \circ f \in \mathscr{E}(G, \Delta)$, by definition of squeezing functions the theorem follows. $\square$

Now we are ready to describe the boundary behaviour of squeezing functions on planar domains.

Theorem 3.2.3 *([10]) Let G be a finitely connected bounded planar domain. Let $\Gamma = \{p\}$ be an isolated boundary component of G. Then*

$$\lim_{G \ni z \to \Gamma} S_G(z) = 0.$$

Proof. As in the case of the punctured disc, every analytic function on G extends analytically to $\tilde{G} := G \cup \{p\}$. Let $f_z \in \mathscr{E}(G, \Delta)$ satisfying $f_z(z) = 0$ and $f_z(G) \supset \Delta_{S_G(z)}$. Then by distance decreasing property of Carathéodory metric we have that

$$d^C_{\tilde{G}}(z,p) \geq d^C_{f_z(\tilde{G})}(0, f_z(p)) \geq d^C_\Delta(0, f_z(p)). \tag{9}$$

As $\tilde{G}$ is bounded its Euclidean topology is equivalent to its topology induced from the Carathéodory metric. Hence as $z \to p$, $d^C_{\tilde{G}}(z,p) \to 0$. Thus, $d^C_\Delta(0, f_z(p))$ tends 0 by (9). By Corollary 2.4.4 we have that

$$\frac{1}{2}\log\left(\frac{1+S_G(z)}{1-S_G(z)}\right) \leq d^C_\Delta(0, f_z(p)).$$

Thus as $z \to p$ we have that $S_G(z) \to 0$. □

Lemma 3.2.1 *([10]) Suppose that $G = \Delta \setminus L$ where L is any relatively compact subset of Δ whose components don't reduce to points. Then we have*

$$\lim_{G \ni z \to \partial\Delta} S_G(z) = 1$$

Proof. By Corollary 2.4.4 and biholomorphic invariance of Poincaré metric, considering the automorphism of the unit disc $f(x) = \dfrac{x-z}{1-\overline{z}x}$ for a candidate for the squeezing function problem at $z \in G$ yields that $S_G(z) \geq \tanh(d^P_\Delta(z, \partial L))$.
Now, as $L \subset \Delta$ is relatively compact, there exists an $r \in (0,1)$ such that L is also relatively compact in Δ_r. Let $p \in \partial L$ be an arbitrary point and $r := (r, 0, ...0)$. Triange inequality on Poincaré distance gives

$$|d^P_\Delta(p,r) - d^P_\Delta(r,z)| \leq d^P_\Delta(z,p). \tag{10}$$

As $|z| \to 1$, $d^P_\Delta(r,z) \to \infty$. Thus $d^P_\Delta(p,z) \to \infty$. Since (10) holds for an arbitrary $p \in \partial L$, we see that $d^P_\Delta(z, \partial L)$ tends to ∞.
Therefore by Corollary 2.4.4, as $z \to \partial\Delta$, $S_G(z) \geq \tanh(d^P_\Delta(z, \partial L))$ tends to 1. □

Theorem 3.2.4 *Let G be a planar domain and $\Gamma \subset \partial G$ be a boundary component that doesn't reduce to a point. Then*

$$\lim_{G \ni z \to \Gamma} S_G(z) = 1.$$

Proof. Let L be the component of $\mathbb{C} \setminus G$ containing Γ. Let $K := \mathbb{CP}^1 \setminus L$ where $\mathbb{CP}^1$ is the Riemann sphere. By the uniformization theorem there exists a biholomorphic mapping $f : K \longrightarrow \Delta$ such that Γ is mapped to $\partial\Delta$. Let $z_i \subset G$ be a sequence converging to Γ and set $y_i := f(z_i)$. Then clearly $y_i \to \partial\Delta$. So combining biholomorphic invariance of squeezing functions with the last lemma yields

$$\lim_{i\to\infty} S_G(z_i) = \lim_{i\to\infty} S_{f(G)}(y_i) = 1.\square$$

Combining theorems 3.2.3 and 3.2.4 gives us the following.

Corollary 3.2.1 *Let G be a bounded finitely connected planar domain.*

1. *If none of the boundary components of G reduce to a point, then G is HHR.*
2. *If any of the boundary components of G reduce to a point, then G is not HHR.*

Proof. For the first part, it is clear that if none of the boundary components of G reduce to a point,

$$\lim_{G\ni z\to\partial G} S_G(z) = 1.$$

So, we can find a neighbourhood N of ∂G and some $C_1 > 0$ such that $\forall z \in N \cap G$ we have that $S_G(z) \geq C_1$. On the other hand, $G \setminus N$ is compact. Hence, $\exists C_2 > 0$ such that $\forall z \in G \setminus N$, $S_G(z) \geq C_2$. We see that the squeezing function of G is uniformly bounded by $\min\{C_1, C_2\} > 0$ and G is HHR.
For the second part, if G has a boundary component that is a point, say $\Gamma = \{p\}$, we can find a sequence $\{z_i\} \subset G$ such that $S_G(z_i)$ tends to 0. In this case, G is not HHR.
$\square$

One may ask that if finite connectivity is a necessary condition for being HHR for planar domains. In [10], the authors showed that the answer to this question is negative by giving an example of an infinitely connected planar domain which is HHR.

Corollary 3.2.1 and the example of the punctured disc shows that there are pseudoconvex domains that are not HHR.

3.3 Boundary Behaviour of Squeezing Function on Higher Dimensions

The notion of tubular neighbourhoods will be useful in the chapter. We give its definition below.

Definition 3.3.1 *([2], Definition 1.64.) Let M be a C^2-smooth submanifold of $\mathbb{R}^n$ and $\eta : M \to \mathbb{R}^n$ a smooth unit normal vector field on M. We say that M has a tubular neighbourhood if for some $\varepsilon > 0$ the set*

$$U_\varepsilon := \cup_{x \in M}\{x + \eta(x)r : |r| < \varepsilon\}$$

is a topological neighbourhood of M.

Lemma 3.3.1 *([30],Theorem 10.19) Every C^2-smooth submanifold of $\mathbb{R}^n$ has a tubular neighbourhood.*

The following theorem relies on an interesting geometric observation. We will follow the proof given in [44].

Theorem 3.3.1 *([44]) Let G be a bounded strongly convex domain in $\mathbb{C}^n$. Then G is HHR.*

Proof. For convenience, let us first introduce our argument in one dimension.
Suppose that $\Gamma_1 = \partial\Delta_{r_1}(z_1)$ and $\Gamma_2 = \partial\Delta_{r_2}(z_2)$ are two circles of radius r_1 and r_2 respectively and that they meet tangentially at a point $\{p\}$. Without loss of generality assume that $\Delta_{r_1}(z_1) \subset \Delta_{r_2}(z_2)$. Further, by taking an affine transformation of $\mathbb{C}$ assume that $z_1 = 0$, $r_1 = 1$ and $p = 1$ and $z_2 = 1 - r_2$ lies on the real axis. Then for $w \in (0, 1)$ the map

$$f(z) = \frac{z - w}{1 - wz}$$

is an injective holomorphic mapping on $\Delta_{r_2}(z_2)$ which fixes $\Delta_{r_1}(z_1)$ and takes the point w to the origin. We claim that $f(\Delta_{r_2}(z_2)) \subset \Delta_{2r_2}(0)$. To see this, we identify $z = x + iy$, with $(x, y) \in \mathbb{R}^2$. Then we have that

$$\Gamma_2 = \Delta_{r_2}(z_2) = \{(x, y) \in \mathbb{R}^2 : (x - r_2 + 1)^2 + y^2 = {r_2}^2\}.$$

Now, consider the function

$$h(x,y) := |f(z)|^2 = \frac{(x-w)^2 + y^2}{(1-wx)^2 + (wy)^2}$$

on $\Gamma_2 = \partial\Delta_{r_2}(z_2)$. Elementary calculus gives that the maximum is achieved at the point $(1-2r_2, 0)$ and as $0 < w < 1 < r_2$ we have that

$$h(1-2r_2, 0) = \frac{(2r_2 - 1 - w)^2}{(1 - w(1-2r_2))^2} < (2r_2 - 1 - w)^2 < (2r_2)^2.$$

So, $\forall z \in \Gamma_2$ we have that $|f(z)| < (2r_2)$. Further, as $h(x,y) = |f(z)|^2$ is subharmonic, $\forall z \in \Delta_{r_2}(z_2)$ we have that $|f(z)| < 2r_2$. Thus, $f(\Delta_{r_2}(z_2)) \subset \Delta_{2r_2}$.

