THE RECONCILABILITY OF NON-EUCLIDEAN GEOMETRIES WITH KANT'S PHILOSOPHY OF MATHEMATICS

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ABSTRACT

THE RECONCILABILITY OF NON-EUCLIDEAN GEOMETRIES WITH KANT'S PHILOSOPHY OF MATHEMATICS

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This thesis examines Kant's philosophy of geometry, and the possibility of reconciling non-Euclidean geometries with Kant's philosophy of geometry. Kant believed that the propositions of Euclidean geometry are necessary and universal. In addition to that, he embraced the view that the character of space is Euclidean and he did not give any credence to the possibility of determining the character of space by using another geometrical structure. He also propounded the view that experience plays no positive role in the acquisition of geometrical knowledge. In this thesis, the views of Helmholtz, Poincaré and Reichenbach as to the positive role experience plays in the genesis of geometry are elaborately discussed. In the light of their views, it is shown that different environmental conditions have the potency to compel sentient beings like us to adopt non-Euclidean geometries. These geometries, in turn, has a proper intuitive content in contradistinction to Kant's claim that they are only possible logically, not intuitively. Under these considerations, this thesis shows that it is not possible to reconcile Kant's theory of geometry with non-Euclidean geometries even if undergoes appropriate modifications offered by certain philosophers such as Strawson, who tried to reduce the scope of Kant's theory of geometry to visual space by arguing that visual space cannot be non-Euclidean. For Strawson, the propositions of Euclidean geometry are necessary and universal as was propounded by Kant, but its validity its limited to our visual space. This thesis also shows the possibility of visualizing non-Euclidean geometries by considering the views of abovementioned philosophers in contradistinction to Strawson's arguments in support of Kant's theory of geometry.

Keywords: pure intuition, non-Euclidean geometry, visual space, Poincaré, Reichenbach

ÖZ

THE RECONCILABILITY OF NON-EUCLIDEAN GEOMETRIES WITH KANT'S PHILOSOPHY OF MATHEMATICS

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Bu tez genel hatları ile Kant'ın geometri felsefesini ve Öklid-dışı geometrilerin Kant'ın geometri felsefesi ile uzlaştırılabilirliğinin olanaklılığını araştırmaktadır. Kant Öklid geometrisinin önermelerinin zorunlu ve evrensel olduğunu savunmuştur. Buna ek olarak uzayın karakterinin Öklidyen olduğunu ve uzayın geometrik karakterinin farklı bir geometrik yapı kullanarak belirlenemeyeceği görüşünü benimsemiştir. Kant'ın ortaya attığı bir başka görüş ise geometrik bilgimizin kökeninde deneyimin asla bir payı olmadığıdır. Geometrik bilgimizin kökeninde deneyimin pozitif bir rolünün olduğuna ilişkin Helmholtz, Poincaré ve Reichenbach tarafından savunulan görüşler detaylı bir şekilde tartışılmıştır. Bu görüşler ışığında, farklı çevresel koşulların, bizim gibi canlıları farklı geometrik yapıları seçmeye itebileceği gösterilmiştir. Öklid-dışı geometrilerin bunun sonucunda duyumsal bir içeriğe sahip olabileceği Kant'ın bu tarz geometrik sistemlerin ancak mantıksal olarak mümkün olabileceği fakat duyumsal olarak mümkün olamayacağı görüşünün aksine gösterilmiştir. Bütün bunlar hesaba katıldığında, bu tez Kant'ın geometri kuramının Öklid-dışı geometriler ile uzlaştırılamayacağı gösterilmiştir. Strawson gibi Kant sonrası filozoflar, Kant'ın geometri kuramının geçerliliğini görsel uzayı kapsayacak şekilde modifiye etmeye çalışmışlardır. Strawson'a göre Öklidyen geometri Kant'ın savunduğu gibi zorunlu ve evrenseldir, fakat geçerliliği görsel uzay ile sınırlıdır. Fakat bu tezde görsel uzayımızın da Öklid-dışı bir içeriğe sahip olabileceği yine aynı filozofların görüşleri göz önünde tutularak tartışılmıştır.

Anahtar Kelimeler: arı görü, görsel uzay, Öklid-dışı geometir, Poincaré, Reichenbach

To My lovely mother and grandmother

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CHAPTER 1

INTRODUCTION

Mathematics, without doubt, is a fascinating topic to reflect upon for philosophers. Since the times of antiquity, philosophers have always enquired into the nature of this peculiar knowledge. What are mathematical entities? Do they exist? If they do exist, then where and how do they exist? Also, how do we know mathematics? From what source did we attain such knowledge? The early philosophers have been burdened by these seemingly insurmountable ontological and epistemological questions. In time, this burden became the peculiar fate of philosophers; being always dissatisfied with partial accounts as to the reality and the origins of mathematical knowledge, they always tried to advance further and improve their understanding of these matters.

Perhaps what gave rise to the contentious views as regards the epistemological and ontological status of mathematical knowledge is, without the slightest doubt, the effect Euclid's Elements brought about on philosophers. Even though Euclid's Elements, and along with it geometry and geometrical reasoning, became the paradigmatic source of apodictic certainty, necessity and universality. The origin of such necessity and universality daunted philosophers for centuries. Many philosophers tried to account for the origins of geometry according to their ideologies. Empiricists and rationalists offered their solutions, but not before Immanuel Kant has appeared on the scene, the disputes have been successfully settled.

Kant settled the disputes by successfully synthesizing empiricism and rationalism, and offered a fresh philosophical ground for Euclidean geometry. Owing to Kant, Euclidean geometry gained its secure and unshakable place as a true science of space. For a very long time, Kant's philosophy of geometry reigned over Europe; it seemed, after a very long time, that Euclidean geometry was successfully and philosophically grounded.

19th and 20th century witnessed revolutionary thoughts that radically altered the way mathematics and physics are understood. The discovery of Non-Euclidean geometries, and the discovery of the special and general theories of relativity and quantum mechanics in the field of physics are, without doubt, the most triumphant discoveries in the history of intellectual thought, the importance and value of which can be compared to the discoveries of The Elements and Newton's laws of motion and universal gravitation. The discovery of Non-Euclidean geometries came into a clashing course with our understanding of geometry and of course with Kant's philosophy of mathematics, because for a considerable long period of time, no one has doubted that the space could be other than the way it was described by Euclid's Elements. But Non-Euclidean geometries have granted the possibility that space could actually be otherwise than the way it was described by Euclidean geometry. Perhaps the final blow to Kant's philosophy of mathematics came from Einstein's discovery of special and general relativity, because Einstein successfully made use of Non-Euclidean geometry to account for spatial relations between objects, which otherwise could not simply be modelled by Euclidean geometry. This, in effect, proved that the non-Euclidean geometries are not just fantasies of mathematicians.

The subject of this thesis is to investigate whether or not non-Euclidean geometries can be reconciled with Kant's philosophy of mathematics. Even though it was first assumed by certain philosophers and scientists that these discoveries both in mathematics and physics rendered Kant's theory of geometry and space obsolete, there appeared, in the subsequent chapters in the history of philosophy, some philosophers who tried to rescue Kant's philosophy by showing that his philosophy can endure these death-blows if appropriate modifications are made in his system. This thesis will begin by a concise exhibition of the historical roots of geometry and then it will make a brief historical survey with regard to the philosophical reflections upon geometry up until Kant has appeared on the scene. In order to obtain a thorough understanding of Kant's philosophy, a brief information as to the grounds which prepared the motivation for Kant to come up with his own thesis must be provided. In the second chapter, a prolonged and more detailed exposition of Kant's philosophy of mathematics, along with his views as regards space and the exact relation between space and geometry will be given. The third chapter will focus on the intellectual climate after Kant, and on the discovery of Non-Euclidean geometries. And lastly, in the final chapter, the impact of Non-Euclidean geometries on Kant's philosophy of mathematics, in the light of numerous interpretations from variety of philosophers, such as Poincaré and Reichenbach, is going to be discussed.

In conclusion, my thesis aims to show that no modification can save Kant's theory of geometry from its demise. First of all, it will be shown that the space could be modelled by non-Euclidean geometries in contradistinction to Kant's views. This brings us to the conclusion that the Euclidean geometry is not the only geometry that can be used in describing the character of space. Second of all, it will be shown that the intuitive comprehension of non-Euclidean geometries, along with its rigorous logical comprehension, is possible. Kant and his followers believed that it is impossible to make sense of non-Euclidean geometries. My thesis is going to tackle the issue of the possibility of the intuitive plausibility of non-Euclidean geometries, and it is going to provide an arena in which conflicting views of the opposing parties are going to battle each other.

CHAPTER 2

A BRIEF HISTORICAL SURVEY

Geometry, as a practice, owes its origins to Egyptians and Babylonians, who used it as an instrument for the measurement and determination of magnitudes. The approach of Egyptians and Babylonian practitioners to geometry was practical; they used it in architecture, surveying, and sky observations and in many more practical fields. The annual rising of the Nile River, for example, necessitated the incorporation of geometry and engineering; for without the aid of geometry, it would have been difficult, if not impossible, for the Egyptian people to cope with the consequences of this yearly flood. Proclus gave a brief comment as to the attitude of the Egyptian practitioners towards geometry with his following words:

> According to most accounts, geometry was first discovered among the Egyptians, taking its origin from the measurement of areas. For, they found it necessary by reason of the flooding of the Nile, which wiped out everybody's proper boundaries. (Proclus, 1970, p. 52)

This, however, does not mean that the minds of the Egyptian and Babylonian practitioners of geometry were not occupied with certain geometrical problems. It is well known today that these ancient practitioners contributed a lot to the field of geometry by discovering certain geometric relations. Egyptians, for example, discovered how to calculate the area of a given triangle and circle (Schreiber, 2015). Various areas and volumes were calculated, but this, again, was carried out with respect to the practical engineering problems that had needed to be solved back then (Schreiber, 2015).

The transition from Egyptian and Babylonian geometry to ancient Greek geometry is remarkable in the sense that in the latter, geometry was gradually stripped off of its empirical character and vulgar origins, and it gained a new rigorous and scientific outlook. For Greek philosophers, it was as though geometry, as a practice, was carried out only for the sake of the spirit of geometry and nothing else. The comment provided above that belongs to Proclus continues as follows:

> Nor is there anything surprising in that discovery both of this and of the other sciences should have had its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect. (Proclus, 1970, p. 52)

Schreiber states that it was with Pythagoras that geometry had been started to be practiced for its own sake; that is, completely detached from the practical affairs (Schreiber, 2015). Also later on, we see with Plato, a distinction between the geometry practiced by merchants and builders, and that practiced by philosophers. (Plato, 1997).

Around 300 B.C. Euclid appeared on the scene. Without doubt, one of the most brilliant and remarkable achievements in the history of the intellectual thought is *Elements of Geometry*. This monumental edifice was compiled by the great ancient Greek mathematician, Euclid. The compilation was remarkable in that Euclid put together the findings of Babylonian, Egyptian and Greek geometricians and organized them into a single and consistent system. What is new in the Greek system is the *axiomatic approach* so as to establish a firm theory of space. The system is composed of *axioms*, *postulates* and *definitions*, each of which is then used to prove certain *propositions*. Axioms actually go by the name of *common principles*. These principles are nothing more than the principles of logic which are common to all scientific disciplines unlike *postulates*, which are bodies of premises taken to be self-evidently true and specific to the field of geometry. Postulates are special in that they function as the determination of or a set of procedures for constructing a well-defined geometric figure. Through the postulates, one obtains information as to the most elementary figures that can be constructed in geometry. Propositions are generally stated in natural languages; they function as statements which are to be shown through certain constructions and then be proven accordingly. The Elements is the prototype of a first deductive system in which theorems can be deduced by virtue of the proper utilization of the axioms, postulates and definitions. One of the most remarkable aspects of Euclid's system is that the soundness of the theorems need not go through validation which involves processes of measurement and experimentation; these theorems are rather shown to be true with an unprecedented rigor and assuredness through deduction.

