A Survey of Ulrich Bundles

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Abstract

The purpose of this article is to serve as an introduction to Ulrich bundles for interested readers. We discuss the origins of the study of Ulrich bundles, their elementary properties, and we give an exposition of the known results in their classification on various kinds of smooth projective varieties. As an illustration of a method recently used by Marta Casanellas and Robin Hartshorne for the classification of Ulrich bundles on a cubic surface, we classify the stable Ulrich bundles on a smooth quartic surface in $\mathbb{P}^3$ of Picard number 2 by first Chern class.

1. Introduction and History

The study of *Ulrich bundles* originates in an article by Bernd Ulrich that appeared in 1984. In [42], Ulrich investigated criteria for a local Cohen-Macaulay ring to be Gorenstein. The second of these criteria involved finitely generated modules M of positive rank over a local Cohen-Macaulay ring R. It can be easily shown that, in this case, the minimal number of generators $\nu(M)$ of M is bounded above by the product of the multiplicity of R and the rank of M. [42, Theorem 3.1] gives necessary conditions for R to be Gorenstein; one condition is the existence of a finitely generated R-module M of positive rank such that the minimal number of generators is greater than half of the product mentioned above. In view of this result, Ulrich asked the following question ([42, p. 26]):

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†The author was supported by TÜBİTAK scholarship 2232. He would also like to thank the Tata Institute of Fundamental Research and the Institut des Hautes Études Scientifiques, where he conducted research on Ulrich bundles on K3 surfaces.

> Let R be a local Cohen-Macaulay ring with positive dimension and infinite residue class field. Does there exist a Cohen-Macaulay R-module M with positive rank such that $\nu(M) = e(R)\mathrm{rank}(M)$?

Three years later Joseph P. Brennan, Jürgen Herzog and Bernd Ulrich investigated these modules, which they called 'Maximally Generated Cohen-Macaulay Modules' (MGMCM) in [5].[1] Specifically, they proved that a homogeneous, two-dimensional Cohen-Macaulay domain R with infinite residue class field admits an MGMCM module. They also raised a number of questions on the existence of MGMCM modules on various kinds of rings.

The term 'Ulrich module' as a synonym for a MGMCM was coined by Jörgen Backelin and Jürgen Herzog in [3]. In that article, they proved the existence of Ulrich modules on hypersurface rings. That result added to the previously known existence results.

It was Arnaud Beauville who in [4], as far as we can ascertain, made a systematic exposition of the relationship between Ulrich bundles and the determinantal or Pfaffian representations of hypersurfaces; and he used these results to reprove a number of classical results (plane curves, cubic surfaces . . .), as well as new results on threefolds and fourfolds.

In the meantime, there had been another line of inquiry that would lead to the investigation of Ulrich bundles. The *generalized Clifford algebra* and its matrix representations (see Section 2, *(ii)* for definitions) had been the subject of a number of papers. For instance, in [27], Darrell Haile proved that the center of the generalized Clifford algebra C_f of a binary cubic form f modulo the intersection of its three-dimensional representations was an Azumaya algebra over its center, and that its center was the coordinate ring of an affine elliptic curve. Three years later, Michel van den Bergh proved in [43] that the rd-dimensional representations[2] of the generalized Clifford algebra C_f of a binary form f of degree d are in one-to-one correspondence with vector bundles on a plane curve X associated to f whose pushforward under a linear projection $X \to \mathbb{P}^1$ are trivial. (Ulrich bundles are precisely the vector bundles satisfying an analogous property; see Definition 3.2.) Rajesh Kulkarni used van den Bergh's results to prove an analogue of Haile's result mentioned above; he proved in [34] that the reduced Clifford algebra is Azumaya over its center; and the center is the affine coordinate ring of the complement of a Θ-divisor in $\mathrm{Pic}^{d+g-1}(X)$ for a suitably defined plane curve X of genus g. Later on, using similar techniques, the present author proved in [11] that the rd-dimensional representations of

[1] The term 'Ulrich module' has been coined by Jürgen Herzog and Michael Kühl in the article [30].

[2] Van den Bergh proves that the matrix representations of C_f must have dimensions that are multiples of d; the same result is also proved by Darrell Haile and Steven Tesser in [28].

the generalized Clifford algebra C_f of a binary form have a fine moduli space that is a nonempty open subset of the moduli space of vector bundles over a suitable plane curve X.

This paper is intended as a survey of known results of Ulrich bundles. In Section 2, we discuss the beginnings of Ulrich bundles; this section is divided into three parts. In the first part *(i)*, we discuss the relations between Ulrich bundles and the problem of expressing the defining equation of a hypersurface as the determinant or the Pfaffian of a matrix linear forms. In the second part *(ii)*, we discuss the relations between Ulrich bundles and the representation theory of generalized Clifford algebras; the material in this part is due to van den Bergh ([43]). In the third part *(iii)*, we discuss the relations between Ulrich bundles on del Pezzo surfaces and the Minimal Resolution Conjecture for a finite collection of sufficiently many points lying on the del Pezzo surface. In Section 3, we give two equivalent definitions of Ulrich bundles and collect some elementary properties of Ulrich bundles that will be used later. In Section 4, we outline known results in the classification of Ulrich bundles. This section is arranged according to the dimension of the underlying variety. Finally, in Section 5, we use a recent technique due to Casanellas and Hartshorne to give a classification of Ulrich bundles according to the first Chern class on a determinantal quartic surface in $\mathbb{P}^3$ with Picard number 2.

Notation and Conventions.

- We work over an algebraically closed base field $\Bbbk$ of characteristic 0.
- All varieties mentioned in the text are implicitly assumed to be smooth and projective.
- $X \subset \mathbb{P}^N$ denotes a variety, of degree d and dimension n. Note that X is polarized by $\mathcal{O}_X(H)$, where H denotes the hyperplane section of X. The inclusion map is $i : X \hookrightarrow \mathbb{P}^N$.
- We use the term *vector bundle* for a *locally free sheaf.*
- We denote $\mathcal{O}_X(H)^{\otimes t}$ by simply $\mathcal{O}_X(t)$ for $t \in \mathbb{Z}$.
- Stability and semistability are used in the sense of Gieseker.

2. Origins

(i) How can we determine whether a *hypersurface* $X \subset \mathbb{P}^{n+1}$ is given by a determinantal or a Pfaffian form?

Theorem 2.1 ([4, Sections 1 and 2]). *Assume that $X \subset \mathbb{P}^{n+1}$ is a hypersurface.*

(a) X is defined by an equation of the form det(M) for a $d \times d$ matrix M of linear forms if and only if there exists a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^d \xrightarrow{M} \mathcal{O}^d_{\mathbb{P}^{n+1}} \to i_*L \to 0$$

where L is a line bundle on X. (If one wants M to be symmetric, one imposes the condition that $L^{\otimes 2} \cong \mathcal{O}_X(d-1)$.)

(b) X is defined by an equation of the form pf(M) for a skew-symmetric $2d \times 2d$ matrix M of linear forms if and only if there exists a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{2d} \xrightarrow{M} \mathcal{O}^{2d}_{\mathbb{P}^{n+1}} \to i_*E \to 0$$

where E is a vector bundle of rank 2 on X with det$(E) \cong \mathcal{O}_X(d-1)$.