Now, let $G \subset \mathbb{C}$ be a bounded strongly convex domain with C^2-smooth boundary. Due to strong convexity of ∂G, for any $p \in \partial G$ we can find discs $\Delta_{r_1(p)}(z_1(p))$, $\Delta_{r_2(p)}(z_2(p))$ whose boundaries meet tangentially at p such that $\Delta_{r_1(p)}(z_1(p)) \subset G \subset \Delta_{r_2(p)}(z_2(p))$.

$\forall p \in \partial G$ we define the line segment $\mathbb{L}_p$ in G which connects p with $r_1(p)$. By the discussion above for all $w \in \mathbb{L}_p$, we can find an $f \in \mathscr{E}(G, \Delta_{2r_2})$ such that $f(w) = 0$ and $f(G) \supset \Delta_{r_1}$. Thus $\forall w \in \mathbb{L}_p$ we have

$$S_G(w) \geq \frac{r_1(p)}{2r_2(p)}.$$

Let N_p denote a neighbourhood of p in $\bar{G}$. As the boundary G is strongly convex, about each point $q \in N_p \cap \partial G$ we can find discs $\Delta_{r_1(q)}(z_1(q)) \subset G \subset \Delta_{r_2(q)}(z_2(q))$ meeting tangentially at q. Also, since the boundary of G is C^2-smooth, the curvature of $N_p \cap \partial G$ is continuous. This implies that by taking a smaller neighbourhood if necessary, we can have that

$$r := \inf_{q \in N_p \cap \partial G} r_1(q) > 0 \text{ and } R := \sup_{q \in N_p \cap \partial G} r_2(q) < \infty \tag{11}$$

Now, as the boundary of G is C^2-smooth, by Lemma 3.3.1 it admits a tubular neighbourhood.

$$U_\varepsilon := \bigcup_{x \in \partial G} \{x + s\eta(x) : s < \varepsilon\},$$

for some $\varepsilon > 0$.

By (11), if we take $\varepsilon < r$, we have that for each point $z \in N_p \cap U_\varepsilon \cap G$, there exists a $q \in \partial N_p \cap \partial G$ such that $z \in \mathbb{L}_q$. So,

$$S_G(z) \geq \frac{r_1(q)}{2r_2(q)} \geq \frac{r}{2R} > 0.$$

Hence about each boundary point $p \in \partial G$, the squeezing function of G is bounded below by a positive constant, say $c(p)$. Due to the compactness of ∂G, we can find a finite set $\{p_1, ..., p_k\}$ such that

$$\bigcup_{i=1}^{k} N_{p_i} \supset \partial G. \tag{12}$$

Then $\forall z \in U := \cup_{i=1}^{k} N_p \cap G$, we have that

$$S_G(z) \geq \min_{i \in 1,2,...,k} c(p_i) =: C > 0. \tag{13}$$

Now, suppose that G is not HHR. Then there exists a sequence $\{z_i\} \subset G$ such that $S_G(z_i) \to 0$ as $i \to 0$. As $S_G(z) > 0$ for any $z \in G$, we see that this sequence must converge to ∂G. In particular, as U is a neighbourhood of ∂G in G, all but finitely many element of $\{z_i\}$ must lie on U. So for all but finitely many $x \in \{z_i\}$ we have that $S_G(x) \geq C > 0$. In particular $S_G(z_i)$ does not tend to 0. This is a contradiction, so G must be HHR.

Now let us see how our argument can be generalized to higher dimensions.

Let $G \subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2-smooth boundary. As in the case of planar domains, for each boundary point $p \in \partial G$, we can find two balls $\mathbb{B}^n_{r_1}(z_1) \subset G \subset \mathbb{B}^n_{r_2}(z_2)$ meeting tangentially at p. Without loss of generality we may assume that $z_1 = 0 := (0, ..., 0)$, $r_1 = 1$, $p = 1 := (1, 0, ..., 0)$ and $y = (1 - r_2, 0, ..., 0)$. As above let $\mathbb{L}_p$ be the line segment connecting z_1 and p. For $w := (w, 0..., 0) \subset \mathbb{L}_p$, we will consider the automorphism of the unit ball

$$f(z_1, z_2, ..., z_n) = \left(\frac{z_1 - w}{1 - wz}, \frac{\sqrt{1 - w^2} z_2}{1 - wz_1}, ..., \frac{\sqrt{1 - w^2} z_n}{1 - wz_1} \right).$$

This function is injective on $\mathbb{B}^n_{r_2}(z_2)$ and it maps w to 0. Also by a calculation similar to the one in the planar case, we have that

$$\mathbb{B}^n_{r_1} \subset f(G) \subset \mathbb{B}^n_{2r_2}.$$

So, for $w \in \mathbb{L}_p$, $S_G(w) \geq \dfrac{r_1}{2r_2}$. Again, we can find a sufficiently small neighbourhood N_p of p in $\overline{G}$ and a sufficiently small tubular neighbourhood U_ε of ∂G such that $\forall z \in N_p \cap U_\varepsilon \cap G$ we have that $S_G(z) \geq C(p)$ for some $C(p) > 0$. By the same compactness argument, this shows that G is HHR. $\square$

Following result will enable us to extend this theorem to strongly pseudoconvex domains.

Theorem 3.3.2 *([12]) Let G be a bounded domain and $p \in \partial G$ be strongly pseudoconvex boundary point such that ∂G is C^2-smooth near p. Then there exists a holomorphic embedding $f : \overline{G} \longrightarrow \overline{\mathbb{B}^n}$ satisfying*

1. $f(p) = (1, 0, ..., 0)$.
2. $\{z \in \overline{G} : f(z) \in \partial\mathbb{B}^n\} = \{p\}$.

Theorem 3.3.3 *Let G be a bounded strongly pseudoconvex domain with C^2-smooth boundary. Then G is HHR.*

Proof. Suppose that G is not HHR. Then we can find a sequence of points $\{z_i\} \subset G$ such that $S_G(z_i) \to 0$. Then as above, $\{z_i\}$ must converge to ∂G.
Let $p \in \partial G$ be an arbitrary strongly pseudoconvex boundary point. Due to Theorem 3.3.2, we can find a map $f \in \mathscr{E}(G, \mathbb{B}^n)$ such that f extends to a biholomorphism of closures (the extension is also denoted by f) and $f(p)$ is a strongly convex boundary point of $f(G)$. As the map f is a biholomorphism of closures, the boundary of $f(G)$ is also C^2-smooth. Thus, by continuity of partial derivatives, the boundary of $f(G)$ is strongly convex near $f(p)$.
Following from the proof of Theorem 3.3.1, we can find a neighbourhood of $f(p)$ in $f(\overline{G}) = \overline{f(G)}$, say $N_{f(p)} \cap f(G)$, and $C_p > 0$ such that $S_{f(G)}(x) > C_p$ for all $x \in N_{f(p)}$. By the biholomorphic invariance of squeezing functions $\forall z \in G \cap N_p := f^{-1}(N_{f(p)})$ we have that $S_G(z) > C_p$.
As any boundary point of G is C^2-smooth and strongly pseudoconvex the argument above can be applied to any boundary point. Thus by compactness of ∂G, we can find a finite set $\{p_i\}_{i \leq I}$, and neighbourhoods $N_{p_i} \subset \overline{G}$ of $\{p_i\}$ defined as above, which covers ∂G such that $\forall z \in N_{p_i} \cap G$ we have a positive constant $C(p_i)$ such that

$$S_G(z) \geq C(p_i).$$

Due to construction, on a neighbourhood of the boundary we have that

$$S_G(z) \geq C := \min_{i \in I} C(p_i) > 0.$$

Thus, if $\{z_i\} \subset G$ is a sequence converging to boundary, we must have that

$$\liminf_{i \to \infty} S_G(z_i) \geq C > 0.$$

In particular we see that $S_G(z_i)$ does not tend to 0. This is a contradiction and G must be HHR. $\square$

There is more to say on squeezing functions on strongly pseudoconvex domains.

Theorem 3.3.4 *([9],[22]) Let G be a bounded strongly pseudoconvex domain with C^2-smooth boundary. Then $S_G(z) \to 1$ as $z \to \partial G$.*

There are two different proofs of this theorem given in [9] and [22]. We do not present them as they are highly detailed. The proof in [22] is based on the scaling method described in Chapter 9 and Chapter 10 of [18] whereas the proof in [9] is heavily inspired from [14] where the authors studied a problem about exhaustion of domains. Theorem 3.3.4 can be considered as a higher dimensional analogue of Theorem 3.2.4.