The Elements of Geometry has become the paradigmatic example of the mathematical method and an axiomatic system, and along with its influential spread across centuries, philosophical and mathematical problems associated with it have begun to surface. Philosophers and mathematicians did not refrain themselves from reflecting upon the nature of geometry and geometric reasoning. Daunted and perplexed, perhaps by the compelling force with which the propositions of Euclidean geometry impose themselves upon the human mind, many of the philosophers naturally questioned the origins of the geometrical knowledge and from whence it derives its necessity and certainty. What was, after all, the proper subject-matter of geometry? Did it study the visible shapes and figurative properties of concrete objects? Did it study the spatial relations between objects? Did it study the space itself? Or was it about something more abstract and ethereal, as perhaps had been thought by Plato?

Plato was the forerunner of the idea that the geometrical objects were not to be confused with sensible objects (Plato, 1968, 529c-530a). A line, for example, as defined by the Elements, is that which lies evenly with its points (Heiberg, 2007, p. 6). But from whence we could know that such a property belongs to the concept of line if the objects of sensation are not able to instantiate that concept accordingly? This shows that he was aware of the difficulty of reconciling the abstract entities of geometry with their sensible counterparts. Thus, in seeking the true origins of geometry, Plato had recourse to the existence of the world of forms, eternal and unchanging, which is revealed to us through rational contemplation. Plato had a point, for Euclid’s Elements seem to have been unrelated with the study of the practical problems related with the measurement of concrete objects. The unrelatedness of Euclid’s Elements with the measurement of concrete objects was exemplified by Stephen Barker, a philosopher who was keenly interested in these issues. In his book, *Philosophy of Mathematics*, he claims that a straight line cannot be drawn between two points on the surface of the earth, for there are various factors which have the potential to render the activity of drawing a straight between two points almost impossible. (Barker, 1964) So whatever the subject matter of *The Elements* was, it surely was not concrete figures and their measurable properties. This is why Plato sought a refuge to the divine and eternal forms and deemed geometry as an extra-mundane endeavor which transcends the world seen, heard and touched.

Plato’s position is characterized today as realism about mathematical entities which also goes by the name of Platonism. This, however, is not the only interpretive solution to Euclid’s Elements. The solution offered by Plato comes with a burden of ontological commitment; a commitment to the existence of a realm which transcends this world and is forever hidden from our perceptual faculty. This ontological commitment, which is in a sharp contrast with the core tenets of empiricism, prepared the ground for the disputes as regards the origins of geometrical knowledge and these disputes have not been settled even up to the present day.

Empiricists, such as John Locke, George Berkeley and David Hume, were not eager to dispense with the sensible aspect of geometrical reasoning; geometry, after all, is using figures, such as points, lines, surfaces, etc. with which we are closely acquainted in our everyday experience. So it is not a daring assumption that we come to know these objects of geometry through our experience.

Berkeley found the abstract geometrical entities as *inconceivable*, or *unimaginable.* To give an example, no particular line seems to be able to instantiate the concept of a straight line properly as was defined by Euclid, for neither are we able to imagine a breadthless line as it was defined within the Elements, nor we are able to see and inspect any in our experience. In Berkeley's words: "Extension without breadth i.e., invisible, intangible length is not conceivable tis a mistake we are led into by the Doctrine of Abstraction." (Berkeley, 2019, 365a). Berkeley concluded that the proper subject matter of geometry is not "*extension in abstract*" (Jesseph, 2009). The object of geometry, for Berkeley, is "the sensible extension, composed of sensible minima." (Brook, 2012, p. 2)

Another seemingly insurmountable problem was related with the proof-procedures in which the particular objects are used as universals that quantify over all the others. After all, for Berkeley, the subject matter of geometry is the particular figures constructed on a canvas or imagined. How is it, then, that a particular geometric figure, such as a constructed triangle, or a line, is able to convey general information as to all other triangles, if the object of geometry is nothing other than sensible extension? After all, no two triangles could be assuredly held to be equal in terms of their magnitude. This point is also stressed by Hume. David Hume was troubled by the granular and irregular nature of apparent bodies, for no body that we measure is able to yield an exact information as to its length, area or volume. He said “appearance can never afford us any security, when we examine, the prodigious minuteness of which nature is susceptible.” (Hume, 1960, p.70). This would mean that, under the empiricist view, no two triangles could have been expected to possess exactly the same properties. A triangle, for example, is constructed *in concreto* when one attempts to prove a given proposition about all triangles. That triangle, then, serves as a universal in that every property that is discovered by virtue of an appeal to that particular triangle is also valid for all the other triangles. This seemed to be an oxymoron for empiricists such as Berkeley, for how is it that the universal is assumed by a mere inspection of the particular? The universality of the propositions of geometry must then, at best, be comparatively universal, a type of universality which is achieved through induction. But this was totally at odds with the deductive structure of Euclid’s Elements. This is why Berkeley thought “that propositions and demonstrations in geometry might be universal, though they who make them never think of abstract general ideas of triangles or circles.” (Berkeley, 2020, p. 209).

Rationalists, on the other hand, seemed to be more content with Platonism compared to the empiricists. Rationalists such as Leibniz and Descartes, believed that it is the intellect which is able to grasp the essence of these propositions and confer to those propositions strict necessity and universality. So rationalists accused empiricists for “explaining away” the apodictic certainty of the propositions of geometry and treating the totality of the system as a mere contingency. Descartes, for example, believed that the propositions of geometry are comprehended *clearly* and *distinctly* in the *natural light of reason* which made any doubt cast on their soundness irrelevant and preposterous.

Descartes held that "the nature of a triangle appears utterly evident" (Descartes, 2008, p. 50). In his mind, he continued, he can determine every property that follows from the essence of the concept of the triangle, clearly and distinctly. Descartes also believed that the presence of a triangle in his mind is not dependent at all to any particular triangle that he has come to know through his senses. One of the reasons that he put forward to support this thesis is that there exists in his mind "innumerable other shapes that it is impossible to suspect ever reached me via the senses." (Descartes, 2008, p. 46) This means that we can conceive of, clearly and distinctly, a shape which we need not have been confronted prior to our contemplation of it in our experience. This, on the part of rationalists, is "explaining away" the connection between the geometry of sensibles and the geometry that is purely contemplated, for Descartes rigidly held the view that the idea of a triangle must not have arisen in him through his sense organs. Descartes, in his *Dioptrics*, developed a theory which he called *natural geometry*, to try to account for how the perceived geometrical character of objects and their relations are also *innate* and has been all along existed in the perceiver prior to one's exposure to the world of senses. His thesis later confronted with series of criticisms raised by Berkeley and others.

Even though rationalism is not necessarily affiliated with Platonism, it nevertheless remained loyal to the core tenets of Platonic thought that the intellect is somehow able to comprehend the propositions of geometry independently of the intervention of our faculty of sensibility. And this is exactly why the propositions of geometry must be universal and necessary, for the comprehension of the propositions begins from within and not bound to the knowledge attained from without, which is contingent and fallible. Rationalists were well aware of the fact that no empirical proposition can impose itself upon the mind with such necessity and universality as that of geometry.

The apparent dichotomy as regards the possible origins of geometry was on the scene. The contention between the members of the opposing school of ideologies is obvious. Both parties had their own reason to insist upon their view, and posed certain problems related with the views of the opposing side. It seems that rationalists were able to account for the apodictic certainty of the propositions of geometry by locating the seat of the geometrical knowledge within the pure intellect. But this in turn made the applicability of geometry to nature problematic and left other problems, such as the conceivability of the abstract figures in imagination and the universality of the propositions unsettled. Empiricists, on the other hand, located the true origins of geometry within sense perception at the expense of giving up on certainty and necessity of its propositions.

Perhaps the most outstanding turn in the philosophy of geometry took place with Immanuel Kant. Kant was well aware of the problems of both schools of thought and his *transcendental idealism* can be crudely described as a synthesis of empirical and rational cognition. Kant, as pointed out by Henry Allison, accused empiricists for sensitivizing the intellectual concepts that belong to the field of geometry, and accused rationalists for intellectualizing appearances what properly belongs to sensibility (Allison, 2015). Owing to Kant’s outstanding work on human understanding and the elaborate picture that he provided as to how the co-operative work between the understanding and sensibility take place to account for condition of the possibility of sciences in general, geometry reclaimed and secured its indisputable proper place along with other sciences, at least for a while. With Kant, geometry became a body of *synthetic a-priori* truths; which are apodictically certain, necessary and universal.

CHAPTER 3

KANT'S PHILOSOPHY OF MATHEMATICS

As was laid down earlier in the previous chapter, Kant's genius lies in his successful synthesis of rationalism and empiricism. Kant did not agree with empiricist philosophers with regard to the *origin* of our mathematical knowledge, nor did he agree with rationalists as to the *content* of mathematics. He was well aware of the problems associated with both schools of thought.

The secure progress of physics and geometry was put into danger by the skepticism raised by radical empiricists such as David Hume. Kant, notwithstanding the skepticism of Hume, was assured by the secure progression of geometry and physics because they are not as frequently renewed and "brought to a stop as they near their goal" (Kant, 2007, Bvii/Bviii). Kant witnessed the coming and going of many metaphysical systems, each of which was in contradiction with the other and strove in vain to claim an eminent place. But mathematics, he observed, never halted its progress and advanced without any breaks in the history. So mathematics is not just a *random-groping,* as is metaphysics, and the aim of the *Critique* is to prove that the ground upon which mathematics travel is secure.

Kant's solution to rescuing geometry from the skeptical assaults of empiricism and the dogmatic tenets of rationalism was his introduction of the philosophical system which goes by the name of *Transcendental Idealism*. The cardinal tenet of transcendental idealism is that the objects must conform to the *forms of our intuition*;

which Kant deemed as *the pure intuition of space* and *pure intuition of time,* and then must be determined according to the *pure concepts of the understanding.* Intuition is a term which is muddled with conflicting interpretations throughout the history of philosophy. The original German term is *'anschauung';* which means *'to behold',* or, *'to grasp directly or immediately'*. Space and time, for Kant, are intuitions as to which we have an *immediate* direct access. Kant believed that our intuition of space and time are *a priori*[1] frameworks (of space and time) which we impose upon experience and which act as the condition of the possibility of experiencing objects in the first place. Those frameworks are *conditio sine qua non*[2] for experience, that is, by virtue of them the experience becomes possible. Not only must the objects conform to this pure framework, but also to the concepts of the understanding which are not derived from experience. Our knowledge, for Kant, cannot be obtained if we rest on intuitions alone, they must also be determined according to the a-priori concepts of the understanding. Through the former, the objects are *given* to us, through the latter, they are thought (Kant, 2007, B74/B75). To display the collaborative work of our pure forms of sensibility and the concepts of the understanding in producing knowledge, Kant famously asserted that "thoughts without content are empty, and intuitions without concepts are blind" (Kant, 2007, A51/A52). The content is provided to us by our intuitions, and understanding acts on these intuitions and *determines* them accordingly, by subsuming them under concepts and relating them to one another in a possible judgment.

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[1] *A-priori,* in Latin, means prior to any given experience. A proposition is knowable a priori if it can be known without experience of the specific course of events in the actual world. (The Oxford Dictionary of Philosophy, 2008). A detailed exposition is going to be given in the subsequent section.

[2] In Latin, it means a necessary condition for something to exist or happen (Oxford Dictionary of Philosophy, 2008)

This revolutionary thought, that the objects must conform to our cognition, completely turned upside down the philosophical method which had been implemented before Kant. Before Kant, it has been assumed that all our knowledge must conform to objects, but Kant, by turning upside down the traditional conception, required that the objects now must conform to our cognition. The emphasis given to the experiencing subject than to the experienced object has been held to be analogous to the revolution brought about by Copernicus, where the astronomy has been turned "inside-out" by the replacement of the position of our Earth with the sun within his heliocentric system. The revolution brought about by Kant, since then, has gone by the name of the *Copernican revolution* in the history of philosophy.