Note that the existence of these short exact sequences implies that $H^i(X, L(t)) = 0$ for $t \in \mathbb{Z}$ and $0 < i < n$; a similar statement holds for E. Note also that the Hilbert polynomials of L and E are given by $dr\binom{t+n}{n}$, where $r = 1, 2$ is the rank of the corresponding vector bundle.

Definition 2.2 (First definition of Ulrich bundles). An *Ulrich bundle* on X is a vector bundle E of rank r on X such that $H^i(X, E(t)) = 0$ for $t \in \mathbb{Z}$ and $0 < i < n$, and whose Hilbert polynomial $P_E(t)$ is equal to $dr\binom{t+n}{n}$.

We now see that the representation of a hypersurface as a (symmetric) determinantal or Pfaffian variety depends on the existence of Ulrich bundles with certain properties.

The defining exact sequence above gives $h^0(X, L) = d$ and $h^0(X, L(-1)) = 0$. Suppose now that $L \cong \mathcal{O}_X(n_0)$ for some $n_0 \in \mathbb{Z}$. It follows that $n_0 = 0$ and $L \cong \mathcal{O}_X$. This forces the trivial case $d = 1$. Hence if Pic(X) is generated by H, X cannot be a determinantal variety unless it is linear. By the Noether-Lefschetz and the Grothendieck-Lefschetz theorems, this leaves curves and general surfaces of degree at most 3 as the nontrivial cases for determinantal varieties (there are nongeneral surfaces of higher degree that are determinantal, as we will see in Section 5).

Theorem 2.3 (Noether-Lefschetz, adapted statement from [24, Theorem (*)]). *If $d \geq 4$, then a general surface $X \subset \mathbb{P}^3$ has Pic(X) generated by H.*

Theorem 2.4 (Grothendieck-Lefschetz, adapted statement from [26, Corollaire 3.6]). *Let $X \subset \mathbb{P}^N$ and let H be a general hyperplane section. Then the canonical morphism Pic$(X) \to$ Pic$(X \cap H)$ is an isomorphism if $n > 3$.*

Remark 2.5. The Grothendieck-Lefschetz theorem implies in particular that, if $N > 3$, the Picard group $\mathrm{Pic}(X)$ of the general hypersurface $X \subset \mathbb{P}^N$ is generated by the hyperplane section H.

Example 2.6. *It is a classical result that a cubic surface $X \subset \mathbb{P}^3$ is determinantal; and there are 72 different ways of writing its defining equation as a 3×3 determinant of linear forms in four variables. We will see later that the Ulrich line bundles on X are precisely the line bundles of the form $\mathcal{O}_X(T)$ where T is a twisted cubic on X; it is easily shown that there are 72 of them.*

(ii) Let $f(x_1, \dots, x_{n+1})$ be a form of degree d in $n+1$ variables $x_1, \dots, x_{n+1}$. Define

$$C_f := \Bbbk\{x_1, \dots, x_{n+1}\}/I,$$

where $I \subset \Bbbk\{x_1, \dots, x_{n+1}\}$ is the two-sided ideal generated by elements of the form $(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1})^d - f(\alpha_1, \dots, \alpha_{n+1})$ with $\alpha_1, \dots, \alpha_{n+1} \in \Bbbk$. C_f is called the *generalized Clifford algebra* of f.

We are interested in equivalence classes of matrix representations of C_f. A matrix representation of C_f of dimension m is a $\Bbbk$-algebra homomorphism $\phi : C_f \to M_m(\Bbbk)$. Two matrix representations ϕ, ψ of C_f of the same dimension m are called *equivalent* if there exists an invertible matrix $A \in M_m(\Bbbk)$ such that $\psi = A\phi A^{-1}$. It turns out that m must be divisible by d; see [12, Proposition 2.3] and [28, Proposition 1.1]. We write $m = rd$.

Using f, we construct a hypersurface $X_f \subset \mathbb{P}^{n+1}$ of degree d by the equation $w^d = f(x_1, \dots, x_{n+1})$. Assume that f is chosen so that X_f is smooth. The general linear projection $\pi : X_f \to \mathbb{P}^n$ is a finite map of degree d.

Theorem 2.7 ([43, Proposition 1]). *There is a one-to-one correspondence between the equivalence classes of rd-dimensional matrix representations of C_f and isomorphism classes of rank r vector bundles E on X_f such that $\pi_* E \cong \mathcal{O}_{\mathbb{P}^n}^{rd}$.*

Sketch. For a given representation $\phi : C_f \to M_{rd}(\Bbbk)$, let $A_i = \phi(x_i) \in M_{rd}(\Bbbk)$ for $i = 1, \dots, n+1$. Using these matrices, we construct a ring homomorphism

$$S = \Bbbk[x_1, \dots, x_{n+1}]/(w^d - f(x_1, \dots, x_{n+1})) \to M_{rd}(\Bbbk[x_1, \dots, x_{n+1}])$$

by sending x_i to $x_i I_{rd}$ and w to $x_1 A_1 + \cdots + x_{n+1} A_{n+1}$. This makes $\Bbbk[x_1, \dots, x_{n+1}]^{rd}$ into a graded S-module; hence we obtain a coherent sheaf $\mathcal{F}$ on X such that $\pi_* \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}^{rd}$. It turns out that this is a rank r vector bundle on X. □

Let us now assume that $X \subset \mathbb{P}^N$ is *any* variety. Recall that we denote $n = \dim X$. Let $\pi : X \to \mathbb{P}^n$ be a linear projection.

Definition 2.8 (Second definition of Ulrich bundles). An *Ulrich bundle* on X is a vector bundle E of rank r on X such that $\pi_* E \cong \mathcal{O}_{\mathbb{P}^k}^{rd}$.

Remark 2.9. The two definitions of Ulrich bundles are equivalent. See Theorem 3.3.

(iii) The *Minimal Resolution Conjecture (MRC)* concerns the minimal free resolutions of the ideal sheaves of collections Γ of points lying on subvarieties $Y \subset \mathbb{P}^N$. The MRC was first studied by Anna Lorenzini in [36]; she considered the minimal resolutions of sufficiently general sets of points in $\mathbb{P}^N$. Gavril Farkas, Mircea Mustaţă and Mihnea Popa generalized this conjecture in [22] to sets of points that lie on varieties $X \subset \mathbb{P}^N$.

For a subvariety $Y \subset \mathbb{P}^N$, write

$$0 \to \oplus_{j_k=1}^{l} \mathcal{O}_{\mathbb{P}^N}(-j_k)^{b_{k,j_k}(Y)} \to \cdots \oplus_{j_1=1}^{l} \mathcal{O}_{\mathbb{P}^N}(-j_1)^{b_{1,j_1}(Y)} \to I_{Y|\mathbb{P}^N} \to 0$$

for the minimal free resolution of $I_{Y|\mathbb{P}^N}$. The number $l+1$ is the Castelnuovo-Mumford regularity $\operatorname{reg}(I_{Y|\mathbb{P}^N})$.