The following is a well-known result from ([42],[37]). We will present a proof based on the squeezing function given in [9].

Corollary 3.3.1 *([42],[37],[9]) Let G be a strongly pseudoconvex domain with C^2-boundary. If the automorphism group of G is non-compact, then G is biholomorphic to $\mathbb{B}^n$.*

Proof. Suppose that the automorphism group of G is non-compact. Then by Corollary 2.2.5, there exists a $p \in G$ such that the set

$$\{f(p) : f \in \mathtt{aut}(G)\}$$

is non-compact in G. That is, we can find a sequence $\{f_i\}_{i=1}^{\infty} \subset \mathtt{aut}(G)$ and a point $q \in \partial G$ such that

$$\lim_{i\to\infty} f_i(p) = q.$$

Thus, by Theorem 3.3.4 and biholomorphic invariance of squeezing functions we get that

$$S_G(p) = \lim_{i\to\infty} S_G(f_i(p)) = 1.$$

By the corollary of Theorem 3.1.2, G is biholomorphic to $\mathbb{B}^n$. $\square$

Fornæss and Wold [13] showed that if the domain has further boundary regularity, then we can estimate the squeezing function in terms of the boundary distance.

Theorem 3.3.5 *([13]) Let G be a bounded strongly pseudoconvex domain with C^3-smooth boundary, then there exists a constant $C > 0$ such that the following estimate holds.*

$$S_G(z) > 1 - C\sqrt{\delta_G(z)}.$$

Further, if the boundary of G is C^4-smooth, this estimate can be improved as

$$S_G(z) > 1 - C|\delta_G(z)|.$$

No analogous estimate for bounded C^2-strongly pseudoconvex domains are known. However, as a partial converse Diederich, Fornæss and Wold gave a characterization of the unit ball purely by its squeezing function.

Theorem 3.3.6 *([11]) Let G be a domain with C^2-smooth boundary such that there does not exist any $c > 0$ such that the squeezing function of G satisfies the estimate*

$$S_G(z) < 1 - c\delta_G(z).$$

Then G is biholomorphic to $\mathbb{B}^n$.

This is equivalent to the following.

Theorem 3.3.7 *(Theorem 3.3.6 restated) Let G be a domain with C^2 smooth boundary such that there exists a sequence $\{p_i\} \subset G$ converging to boundary, and a sequence $\{\varepsilon_i\} \subset (0,1)$ converging to 0 such that the estimate*

$$S_G(z) \geq 1 - \varepsilon_i\delta_G(p_i)$$

is satisfied for all $n \in \mathbb{N}$. Then G is biholomorphic to $\mathbb{B}^n$.

To prove this theorem we need some lemmas.

Lemma 3.3.2 *([20]) If G is a bounded domain with C^2-smooth boundary, containing 0 and a sequence of points $\{p_i\}$ approaching to ∂G. Then there exists a constant C such that*

$$d_G^K(0, p_i) \leq -\frac{1}{2}\log(\delta_G(p_i)) + C. \tag{14}$$

Proof. Because of C^2-smoothness of ∂G, by Lemma 3.3.1 there exists a $\delta > 0$ such that ∂G has the tubular neighbourhood

$$U_\delta := \cup_{x\in\partial G}\{x + \eta(x)r : |r| < \delta\}.$$

Due to this, by taking a small enough $\varepsilon > 0$, we can assure that we have that $\forall z$ with $\delta_G(z) < \varepsilon$ there exists a unique $\zeta(z) \in \partial G$ with $z = \zeta(z) - \delta_G(z)\eta(\zeta(z))$ where $\eta(\zeta(z))$ is the outward unit normal vector of ∂G at $\zeta(z)$. Further, by taking ε even smaller if necessary, we can have that

$$z \in \mathbb{B}^n_\varepsilon(\sigma(z)) \subset G$$

where $\sigma(z) := \zeta(z) - \varepsilon\eta(\zeta(z))$. Applying triangle inequality on the Kobayashi metric gives

$$d^K_G(0, z) \le d^K_G(\sigma(z), z) + d^K_G(0, \sigma(z)).$$

On the other hand by considering the inclusion mapping and distance decreasing property of Kobayashi metric we obtain

$$\begin{aligned} d^K_G(\sigma(z), z) \le d^K_{\mathbb{B}^n_\varepsilon(\sigma(z))}(\sigma(z), z) &= \frac{1}{2}\log\left(\frac{\varepsilon + \|\sigma(z) - z\|}{\varepsilon - \|\sigma(z) - z\|}\right) \\ &= \frac{1}{2}\left(\log(\varepsilon + \|\sigma(z) - z\|) - \frac{1}{2}\log(\varepsilon - \|\sigma(z) - z\|)\right) \\ &\le \frac{1}{2}\log(2\varepsilon) - \frac{1}{2}\log(\delta_G(z)). \end{aligned}$$

Now by taking $\varepsilon > 0$ is even smaller, we can find a relatively compact $K \subset G$ such that the set $\{\sigma(z) : z \in G,\ \delta_G(z) < \varepsilon\} \subset K$. Hence there exists a constant $C_1 > 0$ such that $\forall z$ such that $\delta_G(z) < \varepsilon$ we have $d^K_G(0, \sigma(z)) < C_1$.
Thus $\forall z \in G$ with $\delta_G(z) < \varepsilon$ the estimate

$$d^K_G(0, z) \le -\frac{1}{2}\log(\delta_G(z)) + \frac{1}{2}\log(2\varepsilon) + C_1$$

is satisfied. Now as the sequence $p_i \to \partial G$, all but finitely many lie on the set $\{z \in G : \delta_G(z) < \varepsilon\}$. We note that the rest lie on on a compact subset of G. Thus, for all i, we can find a constant C satisfying

$$d^K_G(0, p_i) \le -\frac{1}{2}\log(\delta_G(p_i)) + C. \square$$

Now, let $\{\Phi_i\} \in \mathscr{E}(G, \mathbb{B}^n)$ be a sequence such that $\Phi_i(p_i) = 0$, where $\{p_i\}$ tends to ∂G and $\Phi_i(G) \supset \mathbb{B}^n_{1-\delta_G(p_i)}$, as in Theorem 3.3.7. We will show the following.

Lemma 3.3.3 *([11], Lemma 2.4.) For all i, we have that $\Phi_i(0) \in \mathbb{B}^n_{1-\frac{\delta_G(p_i)}{e^{2C}}}$ where C satisfies the condition in Lemma 3.3.2.*

Proof. Let $|z| = 1 - \frac{\delta_G(p_i)}{e^{2C}}$. Then by distance decreasing property of Kobayashi metric we have

$$d^K_{\Phi_i(G)}(0,z) \geq d^K_{\mathbb{B}^n}(0,z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+(1-\dfrac{\delta_G(p_i)}{e^{2C}})}{1-(1-\dfrac{1}{2}\dfrac{\delta_G(p_i)}{e^{2C}})}\right) = \frac{1}{2}\log\left(\frac{2-\dfrac{\delta_G(p_i)}{e^{2C}}}{\dfrac{\delta_G(p_i)}{e^{2C}}}\right) \geq \frac{1}{2}\log(\frac{1}{\delta_G(p_i)}) + C. \tag{15}$$

We note that the last inequality in (15) follows by taking C in (14) large enough. Then by Lemma 3.3.2 and the biholomorphic invariance of Kobayashi metric we get that

$$d^K_{\Phi_i(G)}(0,z) > \frac{1}{2}\log\left(\frac{1}{\delta_G(p_i)}\right) + C \geq d^K_G(0,p_i) = d^K_{\Phi_i(G)}(0,\Phi_i(0)).$$

Thus, we have obtained that the Kobayashi distance on $\Phi_i(G)$ from the origin to $\partial\mathbb{B}^n_{1-\frac{\delta_G(p_i)}{e^{2C}}}$ is longer than the Kobayashi distance on $\Phi_i(G)$ from the origin to $\Phi_i(0)$. So, the lemma follows.$\square$

By taking a rotation if necessary, we can assume that $\Phi_i(0) = r_i := (r_i, 0, ..., 0)$ where $0 \leq r_i < 1 - \dfrac{\delta_G(p_i)}{e^{2C}}$. Consider $\Psi_i \in \mathtt{aut}(\mathbb{B}^n)$ given by

$$\Psi_i(z_1, z_2, ..., z_n) = \left(\frac{z_1 - r_i}{1 - z_1 r_i}, \frac{z_2\sqrt{1-{r_i}^2}}{1 - r_i z_1}, ..., \frac{z_n\sqrt{1-{r_i}^2}}{1 - r_i z_1}\right).$$

Then we have that $F_i = \Psi_i \circ \Phi_i \in \mathscr{E}(G, \mathbb{B}^n)$ and $F_i(0)$.