Transcendental idealism enabled Kant to refrain from believing in a mind independent world in which mathematical entities reside. Going back to Plato, mathematical objects had been believed to be residing in a non-spatio temporal and non-mental realm, completely resilient to all kinds of alteration and change. With transcendental idealism, Kant successfully avoided an ontological commitment to the mind-independence of mathematical objects. The origin of mathematical knowledge was now located in the pure intuition of space and time. It is by virtue of the peculiar and subjective constitution of our minds that we are able to do mathematics; and the construction of every mathematical object takes place in it in a-priori fashion.

Transcendental idealism also enabled Kant to relate the mathematical knowledge to our sensibility without depriving the propositions of it from their apodictic certainty, necessity and universality. It has been commonly held before Kant that any kind of knowledge that has its seat on sensibility is *contingent* and is gained from experience. But with Kant, this sensible faculty was no longer related only with the mode of representation by virtue of which the qualitative properties, such as the color, sound, etc., of the objects of senses are given to us. It is true that Kant stated that “the mode in which we are affected by objects, is entitled sensibility” (Kant, 2007, A19/A20), but the distinction that Kant had drawn between the *form* and the *matter* of our representations enabled him to differentiate between *pure intuitions* and *empirical intuitions*. The matter of our representations (viz. their qualitative properties such as their color, sound, etc.) are that which we receive from our sense organs, which alone yields us *sensations*. The form, on the other hand, is the framework in which the manifold of sensations is organized and ordered. These frameworks are *space* and *time,* and they are *forms of intuition*. The pure form of sensibility also goes by the name of *pure intuition*. What remains when we take away from all the content of our representation of a body; such as its color, hardness and other sensible properties, is *pure extension*, which belongs to pure intuition. (Kant, 2007, A20/A21)

Since he located the true origin of mathematics in our sensibility, he did not agree with rationalists, such as Leibniz and Wolff[3] with regard to the *content* of geometry, for neither the sole inspection of any concept nor their relations carried out in chain of syllogisms in purely in a logical manner was capable displaying the peculiar nature of mathematical knowledge. Kant used the proof-procedures in Euclid’s Elements in setting up a counter-example to the methodology used by Wolff to display that mathematics required more than setting concepts into relations. His break with Wolffian tradition enabled Kant to relate our mathematical knowledge to our pure intuition of space. This is why Kant waged a war against analytical treatment of the truths of mathematics which deprived mathematics from its sensible content, which,

[3] Christian Wolff, a rationalist philosopher who had a huge impact on pre-critical Kant, held that mathematical method consists of “chain of syllogisms guided that proceed from axioms and definitions to theorems” (Frketich, 2019).

for Kant, in its *pure form*, space and time. At the heart of his rejection there lies the connection between mathematics, and space and time, and as a *proto-intuitionist*, Kant was perhaps the first philosopher to tackle the origin and content of our mathematical knowledge to space and time.

In conclusion, the proper subject matter of geometry is nothing but pure figures that are either realized by a process akin to *abstraction*[4] as Kant have put it, or *constructed* in the pure intuition. So the propositions of geometry are neither *synthetic a-posteriori* nor *analytic a-priori* truths. The former is related to the empirical and contingent truths, whereas latter to the truths of reason, which are attained by virtue of pure reason alone, detached from our faculty of sensibility. They are *synthetic a-priori* truths, a novel category introduced by Kant to philosophy.

In order to make a thorough understanding of what synthetic a-priori means, the distinction between a-priori/a-posteriori and analytic/synthetic judgments must be elaborately discussed. The next sections are devoted to the elaboration of these two critical concepts.

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[4] By *abstraction*, I meant the process of taking away all that belongs to the content of the given representation. When everything as regards its content is abstracted from a representation, what remains is its *form*, or *extension* (Kant, 1929, B35). So, a *triangularity* of a triangle can either be realized by abstracting all the relevant features from a given empirical intuition, or it can be purely constructed in the imagination. It does not matter, for Kant, whether the form is realized in the sensible object or in the imagination, for the determinative form common in both representations is their spatial form, which is known a-priori. More on this will be discussed in the subsequent sections.

3.1. The Distinction between A-priori and A-posteriori

Kant stated in *Preface to the First Edition* of his *Critique of Pure Reason* that "the subject of the inquiry is the kindred question, how much we can hope to achieve by pure reason, when all the material and assistance of experience is taken away" (Kant, 2007, Axiv/Axv). As it is roughly discussed in the preceding sections, Kant believed that the lawful aspect of reality is a product of our faculties of understanding and sensibility, and it results from the collaborative work of our faculties of sensibility and understanding.

A-priori, in general means the kind of knowledge that is independent of our experience. *A-posteriori* means the kind of knowledge that is obtained through experience. Two essential properties of a-priori judgments are *necessity* and *universality*; all a-priori judgments are necessary and universal. The necessary judgment is that the negation of which does not make any sense and therefore not possible. Similarly, if a judgment is universal, it means that no exception to that can be provided. These two criteria go hand in hand with one another and, for Kant, are not separable. A-posteriori judgments, on the other hand, are *contingent* and *comparatively universal*. Contingent judgments are that the negation of which are possible both in thought and in reality. Comparative universality, on the other hand, is the criterion which enables the possibility of the occurrence of certain exceptions to those judgments. Comparative universality can only be achieved through *induction*, but a strict universality through *deduction*.

Certain propositions that belong to natural sciences can only be justified on contingent grounds and can only achieve a comparative universality. The proposition, our solar system has eight planets, is a contingent proposition. It is contingent because there remains no reason not to think of the possibility of our solar system having more or less planets. Who knows what is going to happen to our solar system and the planets in it in the future? This is also why it can be justified inductively; we can only assumethat the future is going to resemble the past and thereby conclude that our solar system is going to have the number of planets that it currently has in the future. But we cannot make this statement with hundred percent certainty.

Mathematics, on the other hand, is considered to be an a priori science by Kant. An elaborate discussion as to why mathematics is a-priori will be given in the subsequent sub-sections of this chapter.

3.2. The Distinction between Analytic and Synthetic Judgments

Perhaps what needs to be laboriously scrutinized, so as to achieve a thorough understanding of the nature of the *synthetic a-priori* propositions, is the famous distinction that Kant made between analytic and synthetic judgments. There are two criteria, which can be indirectly inferred from Kant, that exist for distinguishing analytic judgments from synthetic judgments.

The first criterion can be entitled as *containment criterion.* All judgments come in the standard subject-predicate form. If the predicate is necessarily thought, or in other words, contained in the subject, then the judgment is entitled as analytic. This containment relation between the predicate and the subject is the identity relation that takes place between them. In other words, the predicate becomes nothing but a restatement of the subject term through concepts that are already contained *within* itself. These concepts, through which the subject is rephrased, are concepts that collectively constitute the subject. If we imagine a taxonomy in which our subject has a definite place among other taxa, then the concepts that belong to the predicate belong to a higher-level in the same conceptual hierarchy, that is to say, they represent a more general taxa under which the subject becomes a species. Consider the following example *"all bachelors are unmarried men".* Let's denote the concepts by using brackets, so <bachelor> refers to the concept of bachelor. It is clear that in the taxonomy of concepts, <bachelor> and <men> belong to a more general (higher) level. So the formation of <bachelor> necessarily requires first the formation of <men> and <unmarried>. Only with the combination of those two concepts, <bachelor> can be formed.

In addition, in the synthetic judgments, the predicate is not analytically contained within the subject; it is only connected with the subject. This indicates that no matter how much the subject is analyzed into its constituent concepts, the predicate which is connected to it can never be found *within* it; the predicate constitutes wholly and addition to the given subject. The example *"the sky is blue"* is an instance of a synthetic judgment; the predicate *"is blue"* is connected to the concept *"sky"* which is not originally thought within it. In this example, the connection between the subject and the predicate is learned through experience. It is the experience which forms the ground of such connection. We learned that the sky is blue through observation.

The second criterion is the reducibility of judgments to the principle of non-contradiction. In fact, the second criterion is just a way to be assured of the first criterion through subjecting the containment relation to the principle of non-contradiction and see whether the containment relation is analytic or synthetic. Analytical judgments can be known through the principle of non-contradiction. This implies that in analytic judgments, the denial of the predicate and the affirmation of the subject always yield a contradiction. This makes sense; for the subject is necessarily constituted through the concepts that belong to the predicate in analytic judgments, so the negation of the predicate amounts to the negation of the concepts that necessarily and collectively constitute the subject. This inevitably yields a contradiction. But the same principle; the principle of non-contradiction, is not the sole criterion through which the knowledge is attained in the synthetic judgments; the comprehension of synthetic judgments requires more than the principle of non-contradiction. The same example, *"the sky is blue"* can be given to explain why it is the case. The predicate *"is blue"* does not stand in a necessary connection with the subject *"sky"*. This is why, the negation of the predicate does not provide any cue for anyone who is not acquainted with the connection between the predicate and the subject in one's experience; it is merely a contingent truth that the sky is blue. The sky could have been red as well. The truth-value of this statement depends on a variety of contingent conditions.

All analytical judgments are *explicative*, that is, unable to add anything new to our knowledge of the subject through the predicates attached to it. Because, as it was discussed above, all the concepts inherent in the predicate are already contained within the subject. But synthetic judgments, in contrast to analytic judgments, are *ampliative*. This means that they expand our knowledge and avail to us new connections.

In conclusion, the distinction between a-priori/a-posteriori concerns the *origin* of our knowledge. The distinction between analytic/synthetic, on the other hand, concerns the *content* of our knowledge. A-priori judgments have their origin in the mind. Kant clearly stated that "we can know a-priori of things only what we ourselves have put into them" (Kant, 2007, Bxviii/Bxix). It has already been discussed in the previous sections that Kant resorted to the faculty of our pure intuition to locate the true origin of mathematical knowledge. So mathematical judgments are a-priori; they are not derived from experience, they carry with them a strict necessity and universality which are not to be found in the empirical judgments. But what about the content of mathematical knowledge? As it was briefly discussed at the outset of this chapter, he was not sided with rationalists as to the content of mathematical knowledge; he did not accept the view that the propositions of geometry, for example, could be discovered through *discursion*, that is to say, through conceptual analysis. So the propositions of mathematics must be synthetic.

Thus the indispensable conclusion that is to be drawn from it is the following: mathematics is synthetic a-priori. But what exactly does it mean for mathematical propositions to be synthetic a-priori? The following section is mainly focused on this particular question.

3.3. Mathematics as a Synthetic A-Priori Science

Mathematical judgments are both a-priori and synthetic. They provide us with ample examples of a-priori judgments, for they are necessary and universal, that is to say, the negation of which are not possible and there occurs no exception to them. According to Kant, the proposition, for instance, the sum of the interior angles of a triangle is equal to the sum of two right angles, is a necessary and universal proposition. It is necessary, for the negation of it cannot be comprehended and therefore not possible, it is universal, for there exists no triangle, the sum of its interior angles of which are larger than, or smaller than the sum of two right angles.

Mathematical judgments are also synthetic; this means that no matter how hard one analyzes, for example, his concept of triangle, one can never find that its interior angles add up to two right angles. The mathematician needs to go beyond the given concept and make some *constructions* to be able to see that the concept of triangle can be predicated of the given property. Kant explained this procedure followed by the geometer as follows:

> He at once begins by constructing a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained external adjacent angle which is equal to an internal angle. In this fashion, through a chain of inferences guided by throughout by pure intuition he arrives at a fully evident and universally valid solution of the problem. (Kant, 2007, A716/B744)

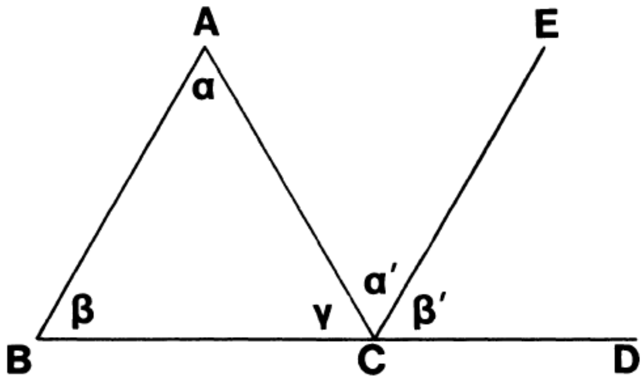

Figure 1

The logical analysis of the concepts cannot account for the connection between the subject and the predicate of the given proposition; the relation between the subject and the predicate can only be discovered through chain of inferences guided by pure intuition. The constructions are necessary for one to make the necessary synthesis between the subject and the predicate and to connect them.