Definition 2.10 ([14, Definition 4.1]). Let Γ be a zero-dimensional subscheme of Y. We say that Γ *satisfies the MRC for* Y if

$$b_{i+1,q-1}(\Gamma) \cdot b_{i,q}(\Gamma) = 0$$

for all i and whenever $q \geq \operatorname{reg}(I_{Y|\mathbb{P}^N}) + 1$.

For general Γ consisting of sufficiently many points, the Betti diagram of Γ contains the Betti diagram of Y together with two rows at the bottom. Also, it was proved in [22, Theorem 1.2 (iv)] that the difference $b_{i+1,q-1}(\Gamma) - b_{i,q}(\Gamma)$ is known. Therefore, knowing that the MRC holds would let us determine the Betti diagram of Γ completely.

For a variety $X \subset \mathbb{P}^N$, define the *Ulrich semigroup* $\mathfrak{Ulr}(X)$ to be the semigroup consisting of the first Chern classes of Ulrich bundles on X. Since the direct sum of two Ulrich bundles is again Ulrich by [12, Proposition 2.14], this set is indeed a semigroup.

On a del Pezzo surface, this conjecture is related to the existence of Ulrich bundles in the following way:

Proposition 2.11 ([14, Theorem 1.5]). *Let $X_d \subset \mathbb{P}^d$ be a del Pezzo surface of degree d. Suppose that for every generator Q of $\mathfrak{Ulr}(X_d)$, the MRC holds for a general smooth curve in $|Q|$. Then for an effective divisor $D \subset X_d$ which does not lie on a hyperplane, the following are equivalent:*

1. *There exists an Ulrich bundle E of rank r on X_d with $c_1(E) = D$.*
2. *The degree of D is dr, and the MRC holds for a general smooth curve in the linear system $|D|$.*

Remark 2.12. In [14, Proposition 2.19], it was shown that the Ulrich line bundles on X_d are the line bundles of the form $\mathcal{O}_{X_d}(Q)$, where $Q \subset X_d$ is the class of a rational normal curve; this also follows from [40, Theorem 1.1]. Using [22, Corollary 1.8], it was shown in [14, Lemma 4.3] that a rational normal curve $Q \subset X_d$ satisfies the MRC. Now, if $\mathfrak{Ulr}(X_d)$ is generated by the classes of the rational normal curves, the hypotheses of Proposition 2.11 are satisfied. This is true for $d = 3$ by [6, Theorem 3.9], but false for $d = 6, 7$. This is a consequence of [14, Proposition 3.7] that states that there is a rank-2 Ulrich bundle E with $c_1(E) = H + Q$ and the fact that for $d = 6, 7$ it is not possible to write $H + Q$ as the sum of two rational normal curves. For $d = 4, 5$ it is possible to write $H + Q$ as the sum of two rational normal curves; but we do not yet know whether $\mathfrak{Ulr}(X_d)$ is generated by the classes of the rational normal curves.

3. Properties of Ulrich Bundles

In this brief section, we recall certain properties of Ulrich bundles, some of which will later be used.

We invite the reader to recall Definitions 2.2 and 2.8 now. We point out that the definitions can be trivially generalized to any variety $X \subset \mathbb{P}^N$ of dimension n. For convenience, we restate the definitions here. Note that in the second definition, $\pi : X \to \mathbb{P}^n$ is *any* linear projection.

Definition 3.1 (First definition of Ulrich bundles). An *Ulrich bundle* on X is a vector bundle E of rank r on X such that $H^i(X, E(t)) = 0$ for $t \in \mathbb{Z}$ and $0 < i < n$, and whose Hilbert polynomial $P_E(t)$ is equal to $dr\binom{t+n}{n}$.

Definition 3.2 (Second definition of Ulrich bundles). An *Ulrich bundle* on X is a vector bundle E of rank r on X such that $\pi_* E \cong \mathcal{O}_{\mathbb{P}^n}^{rd}$.

These definitions are equivalent by the following result.

Theorem 3.3. *If $n \geq 2$, the two definitions of Ulrich bundles are equivalent. If X is a curve, a vector bundle E of rank r on X is Ulrich if and only if $\deg(E) = r(d + g - 1)$ and $h^0(X, E(-1)) = 0$.*

Proof. If X is a curve, the proof of [43, Lemma 2, (a)] can easily be generalized. If $n \geq 2$, the result is proved in [14, Proposition 2.3]. □

Remark 3.4. Definition 3.2 implies that, on a linear subvariety $\mathbb{P}^n \subset \mathbb{P}^N$, the only Ulrich bundles are the trivial bundles.

As is well known, the theory of moduli spaces of vector bundles requires some sort of stability condition. The two main definitions of stability are

Gieseker stability (using the reduced Hilbert polynomial) and μ-stability or slope-stability (using the slope). It turns out that Ulrich bundles have desirable stability properties.

Proposition 3.5 ([12, Proposition 2.11]). *An Ulrich bundle is semistable.*

Proposition 3.6 ([12, Corollary 2.16]). *The Jordan-Hölder factors of an Ulrich bundle are again Ulrich.*

Remark 3.7. The proofs of [12, Proposition 2.11] and [12, Corollary 2.16], even though stated for hypersurfaces, are trivially generalized.

These propositions imply that any Ulrich bundle is obtained from a finite collection of *stable* Ulrich bundles by consecutive extensions. Therefore, from now on we will focus on stable Ulrich bundles.

4. Classification of Ulrich Bundles

4.1. Curves.

Suppose that $X \subset \mathbb{P}^N$ is a curve of degree d and genus g. We use Theorem 3.3 for the classification of Ulrich bundles on X.

4.1.1. $g = 0$. Let E be an Ulrich bundle on X of rank r. Then $E \cong L_1 \oplus \cdots \oplus L_r$ by the theorem of Grothendieck[3] that states that any vector bundle on $\mathbb{P}^1$ splits as a sum of line bundles ([25, Théorème 2.1]). Proposition 3.5 implies that E is semistable and hence that the L_i all have the same degree, say m.

We now get

$$\deg(E) = rm = r(d-1),$$

and hence $m = d - 1 \geq 0$. It follows that E is simply the direct sum of r line bundles of degree $d - 1$.

Conversely, we will show that a line bundle L of degree $d-1$ on X is Ulrich; it will then follow that direct sums of such line bundles are also Ulrich. Let $\pi : X \to \mathbb{P}^1$ be any linear projection. Write

$$\pi_* L = F_1 \oplus \cdots \oplus F_d,$$

where the F_i are line bundles on $\mathbb{P}^1 \subset \mathbb{P}^N$. Since the degree of L is $d - 1$, $h^0(X, L(-1)) = 0$ and hence $\sum h^0(\mathbb{P}^1, F_i(-1)) = 0$. This implies that $\deg(F_i) \leq 0$. Since $h^0(X, L) = d$ by the Riemann-Roch theorem, we have $\sum h^0(\mathbb{P}^1, F_i) = d$ and this implies that $\deg(F_i) = 0$ and therefore $F_i \cong \mathcal{O}_{\mathbb{P}^1}$, i.e. that L is Ulrich.

[3] In fact, the content of this theorem was known before Grothendieck; see a note of Winfried Scharlau at http://wwwmath.uni-muenster.de/u/scharlau/scharlau/grothendieck/Grothendieck.pdf.