After the following lemma, we will establish Theorem 3.3.6.

Lemma 3.3.4 *([11], Lemma 2.5) For each i, we have that $F_i(G) \supset \mathbb{B}^n_{1-6C\varepsilon_i}$.*

Proof. We have that $\Phi_i(G) \supset \mathbb{B}^n_{1-\varepsilon_i\delta_G(p_i)}$. So, we have that $\Phi_i(G) \supset \mathbb{B}^n_{1-2\varepsilon_i\delta_G(p_i)}$. By a direct calculation ([11],proof of Lemma 2.5) it follows that $\forall z \in \partial\mathbb{B}^n_{1-2\varepsilon_i\delta_G(p_i)}$,

$\|\Psi_i(z)\| \geq 1 - 6C\varepsilon_i$. So, as the function $\|\Psi_i(z)\|$ is plurisubharmonic we have that $\Psi_i(\mathbb{B}^n_{1-2\varepsilon_i\delta_G(p_i)}) \subset \mathbb{B}^n_{1-6C\varepsilon_i}$. Hence, $F_i(G) \supset \Psi_i(\mathbb{B}^n_{1-2\varepsilon_i\delta_G(p_i)}) \supset \mathbb{B}^n_{1-6C\varepsilon_i}$. $\square$

We are now ready to prove Theorem 3.3.6.

Proof of Theorem 3.3.6 The last lemma implies that $S_G(0) > 1 - 6C\varepsilon_i$. Letting $i \to \infty$ gives $S_G(0) = 1$. By existence of extremal maps 3.1.2, it follows that G is biholomorphic to $\mathbb{B}^n$. $\square$

After we observed this interesting phenomenon, let us give some generalizations of Theorem 3.2.3 to $\mathbb{C}^n$.

Definition 3.3.2 *Let $G \subset \mathbb{C}^n$ be a bounded domain. $A \subset G$ is said to be an analytic subset if $\forall p \in A$, there exists a neighbourhood of p, $N_p \subset G$ and a family of analytic functions $\{f_i\}_{i\in I} \subset \mathscr{O}(N_p, \mathbb{C})$ such that*

$$N_p \cap A = \{z \in N_p : \forall i \in I \; f_i(z) = 0\}.$$

As a remark, we would like to note that due to the Wierstrass factorization theorem [26], the family of holomorphic functions in the definition above can be taken as a finite set.

Note that, if $n = 1$ and G is a planar domain, any analytic subset A of G is a set of discrete points. In particular, by Riemann's removable singularity theorem, any analytic function on $G \setminus A$ extends analytically to G. The same assertion holds in higher dimensions as well, see [26].

Theorem 3.3.8 *([10]) Let $A \subset \mathbb{B}^n$ be a proper analytic set and $G := \mathbb{B}^n \setminus A$. Then*

$$S_G(z) = \tanh(d^K_{\mathbb{B}^n}(z, A)).$$

Proof. By considering automorphisms of the unit ball as candidate maps for the squeezing function problem of G at z, biholomorphic invariance of Kobayashi metric gives that

$$S_G(z) \geq \tanh(d^K_{\mathbb{B}^n}(z, A)).$$

Now, by Riemann's extension theorem [26] every analytic function on G extends to $\mathbb{B}^n$.

Let $f \in \mathscr{E}(G, \mathbb{B}^n)$ such that $f(z) = 0$. Then it extends to an analytic function on $\mathbb{B}^n$. With an abuse of notation we denote the analytic extension by f as well. Then by the distance decreasing property of Kobayashi metric, for any $x \in A$ we have that

$$d^K_{\mathbb{B}^n}(z, x) \geq d^K_{f(\mathbb{B}^n)}(0, f(x)) \geq d^K_{\mathbb{B}^n}(0, f(x)). \tag{16}$$

By definition

$$S_G(z) = \sup\{dist(0, \partial f(G)) : f \in \mathscr{E}(G, \mathbb{B}^n) \ \ f(z) = 0\}.$$

Then by Corollary 2.4.4 it follows that

$$S_G(z) = \sup\{\tanh d^K_{\mathbb{B}^n}((0, \partial f(G))) : f \in \mathscr{E}(G, \mathbb{B}^n)\}. \tag{17}$$

By (17) we see that

$$S_G(z) \leq \tanh(d^K_{\mathbb{B}^n}(z, A)).$$

This establishes the theorem.□

The following is an interesting generalization of this theorem.

Theorem 3.3.9 *([5]) Let $D \subset \mathbb{C}^n$ be a bounded domain and K be a relatively subset of D such that $G := D - K$ is connected. Then*

$$S_G(z) \leq \tanh(d^K_D(z, \partial K)).$$

Proof. Let $f \in \mathscr{E}(G, \mathbb{B}^n)$ arbitrary such that $f(z) = 0$. Theorem 2.1.6 asserts that f extends analytically to D. With an abuse of notation we let f denote its analytical extension as well.

Let N be a neighbourhood of $\bar{G}$ in D. Applying maximum principle to $\|f(z)\|$ on N shows that $\forall z \in D$ we have that $\|f(z)\| < 1$, hence $f \in \mathscr{O}(D, \mathbb{B}^n)$.

We claim that $f(\partial K) \cap f(G) = \emptyset$. To see this suppose not. Then $\exists x \in \partial K$, $\exists y \in G$ such that $f(x) = f(y)$. Then we take a sequence $\{z_i\} \subset G$ tending to x. Due to continuity of f in G

$$\lim_{i \to \infty} f(z_i) = f(x) = f(y) \in f(G).$$

This contradicts with properness of f. Therefore we must have that $f(G) \cap f(\partial K) = \emptyset$. Let $x \in \partial K$ be an arbitrary point. Then by Corollary 2.4.3 and the distance decreasing property of Kobayashi metric we get

$$d_D^K(z,x) \geq d_{f(D)}^K(0, f(x)) \geq d_{\mathbb{B}^n}^K(0, f(x)).$$

Taking the infimum over $x \in \partial K$ we get that

$$d_{\mathbb{B}^n}^K(0, \partial f(K)) \leq d_D^K(z, \partial K)$$

Since $f \in \mathscr{E}(G, \mathbb{B}^n)$ was arbitrary, by (17) we see that

$$S_G(z) \leq \tanh(d_D^K(z, \partial K)).\square$$

Corollary 3.3.2 *Let $K \subset \mathbb{B}^n$ be a compact set. Set $G = \mathbb{B}^n \setminus K$. Then we have that*

$$S_G(z) = \tanh(d_{\mathbb{B}^n}^K(0, K)).$$

Proof. By Theorem 3.3.9 we have that

$$S_G(z) \leq \tanh(d_{\mathbb{B}^n}^K(0, K)).$$

On the other hand, due to the biholomorphic invariance of Carathéodory metric, by considering the automorphism of $\mathbb{B}^n$ mapping z to 0 we have that

$$S_G(z) \geq \tanh(d_{\mathbb{B}^n}^K(0, K)).$$

Thus the corollary follows. $\square$

Corollary 3.3.3 *Let $\mathbb{G}_r := \mathbb{B}^n \setminus \overline{\mathbb{B}_r^n}$ for $r \in (0, 1)$. Then*

$$S_{\mathbb{G}_r}(z) = \tanh\left[\frac{1}{2}\log\left(\frac{1+\|z\|}{1-\|z\|}\right) - \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)\right].$$

Proof. In the view of Corollary 2.4.3 and Corollary 2.4.4, this result trivally follows from the Corollary 3.3.2. $\square$

In $\mathbb{C}^n$ with $n \geq 2$, the Hartogs extension formula allowed us to get the explicit form of the squeezing function on certain subsets of the unit ball including the generalized annuli, the domains $\mathbb{G}_r$ given above. Whereas in $\mathbb{C}$, the Riemann mapping theorem easily allowed us to understand the boundary behaviour of squeezing functions of finitely connected domains with non-degenerate boundaries.