This makes mathematics, a true science, for every proposition is achieved through a synthesis guided by pure intuition which expands our knowledge. Mathematics is more than explicating and making clear what has already been given to us; had it been the case, it would have been nothing other than a gigantic tautology in which nothing new is said.

But what is this *construction in pure intuition?* Why does geometry require such an act? The answers to these questions are immediately tied with the relation between our intuition of space, time and mathematical practice. Indeed; the very possibility of the knowledge of mathematics is dependent upon the faculty of sensibility. That being said, two important questions need to be addressed at this point; first, how are synthetic a-priori judgments possible? And second, what exactly does construction in pure intuition mean? Our inquiry is going to begin with the former question.

3.4. Forms of Sensibility: Pure Intuition of Space and Time

Kant believed that we have forms of sensibility (pure intuition of space and time) by virtue of which the various properties of appearances are *determined* completely a-priori. "In space", Kant states, "…their shape, magnitude and relation to one another are determined." (Kant, 2007, A23). Every object of experience must be located in space and time necessarily. There simply exists no object which do not appear as not belonging to a particular space or time. So space and time are responsible from ordering and organizing our manifold of sensations; therewith, the various spatio-temporal relations among appearances are possible only in virtue of this pure faculty of sensibility.

Kant aimed to show, in *The Critique of Pure Reason*, that the necessity and the universality of the propositions of geometry, and the very possibility of the construction of the concepts of geometry, strictly follows from the a-priority of our pure intuition of space. Geometry "is a science which determines the properties of space synthetically, and yet a-priori" (Kant, 2007, B40/B41). To do that, Kant adopted *the synthetic method* in *The Critique of Pure Reason*. Synthetic method, as utilized by Kant, is that which derives the possibility of the necessity [5]and universality of the judgments of a particular domain of knowledge, such as geometry, from the faculties of the mind which are a-priori. So it is a *progressive method*[6] in which one starts from the original resources of cognition (such as from pure intuitions and understanding) towards a particular domain of knowledge through the synthesis of the elements that belong to that cognition. This is why he claims that the modal status of the knowledge of Euclidean geometry is dependent upon that of our pure intuition of space. The progression from our pure forms of intuition to the necessity and universality of the geometrical knowledge is in accordance with the synthetic method; for without this pure intuition of space, it is not possible to account for from whence the necessity and universality of the propositions of geometry arise. This is supported by Kant stated that "the apodictic certainty of all geometrical propositions, and the possibility of their a-priori construction, is grounded in this a-priori necessity of space." (Kant, 2007, B39)

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[5] The modal ambiguity may strike the reader here. But the reader must not forget that Kant was mainly interested in the conditions of the possibility of knowledge in general. So, what must be shown first is the possibility of having a capacity or a faculty by virtue of which the necessity and the apodictic certainty of the propositions of geometry can be sown.

[6] In *Prolegomena,* however, Kant uses the regressive argument to show that there must be an a-priori source of cognition of space given that geometry is a science which demonstrates its results necessarily and a-priorily through constructions. The regressive method goes by the name of the *analytic method.* Kant can be accused of using the analytic method to justify the a-priority of our intuition of space and thereby committing a fallacy by reasoning in a vicious circle, for, in *The Critique of Pure Reason,* he originally demonstrated the a-priority of our intuition of space as a necessary condition for the science of geometry in the first place. In the literature, this argument goes by the name of *'Argument from Geometry'.* A detailed analysis of this methodological distinction and how should Kant's argument from geometry be correctly treated can be found in Lisa Shabel's article *Kant's Argument from Geometry*.

The pure frameworks of space and time, postulated in *The Transcendental Aesthetic*, account both for the possibility of geometry as a synthetic a-priori science, and the possibility of the necessary applicability of that science to the objects of senses. By this reason, the postulation of an a-priori framework enabled Kant to tackle the long unsettled question of how mathematics can be applied to the nature. The proper mathematization of objects of senses is said to be accomplished with respect to these forms of intuition. The geometric determination of the relations among appearances, and the determination of their figurative properties (e.g., their geometric form) is said to be accomplished in the pure intuition of space and time. What follows from this is that geometry find its *transcendental applicability* to the objects of senses. In brief, Kant offered a *"transcendental explanation of the mathematical nature of the world"* (Cantu, 2018), and he achieved this by locating the seat of the possibility of the geometrization of the nature within the subjective constitution of the human mind, that is, in pure intuitions.

The reader, at this point, must be mindful of the modern distinction between *pure geometry* and *applied geometry* and how it relates to Kant's theory of geometry even though this distinction has not been explicitly stated within the works of Kant and can only be inferred indirectly. Had Kant limited his discussions solely to the possibility of the science of geometry and its a-priori and synthetic nature, the application of geometry to experience, and therefore its *objective validity (reality)* would have begged and explanation. Consider the following passage:

> Through the determination of pure intuition we can acquire a-priori knowledge of objects, as in mathematics, but only in regard to their form, as appearances; whether there can be things which must be intuited in this form, is still left undecided. Mathematical concepts are not, therefore, by themselves knowledge, except on the supposition that there are things which allow of being presented to us only in accordance with the form of that sensible intuition. (Kant, 2007, B147)

In brief, Geometry determines space both a-priori and synthetically. The determination is a-priori, for the intuition upon which the science of geometry is predicated is a-priori. It is synthetic, for the propositions of geometry, as was shown in the previous chapter, can only be obtained by going beyond the given concepts, and this is achieved through certain procedures involving constructions[7] that takes place in a-priori intuition.

Having displayed the tripartite modal relation between geometry and space and spatial perception, what needs to be shown is the transcendental ideality of space as being a pure form of sensibility, for only if the space is transcendentally ideal, the attainment of synthetic a-priori knowledge with regard to it becomes possible.

The philosophy of space before the time of Kant had long been occupied by and centered on two overarching conceptions: *absolutism* and *relationism*. The former is the view that space and time exist independently of all possible objects and object relations, and the latter is the view that space and time depend for their existence on possible objects and relations. The question put forward by Kant, in the *§2* of *Transcendental Aesthetic,* as to the origin of space and time is given as follows:

> What, then, are space and time? Are they real existences? Are they only determinations or relations of things, yet such as would belong to things even if they were not intuited? Or are space and time such that they belong only to the form of intuition, and therefore subjective constitution of our mind apart from which they could not be ascribed to anything whatsoever?" (Kant, 2007, A23/B38)

Here, Kant started his investigation as to the nature of space and time with an ontological question; he asked what kind of entities space and time are. He wanted to

[7] In fact, the construction that takes place in pure intuition of space is not only a *spatial construction*; it is rather a *spatio-temporal construction* which requires the transcendental ideality of both space and time to yield a-priori synthetic knowledge. This will be discussed more elaborately in the subsequent section.

know whether they *subsist* on their own, or *inhere* in things, or none of them. So the opening of the passage suggests that the discussion was mainly concerned, at this stage, with the ontological status of space. The ontological concerns of Kant can be restated as his concerns as to the *origin* of our representation of space. By locating the space and time in our minds as pure forms of sensibility, Kant avoided any commitment to the existence of an absolute space and time, a view propounded by Newton and Newtonians, and to the existence of space and time as relations between things in themselves, a view purported by Kant to have been propounded by Leibniz. He did not think that this origin lies outside of the faculty of pure sensibility. He clearly stated in the conclusion part of *§3* that space "does not represent any property of things in themselves, nor does it represent them in their relation to one another" (Kant, 2007, B42/B43). So all the doors for the *transcendental reality* of space in the form either as an absolute empty container of things, or as relation between things are closed.

Both standpoints were criticized by Kant for several reasons. According to the former view; entitled as *absolutism*, space is a totally mind-independent entity, capable of subsisting on its own. The question, then, naturally arises: how is it that one knows with indubitable certainty that points of space exist independent of any material object? The question can be evaluated both from an ontological and an epistemological standpoints. From an ontological standpoint, the claim that space, as *no-thing,* exists is a bizarre claim. Kant states that the proponents of this view "have to admit two eternal and infinite self-subsistent non-entities, which are there yet without there being anything real." (Kant, 2007, B56/A40) From an epistemological standpoint, given that one cannot obtain any empirical information by any means about these points, it is a rightful question to ask. There seems to be an insurmountable epistemological barrier that needs to be overcome by the proponents of the view of space, as an *empty* *container*. Empirically speaking, the problem is that space is *causally inert,* it does not *affect* our sensory organs and consequently it is not causing any sensible effect. This is a view not acceptable by Kant, for it is committed to the belief that the knowledge of a thing as it is in itself is possible even though there seems to be no way to be acquainted with it in a possible experience. The proponents of this view, notwithstanding the insurmountable ontological and epistemological problems they are facing, were committed to the transcendental reality of space and time by locating the origin of space outside our form of intuition.

The latter view, entitled as *relationism,* is the view that space and time are nothing but relations among objects. According to this view, the existence of space is dependent upon the relations of objects, so without there being the experience of the objects first, the idea of space cannot arise in us. Thus, it can be said that the perceiving of appearances is prior to the existence of space. This means that space is not something over and above the objects of experience. In brief, without there being objects; space would lose its meaning and could not exist.

That the space is *transcendentally real* as a system of relations between things as they are in themselves was propounded by Kant to be the position of Leibniz. Leibniz held that space and time are *phenomena bene fundata.* Phenomena bene fundata, when translated from Latin to English, means *well-founded phenomena*. Well-founded phenomena are the ways in which the various activity of *monads*[8] appears to us in a confused manner. Space and time, as being confused representations of *monads,* are in fact real and thus representative of things in themselves. The only

[8] Monads are mind-like simple substances which are the ground of all corporeal phenomena in Leibniz's philosophy. A detailed discussion as to the nature of monads takes place in Leibniz's *Monadology*.

difference between *monads* and *phenomena* is that there exists a degree of clarity and distinctness in the idea of them; monads are the *supreme reality*, but *phenomena* are the *confused representations* of these monads and their various activity. But they are the one and the same reality for Leibniz; the difference between them comes only in degrees, not in kind, as Kant propounded. So space and time, under Leibniz's treatment, becomes a real, yet confused representation of the activity of the monads, which is interpreted by the subject as relations between them, and the transcendental distinction between appearances and the reality is lost. Kant, by locating the origin of space in our faculty of pure sensibility, secured the transcendental distinction between appearances and reality and thereby granted that our sensibility is not a confused representation of things as they are in themselves. It is clearly explicated by Kant in the following passage taken from *§8* in *The Critique of Pure Reason*:

> …all our intuition is nothing but the representation of appearance; that the things which we intuit are not in themselves what we intuit them as being, nor their relations so constituted in themselves as they appear to us, and that if the subject, or even only the subjective constitution of the senses in general, be removed, the whole constitution and all the relations of objects in space and time, nay space and time themselves, would vanish. (Kant, 2007, A42/B60)

This, as a consequence, do secured the necessary and universal progression of the science of geometry. Had space and time had been entities in themselves, either as *substances* or as *relations*, how could we legitimately confer a-priority and apodictic certainty to the geometric propositions? Kant rightly raised this question. Under this view, the apodictic certainty and necessity of geometrical propositions could not have been justified; for they would have been nothing but a set of general relations abstracted from experience, which can only grant us a-posteriori knowledge. Even if we do concede that the geometry is a necessary and universal science; it's relation with appearances would remain problematic. This is brilliantly summarized by Kant as follows:

> …But since they are unable to appeal to a true and objectively valid a priori intuition, they can neither account for the possibility nor bring the propositions of experience into necessary agreements with it. (Kant, 2007, A41)

It is only in virtue of having their seat in the subject, as was discussed in the section about the nature of a-priori judgments, could the universality, necessity and apodictic certainty of the propositions of geometry have emerged along with its necessary agreement with the propositions of experience. To show that space actually has its seat in the subject and is in fact transcendentally ideal, Kant also stated the following:

> Space is not an empirical concept which has been derived from outer experiences. For in order that certain sensations be referred to something outside me (that is, to something in another region of space from that in which I find myself), and similarly in order that I may be able to represent them as outside and alongside one another, and accordingly as not only different but as in different places, the representation of space must be presupposed. (Kant, 2007, B38/B39)

This means that we cannot read from any appearance anything spatial; the spatiality is something that we bring into the appearances. The assumption that those spatial relations are derived from experience *begs the question* for Kant; for any ascription of polyadic relational predicates to appearances already presupposes the idea of space, so the spatiality of appearances cannot be mentioned without first having an idea of space, thus this idea is not derived from experience. In Patricia Kitcher's words:

> Kant may also be noting that Leibniz's own position in the correspondence with Clarke suggests that our representation of space involves a priori elements. Leibniz claims that we perceive objects in various positions relative to one another. We then abstract from the objects and think of the positions themselves, filling in the currently unoccupied places in the perception, to reach the intellectual idea of space as a system of positions for actual and possible objects. Thus Leibniz seems committed to the view that the creative subject is responsible for elements in our representation of space. So Kant's point may also be that it is inconsistent for Leibniz to characterize [the representation of] space as a product of the creative

> activity of the subject and then to claim to have shown that it depends on actual objects encountered in perception. (Kitcher, 1987, p.234)

It is hard to be oblivious to the apparent symmetry between the content of our geometrical knowledge and the content of our representation of space. Geometry, after all, is the science which study space. Just as the geometrical knowledge could not be obtained discursively, that is, through the analysis of the given concepts, so our representation of space could not be obtained in the same way, so by virtue of its relation to our pure intuition the content of geometry becomes synthetic. It is within that pure framework that we construct the objects of geometry. According to another view affiliated with relationism[9] our representation of space could in fact be an idea that belongs to reason itself to give the phenomena a *spatial and temporal order*. According to this view, the mind generates notion of *place*, or *distance* to make an ordered, *conceptual representation* of the manifold of appearances perceived by the sense organs. At this particular juncture, Kant raised his second criticism towards relationist accounts of space. His second criticism is about the *content* of those relations. He objected to the view that the general concept of space could be a concept (or idea) that belong to reason, and thereby rejected the view that space is nothing but a general concept of relations.

It is not, however, altogether clear what it really means for space to be a non-conceptual representation, and how Kant support his thesis that space is not discursive. To understand why exactly the origin of our representation of space is located under

[9] Leibniz may said to have held two distinct conceptions about space which may be overlooked by Kant. Kant accused Leibniz, on the metaphysical grounds that he equated the representation of space and time as confused representations of things-in-themselves, therefore from Kant's lenses, he committed a transcendental fallacy. But Leibniz also held that space is an idea that belongs to the pure understanding. In several passages, in his *New* Essays, he held that space is an *idea of relation* that belong to the pure understanding. *See* Gottfried Leibniz, *New Essays* for more.

the faculty of pure sensibility rather than in understanding, the differences between *intuitions* and *concepts* must be elaborately discussed. Two of the most important features of intuitions are that they are *singular*, in opposition to being general, like concepts, and *immediate*, that is, their knowledge is not mediate and dependent upon the knowledge of other concepts. Intuitions are singular in the sense that they can be ostensibly referred to as something that is out there; as outside of our bodies in a particular spatio-temporal region. Space is the condition of the possibility of any kind of delineation or ostension; it is by virtue of our *outer sense* that we are able to point towards things and refer to them as *out there*. There is only one unique space in which every object appears. So it is a singular framework. When we talk about diverse spaces, what we actually think of is the parts of the same unique space. Intuitions are also immediate in that being aware of their presence do not require any mediation; that is to say, the object is no longer indirectly referred to through concepts. In brief, immediacy is related to the awareness of the actual presence of any object. Our knowledge of space and the parts of space is not known mediately, that is, through the mediation of other concepts. We are immediately aware of the presence of all possible locations in space. Through these two important criteria, Kant was able to relate our representation of space, and as a consequence of it, our knowledge of mathematics, to non-conceptual elements.

What remains to be shown is how space and time, as pure forms of our intuition, satisfy these two criteria. The right place to begin this proof is to point out the discrepancy between intuitive ways of knowing and conceptual ways of knowing. The laws of intuitive knowledge comes into friction with the laws of understanding and this poses certain difficulties when an intuitive representation is forced by the understanding to be represented conceptually. What is evidently different in between intuitive representation and conceptual representation is that the *mereological*[10] *structure* of the former is exactly the opposite of that of the latter. This difference with regard to their mereological structure is explicated both in *The Metaphysical Exposition of Space*, and in *The Form and Principles of The Sensible and Intelligible World.* When the part-whole relation of these two different kinds of representation is considered, it is seen that the whole precedes its parts in an intuitive representation, whereas the parts precede the whole in a conceptual representation. So the totality of an intuitive representation is given prior to its parts; whereas the totality of a conceptual representation demands to be constructed from its parts. The third proposition of *The Metaphysical Exposition of Space* goes as follows:

> Space is not a discursive or, as we say, general concept of relations of things in general, but a pure intuition. For, in the first place, we can represent to ourselves only one space; and if we speak of diverse spaces, we mean thereby only the parts of one and unique space. Secondly these parts cannot precede the one all-embracing space, as being, as it were, constituents out of which it can be composed; on the contrary, they can only be thought as in it. Space is essentially one, and the manifold in it, and therefore the general concept of spaces, depends solely on the introduction of limitations… (Kant, 2007, A25)

To support this claim that the all-embracing space is given prior to its parts, Kant immediately recurs to the problems that occur in trying to represent it conceptually:

> Space is represented as an infinite given magnitude. Now every concept must be thought as a representation which is contained in an infinite number of different possible representations (as their common character), and which therefore contains these *under* itself; but no concept, as such, can be thought as containing an infinite number of representations *within* itself. It is in the latter, however, that space is thought; for all the parts of space co-exist *ad infinitum.* Consequently, the original representation of space is a-priori intuition, not a concept. (Kant, 2007, B40)

[10] Mereology is a branch of philosophy which studies the relationship between the parts and the whole.

The understanding of these passages is difficult even by Kantian standards. The decryption of the passage requires a clear understanding of what Kant possibly have meant by two different kinds of containment relations; it seems that to be contained *within* something is not to be confused with to be contained *under* something. One of the most illuminating interpretations to make sense of the difference between those two distinct containment relations comes from Michael Friedman. According to Friedman, "B40 operates with Kant's particular notions of extension and intension" (Friedman, 1992, p. 67). By extension, it should not be understood, however, the modern usage of the term. Friedman states that "the modern notion of the extension of a concept was completely foreign to Kant" (Friedman, 1992, p.68). Extension, according to the way it is generally used and accepted in the modern literature, is a set of particular objects that fall under a given *concept*. So, according to this definition, the particular tables are the *extension* of the concept of table. But here, extension of a concept is not the particular objects which partake under a concept because they share a certain property; the extension is itself a concept which falls *under* another in a given conceptual taxonomy. The concepts; <bachelor> and <unmarried>, for example falls under the more general concept, <men>. So those concepts are extensions of <men>. The *intension* of a concept, however, consists of those concepts which constitutes it. So the same concept, the concept of man has this intension: <rational>, <animal>, <material> and <created being>. In other words, the intension of a concept is the definition of a concept with reference to the higher concepts under which the first concept is subsumed. So the concept of men contains *within* itself <rational>, <animal>, <material>, and <created being>.

No concept contains *within itself*, as do spatio-temporal quantities, infinitely many representations. If <bachelor> contained within itself infinitely many constituents, the analysis would never terminate. This would practically render our concept unintelligible, for in order to comprehend a concept, we must first comprehend the concepts which collectively constitute it. So, in order for our concept to be rendered intelligible, the number of concepts that collectively constitute it (intensions) must be finite. If space is to be represented conceptually, then the particular instances (regions) of space must be considered as intensions of the general concept of space. But every instance of space, due to the fact that space is *infinitely divisible*, already contains an infinitely many sub-regions, and those sub-regions infinitely many other regions, and this goes on *ad infinitum*. Accordingly, neither the synthesis of those quantities, that is, the progress from the given regions, would give us the totality of space, nor the analysis of them, that is, the regress from the given part to its constituents, would come to an end and terminate in a simple part which is not a part of anything else, if we were to represent space conceptually. Therefore, when it comes to the representation of *continuous quantities*[11], the only way to represent them is through intuition. Only our intuition is able to give us an object *immediately* and in a *singular* way. According to Kant, his *argument from infinity* provides us with a clear intuitive grasp of the one and unique space, so he argues that space must be a form of intuition. Because when we attempt to represent diverse spaces, we generally assume those diverse spaces, as Kant states, a part of a one and all-embracing unique space. What follows from this is the following: space is both *singular* and *immediate*. It is immediate, for no mediation is required for us to intuit space; we have a direct relation to it; it is presented to us directly. It is singular, for its parts is not given prior to the whole of it; on the contrary,

[11] Space and time, for Kant, are *quanta continua,* which are given as enclosed within *limits*. They are given within limits because "any portion of space must be composed of smaller portions, and therefore can't be 'simple' in the sense of not having parts" (Kant, 2017). Therefore, the only simple items in space are not portions, but limits, which can be divided further into simpler elements.

its parts can only be thought within it. When it comes to the sense impressions that we receive through our sense organs, that we have an *immediate* relation to them and that they are singular seem to be so obvious that we are not obliged by an internal mechanism to offer any kind of proof. But when it comes to our representation of space; since the form itself does not reach us as impressions do, certain justifications seem quite indispensable. And the argument was provided by Kant to ease the doubts.

In conclusion, the space is neither absolute, nor relational; the emphasis on the absolute/relational debate has been shifted into ideal/real. It is also not a conceptual representation, but an intuition. Geometry, as an action of the geometer, takes place in this pure intuition completely a-priori. But as was laid out in the previous sections, we cannot rest in these pure intuitions if we are to produce mathematical knowledge; we also have to recognize what we actually intuit under concepts. Kant famously asserted that "to construct a concept is to present the intuition corresponding to it a priori" *(A713/B74).* If the concept has no intuitive content, it is empty, therefore it must find its referent in the pure intuition. The process of the formation of geometrical knowledge was brilliantly summarized by Kant in *The Critique of Pure Reason, Transcendental Deduction.* The construction procedure is what remains to be fully explicated in this chapter in the following section.

3.5. Construction in Pure Intuition

In *Transcendental Deduction,* Kant speaks of a *three-fold synthesis* in order to account for what actually takes place within the mind during the process of synthesis and the generation of concepts. Even though *TD* aimed at providing a transcendental ground for the deduction of the pure concepts of the understanding, it nonetheless provides the reader with illuminating insights as to how construction of the geometrical concepts in pure intuition takes place. So the analysis of this three-fold synthesis which essential to obtain any knowledge whatsoever will shed a light into our understanding of what necessarily takes place in the mind of a geometer during the procedure of construction.