Since line bundles are completely classified by degree on a genus 0 curve, we conclude that there is a unique Ulrich line bundle on X up to isomorphism, and all Ulrich bundles on X are direct sums of copies of this line bundle.

4.1.2. $g = 1$. One can use Atiyah's classification (see [2]) of vector bundles on elliptic curves to classify Ulrich bundles on an elliptic curve.

Let $X \subset \mathbb{P}^N$ be a curve of genus $g = 1$, and let E be a vector bundle of rank r on X. By Theorem 3.3, E is Ulrich if and only if $\deg(E) = rd$ and $h^0(X, E(-1)) = 0$. Since E and hence $E(-1)$ is semistable by Proposition 3.5, we can write

$$E(-1) = \oplus E_i,$$

where the E_i are indecomposable vector bundles of rank r_i, degree 0 and no global sections. By [2, Theorem 5, (ii)] we can write $E_i \cong F_{r_i} \otimes L_i$, where F_{r_i} is the unique indecomposable vector bundle on X of rank r_i, degree 0 and nonzero global sections, and L_i is a line bundle of degree 0 determined uniquely by E_i. The fact that $h^0(X, E_i) = 0$ implies that $L_i \not\cong \mathcal{O}_X$. Summing up, we can write

$$E = \oplus(F_{r_i} \otimes L_i(1)),$$

where the L_i are nontrivial line bundles of degree 0.

Conversely, it can be easily seen that any vector bundle E of the form

$$\oplus(F_{r_i} \otimes L_i(1))$$

for various F_{r_i} and nontrivial line bundles L_i of degree 0 has rank $r = \sum r_i$, degree rd and $h^0(X, E(-1)) = 0$. In other words, it is Ulrich.

4.1.3. $g \geq 2$. Recall that there is a divisor $\Theta \subset \mathrm{Pic}^{d+g-1}(X)$, called the *theta-divisor*, consisting of line bundles L on X such that $H^0(X, L(-1)) \neq 0$. It is therefore clear that Ulrich line bundles on X are precisely those line bundles outside of the theta-divisor.

Let us now consider a given arbitrary rank $r \geq 2$. In the case where $X = X_f \subset \mathbb{P}^2$ is the smooth curve associated to a binary form $f(u, v)$ that is not divisible by a square, the existence of stable Ulrich bundles of rank r on X was established by van den Bergh in [43, Theorem 4].

We would like to give a proof that there exist stable Ulrich bundles on X of any given rank $r \geq 2$ using a method that was first used by Marta Casanellas and Robin Hartshorne in [6] to classify stable Ulrich bundles on a cubic surface.

Theorem 4.1. *There exist stable Ulrich bundles of any rank $r \geq 2$ on X.*

Proof. Since Ulrich line bundles on X correspond to points outside a divisor in $\mathrm{Pic}^{d+g-1}(X)$, we can take two nonisomorphic Ulrich line bundles L_1 and L_2. By the Riemann-Roch theorem, we have

$$\begin{aligned}\mathrm{ext}^1(L_1, L_2) &= \mathrm{h}^1(X, L_1^\vee \otimes L_2)\\ &= -\chi(L_1^\vee \otimes L_2)\\ &= g-1 > 0.\end{aligned}$$

Hence, there exist nonsplit extensions of L_1 by L_2; and these are simple by [6, Lemma 4.2]. As Ulrich bundles, they are semistable by Proposition 3.5.

Let E be one such extension. The dimension of the moduli space of vector bundles near $[E]$ is

$$\begin{aligned}\mathrm{h}^1(X, E \otimes E^\vee) &= 1 - \chi(E \otimes E^\vee)\\ &= 1 - (4(1-g))\\ &= 4(g-1)+1.\end{aligned}$$

Now let us consider the modular dimension of *strictly semistable* Ulrich bundles of rank 2. These are obtained by extensions of two Ulrich line bundles L_1 and L_2 as above; and their dimension count is

$$g + g + \mathrm{ext}^1(L_1, L_2) - 1 = 3g - 2.$$

It is clear that $4(g-1)+1 > 3g-2$. Hence the dimension of Ulrich bundles of rank 2 on X exceeds the dimension of the *strictly semistable* ones; and that implies the existence of *stable* Ulrich bundles of rank 2.

For higher ranks $r \geq 3$, we proceed similarly. Assume that there is a stable rank $r-1$ Ulrich bundle F on X. Let L be an Ulrich line bundle on X. Note that $h^0(X, F^\vee \otimes L) = 0$ since F and L are both stable of the same slope and they are not isomorphic; any nonzero homomorphism between them would have to be an isomorphism by [31, Proposition 1.2.7]. Therefore by the Riemann-Roch theorem, we have

$$\begin{aligned}\mathrm{ext}^1(F, L) &= \mathrm{h}^1(X, F^\vee \otimes L)\\ &= -\chi(F^\vee \otimes L)\\ &= (r-1)(g-1) > 0.\end{aligned}$$

Just as before, there exist nonsplit extensions of F by L; and these are simple by [6, Lemma 4.2]. If E is one such extension, then the dimension of the moduli space of vector bundles near $[E]$ is

$$\begin{aligned}\mathrm{h}^1(X, E \otimes E^\vee) &= 1 - \chi(E \otimes E^\vee)\\ &= 1 - (r^2(1-g))\\ &= r^2(g-1)+1.\end{aligned}$$

Next, we have to consider the modular dimension of *strictly semistable* Ulrich bundles of rank r. There are various ways to obtain such bundles as consecutive extensions of stable Ulrich bundles of strictly smaller rank, but by the discussion that follows [6, Remark 4.6] it is enough to consider extensions of a stable Ulrich bundle F of rank $r-1$ and an Ulrich line bundle L. And their dimension count is

$$(r-1)^2(g-1)+1+g+(r-1)(g-1)-1.$$

Again, it is not too difficult to show that

$$r^2(g-1)+1>(r-1)^2(g-1)+1+g+(r-1)(g-1)-1,$$

establishing the existence of Ulrich bundles of rank r on X that are not strictly semistable, hence stable. □

Thus, we know the existence of stable Ulrich bundles on any rank on curves. Since semistability and cohomology vanishing are open conditions, Ulrich bundles are in one-to-one correspondence with the geometric points of a nonempty open subset of $M_X(r, r(d+g-1))$.

4.2. Surfaces.

Suppose now that $X \subset \mathbb{P}^3$ is a surface.

4.2.1. Cubic surfaces ($d = 3$). In this case, the Ulrich bundles of ranks 1 and 2 on X can be classified using the results in the article [20] by Daniele Faenzi. Ulrich line bundles on X turn out to be $\mathcal{O}_X(T)$ for T a twisted cubic in X. (Recall that in general, the Ulrich bundles on a del Pezzo surface are the line bundles corresponding to the rational normal curves by [14, Proposition 2.19].) As there are 72 such classes, there are 72 Ulrich line bundles on X; and hence there are 72 different representations of X as a determinantal variety, agreeing with the classical result[4] in [23]. One can also construct Ulrich bundles of any rank on X: Casanellas and Hartshorne constructed rank r stable Ulrich bundles on X with first Chern class rH in [6]; in particular, X can be seen to be nontrivially Pfaffian since a stable Ulrich bundle of rank 2 cannot be written as the direct sum of two Ulrich line bundles. Casanellas and Hartshorne also classified the possible first Chern classes of stable Ulrich bundles. These turn out to be sums of r twisted cubics satisfying a numerical condition; see [6, Theorem 1.1] for details.