Unfortunately, determining the explicit formula of squeezing function subsets of the unit disc are way more complicated. Several authors [19, 34, 41] have determined the squeezing function of $\mathbb{A}_r := \Delta \setminus \Delta_r$ by advanced tools of complex analysis in one variable.

On the next chapter, by using generalizations of Riemann mapping theorem, we will see that the squeezing function problem on higher connected planar domains can be considered as an analogues of the famous Schwarz lemma. Based on this fact we will give our simple proof of the explicit formulas of the squeezing functions on annuli and provide bounds to squeezing functions of higher connected domains.

CHAPTER 4

SQUEEZING FUNCTIONS ON PLANAR DOMAINS

4.1 Equivalence of Finitely Connected Domains and Circularly Slit Discs

Let us recall some concepts from complex analysis of one variable.

Definition 4.1.1 *We say that the domain $G \subset \mathbb{C}$ is n-connected if $\mathbb{CP}^1 \setminus G$ has n components.*

Clearly $\mathbb{C}$ and Δ is 1-connected, whereas the annulus $\mathbb{A}_r$ is 2-connected.

If $f : G_1 \to G_2$ is a biholomorphism between two planar domains, it is not necessary that f extends to a map of closures. Let us recall the notion of boundary correspondance.

Definition 4.1.2 *Let $f : G_1 \to G_2$ be a biholomorphism between two planar domains. Let Γ_1, Γ_2 be boundary components of G_1 and G_2 respectively. We say that under f, Γ_1 and Γ_2 correspond to each other if for any $\{z_i\} \subset G_1$ converging to Γ_1 and for any $\{w_i\} \subset G_2$ converging to Γ_2 we have that $\{f(z_i)\} \subset G_2$ converges to Γ_2 and $\{f^{-1}(w_i)\}$ converges to Γ_1. In this case we write $f(\Gamma_1) = \Gamma_2$.*

Definition 4.1.3 *Let $G \subset \mathbb{C}$ be a bounded domain. We say that the boundary component $\Gamma \subset \partial G$ is the outer boundary of G if Γ is the boundary of the unbounded component of $\mathbb{C} \setminus G$.*
If G is 2-connected, we call the non-outer boundary component of G as the inner boundary.

Definition 4.1.4 *Let G be a bounded planar domain with non-degenerate boundary components $\Gamma_1, \Gamma_2, ..., \Gamma_n$. We define*

$$\mathscr{E}_i(G, \Delta) := \{f \in \mathscr{E}(G, \Delta) : \ \Gamma_i \text{ corresponds to outer boundary of } f(G)\}.$$

Clearly the class $\mathscr{E}_i(G, \Delta)$ is non-empty.

Recall the notation in (6), we denote

$$\mathscr{S}\mathscr{E}(G, \Delta) = \{f \in \mathscr{E}(G, \Delta) : \partial\Delta \subset \partial f(G)\}.$$

Definition 4.1.5 *Let $G \subset \mathbb{C}$ be a n-connected domain with non-degenerate boundary components $\Gamma_1, \Gamma_2, ..., \Gamma_n$. For each $i \in \{1, 2, ..., n\}$ we define*

$$\mathscr{S}\mathscr{E}_i(G\Delta) := \mathscr{S}\mathscr{E}(G, \Delta) \cap \mathscr{E}_i(G, \Delta) = \{f \in \mathscr{S}\mathscr{E}(G, \Delta) : f(\Gamma_1) = \partial\Delta\}.$$

By definition, we have the following lemma.

Lemma 4.1.1 $\mathscr{E}(G, \Delta) = \cup_{i=1}^{n} \mathscr{E}_i(G, \Delta)$ *and* $\mathscr{S}\mathscr{E}(G, \Delta) = \cup_{i=1}^{n} \mathscr{S}\mathscr{E}_i(G, \Delta)$.

Definition 4.1.6 *Let G be a planar n-connected domain with non-degenerate boundary components $\{\Gamma_1, \Gamma_2, ..., \Gamma_n\}$. For $i \in \{1, 2, ..., n\}$, we denote the squeezing function of G corresponding to the boundary component Γ_i as*

$$S_{G,\Gamma_i}(z) := \sup\{S_G^f(z) : f \in \mathscr{E}_i(G, \Delta), f(z) = 0\} = \sup\{S_G^f(z) : f \in \mathscr{S}\mathscr{E}_i(G, \Delta), f(z) = 0\}$$

We note that the last equality in the definition above follows from Lemma 3.2.2.

Lemma 4.1.2 *([10]) Let $G \subset \mathbb{C}$ be an n-connected domain with non-degenerate boundary components $\Gamma_1, \Gamma_2, ..., \Gamma_n$. Then*

$$S_G(z) = \max_{i=1,2,...,n} S_{G,\Gamma_i}(z).$$

Proof. The proof follows from Lemma 4.1.1. □

The Riemann mapping theorem is very interesting because it enables us to obtain information about complex analytical structure of a domain by looking at its topology. We will give well-known results about generalizations of the Riemann mapping theorem to domains of higher connectivity.

Theorem 4.1.1 *([7], Theorem 5.1.) Let G be 2-connected planar domain with non-degenerate boundary components. Then there exists an $r \in (0,1)$ such that G is biholomorphic to the annulus $\mathbb{A}_r = \Delta \setminus \overline{\Delta_r}$.*

Theorem 4.1.2 *([7], Lemma 5.2.) If $r \neq R$, then $\mathbb{A}_r$ and $\mathbb{A}_R$ are not biholomorphically equivalent.*

So, we have the complete characterization of complex analytical structure of bounded 2-connected domains with non-degenerate boundary.

Definition 4.1.7 *A domain G is said to be a circularly slit annulus if there exists an annulus $\mathbb{A}_r$ such that $\mathbb{A}_r \supset G$ and the components of $\mathbb{A}_r \setminus G$ are proper arcs of circles centered at the origin.*

The following result is a generalization of the Riemann mapping theorem and Theorem 4.1.1.

Theorem 4.1.3 *([7], Theorem 5.1.) Let G be an n-connected domain planar domain with non-degenerate boundary where $n \geq 3$. Then for any two boundary components Γ_1,Γ_2 of G and $p \in G$ we can find a $f \in \mathscr{E}(G, \Delta)$ that takes G to a circularly slit annulus such that $f'(p) > 0$ and Γ_1 corresponds to the outer boundary of $f(G)$ and Γ_2 corresponds to the inner boundary of $f(G)$, that is $f(\Gamma_1) = \partial\Delta$ and $f(\Gamma_2) = \partial K$ where K is the component of $\Delta \setminus f(G)$ containing 0.*

Theorem 4.1.4 *([7], Lemma 5.2.) Let $f : G_1 \longrightarrow G_2$ be a biholomorphism between two circularly slit annuli such that under f, the outer boundary of G_1 corresponds to the outer boundary of G_2 and inner boundary of G_1 corresponds to the inner boundary of G_2. Then f is a rotation.*

Combining Theorem 4.1.3 and Theorem 4.1.4 shows that for any n-connected planar domain with non-degenerate boundary (where $n \geq 3$) and any $a \in G$, we can find $n(n-1)$ canonical conformal maps, say ψ_i for $i \in \{1, 2, ..., n(n-1)\}$, which are unique up to a rotation, taking G to a circularly slit annuli and satisfying $\psi_i'(a) > 0$.

Now, we will introduce a different type of planar domain which will be useful in the proof of our main result.

Definition 4.1.8 *Let $G \subset \Delta$ be an n-connected domain with non-degenerate boundary where $n \geq 2$. We say that G is a circularly slit disc if components of $\Delta \setminus G$ consists of proper arcs of circles centered at the origin.*

The following an analogous result to Theorem 4.1.2 and Theorem 4.1.4.

Theorem 4.1.5 *([7], Lemma 6.3.) Let $f : G_1 \longrightarrow G_2$ be a biholomorphism between two circularly slit discs. Suppose that $f(0) = 0$ and $f(\partial\Delta) = \partial\Delta$. Then f is a rotation.*

Now, as an analogous result to Theorem 4.1.1 and Theorem 4.1.3 we have the following result.