The mentioned three-fold synthesis is composed of *synthesis of apprehension*, *synthesis of reproduction* and lastly, *the synthesis of recognition*. All these three-syntheses take place a-priori in mind, therefore provide the transcendental ground for the possibility of knowledge. Every synthesis; be it a synthesis of outer representations or inner representations, must be carried out according to the conditions of our inner sense, which is entitled as the pure intuition of time by Kant. Therefore, what must be considered first is the temporal nature of our consciousness; for all representations, be it outer or inner, necessarily belong to inner sense as Kant states in *Transcendental Aesthetic*. Thus all our representations are modifications of inner sense; by means of our inner sense, representations are intuited successively, one succeeding the other in time. First and foremost, what is needed for a proper cognition is to unify our representations that succeed each other into a whole so that they be represented as a single representation. *The synthesis of apprehension* aims at providing *unity* for our representations; due to the temporal nature of our consciousness, the processing of every representation requires time, and through the synthesis of apprehension, the synthetic unity of the manifold, given successively in time as a modification of inner sense, is constituted. A spatio-temporal manifold, which we are receptive of due to our pure forms of intuition, is unified under a single representation and represented as a whole through this synthesis. Kant states that we should never have a-priori representations of space and time without the synthesis of apprehension. Because a geometer constructs objects in the pure intuition of space through *delimiting* the one and unique space. This means that the space has now become itself the object so as to be *determined*; the manifold in it must be subjected to the conditions of the inner sense and unified.

Moving onto the *synthesis of reproduction,* we witness that Kant divided the synthesis of reproduction into two; the one which is carried out in the *empirical imagination*, and the other which is carried out in the *transcendental imagination*. Since the geometry proper is predicated upon the one which is exercised a-priori, the focus of our attention must be directed to the analysis of the synthesis that takes place in the transcendental imagination. For Kant, there must be something which, as the a-priori ground of the necessary synthetic unity of appearances, makes their reproduction possible. That ground is none other than the transcendental imagination. In the transcendental imagination, *the synthesis of production* takes place. It is that which enables us to retain previously constructed representations in our imagination in order for us to be able to connect them with others that comes after them in the given sequel. Kant provides an example as to what really takes place in our imagination when we are making the synthesis of production:

> I seek to draw a line in thought… obviously the various manifold representations that are involved must be apprehended by me in thought one after the other. But if I were always to drop out of thought the preceding representations (the first parts of the line …), and did not reproduce them while advancing to those that follow, a complete representation would never be obtained… (Kant, 2007, A102/103)

So a geometer successfully keeps in his mind the previous parts of a line (points) that he constructed in order to carry out one's construction and be able to represent oneself a line. Without this power of imagination, he would always drop out of thought and never be able to represent a particular whole (say, a line). This activity of *drawing* was given a different nuance in the *B-Deduction*. It was stated as this drawing is none other than the *figurative representation of time*, whereby the manifold of outer intuition is determined by means of our inner sense:

> We cannot think a line without drawing it in thought… Even time itself we cannot represent, save in so far as we attend, in the *drawing* of a straight line (which has to serve as the outer figurative representation of time), merely to the act of synthesis of the manifold whereby we successively determine inner sense, and in so doing attend to the succession of this determination in inner sense. Motion, as an act of the subject (not as determination of an object), and therefore the synthesis of the manifold in space, first produces the concept of succession. (Kant, 2007, B155)

Kant confronts the reader with a very interesting and puzzling part; he talks about *motion* but apparently in a different sense. To a reader who is ready to come to a hasty conclusion, it might sound as if Kant has blended certain empirical elements into the construction of a figure. But as it was carefully explained in the footnote below the passage; *the motion as an act of a subject* is not the same thing as *motion as a determination of an object*, the latter belongs to an empirical science, but former to a pure science, which is entitled as *phoronomy* in Kant's *Metaphysical Foundations of Natural Sciences (1786)*. Kant noted the following: "Motion, however, considered as the describing of space, is a pure act of the successive synthesis of the manifold in outer intuition in general by means of the productive imagination." (Kant, 2007, B156) This proves that the construction of the geometrical concepts are not only spatial, but spatio-temporal. The temporal element is the necessary ingredient in every spatio-temporal construction, for it alone makes possible the act of drawing in the first place and the synthesis of the manifold in intuition.

The last and the most important ingredient of the three-fold synthesis is the recognition of the manifold under a concept. In *A-Deduction,* Kant opens up the section as follows: "If we were not conscious that what we think is the same as what we thought a moment before, all reproduction in the series of representations would be useless." (Kant, 2007, A103) The passage seems to be solely concerned with the identity of our representations; for without it, neither apprehension, nor reproduction would make any sense, for the representations would crowd up in the soul, as put by Kant, without being in a thorough connection with each other. Every representation thereby generated in time would seem to be a new representation without such a function of the understanding. The *rule* whereby we connect all representations with one another, and become conscious of the identity of them in time is none other than the synthetic unity of consciousness in the synthesis of the manifold of representations. This synthetic unity of consciousness is the transcendental ground of the unity of the synthesis of all manifold of intuition. Kant defines this synthetic unity as "pure original unchangeable consciousness" which goes by the name of *transcendental apperception*. The act through which the mind is capable of becoming conscious of the identity of a function whereby it synthetically combines the manifold into a single general representation which is generically identical to itself is the same act through which the consciousness of a manifold of intuition that pertains to its identity through time is made possible. So every act of unification and identification (be it the act of bringing various representations under a general representation, or the act of bringing together various intuitions) is necessarily predicated upon the original transcendental unity of consciousness. Through transcendental apperception, the awareness of the unity of the concept under which particular representations are subsumed, and the unity of the manifold of intuition which is successfully reproduced and apprehended made possible.

The *analytic unity of consciousness* is the same consciousness which is found in many distinct representations (one-in-many). It is the same "I" that we find in distinct representations, therefore the original transcendental apperception has an analytic unity. But in order for us to be able to represent to ourselves the identity of this consciousness present in distinct representations, we must unite the given manifold of representations synthetically in one consciousness due to the temporal nature of the consciousness. The temporality of our consciousness demands that we do this necessary synthesis of the manifold given successively in time in order to obtain a unitary and single consciousness of it. This is why the analytic unity of consciousness, and along with it the analytic unity of concepts and logical forms of judgment, requires an active synthesis in time and brought into a synthetic unity (many-in-one).

The synthetic unity of apperception is therefore the highest principle of all understanding and precedes all concepts of the understanding. In fact, it is by virtue of the synthetic unity of apperception that the unity of the pure form of logical judgments and the unity of given concepts that enter into possible judgments made possible. It simply is the supreme principle of all understanding.

The consciousness of the *homogeneity* of the successive parts of a line, produced by the geometer in the transcendental imagination, has its ground in *the synthesis of recognition*, and therefore, in the *transcendental unity of apperception*. It is by virtue of the identification of the temporal parts as *homogeneous* the geometer can prove certain propositions in geometry, for without *congruence relations* in geometry, nothing can be proven. This is explicated by Kant along these lines:

> Consciousness of the synthetic unity of the manifold and homogeneous in intuition in general, in so far as the representation of an object first becomes possible by means of it, is, however, the concept of magnitude *(quantum)*. (Kant, 2007, B203)

The pure concepts of the understanding are, then, used as a rule of the unification of the manifold of intuitions, and through that rule we are able to recognize every temporal part of a manifold, given successively, as an identical instance of the general concept of magnitude. The rule describes the identity of the function whereby the synthesis of the manifold is iterated completely in an identical manner to generate series of homogenous syntheses in time in the production of the instances of the concept of magnitude. Thereupon, the construction of all of the objects of geometry must be in conformity with the *category of quantity* which serves as a rule whereby the geometer become conscious of a homogeneous intuition.

The necessary conformity between the produced manifold of intuitions in the imagination of the geometer and the concept whereby the same manifold in the imagination is unified under a generic whole is predicated upon the possibility of the subsumption of intuitions under concepts. It is by virtue of *the pure schemata of the concepts* that the intuitions are glued to concepts. A *schema* is nothing other than the transcendental determination of time which is both homogeneous to concepts and intuitions. This is because each category, when schematized, represent a different determination of time. The schemata, for Kant, “are thus nothing but a-priori determination of time in accordance with rules. These rules relate in the order of the categories to the time-series, the time-content, the time-order, and lastly, to the scope of time.” (Kant, 2007, B185). With this last gluing link, Kant was able to show, on one hand, the possibility of the subsumption of appearances under the respective geometric concepts[12] (viz., <triangle>, <circle>, et. al) and on the other hand, the possibility of

[12] In fact, Kant opens this passage with an example from geometry to show that the construction of the geometrical concepts can be taken as a paradigm case of *schematism*. It is through constructing a figure in pure imagination (such as drawing a line in thought) that we make possible any geometrical concept. So, as Jørgensen put it brilliantly; “our capacity for producing images by means of schemata

subsuming them under the pure concepts of the understanding; especially under the concept of quantity.

Taken this way, the schema of a given concept becomes the condition for the possibility of the construction of the very concept. Through schematization, images are subsumed under their respective geometric concepts. The following passage brilliantly summarizes the point made by Kant:

> The image is a product of the empirical faculty of reproductive imagination; the *schema* of sensible concepts, such as figures in space, is a product and, as it were, a monogram, of pure a-priori imagination, through which, and in accordance with which, images themselves first become possible. These images can be connected with the concept only by means of the schema to which they belong. (Kant, 2007, A142)

By 'image', what Kant simply equates the particular image of a triangle that we can form in our imagination with the image received through our sense organs. At the outset of this chapter, we entitled the former as *mental image* and the latter as *sensible image*. They both stand under the same rules prescribed by schematism. *Schema*, is the function which acts as a *norm* in prescribing the law-governed connection between the image and the concept. It is, to an important extent, the condition of the possibility of the image. Schema enables us to relate two images (as intuitions) as 'homogeneous', for both share the same spatio-temporal content, for both are constructed according to the same set of operations. The image is nothing other than the object, which is realized in the experience.

The geometry is thus established as a pure a-priori science; *Transcendental Aesthetic* provided space as the necessary content of geometry, and *Transcendental Logic* provided the pure concepts which function as necessary rules in constructing

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can be seen as a transcendental condition for knowledge and objective representation." (Jørgensen, 2005, p. 3) geometric concepts in the pure intuition of space. The mediation between *sensibility* and *understanding* is accomplished via *imagination*; it is through the transcendental imagination that the pure concepts of the understanding are *schematized* and made possible to be used in the science of geometry.

The three-partite relation between space, geometry and our spatial perceptions are thus finally established. Space stands as the necessary pre-condition for both geometry and the geometrization of appearances. Without having first been acquainted with such a-priori framework, neither construction of any geometrical concept, nor the recognition of any appearance under those concepts be made possible. Through schematism of the pure understanding, appearances are subsumed under respective geometrical concepts; without concepts no knowledge is possible, therefore schematism of pure understanding, along with our pure intuition of space, is the necessary condition for the possibility of geometry.

3.6. Summary

Kant's theory of geometry, as was tried to be shown in this lengthy chapter, is standing on three pillars. The first pillar is Kant's view as to the *origin* of our representation of space, it is formed as a solution offered to two dichotomous options; reality and ideality. Kant has chosen the latter and viewed space as ideal. As regards the *content* of our representation of space, the second pillar is formed as a solution to the dichotomy between intuitions and concepts. Kant located space and thereby the objects of geometry in intuition. So his answer to this dichotomy was that space is not a concept but an intuition. Finally, as regards the *modality* of our representation of space, the third pillar is predicated upon the dichotomy between a-priori and a-posteriori. Kant held that space is an a-priori and an intuitive framework.

Without the collaborative work of our faculty of sensibility and understanding, no geometrical knowledge can arise. The construction of the concepts in pure intuition demands the subsumption what is manifold in our intuition under concepts. Along with the role played by our pure forms of sensibility, what must not be overlooked is the peculiar role played by schematism of pure concepts of understanding, for the construction of the geometrical concepts in pure intuition is the paradigmatic exemplar of how schematism works in producing a-priori knowledge of geometry.

In the following chapters, it will be seen that how the opponents of Kant's theory of geometry tackled these dichotomies, in the light of the discovery of non-Euclidean geometries, in their rejection of Kant's theory of geometry. The discovery of non-Euclidean geometries, and along with it its wide ranging applications in astronomy and cosmology posed serious threat to Kant's theory of geometry. The monumental edifice, upon which is constructed upon these three pillars seemed to be on the verge of collapse. In the face of the advent of non-Euclidean geometries, Kant's theory of geometry required certain modifications and reconsiderations in order that it be reconciled with these new geometries.