[4]For more information about the history of research on the geometry of cubic surfaces, we invite the reader to consult the article [19] by Igor Dolgachev.

4.2.2. Quartic surfaces ($d = 4$). Results are limited.

According to [4, Proposition 7.6], the *general* surface $X \subset \mathbb{P}^3$ of degree d is Pfaffian if and only if $d \leq 15$. By Theorem 2.1, this implies in particular that the general quartic surface $X \subset \mathbb{P}^3$ carries a rank 2 Ulrich bundle. However, there are no Ulrich line bundles on X as $\mathrm{Pic}(X)$ is generated by H and any Ulrich line bundle on X would have degree 6 by [15, Proposition 2.8].

Using the method of Casanellas and Hartshorne, and the existence of rank 2 simple Ulrich bundles with $c_1 = 3H$ on any smooth quartic ([13]), the present author showed that for every even rank $2r$, there exists a *stable* Ulrich bundle of rank $2r$ and first Chern class $3rH$ on the general quartic surface $X \subset \mathbb{P}^3$. (See [15, Theorem 3.1].) This result was later generalized in [1, Theorem 0.5] to any projective K3 surface.

It was proved by the present author, Rajesh Kulkarni and Yusuf Mustopa in [13, Theorem 1.1] that *any* quartic surface is Pfaffian by constructing a 14-dimensional family of simple Ulrich bundles of rank 2 and first Chern class $3H$. The idea is to construct this bundle E as in the short exact sequence

$$0 \to E^\vee \to \mathcal{O}_X^2 \to L \to 0,$$

where L is a line bundle with suitable properties on a smooth curve $C \in |3H|$.

If the rank ρ of $\mathrm{Pic}(X)$ is 2, we will prove a limited result below in Section 5. We will show that there are Ulrich line bundles on X if and only if $\mathrm{Pic}(X)$ has a basis consisting of H and C, where $C.H = 6$ and $C^2 = 4$. The Ulrich line bundles are then $\mathcal{O}_X(C)$ and $\mathcal{O}_X(3H - C)$. In this case, we will also classify the first Chern classes of *stable* Ulrich bundles of any rank r.

4.2.3. $d \geq 5$. There are no results.

4.2.4. Del Pezzo Surfaces. The work of the present author, Rajesh Kulkarni, and Yusuf Mustopa about the existence of Ulrich bundles on del Pezzo surfaces and its relation to the Minimal Resolution Conjecture has already been mentioned in *(iii)*, Section 2.

As mentioned in Remark 2.12, the work of Joan Pons-Llopis and Fabio Tonini implies a classification of Ulrich line bundles on a del Pezzo surface $X \subset \mathbb{P}^d$ of degree d. (See [40, Theorem 1.1].) For degrees up to 6, they also construct, for any rank $r \geq 2$, a family of simple ACM bundles of rank r by considering consecutive extensions ACM line bundles of the form $\mathcal{O}_X(Q)$ where $Q \subset X$ is a rational normal curve. Since these line bundles are Ulrich, any vector bundle obtained by their extensions is also Ulrich. However, as the authors point out, these vector bundles are not stable. (See [40, Proposition 5.7].)

Rosa M. Miró-Roig and Joan Pons-Llopis came up with similar results in [38]. They prove that, for any del Pezzo surface $X \subset \mathbb{P}^d$ of degree d and for any $r \geq 2$, there exists an (r^2+1)-dimensional family of simple Ulrich bundles with $c_1 = rH$.

For a given del Pezzo surface $X \subset \mathbb{P}^d$, there exists no result classifying the first Chern classes of *stable* Ulrich bundles completely.

4.3. Threefolds and higher dimensional varieties.

4.3.1. Cubic Hypersurfaces. The work of Casanellas and Hartshorne related to Ulrich bundles on a cubic surface $X \subset \mathbb{P}^3$ has been mentioned above. They also constructed a rank 2 Ulrich bundle on any smooth cubic 3-fold in $\mathbb{P}^4$ and a rank 3 Ulrich bundle on the *general* cubic 3-fold in $\mathbb{P}^4$. Recently, using derived category techniques, Martí Lahoz, Emanuele Macrì and Paolo Stellari proved in [35, Theorem B] that there exist stable Ulrich bundles of any rank $r \geq 2$ on any smooth cubic 3-fold $X \subset \mathbb{P}^4$.

4.3.2. Quintic Threefolds. In [10], Luca Chiantini and Carlo Madonna proved the following result. Every stable rank 2 ACM bundle E on a general quintic threefold X in $\mathbb{P}^4$ is infinitesimally rigid. In other words, E satisfies $\mathrm{H}^1(E^\vee \otimes E) = 0$. This implies that the moduli space of Ulrich bundles of rank 2 on X is expected to be discrete.

Beauville proves in [4, Proposition 8.9] that there exists a rank 2 Ulrich bundle on X. The number of such bundles must be finite, as is proved by a dimension counting argument. Let $\mathcal{S}_5$ denote the space of 10×10skew-symmetric matrices of linear forms. The group $GL(10)$ acts on $\mathcal{S}_5$ freely and properly; and the Pfaffian map $pf : \mathcal{S}_5 \to |\mathcal{O}_{\mathbb{P}^4}(5)|$ factors through $\mathcal{S}_5/GL(10)$. One can easily calculate the dimensions of $\mathcal{S}_5/GL(10)$ and $|\mathcal{O}_{\mathbb{P}^4}(5)|$ to be the same. Then it can be verified that (see [4, Appendix]) the map pf is generically surjective. It follows that for a generic quintic threefold $X \subset \mathbb{P}^4$, there must be finitely many ways of expressing its defining equation as a Pfaffian. In other words, there are finitely many rank 2 Ulrich bundles on X. However, as Beauville points out in [4, 8.10], this number is not known.

4.3.3. Other Cases. We finish this section with a quick overview of recent results.

- In [37], Rosa M. Miró-Roig proves that on nonsingular rational normal scrolls (with exceptions) there exist families of simple Ulrich bundles of arbitrarily high rank and dimension. Similarly, in [16], Laura Costa, Rosa M. Miró-Roig and Joan Pons-Llopis prove that on Segre varieties (except the quadric in $\mathbb{P}^3$) there exist families of simple Ulrich bundles of arbitrarily high rank and dimension.

- Write $Gr(k,n)$ for the Grassmannian of k-dimensional linear subspaces of $\mathbb{P}^n = \mathbb{P}(V)$ for an $(n+1)$-dimensional $\Bbbk$-vector space V. In [17], Laura Costa and Rosa M. Miró-Roig classify the Ulrich bundles on the Grassmannian $Gr(k,n)$ that are invariant under the action of $GL(V)$.

- In [32], Yeongrak Kim proves the existence of stable rank 2 Ulrich bundles on rational surfaces with an anticanonical pencil, under additional assumptions. To do this, he uses Lazarsfeld-Mukai bundles, which were also used in [1].