Theorem 4.1.6 *([7], Theorem 6.2.) Let G be an n-connected domain with non-degenerate boundary where $n \geq 2$. Let $p \in G$ and Γ be any boundary component. Then we can find a map $f \in \mathscr{E}(G, \Delta)$ such that $f(p) = 0$, $f(\Gamma) = \partial\Delta$, $f'(a) > 0$ and $f(G)$ is a circularly slit disc.*

As in the case of circularly slit annuli combining Theorem 4.1.6 and Theorem 4.1.5 gives that for an n-connected planar domain with non-degenerate boundary ($n \geq 2$) and for any $p \in G$, we can find n-canonical conformal maps that are unique up to rotation so that p is mapped to 0 and G is mapped into a circularly slit disc.

There are other types of canonical conformal maps. For a deeper discussion about their construction, we refer the reader to [7, 21, 29, 36].

The following theorem shows that the maps that take a domain to a circularly slit discs form a particularly important class of canonical conformal mappings.

Theorem 4.1.7 *([36], Theorem 3, Theorem 7 and Theorem 8) Let G be an n-connected planar domain with non-degenerate boundary. Let $p \in G$ and $\{\Gamma_1, \Gamma_2, ..., \Gamma_n\}$ be the set of boundary components of G. Set*

$$\tau := \sup_{f \in \mathscr{E}_i(G,\Delta)} |f'(p)|$$

There exists a unique $f_i \in \mathscr{E}_i(G, \Delta)$ such that $f_i'(p) = \tau$. Further, $f_i(\Gamma_i) = \partial\Delta$, $f_i(p) = 0$ and f_i maps G onto a circularly slit disc.

Definition 4.1.9 *Schottky-Klein prime function on the annulus $\mathbb{A}_r$, $\omega : \mathbb{A}_r \times \mathbb{A}_r \to \mathbb{C}$ is defined as follows.*

$$\omega(z, y) = (z - y) \prod_{n=1}^{\infty} \frac{(z - r^{2n}y)(y - r^{2n}z)}{(z - r^{2n}z)(y - r^{2n}y)}.$$

Lemma 4.1.3 *([34]) Schottky-Klein prime function on the annulus $\mathbb{A}_r$ satisfies the following properties.*

1.
$$\overline{\omega(\overline{z}^{-1}, \overline{y}^{-1})} = \frac{-\omega(z, y)}{zy}. \tag{18}$$

2.
$$\omega(r^{-2}z, y) = \frac{rz\omega(z, y)}{y}. \tag{19}$$

Lemma 4.1.4 *([34],Theorem 4) Let $y \in \mathbb{A}_r$. Define*

$$f(z, y) = \frac{\omega(z, y)}{|y|\omega(z, \overline{y}^{-1})}.$$

Then $f(., y)$ is an injective holomorphic map taking $\mathbb{A}_r$ onto a circular slit disc, $f(\partial\Delta) = \partial\Delta$ and $y \in \mathbb{A}_r$ is mapped to 0.

So, we have an explicit formula for the maps that take an annulus to a circularly slit disc.

The following lemma will be crucial for the proof of our main theorem. It is proven in [36] by elementary complex analytical reasoning whereas in [34] a proof is given is based on the properties (18) and (19) of the Schottky-Klein prime function.

Lemma 4.1.5 *([34] Lemma 1,[36] Lemma 3) Let $y \in \mathbb{A}_r$. Let $f \in \mathscr{E}(\mathbb{A}_r, \Delta)$ be a map that such that $f(y) = 0$, $f(\partial\Delta) = \partial\Delta$ and further assume that f maps G onto a circularly slit disc. Then the radius of the slit of $f(\mathbb{A}_r)$ is $|y|$, that is $\forall x \in \Delta \setminus f(\mathbb{A}_r)$ we have that $|x| = |y|$.*

4.2 Squeezing Functions on Annuli

Now, let us share some observations of Deng, Guan an Zhang [10] on squeezing function of annuli.

As earlier, let $\mathbb{A}_r := \Delta \setminus \overline{\Delta_r}$ be the annulus. Recall that by Theorem 2.2.5, the automorphisms of the annulus $\mathbb{A}_r$ consists of maps of the form $R \circ f$ where R is a rotation and f is either the reflection $f(z) = \frac{\sqrt{r}}{z}$ or the identity mapping. Due to biholomorphic invariance of squeezing functions, this implies that we can consider the squeezing function on annuli as a continuous function on $[\sqrt{r}, 1) \subset \Delta$. Due to topological considerations, Deng, Guan and Zhang obtained the following result.

Theorem 4.2.1 *([10]) Let $S_r(z) := S_{\mathbb{A}_r}(z)|_{[\sqrt{r},1)}$. Then S_r is a strictly increasing function.*

Let $\Gamma_1 := \partial\Delta_r$, $\Gamma_2 := \partial\Delta$ be the two boundary components of the annulus. The previous result and considering the automorphims of the unit disc as extremal mappings for the squeezing function problem led Deng, Guan and Zhang to conjecture the following.

Conjecture 4.2.1 *([10])*

$$S_{\mathbb{A}_r,\Gamma_2}(z) := \tanh\left[\frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right) - \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)\right].$$

In particular, by Lemma 4.1.2 this would imply that squeezing functions of annuli have the analogous explicit formula to the squeezing functions of generalized annuli.

By applying the map $f \in \mathtt{aut}(\mathbb{A}_r)$ defined as $f(z) = \frac{r}{z}$ this conjecture leads to the following.

Conjecture 4.2.2 *([10])*

$$S_{\mathbb{A}_r,\Gamma_1}(z) := \tanh\left[\frac{1}{2}\log\left(\frac{1+\frac{r}{|z|}}{1-\frac{r}{|z|}}\right) - \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)\right].$$

So

$$S_{\mathbb{A}_r}(z) := \max\{S_{G,\Gamma_1}(z), S_{G,\Gamma_2}(z)\}.$$

This conjecture remained open for seven years. It turned out to be incorrect when Ng, Tang and Tsai[34] established the right formula in early 2020's. Their proof relies on a tool called Löwner's differential equation. It is a very useful yet a very complicated tool. Their result was later confirmed in [19] by potential theoretic reasoning and in [41] by using arguments based on modules and extremal length. We now give the right formula for the squeezing functions on annuli.

Theorem 4.2.2 *([19],[34],[35],[41]) Let $\mathbb{A}_r = \Delta \setminus \Delta_r$. Then we have that*

1.
$$S_{\mathbb{A}_r}(z) = \max\left\{|z|, \frac{r}{|z|}\right\}. \tag{20}$$

2. *The extremal maps for the squeezing function problem on $\mathbb{A}_r$ maps $\mathbb{A}_r$ onto a circular slit disc. Moreover, for $z \in \mathbb{A}_r$, this map is unique up to rotation if $|z| \neq \frac{r}{|z|}$.*

We note that the second part of this theorem is proven in [19], [35] and [41]. In [34], the authors showed that a map that takes $\mathbb{A}_r$ to a circularly slit disc is extremal for the squeezing function problem but they weren't able to check if there exists any other extremal functions.

We now present our proof of the Theorem 4.2.2.

Proof of Theorem 4.2.2 For the first part, let $z \in \mathbb{A}_r$ be an arbitrary point. By Theorem 4.1.6, there exists a $f \in \mathscr{E}(\mathbb{A}_r, \Delta)$ such that $f(z) = 0$, $f(\partial\Delta) = \partial\Delta$ in the sense of boundary correspondence and $G := f(\mathbb{A}_r)$ is a circularly slit disc.
Now, let $\Gamma_2 := \partial\Delta$, $\Gamma_1 := \partial\Delta_r$ be the two boundary components of $\mathbb{A}_r$. and g be an arbitrary embedding in $\mathscr{S}\mathscr{E}_2(\mathbb{A}_r, \Delta)$ that takes z to 0. As $g \in \mathscr{S}\mathscr{E}_2(\mathbb{A}_r, \Delta)$, g fixes

$\partial\Delta$ in the sense of boundary correspondance.

Consider the biholomorphism $h := g \circ f^{-1} \in \mathscr{E}(G, \Delta)$. By Theorem 4.1.7, we have that $|g'(z)| \leq |f'(z)|$. This implies that $|h'(0)| \leq 1$. Moreover due to the Theorem 4.1.5 and Theorem 4.1.7, we have that $|h'(0)| = 1$ if and only if h is a rotation.