Prior to the explication of the views of the opponents of Kant's theory of geometry, a concise history of non-Euclidean geometries is going to be presented in the next chapter.

CHAPTER 4

THE DISCOVERY OF NON-EUCLIDEAN GEOMETRIES

For a considerably long period of time, the mathematicians from the late antiquity to Renaissance tackled the parallel postulate of the Euclidean geometry. The parallel postulate had always been suspected to be a redundant proposition for mathematicians for many reasons. Mathematicians either had been dissatisfied with it as not being self-evident as the other four postulates, or as being capable of deduced from the rest. In the middle of the 18th century, the logical independence of the parallel postulate was discovered by Girolamo Saccheri and it prepared the ground for a fruitful research for alternative geometries carried out by subsequent geometers mentioned above.

Saccheri's approach to the enigma of the fifth postulate was different than his predecessors. All of the mathematicians[13] and philosophers who tried to derive the fifth postulate before Saccheri either tried to deduce it by assuming other premises the truth of which must be taken as self-evident, or tried to deduce from the rest of the postulates and failed in their attempts. Saccheri tried to show that assumption of the negation of the fifth postulate must be incompatible with the rest of the theorems in

[13] I need not go to the details of every attempt made prior to Saccheri, for it is out of the scope of the interest of this thesis. A keen reader can find all the data in *Harold E. Wolfe's* wonderful book, *Introduction to Non-Euclidean Geometry (1945).*

Euclid's Elements. In other words, the negation of the fifth postulate is a threat to the consistency of the entire system.

The implementation of *reductio ad-absurdum*[14] was indeed a novel attempt which hitherto had not been tried. In *The Euclid Vindicated from Every Blemish (2014), Book I,* Saccheri listed, in total, 33 propositions. The first three propositions make use of quadrilaterals which, today, go by the name of *Saccheri quadrilaterals*. Each proof, then, begins by a construction of a specific quadrilateral. Saccheri's quadrilaterals are different from one another in terms of their *summit angles.* It can be seen from the first propositions that the summit angle of the constructed quadrilateral are either equal to, greater than, or less than a right angle. So there are, in total, three quadrilaterals to be considered for each proposition.

[14] In logic, it simply means deriving, from a given proposition to an absurd conclusion by assuming a false premise in the start. It is also a strategy to derive the truth of a given proposition indirectly by demonstrating that an absurd and impossible conclusion follows upon the assumption of the negation of the given proposition.

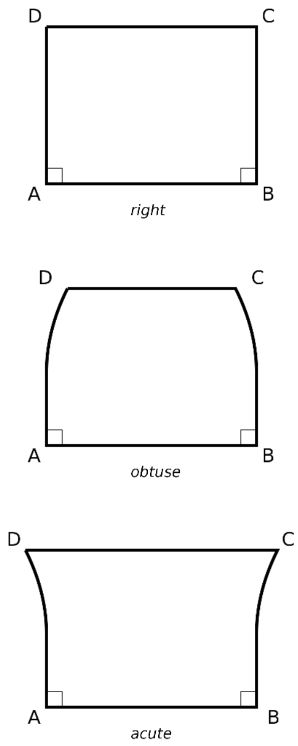

Figure 2

In first proposition, he established that the summit angles must be congruent to one another. In the second proposition, he proved that if the quadrilateral is bisected in the points M and H, then the angles at the joint MH will always be right angles. In the third proposition, he showed that the upper base of the quadrilateral on the joints of which the summit angles are contained must be equal to, greater or less than the base of the quadrilateral according as the summit angles are right, obtuse or acute respectively. In some of the remaining postulates, it can be seen that even though he managed to derive contradictory consequences on the assumption of *HOA*, he could never find any under the assumption of *HAA*. He was able to derive bizarre conclusions that followed from *HAA* which were later to be deemed as theorems of hyperbolic geometry.

It is very probable that Saccheri dismissed his findings due to their intuitive implausibility. Saccheri added that "the hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of the straight line." (Bonola, 1912, p. 43), which is nothing but an extra-logical reaffirmation of the truth of the fifth postulate. The parallel postulate, after all, is a statement about the behavior of straight lines; so rejecting *HAA* on the basis of his pre-conceptions about what straight line actually is gives away his intuitive stance towards parallel postulate from the beginning. Had he realized that he was on the verge of finding a new geometry, the discovery of the non-Euclidean geometries could have been made a century earlier.

After Saccheri, the next person who deserves to be credited in the course of the attempts made to vindicate Euclid, is without doubt Lambert. Lambert's approach was very similar to that of Saccheri. He made use of quadrilaterals and approach the problem through assuming the impossible, that is, the negation of the fifth postulate. The difference between the quadrilateral used by Saccheri and that used by Lambert is that the former included two summit angles whereas the latter had only one summit angle. The consequences that he was able to derive was much richer and exotic. Lambert was able to show that there is a relation between the area of a triangle and the sum of its angles. In *HOA*, the area of a triangle is directly proportional to the sum of its interior angles plus two right angles. In *HAA*, the area of a triangle is directly proportional to the sum of its interior angles minus two right angles[15]. (Wolfe, 1945, p.33). He also noticed that the geometry based on *HOA* resembled *spherical geometry*[16]. The geometry, on the other hand, based on *HAA*, could be modelled on a

[15] Mathematically, they can be expressed as follow: in the case of *HOA*, the formula for the area of a triangle becomes $A\Delta = r^2 (\pi + \alpha + \beta + \gamma)$. In the case of *HAA*: $A\Delta = \pi - (\alpha + \beta + \gamma)$.

[16] Spherical geometry is a branch of geometry which is made on the surface of a sphere. On the surface of a sphere, an area of triangle is directly proportional to the magnitude of the sides of the triangle, as the sides get bigger so does the area. This is what Lambert observed when dealing with *HOA*. It is through the realization of the similarity between the geometry based on *HOA* and spherical geometry, he was able to conclude that the same property holds for the triangles in the geometry based on *HOA.*

sphere with an *imaginary radius*[17] (Wolfe, 1945, p.34). The last remark he made about the geometry based on *HAA* is that the figures, such as triangles and quadrilaterals, generated in it have an *absolute unit of length*[18] (Wolfe, 1945, p.34).

Both Saccheri and Lambert were on the same boat; they already had been pre-occupied with certain kind of tacit assumptions as to the nature of the fifth postulate prior to their investigations and their attempts to show its truth by virtue of reductio arguments. Therefore, neither of them were able to realize that they are on the verge of discovering a new territory in geometrical landscape.

The situation took a completely different course with Bolyai, Lobachevsky, Schweikart and Gauss. They were indebted to the works of Saccheri and Lambert in that both Gauss, Bolyai and Lobachevsky began their investigations with the utilization of the reductio method first tried by Saccheri and Lambert. Unlike, however, Saccheri and Lambert, their mind were more open to embrace the new evolution of geometry. It seems as though these mathematicians were no more under the spell of the dogma of the centuries, and had their gaze fixed on a new landscape. There were many reasons which delayed the admission of a new geometry. One of the important reasons which delayed it was of course the orthodoxy of Kant's theory of space among philosophers and scientists. Nobody was ready to give credence to a

[17] Imaginary sphere is a sphere with a radius of an imaginary quantity, which is usually denoted in mathematics by the symbol 'i'. In connection with the formula for the area of a triangle in the geometry based on *HOA* and *HAA,* one can obtain the formula for the area of an acute triangle by simply substituting 'r' in the formula $r^2 (\pi + \alpha + \beta + \gamma)$ with 'i' the square of which is equal to -1. One obtains, thus, by making the appropriate substitution, the formula for the area of an acute triangle, $\pi - (\alpha + \beta + \gamma)$.

[18] This means that as one is provided an information as to the angles of a given triangle or a quadrilateral, one is likely to find its absolute length. This has certain consequences; unlike in Euclidean geometry, the objects do not have scalable properties, and the congruence relations do not hold for the same kind of object across different scales in the geometry based on *HAA*. In Euclid's Elements the constructed figures are always in a constant relationship independent of their scale. This is no longer true in the latter geometry.

system of geometry which was completely at odds with the picture of space drawn by Kant. Another reason was that the rate of transfer of ideas took place relatively slowly compared to the rate at which they are transferred today. It took years for the discovery made in the one portion of the earth to reach other portion of it. And thirdly, the pre-eminence of Euclidean geometry and its successful inheritance over almost two thousand years. Notwithstanding all these factors, things began to change with Bolyai, Lobatchevsky and Gauss.

Gauss published, compared to the publications made about the possibility of new geometry, almost no substantial work. But his lifelong interest as to the subject matter can be traced from the letters he exchanged with Farkas Bolyai, F. A. Taurinus, and many others. It was Gauss who first recognized, along with Lobatchevsky, Schweikart and Janos Bolyai, the geometry based on *HAA* as a new geometry, and was the first person to call it *Non-Euclidean geometry* (Wolfe, 1945, p.46). In a letter to written at Göttingen on November 8, 1824 to Taurinus, he stated that "the assumption that the sum of three angles is less than 180° leads to a curious geometry, quite different but thoroughly consistent." (Wolfe, 1945, p. 46) And he also mentions about a *constant* he discovered which, when taken infinitely large, makes the new geometry approximates to Euclidean in the same letter to Taurinus. He also states that all his attempts were in vain to discover a contradiction in this new system. His meditations on these issues led him to the idea that space is something utterly mysterious to us and led him to adopt an empiricist theory of space. To quote from the same letter:

> But it seems to me that we know, despite the say-nothing word-wisdom of the metaphysicians, too little, or too nearly nothing at all, about the true nature of space, to consider as *absolutely impossible* that which appears to us unnatural. But if this Non-Euclidean geometry were true, and it were possible to compare that constant with such magnitudes as we encounter in our measurements on the earth and in the heavens, it could then be determined *a posteriori*. Consequently in jest I have sometimes expressed the wish that Euclidean

> geometry were not true, since then we would have a priori an absolute standard of measure. (Wolfe, 1945, p. 47)

The idea that Euclidean geometry may not be the true geometry of space, and in fact, may be a chapter of another geometry which encompass it, was not novel to Gauss. Karl Ferdinand Schweikart, in 1818, already developed a system in which there were two kinds of geometry; Euclidean and *Astral* (Halsted, 1900, p. 251). The sum of the interior angles in the former adds up to two right angles whereas in the latter to less than two right angles. He, like Gauss, talks about the same *constant*[19] that belong to astral geometry which, when taken large enough, yields Euclidean geometry. This, in fact, makes the first explicit description of a non-Euclidean geometry, which is made prior to the discovery of it by Janos Bolyai in the year 1923, and Nikolai Lobachevsky in 1926.

The first publication made that established non-Euclidean geometry as a consistent system of geometry was made by Janos Bolyai in the year 1923. Janos Bolyai was the son of Farkas Bolyai, who also had long been occupied with the problem of parallels. Janos Bolyai also started, like Saccheri and Lambert, his attempts to prove the fifth postulate by negating it. But prior to that, he changed the fifth postulate as had been formulated by Euclid with Playfair's[20]. The denial of the postulate, thus, implies that either *no* parallel lines could be drawn through a given point to a line or *many* parallels could be drawn (Wolfe, 1945, p. 50). Bolyai eliminated

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[19] The letter in which Gauss talked about such a constant is that which was exchanged with Franz Taurinus, who was the nephew of K. F. Schweikart and whose attention was first directed to these matters by his uncle Schweikart. So, Gauss may have first acquainted with the idea of such a constant through Schweikart.