- In a recent preprint, [18], Laura Costa and Rosa M. Miró-Roig classify irreducible homogeneous Ulrich bundles on the flag manifold $\mathbb{F}(0, n-1, n)$.

- In two recent papers, [8] and [7] Gianfranco Casnati, Daniele Faenzi and Francesco Malaspina study rank 2 ACM bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and their moduli spaces. Ulrich line bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ can be obtained by [8, Lemma 2.4]; they are line bundles that correspond to permutations of the triple $(0, 1, 2)$, that is, the last six bundles in their list. Ulrich bundles of rank 2 are have Chern classes that are listed in [8, Theorem A, (3)].

- In another preprint, [9], the same authors work on the classification of rank 2 ACM bundles on $\mathbb{P}^2 \times \mathbb{P}^2$. From their work, it follows that there are exactly two Ulrich line bundles on $\mathbb{P}^2 \times \mathbb{P}^2$; namely the line bundles corresponding to the pairs $(0, 2)$ and $(2, 0)$. The classification of the first Chern classes of rank 2 Ulrich bundles on $\mathbb{P}^2 \times \mathbb{P}^2$ is given in [9, Proposition 4.2], [9, Theorem 5.7] and [9, Theorem 5.8].

- In [21], Daniele Faenzi and Francesco Malaspina prove that on a smooth quartic scroll $X \subset \mathbb{P}^5$, there exists a family of indecomposable Ulrich bundles of rank $2r$ on X, parametrized by $\mathbb{P}^1$. ([21, Theorem A]) They also give a criterion to decide if a given ACM bundle is Ulrich on a smooth surface scroll of degree at least 5. ([21, Theorem B])

5. Quartic Surfaces of Picard Number 2

In this section, we prove the results announced above in 4.2.2. Throughout this section, $X \subset \mathbb{P}^3$ is a quartic - hence, K3 - surface. The Picard number $\rho = \text{rank}(\text{Pic}(X))$ is assumed to be 2.

Proposition 5.1. *(a) An Ulrich bundle $\mathcal{E}$ of rank r on X has degree $6r$ and second Chern class $\frac{c_1(\mathcal{E})^2}{2} - 2r$.*

(b) Let $\mathcal{E}$ and $\mathcal{F}$ be Ulrich bundles on X of ranks r and s respectively, and with first Chern classes C and D respectively. Then

$$\chi(\mathcal{E} \otimes \mathcal{F}^{\vee}) = 6rs - C \cdot D. \tag{5.1}$$

Proof. (a) See [14, Proposition 2.10].

(b) This follows from [6, Proposition 2.12]. □

Remark 5.2. The slope of an Ulrich bundle is 6, and the reduced Hilbert polynomial of an Ulrich bundle is $2(t+2)(t+1)$.

Remark 5.3. If $\mathcal{O}_X(C)$ is an Ulrich line bundle on X, then $C.H = 6$ and $C^2 = 4$.

Lemma 5.4. $\mathrm{Pic}(X)$ *admits a $\mathbb{Z}-$basis containing the hyperplane class H.*

Proof. Assume that $A, B \in \mathrm{Pic}(X)$ form a $\mathbb{Z}-$basis for $\mathrm{Pic}(X)$, and let $a, b \in \mathbb{Z}$ be such that $H = aA + bB$. We claim that a and b are relatively prime. Indeed, suppose that they are both divisible by a prime p. Then for some $D \in \mathrm{Pic}(X)$, we have $4 = H^2 = p^2 D^2$. This is impossible as D^2 is even. Hence, the claim is proved.

Let x and y be integers for which $xa + yb = 1$, and define $H' = yA - xB$. It is immediate that the transition matrix from $\{A, B\}$ to $\{H, H'\}$ is unimodular, and the result follows. □

In light of Remark 5.3, we are interested in quartic K3 surfaces with Picard lattice given by a basis $\{H, C\}$, where $H^2 = C^2 = 4$ and $C.H = 6$. The following proposition implies that there exist quartics with this property.

Proposition 5.5. *There exists a polarized K3 surface (X, H) of genus 3 and a smooth, irreducible curve C on X satisfying $C^2 = 4$ and $C.H = 6$ such that* $\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}C$.

Proof. The result follows from [33, Theorem 1.1 (iv)] as the intersection numbers given above satisfy the required criteria. □

Let us mention the following property of the lattice given above.

Proposition 5.6. *In the Picard lattice given by the basis $\{H, C\}$ with $H^2 = C^2 = 4$ and $C.H = 6$, there do not exist divisor classes D with $D^2 = 0$ or $D^2 = \pm 2$.*

Proof. Write $D = aH + bC$. Then $D^2 = 4a^2 + 12ab + 4b^2$. Since this number is a multiple of 4, it cannot be equal to ± 2. By the quadratic formula, one sees easily that there exist no integer solutions to $4a^2 + 12ab + 4b^2 = 0$. □

Assumption 1. The Picard lattice of X can be given as $\text{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is the hyperplane class, $C.H = 6$ and $C^2 = 4$.

Proposition 5.7. *Assume that Assumption 1 holds. Then one can take C to be smooth and irreducible, and moreover one has $h^1(\mathcal{O}_X(C)) = 0$.*

Proof. By a Riemann-Roch calculation, one finds $h^0(\mathcal{O}_X(C)) = h^1(\mathcal{O}_X(C)) + 4 > 0$, hence C can be assumed to be effective.

We claim that the complete linear system $|C|$ has no fixed components.

Assume that $D = aH + bC$ is such a fixed component. We assume, without loss of generality, that D is integral. Then since $C - D$ must be effective, we have

$$(C - D).H = (-aH - (b-1)C).H = -4a - 6b + 6 > 0$$

which is equivalent to $2a + 3b < 3$. Since D is effective, we have

$$D.H = (aH + bC).H = 4a + 6b > 0.$$

Hence, we have that $2a + 3b = 1$ or $2a + 3b = 2$.

By [29, III, Ex. 5.3], $p_a(D) \geq 0$ and hence $D^2 = 4a^2 + 12ab + 4b^2 \geq -2$.

Assume first that $2a + 3b = 1$. Substituting, we obtain $1 - 5b^2 \geq -2$, which implies $b = 0$ and hence $a = 1/2$, a contradiction.

Assume next that $2a + 3b = 2$. Substituting, we obtain $4 - 5b^2 \geq -2$, which implies that $b = 0$ or $b = 1$. If $b = 0$, then $D = H$; which is a contradiction since H cannot be a fixed component of any complete linear system. If $b = 1$, then $a = -1/2$, a contradiction. Hence, the claim is proved.

The result now follows from [41, Proposition 2.6 (i)]. □

Proposition 5.8. *Assume that Assumption 1 holds. The Ulrich line bundles on X are $\mathcal{O}_X(C)$ and $\mathcal{O}_X(3H - C)$, and these line bundles are very ample.*

Proof. By Proposition 5.7, we have that C is a smooth irreducible curve. Notice also that C is nef. Since by Proposition 5.6 there are no divisors on X with square 0 or ± 2, by [33, Lemma 2.4(b)] it follows that $\mathcal{O}_X(C)$ is very ample. Note also that replacing C by $3H - C$ in the above arguments gives the same results for $\mathcal{O}_X(3H - C)$.