Now, we assume that $|h'(0)| < 1$. Let $K := \Delta \setminus G$ and $\{\varepsilon_i\}$ be a sequence in $(0, 1)$ converging to 0 such that $\forall i$ the set

$$K_i := \{z \in G : d(z, K) \leq \varepsilon_i\}$$

is a relatively compact subset of the unit disc. Let $G_i := \Delta \setminus K_i$. It is clear that the sequence $\{G_i\}$ exhausts G and $h_i := h|_{G_i}$ is a biholomorphism onto it's image and extends to a continuous map of the boundaries.

Consider the map $k_i(z) := \dfrac{h_i(z)}{z}$ on G_i. As h_i is injective and fixes the origin we have that $h_i(z) = {h_i}'(0)z + O(z^2)$, the function k_i extends holomorphically to G_i. With an abuse of notation, we denote the extension by k_i as well.

Since we assumed that $|h'(0)| < 1$, we have that $|h_i'(0)| < 1$. This implies that

$$|k_i(0)| := \lim_{z\to 0} \left|\frac{h_i(z)}{z}\right| = |h_i'(0)| < 1.$$

As h is injective it doesn't have any other zeroes. We note that $\forall x \in K$, $|x| = |z|$. Therefore by the minimum principle, we have that

$$\frac{\min_{z\in\partial G_i} |h_i(z)|}{|z| + \varepsilon_i} \leq \min_{z\in\partial G_i} \left|\frac{h_i(z)}{z}\right| < 1$$

so

$$\min_{z\in\partial G_i} |h_i(z)| < |z| + \varepsilon_i.$$

Then we can find a sequence $\{x_i\} \subset G$ such that $|h(x_i)| = |h_i(x_i)| \leq |z| + \varepsilon_i$. From construction, it is clear that $\{x_i\}$ tends to K. By passing to a subsequence if necessary we assume that $\{x_i\}$ converges to $x \in K$. Then in the sense of boundary correspondence we get that $|h(x)| := \lim_{i\to\infty} |h(x_i)| \leq |z|$. This gives $S_G^h(0) \leq |z|$.

For the circularly slit disc G we set the boundary components $\tilde{\Gamma}_2 := \partial\Delta$ and $\tilde{\Gamma}_1 := K$. As $h \in \mathscr{E}_2(G, \Delta)$ was arbitrary we get that $S_{G,\tilde{\Gamma}_2}(0) \leq |z|$.

On the other hand, considering the identity mapping gives $S_{G,\tilde{\Gamma}_2}(0) \geq |z|$. Therefore, $S_{G,\tilde{\Gamma}_2}(0) = |z|$. By biholomorphic invariance of squeezing functions we get that

$$S_{\mathbb{A}_r,\partial\Delta}(z) = S_{G,\tilde{\Gamma}_2}(0) = |z|.$$

We apply a the reflection of the annulus $f(z) = \dfrac{r}{|z|}$ so that z gets mapped to $\frac{r}{z}$ and $\Gamma_1 = \partial\Delta_r$ gets mapped to $\Gamma_2 = \partial\Delta$. Applying the same reasoning immediately gives that

$$S_{\mathbb{A}_r,\Gamma_1}(z) = \frac{r}{|z|}.$$

Thus by Lemma 4.1.2, we have that

$$S_{\mathbb{A}_r}(z) = \max\{S_{\mathbb{A}_r,\Gamma_1}(z), S_{\mathbb{A}_r,\Gamma_2}(z)\} = \max\left\{\frac{r}{|z|}, |z|\right\}.$$

As $z \in \mathbb{A}_r$ was arbitrary, the first part follows.
For the second part, we assume that G is a 2-connected circularly disc with boundary components $\partial\Delta$ and K. Suppose that the non-identity map $f : G \longrightarrow \Delta$ is a biholomorphic embedding fixing 0 and $\partial\Delta$. Then by we have that $\alpha := |f'(0)| < 1$. Then we consider the map $h(z) = \dfrac{(1+\alpha^{-1})f(z)}{2z}$ on G. We have that $|h(0)| = \dfrac{(1+\alpha^{-1})\alpha}{2} < 1$ and $\forall x \in \partial\Delta$ we have that $|h(x)| = \dfrac{(1+\alpha^{-}1)}{2} > 1$. Again, applying minimum principle to h on G gives that $\exists x \in K$ such that $|h(x)| \leq \dfrac{2\alpha|x|}{1+\alpha^{-1}} < |x|$ in terms of boundary correspondence. Thus in this case, the squeezing number of h is strictly less that the squeezing number of the identity mapping. As h was arbitrary, the identity mapping is the unique (up to rotation) extremal mapping for the squeezing function problem for $0 \in G$ with respect to the boundary component $\partial\Delta$. Consequently by biholomorphic invariance, the only extremal map for the squeezing function problem on annuli are the maps that take annuli onto circularly slit discs. This finishes to proof. $\square$

We note that from the proof of Theorem 3.1.4 one can see that the stability property of squeezing functions still hold when we restrict the problem to the squeezing functions with respect to a fixed boundary component. This observation can be used to slightly refine our proof of Theorem 4.2.2.

We have shown that the solution of the squeezing function problem on annuli can be considered as an analogue of the Schwarz lemma for simply connected domains. Now, let us discuss what can be said about squeezing function problem on planar domains of higher connectivity.

4.3 Squeezing Functions on Higher Connected Domains

Let G be an n-connected domain ($n \geq 2$) with non-degenerate boundary and $z \in G$ be an arbitrary point. Further, let $\Gamma_1, ..., \Gamma_n$ be the boundary components of G. By Theorem 4.1.6, we can find n-canonical conformal maps, say ψ_i for $i \in \{1, 2, ..., n\}$ that take G to a circularly slit disc. Let $\psi_i : G \longrightarrow C_i(z, G)$ denote the biholomorphism where z gets mapped to 0, Γ_i corresponds to $\partial\Delta$ and G gets mapped to the circularly slit disc $C_i(z, G)$.

For each i, let

$$K_i(z, G) = \Delta \setminus C_i(z, G).$$

Further, let

$$r_i(z, G) := \min\{|x| : x \in K_i(z, G)\} \text{ and } R_i(z, G) := \max\{|x| : x \in K_i(z, G)\}.$$

Finally, we set

$$r(z, G) := \max_{i \in \{1,2,...,n\}} r_i(z, G) \text{ and } R(z, G) := \max_{i \in \{1,2,...,n\}} R_i(z, G).$$

We note that the quantities $r(., .)$ and $R(., .)$ are biholomorphically invariant. That is if $f : G_1 \longrightarrow G_2$ is a biholomorphism, $\forall z \in G_1$ we have that $r(z, G_1) = r(f(z), G_2)$ and $R(z, G_1) = R(f(z), G_2)$.

Now, let G be a bounded 2-connected planar domain with non-degenerate boundary. Then by Theorem 4.1.1 we have that G is biholomorphic to an annulus $\mathbb{A}_r$. Thus by the earlier discussion, Theorem 4.2.2 gives the following corollary.

Corollary 4.3.1 *Let G be a bounded 2-connected planar domain with non-degenerate boundary. Then*

$$S_G(z) = r(z, G).$$

The work of Ng, Tang and Tsai [34], led them to conjecture that circular slit maps are also the extremal maps for the squeezing function problem on higher connected domains. More explicitly, they gave the following conjecture.

Conjecture 4.3.1 *([34]) Let G be a bounded n-connected planar domain with non-degenerate boundary where $n \geq 3$. Then*

$$S_G(z) = r(z, G).$$

By purely analytical reasoning, Gumenyuk and Roth [19] quickly constructed a counter-examples to Conjecture 4.3.1. Explicitly, they obtained the following theorem.

Theorem 4.3.1 *[19] For any $n \geq 3$, there exists a bounded n-connected domain with non-degenerate boundary and there exists a $z \in G$ such that*

$$S_G(z) > r(z, G).$$

Our method in the proof of Theorem 4.2.2 allows us to obtain upper and lower bounds to squeezing functions on n-connected planar domains with non-degenerate boundaries with $n \geq 2$.