To prove that $\mathcal{O}_X(C)$ is Ulrich, we need to prove that it has Hilbert polynomial $P(t) = 2(t+2)(t+1)$ and that it is ACM. The Hilbert polynomial is easily computed. To prove that it is ACM, we use the following short exact sequence:

$$0 \to \mathcal{O}_X(C) \to \mathcal{O}_X(C + tH) \to \omega_{tH}(C) \to 0,$$

where $t \geq 0$ is an integer. Recall that, by Proposition 5.7 we have $h^1(\mathcal{O}_X(C)) = 0$. From the long exact sequence of the above short exact sequence, we obtain

$h^1(\mathcal{O}_X(C+tH)) = h^1(\omega_{tH}(C)) = 0$; where the second equality follows because the degree of $\omega_{tH}(C)$ is larger than $2g(tH) - 2$.

Now recall that $h^1(\mathcal{O}_X(3H - C)) = 0$ as well. The argument in the above paragraph gives $h^1((3+t)H - C) = 0$ for an integer $t \geq 0$. Using Serre duality, we have that $h^1(C + tH) = 0$ if $t \leq -3$.

We claim that $h^1(\mathcal{O}_X(C-H)) = 0$. By Riemann-Roch, we have $h^0(\mathcal{O}_X(C-H)) = h^1(\mathcal{O}_X(C-H))$ and hence it suffices to show that $C-H$ is not effective. If $C - H$ were effective, since $(C - H)^2 = -4$, there would be an irreducible curve of negative self-intersection on X. This is impossible by Proposition 5.6. Hence the claim is proved. Replacing C by $3H - C$ and applying the same arguments gives $h^1(\mathcal{O}_X(2H - C)) = h^1(\mathcal{O}_X(C - 2H)) = 0$.

It remains to show that there are no other Ulrich line bundles on X. Consider an Ulrich line bundle $\mathcal{O}_X(D)$ where we write $D = aH+bC$. From $H.D = 6$ we obtain $4a + 6b = 6$ and from $D^2 = 4$ we obtain $4a^2 + 12ab + 4b^2 = 4$. Since a quadratic equation can have at most two solutions; and $(a, b) = (0, 1)$ and $(a, b) = (3, -1)$, which correspond to $\mathcal{O}_X(C)$ and $\mathcal{O}_X(3H - C)$, are solutions, it follows that these are the only solutions. □

We now prove that the only linear determinantal quartics of Picard number two are the ones considered in this section.

Proposition 5.9. *If X is a quartic of Picard number two, and if there exists an Ulrich line bundle on X, then Assumption 1 holds for X.*

Proof. If $\mathcal{L} = \mathcal{O}_X(D)$ is an Ulrich line bundle, then the lattice $M = \langle H, D\rangle$ is contained in $\mathrm{Pic}(X)$. By a well-known result in lattice theory, we have $[\mathrm{Pic}(X) : M]^2 = d(M)/d(\mathrm{Pic}(X))$ where $d(M)$ denotes the discriminant of M and similarly for $\mathrm{Pic}(X)$. We have $d(M) = -20$, and hence $d(\mathrm{Pic}(X))$ has to divide -20 and the quotient must be a square. Therefore, the only cases are $d(\mathrm{Pic}(X)) = -20$ or $d(\mathrm{Pic}(X)) = -5$. We prove the statement by showing that the second case is not possible. Consider the matrix representation

$$\mathrm{Pic}(X) = \begin{pmatrix} 4 & a \\ a & b \end{pmatrix}$$

of $\mathrm{Pic}(X)$. We have $4b - a^2 = -5$. Note that b is even, so $a^2 \equiv 5 \bmod 8$. But this is impossible since 5 is not a square modulo 8. □

From now on, we assume that Assumption 1 holds.

Proposition 5.8 guarantees the existence of Ulrich line bundles on X. Using these, we can now attempt to classify all stable Ulrich bundles on X. We start with the following lemma.

Lemma 5.10. *Let $\mathcal{E}$ be a stable Ulrich bundle of rank r with $c_1(\mathcal{E}) = aH + bC$. Then we have $2a + 3b = 3r$, and $-\frac{3r}{5} \leq b \leq \frac{3r}{5}$.*

Proof. The first equality follows from the fact that, by Proposition 5.1, (iii), $\mathcal{E}$ has degree $6r$.

Since $\mathcal{E}$ and $\mathcal{O}_X(C)$ are nonisomorphic stable Ulrich bundles with the same reduced Hilbert polynomial, by [31, Proposition 1.2.7], we have $\mathrm{h}^0(\mathcal{E}^\vee \otimes \mathcal{O}_X(C)) = \mathrm{hom}(\mathcal{E}, \mathcal{O}_X(C)) = 0$. Similarly, $\mathrm{h}^2(\mathcal{E}^\vee \otimes \mathcal{O}_X(C)) = 0$. Hence, $\chi(\mathcal{E}^\vee \otimes \mathcal{O}_X(C)) \leq 0$. By Proposition 5.1, (b), we have $\chi(\mathcal{E}^\vee \otimes \mathcal{O}_X(C)) = -(aH + bC).C + 6r = -3r + 5b \leq 0$. Using the same argument with $\mathcal{O}_X(C)$ replaced by $\mathcal{O}_X(3H - C)$ gives the inequality in the statement. □

We now classify all stable Ulrich bundles on X by their first Chern class.

Theorem 5.11. *Let $r \geq 2$ be an integer. Denote $C_r = \{aH + bC \in \mathrm{Pic}(X) \,|\, 2a + 3b = 3r, -\frac{3r}{5} \leq b \leq \frac{3r}{5}\}$. Given $D \in \mathrm{Pic}(X)$, there exists a stable Ulrich bundle $\mathcal{E}$ of rank r on X with first Chern class $c_1(\mathcal{E}) = D$ if and only if $D \in C_r$.*

Proof. Lemma 5.10 gives one direction. For the other direction, we proceed by induction on the rank. For $r = 2$, we have

$$\chi(\mathcal{O}_X(C)^\vee \otimes \mathcal{O}_X(3H - C)) = -C.(3H - C) + 6 = -8$$

by Proposition 5.1, (iv). We have

$$\mathrm{h}^0(\mathcal{O}_X(C)^\vee \otimes \mathcal{O}_X(3H - C)) = \mathrm{hom}(\mathcal{O}_X(C), \mathcal{O}_X(3H - C)) = 0$$

since $\mathcal{O}_X(C)$ and $\mathcal{O}_X(3H-C)$ are nonisomorphic stable Ulrich bundles with the same reduced Hilbert polynomial. Similarly, $\mathrm{h}^2(\mathcal{O}_X(C)^\vee \otimes \mathcal{O}_X(3H - C)) = 0$. Hence we obtain $\mathrm{ext}^1(\mathcal{O}_X(C), \mathcal{O}_X(3H - C)) = 8$. Therefore, there exists a nonsplit extension of $\mathcal{O}_X(C)$ by $\mathcal{O}_X(3H - C)$, which is a simple rank-2 Ulrich bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 3H$ by [6, Lemma 4.2]. To prove that this bundle is stable, we consider the modular dimension of the family of simple rank-2 Ulrich bundles with first Chern class $3H$. We claim that this dimension is 14. Let $\mathcal{E}$ be one such bundle. By Proposition 5.1, (iv), we compute

$$\chi(\mathcal{E}^\vee \otimes \mathcal{E}) = -3H.3H + 24 = -12.$$

Since $\mathcal{E}$ is simple, we have

$$\mathrm{h}^0(\mathcal{E}^\vee \otimes \mathcal{E}) = \mathrm{hom}(\mathcal{E}, \mathcal{E}) = 1.$$

Similarly, $\mathrm{h}^2(\mathcal{E}^\vee \otimes \mathcal{E}) = 1$. Therefore, $\mathrm{h}^1(\mathcal{E}^\vee \otimes \mathcal{E}) = 14$. The claim now follows from [39, Theorem 0.1].

By Proposition 3.5, all Ulrich bundles are semistable; and by Remark 3.7 any strictly semistable Ulrich bundle of rank 2 must be an extension of Ulrich line bundles. Now since $\mathrm{ext}^1(\mathcal{O}_X(C), \mathcal{O}_X(3H - C)) = 8$, the extensions of

$\mathcal{O}_X(C)$ by $\mathcal{O}_X(3H-C)$ form a 7-dimensional family; similarly, the extensions of $\mathcal{O}_X(3H-C)$ by $\mathcal{O}_X(C)$ form a 7-dimensional family. As the modular dimension of the family of simple rank-2 Ulrich bundles with first Chern class $3H$ exceeds the modular dimension of semistable Ulrich bundles with the same rank and first Chern class, there exist stable rank-2 Ulrich bundles with first Chern class $3H$.

Now assume that for some $r \geq 3$, every class in C_{r-1} is the first Chern class of a stable Ulrich bundle of rank $r-1$, and let $aH + bC \in C_r$ be given. We claim that at least one of $aH+(b-1)C$ or $(a-3)H+(b+1)C$ is in C_{r-1}. Since $2a+3(b-1) = 2(a-3)+3(b+1) = 3(r-1)$, it suffices to check that at least one of $b-1$ or $b+1$ lies in the interval $[-\frac{3}{5}(r-1), \frac{3}{5}(r-1)]$. The inequalities $b-1 \leq \frac{3}{5}(r-1)$ and $b+1 \geq -\frac{3}{5}(r-1)$ follow immediately from the fact that $-\frac{3}{5}r \leq b \leq \frac{3}{5}r$. If $b-1 < -\frac{3}{5}(r-1)$ and $b+1 > \frac{3}{5}(r-1)$, it follows that $\frac{3r-8}{5} < b < -\frac{3r-8}{5}$, but this can only happen if $r \leq 2$, which is contrary to the assumption. This proves the claim.

Suppose $aH + (b-1)C \in C_{r-1}$. By our inductive hypothesis, there exists a stable Ulrich bundle $\mathcal{E}$ of rank $r-1$ with $c_1(\mathcal{E}) = aH + (b-1)C$. We have, by Proposition 5.1, (iv),

$$\begin{aligned}\chi(\mathcal{E}^\vee \otimes \mathcal{O}_X(C)) &= -(aH + (b-1)C).(C) + 6(r-1)\\ &= -3r + 5b - 2 < 0,\end{aligned}$$

where we have used $2a + 3b = 3r$ to eliminate a. Since $\mathcal{O}_X(C)$ and $\mathcal{E}$ are nonisomorphic stable Ulrich bundles with the same reduced Hilbert polynomial, $\mathrm{h}^0(\mathcal{E}^\vee \otimes \mathcal{O}_X(C)) = \mathrm{h}^2(\mathcal{E}^\vee \otimes \mathcal{O}_X(C)) = 0$. Hence $\mathrm{ext}^1(\mathcal{E}, \mathcal{O}_X(C)) > 0$, and there exist nonsplit extensions of $\mathcal{E}$ by $\mathcal{O}_X(C)$, and these are simple by [6, Lemma 4.2]. The case where $(a-3)H + (b+1)C \in C_{r-1}$ is handled similarly, by considering $\chi(\mathcal{E}^\vee \otimes \mathcal{O}_X(3H-C))$ and extensions of $\mathcal{E}$ by $\mathcal{O}_X(3H-C)$.

We compute the modular dimension of simple, rank-r Ulrich bundles $\mathcal{E}$ with first Chern class $aH + bC$ by the expression $\mathrm{h}^1(\mathcal{E}^\vee \otimes \mathcal{E})$.

$$\begin{aligned}\mathrm{h}^1(\mathcal{E}^\vee \otimes \mathcal{E}) &= 2 - \chi(\mathcal{E}^\vee \otimes \mathcal{E})\\ &= 2 + (aH + bC)^2 - 6r^2\\ &= 2 + 3r^2 - 5b^2,\end{aligned}$$

where we have used the equation $2a + 3b = 3r$ to eliminate a.

The proof will be complete once we prove that the modular dimension of strictly semistable Ulrich bundles with fixed Jordan-Hölder type is strictly smaller than the dimension given above. We make use of the observation that it is enough to consider the extensions of a stable, rank-$(r-1)$ Ulrich bundle $\mathcal{F}$ with first Chern class $aH + (b-1)C$ with the Ulrich line bundle $\mathcal{O}_X(C)$; or the extensions of a stable, rank-$(r-1)$ Ulrich bundle $\mathcal{F}$ with first Chern class

$(a-3)H+(b+1)C$ with the Ulrich line bundle $\mathcal{O}_X(3H-C)$. (We owe this observation to the proof of [6, Theorem 4.3].) We will only consider the first case, as the calculations for the second case are similar.

The extensions of $\mathcal{F}$ by $\mathcal{O}_X(C)$ are parametrized by $\mathbb{P}\mathrm{Ext}(\mathcal{F},\mathcal{O}_X(C))$. We have

$$\begin{aligned} \mathrm{ext}^1(\mathcal{F},\mathcal{O}_X(C)) &= \mathrm{h}^1(\mathcal{F}^\vee \otimes \mathcal{O}_X(C)) \\ &= -\chi(\mathcal{F}^\vee \otimes \mathcal{O}_X(C)) \\ &= C.(aH+(b-1)C) - 6(r-1) \\ &= 3r-5b+2, \end{aligned}$$

where, as before, we have used the equation $2a+3b=3r$ to eliminate a. Therefore, the modular dimension of the extensions of $\mathcal{F}$ by $\mathcal{O}_X(C)$ is $3r-5b+1$. The Ulrich bundle $\mathcal{F}$ can be chosen from a family of dimension $2+3(r-1)^2-5(b-1)^2$. Adding the two dimensions $3r-5b+1$ and $2+3(r-1)^2-5(b-1)^2$, we obtain that this number is smaller than $2+3r^2-5b^2$, which is the modular dimension of simple, rank-r Ulrich bundles with the given first Chern class $aH+bC$. Hence, there exist *stable* Ulrich bundles with rank r and first Chern class $aH+bC$. This finishes the proof. □

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