Lemma 4.3.1 *Let G be an n-connected domain with non-degenerate boundary. Let $z \in G$ and $\Gamma_1, ..., \Gamma_n$ be the boundary components of G. Then we have that*

$$r_i(z, G) \leq S_{G,\Gamma_i}(z) \leq R_i(z, G).$$

Proof. Let G be an n-connected planar domain with non-degenerate boundary such that $n \geq 2$ and $z \in G$ be an arbitrary point. Further, let $\psi_i : G \longrightarrow \Delta$ be a biholomorphism such that $C_i := \psi_i(G)$ is a circularly slit disc, $\psi_i(z) = 0$ and $\psi_i(\Gamma_i) = \partial\Delta$ in the sense of boundary correspondence. Now, let $f \in \mathscr{E}(C_i, \Delta)$ such that $f(0) = 0$ and $f(\partial\Delta) = \partial\Delta$.
As in the 2-connected case, we have that $|f'(0)| \leq 1$ and $|f'(0)| = 1$ if and only if f is a rotation.
Set $h(z) := \frac{f(z)}{z}$. Similarly, h extends to an analytic function on C_i and

$$|h(0)| = \lim_{G \ni z \to 0} \frac{|f(z)|}{|z|} \leq 1.$$

Set $K_i := \Delta \setminus C_i$ and let $\{\varepsilon_j\}$ be a sequence of positive numbers converging to zero such that the sets

$$K_{i,j} := \{z \in \Delta : dist(z, K) \leq \varepsilon_j\}$$

are relatively compact in Δ and have precisely $n-1$ components. Set $C_{i,j} := \Delta \setminus K_{i,j}$. Applying minimum principle to h on $C_{i,j}$ shows that

$$\frac{\min_{x \in \partial K_{i,j}} |f(x)|}{R_i(z,G) + \varepsilon_j} \leq \min_{x \in \partial C_{i,j}} |h(x)| \leq 1.$$

Thus

$$\min_{x \in \partial K_{i,j}} |h(x)| \leq R(z,G) + \varepsilon_j.$$

This implies that $S_{C_{i,j},\partial\Delta}(0) \leq R_i(z,G) + \varepsilon_j$. Again, by biholomorphic invariance and a limiting argument we have that

$$S_{G,\Gamma_i}(z) = S_{C_i,\partial\Delta}(0) \leq R_i(z,G).$$

On the other hand, considering the map ψ_i gives that $S_{G,\Gamma_i}(z) \geq r_i(z,G)$. This establishes the lemma. □

This lemma, combined with Lemma 4.1.2 immediately gives the following theorem.

Theorem 4.3.2 *([35]) Let G be an n-connected domain with non-degenerate boundary. Then the squeezing function of G satisfies the estimate*

$$r(z,G) \leq S_G(z) \leq R(z,G).$$

4.4 Conclusion

Now we will give some further questions about squeezing functions. The first two will be about squeezing functions on planar domains.

Similar to the proof of Theorem 4.2.2, our proof of Lemma 4.3.1 shows the following. Let $f : G \longrightarrow \Delta$ is a biholomorphic map that takes G to the circularly slit disc C_i with $f(0) = 0$ and $f(\Gamma_i) = \partial\Delta$ such that all slits of C_i lie on the same circle, that is $\forall x \in K_i := \Delta \setminus C_i$ we have that $|x| = r$ for some $r > 0$. Then we have that $S_{G,\Gamma_i}(z) = r$ and ψ_i is the unique extremal map in the family $\mathscr{E}_i(G,\Delta)$. To see this, suppose that $f : C_i \longrightarrow \Delta$ is a biholomorphic map that such that $f(0) = 0$ and $f(\partial\Delta) = \partial\Delta$. If f is not a rotation, by the Theorem 4.1.7, $|f(0)| = \alpha < 1$. Applying the same argument in the proof of Lemma 4.3.1 to the function $g(z) = \frac{(\alpha^{-1}+1)f(z)}{2z}$ gives that

$$dist(0, \partial f(C_i)) \leq \frac{2}{(1+\alpha^{-1})} R_i(z,G) < R_i(z,G).$$

This fact is given as Lemma 1 of [41] but proven with a different method. Using this lemma and an argument based on modules which are described in [21], Solynin[41] was able to obtain the following theorem.

Theorem 4.4.1 *([19]) For any $n \geq 2$, there exists an n-connected domain G, which is a circularly slit disc such that the identity mapping is the extremal mapping for the squeezing function problem for $0 \in G$.*

This theorem led him to ask two of the following questions which we find quite interesting.

Question 4.4.1 *([41]) For $n \geq 3$, is there an n-connected planar domain G which is a circularly slit disc such that there exists $x_1, x_2 \in K := \Delta \setminus G$ satisying $|x_1| \neq |x_2|$ and identity mapping is the extremal mapping for the squeezing function problem for $0 \in G$?*

Question 4.4.2 *([41]) For $n \geq 3$, is there an n-connected planar domain G and an open set $U \subset G$ such that $\forall z \in U$, the extremal mapping for the squeezing function problem for $z \in G$ is a map that takes G to a circularly slit disc?*

According to [8] and [34], for an n-connected domain G with non-degenerate boundary where $n \geq 3$, we can express the Schottky-Klein prime function of G in terms of automorphisms of the Riemann sphere. Also, for the maps that take G to a circularly slit disc we have the same representation given in Lemma 4.1.4 for the annuli. We believe that by using this representations, an analytical reasoning as in the proof of Theorem 4.4.1 in [19] can be useful in the study of Question 4.4.1 and Question 4.4.2.

The next two questions are not directly releated to what we studied but they are worthy of noting.

To ask the first one, let us give some background. Interestingly, circularly slit discs also appear on the problem of finding Carathéodory metric of annuli.

As we stated earlier, the extremal map for the Carathéodory metric exists. That is for domains $G \subset \mathbb{C}^n$, $p \in G$ and $v \in \mathbb{C}^n \cong T_pG$ there exists a map $f \in \mathscr{E}(G, \Delta)$ such that

$$|df_p(v)| = C_G(p; v).$$

For the domain $G \subset \mathbb{C}^n$, we call the function $f \in \mathscr{O}(G, \mathbb{B}^n)$ satisfying $J_f(z) \geq J_g(z)$ for all $g \in \mathscr{O}(G, \mathbb{B}^n)$ the Ahlfors function where J denotes the Jacobian determinant. It is known that the Ahlfors functions exist in all cases [29] and it can be easily observed that they map $z \in G$ to the origin. So, in one dimension Ahlfors maps are precisely the maps that give the Carathéodory metric. In the late seventies R.R. Simha [40] obtained the following result using elementary properties of Ahlfors maps given in [29].

Theorem 4.4.2 *([40]) For the annulus $\mathbb{A}_r := \Delta \setminus \overline{\Delta_r}$ and for all $z \in \mathbb{A}_r$, the Ahlfors function of $\mathbb{A}_r$ at z is given by*

$$f(z) := \frac{1}{z} f_1(z) f_2(z)$$

where $f_1(z)$ is the biholomorphic map such that $f_1(\mathbb{A}_r)$ is a circularly slit disc, $f_1(z) = 0$, $f_1'(z) > 0$ and f_2 is the biholomorphic map such that $f_2(\mathbb{A}_r)$ is a circularly slit disc, $f_2(-\frac{r}{z}) = 0$, $f_2'(-\frac{z}{r}) > 0$.

Further, using Schottky-Klein prime function, Simha also provided the explicit form of the Carathéodory metric on annuli.

Theorem 4.4.2 and our proof of Theorem 4.2.2 shows that there is a deep connection between the maps that are extremal for the squeezing function problem and Ahlfors maps. Further, due to a proof of the Riemann mapping theorem given in [16], on simply connected planar domains Ahlfors maps and the maps that give the squeezing functions coincide. Further, in the view of Theorem 2.2.2, on the unit ball the Ahlfors maps and the maps that give squeezing function coincides. In the conclusion part of [9], the authors provided examples of ellipsoids G in $\mathbb{C}^n$, where for $0 \in G$ the Ahlfors maps and the extremal maps for the squeezing function for $0 \in G$

coincide. In view of their discussion and our observations, we pose the following question.

Question 4.4.3 *[9, 10] Is it true that for simply connected proper subsets of $\mathbb{C}^n$, the Ahlfors maps and the maps that give the squeezing function coincide at every point? If not, we would love to know under which conditions the Ahlfors maps and the extremal maps for the squeezing function coincide.*

Finally, we want to note that Theorem 3.3.5 has a generalization given in [33]. The result in [33] implies Theorem 3.3.4. However, it doesn't provide boundary estimates for squeezing functions of C^2-smooth strongly pseudoconvex domains as in Theorem 3.3.5. We now state the following question posed by Fornæss and Wold.

Question 4.4.4 *Is there a similar estimate as in Theorem 3.3.5 for C^2-smooth strongly pseudoconvex domains? If the answer to this question is negative, we would like to understand why we cannot get such an estimate for C^2-smooth strongly pseudoconvex domains.*

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