[20] Playfair's axiom is an axiom which was preferred to the fifth postulate in Euclid's Elements. The axiom is stated by Playfair as follows: "In a plane, given a line and a point not on it, at most one parallel to the given line can be drawn through the point" (Playfair, 1846, p. 29). The term "at most" is added to the original postulate to show that the parallel line drawn is *unique*.

the former hypothesis by arguing that it contradicted with the 27th and 28th propositions[21] in Euclid's Elements (Wolfe, 1945, p. 50). The consequences drawn from the second hypothesis, however, never ceased to amaze Bolyai for a very long time. He concluded that if there can be drawn two lines that are not parallel through a given point to a given line, then there must be infinitely many lines parallel to a given line (Wolfe, 1945, p. 50). As he worked upon the second hypothesis, he realized that a consistent geometry has risen from the assumption of its existence; he could not find any contradiction in this new system he is dealing with. He published all of his ideas and formulations in the book named *Appendix* in the year 1832.

Independently of the discoveries of Janos Bolyai, Nikolai Lobachevsky already reached similar conclusions in the year 1826 in Kazan. The results he obtained was similar to that of Bolyai; he discovered that a geometry exists in which there can be drawn, to a given line and through a point, more than a single line. The interior angles of the triangles constructed within that geometry is also less than 180° (Wolfe, 1945, p. 52).

The methodology used by Bolyai and Lobachevsky was the same as that used by Euclid; they were using what is called synthetic method in which they proceeded from certain axioms, postulates and definitions towards theorems. It is a progressive method developed by Euclid. With Riemann, an analytical method is developed in the treatment of space. The synthetic treatment of space required that the geometrical character of space could either be described by the propositions of Euclidean

[21] In the proposition 27, it is proven that if two straight lines intersected by a third line making the alternate interior angles congruent to each other, then the two straight lines must be parallel. It is provided by the second postulate that two straight lines can be extended indefinitely, therefore the proposition 27 clearly shows the existence of parallels which contradicts with the first assumption that there exist *no* parallels to a given line.

geometry, or by the propositions of the new geometry developed independently by Bolyai, Lobachevsky, Schweikart and Gauss. More and more, especially with Schweikart, Taurinus and Gauss, it is believed that the geometry that is fit to describe the astronomical scales is astral geometry, and Euclidean geometry is nothing but a special case of this geometry. The assumption is grounded upon the discovery of the constant described by Schweikart which, when taken to infinity, yielded Euclidean space. Thus astral geometry seemed to have included Euclidean space in itself. But with Riemann, space is completely rid of a particular geometrical character. In his *On the Hypotheses which lie at the Foundations of Geometry*, he stated the following:

> It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor *a priori*, whether it is possible… The reason of this is doubtless that the general notion of multiply extended magnitudes (in which space-magnitudes are included) remained entirely unworked. I have in the first place, therefore, set myself the task of constructing the notion of a multiply extended magnitude out of general notions of magnitude. (Riemann, 1873, p.1)

Riemann endeavors to construct the space from the general notions of magnitude and he adds to that that the possibility of obtaining measure-relations for a *continuous manifoldness*[22] rests upon *measurement* shortly after this passage. Given that space is a *continuous triply-extended magnitude*[23], what follows strictly from this is the following: the general character of space cannot be obtained through axioms, it must

[22] A collection of points or elements (objects, entities) that has the structure of a multiply extended magnitude

[23] A concept with an associated fixed number of magnitude concepts, each of which must be specified according to its mode in order to individuate and identify an instance of that concept; ordinary physical space, e.g., is a triply extended magnitude, because it needs three spatial lengths (coordinates, say, in a fixed coordinate system)

be obtained through measurement. As was brilliantly said by Reichenbach; "Riemann's extension of the concept of space did not start from the axiom of the parallels, but centered on the concept of metric." (Reichenbach, 1958, p. 7).

Riemann's theory is grounded upon the Gauss' studies on curved surfaces and can be characterized as a brilliant extension of it. According to Gauss' *Theorame Egregium*, the curvature of any surface, which amounts to the deviation of a curve from a plane, can be completely determined within the two-dimensional surface alone without embedding the surface in a higher dimensional space. What Gauss had in mind was a geometry akin to practical geometry in which one could find how much the surface upon which one stands is deviated from being planar by making measurements with rigid rods. This is called as the intrinsic curvature. If one finds that the ratio of the circumference of the circle he measured with ones measuring rods to its diameter is greater than *π,* then one could conclude that one is standing on the surface of a sphere, *if it is less than* π, on the surface with a saddle-shape, if it is exactly π, on a planar surface. So each surface is characterized according to the measurements obtained with these rigid rods. Riemann can said to have advanced Gauss' theory in that he began with manifolds, for there existed possibility, for Riemann, that any two surfaces had different curvature. This is why he abolished the view that the geometric character of the all-inclusive space is given prior to the determination of the measure-relations between manifolds which constitute it. Reichenbach commented on the procedure adopted by Riemann as follows:

> Riemann showed that it is not necessary to develop an axiomatic system in order to find the different types of space; it is more convenient to use an analytic procedure analogous to the method developed by Gauss for the theory of surfaces. (Reichenbach, 1958, p. 9)

Riemann looked like an adherent of empiricism about space, perhaps similar to Schweikart, Taurinus and Gauss. What differentiated the views of Riemann from the others are his statement that there are different measure relations which can hold for space. Euclidean geometry establishes one system of measure-relations; so it is only a hypothesis used by mathematicians. So far, mathematicians preferred to use Euclidean straight lines and segment for measurement, but Riemann thinks that the measurement could take place with totally different lines and segments. This means that there is no true and unique metric which we can employ to characterize the geometry of space; space is *metrically amorphous*. With regard to this, Riemann wrote:

> It will follow from this that a multiply-extended magnitude is capable of different measure relations, and consequently, that space is a particular case of a triply-extended magnitude. But hence flows a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other triply-extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure relations of space may be determined; a problem from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure relations of space… (Riemann, 1873, p.1)

There are many measure-relations which can be used for the determination of the metric of the space, and Riemann thought that it is only through experimentation that one can determine those measure relations. Mathematicians must always expand their system and try different measure-relations. The determination would never be exact, for we are in the domain of empirical science. It is only through the extension of the variety of measure-relations we can get to know, of course again within the range of probability, the true metric of space. We must extend the systems of different measure relations to infinitely small and infinitely big and, in the light of experimentation and test their validity (Riemann, 1873). But possibly contrary to his thesis as to the amorphousness of space in terms of its metric, Riemann held that the infinitesimal distance between two points are given; so, in a sense, as, his views can be taken as a confirmation that at the infinitesimal level, space is Euclidean.

Another remarkable achievement of Riemann is that he discovered another kind of non-Euclidean geometry with a constant curvature. This new geometry is obtained by the negation of the second postulate in addition to the negation of the parallel postulate. The new geometry thus formed is finite but unbounded, for one can walk indefinitely in the same direction without being brought to a halt. In it, every straight line converge at two antipodal points. Therefore, there exists no parallels in Riemann's geometry, whereas in Lobachevski's and Bolyai's geometries, there exists infinitely many parallels. It can be said that Riemann's geometry is a "spherical geometry extended to three-dimensions" (Poincaré, 1905, p. 38). So it naturally follows that the surface of a sphere provides a *model* to Riemannian geometry in two-dimensions. Riemann's geometry serves as the axiomatic foundation for the spherical geometry of two and higher dimensions.

These new geometries throw a serious challenge to Kant's theory of space and geometry. If the axioms of the Euclidean geometry were synthetic a-priori, how is it that alternative geometries can in fact be conceived? The advancements initiated by Saccheri and ended up in Riemann called into the question the soundness of Euclidean geometry; more and more, the mathematicians adopted an empiricist approach, and thought that space could in fact be non-Euclidean. In the next chapter, it is going to be discussed how all of these advancements in the field of geometry impacted Kant's theory of space and geometry.

CHAPTER 5

THE IMPACT OF NON-EUCLIDEAN GEOMETRIES TO KANT'S PHILOSOPHY OF MATHEMATICS

5.1. The Possibility of Non-Euclidean Geometries in Kant's Philosophy

The attractiveness of the thesis that space is transcendentally ideal and is dependent upon the subjective constitution of our mind was lost after the advent of non-Euclidean geometries. The discovery of non-Euclidean geometries almost rendered Kant's theory of space and geometry obsolete. The consistency and the fruitfulness of these alternative geometries provided philosophers and scientists a ground for their presumption that non-Euclidean spaces can be 'conceived', or 'thinkable'. These new geometries are indeed thoroughly thinkable, and free from contradiction. But does this make these new geometries conceivable in the sense that Kant uses the term *'conceivability'*? How should a Kantian react to the thesis that non-Euclidean geometries are as equally conceivable as Euclidean geometries? Apparently, Kant himself did not deny that we can *reason* about everything as long as our reasoning is not brought to a halt by contradictions. He claimed "I can think whatever I please, provided only that I do not contradict myself." (Kant, 2007, Bxxvii/Bxxviii) From the mere *logical possibility*, Kant claimed, the *objective validity* cannot be ascribed to the concept. To ascribe an objective validity to the concept, something more is needed; something in the experience must correspond to the concept. If there exists no object which correspond to the concept, than the concept is empty; it only stands for a possible experience, which is yet to be actualized. The following passage can be given as a contra-thesis to the proponents of the view that non-Euclidean geometries are as equally conceivable as Euclidean geometries:

> It is, indeed, a necessary logical condition that a concept of the possible must not contain any contradiction; but this is not by any means sufficient to determine the objective reality of the concept, that is, the possibility of such an object as is thought through the concept. Thus there is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concepts of two straight lines and of their coming together contain no negation of a figure. The impossibility arises not from the concept in itself, but in connection with its construction in space, that is, from the conditions of space and of its determination. And since these contain a priori in themselves the form of experience in general, they have objective reality; that is, they apply to possible things. (Kant, 2007, B268)

From what Kant said here, it can be inferred that there is a difference between *logical possibility* and *intuitive plausibility*. Every intuitively plausible concept must also be logically possible, but not vice versa. As long as the given concept cannot be constructed in pure intuition, the concept remains empty. It must be recalled from what is laid out in the second chapter that *objective reality* cannot be secured only through constructing the given concept in pure intuition. At the end of the construction procedure, a particular object must be realized in the experience in conformity with the constructive action of the geometer in order for that concept to gain an objective reality. After all, the construction, as a *schema of a given concept,* is a function which acts a *norm* that provides a recipe for subsuming particular objects under their respective geometric concepts; it is in virtue of the normative function that the image is connected to the geometric concept. At the basis of Kant's thesis, there lies the argument that within our experience, neither we can confront non-Euclidean relations within our experience; nor we can construct any geometrical object that deviates from the Euclidean characteristics in our imagination.

5.2. Helmholtz, Poincaré and Conventionalism in Geometry

The possibility of the intuitive plausibility of non-Euclidean geometries were first realized and extensively treated by Helmholtz and Poincaré. The views of these two thinkers were not identical, but what was common in both of them was their effort to provide a psycho-physiological genesis for the foundations of geometry. In a sense, this means a return to the empiricist programme. But even though Helmholtz can said to be committed to the empiricist programme about the genesis of geometry[24] by stressing the importance of the environment in which the species is embedded and its impact on the species in its acquisition of a particular geometry, Poincaré, by partially building up on the accounts of Helmholtz, offered a completely novel epistemological category to account for the foundations of geometry. The new epistemological category introduced by Poincaré is entitled as *conventionalism*.

Conventionalism aimed to incorporate the empirical and rational elements to account for the genesis of geometry; but not in the sense that Kant incorporated them. For Poincaré, the propositions of geometry were not synthetic a-priori truths, for if it were the case, then "they would be imposed upon us with such a force that we could not *conceive* of the contrary proposition." (Poincaré, 1905) Here again, the reader is confronted with the term *conceivability* of the propositions of non-Euclidean geometry. For Poincaré, the comprehension of the propositions of non-Euclidean

[24] Helmholtz shared his views about the foundations of geometry in his short article *On the Origin and Significance of Geometrical Axioms (1870).* The article was included in the book *Epistemological Writings (1921)* which was edited and published by positivist philosophers; Moritz Schlick and Paul Hertz. More about the relevant content about these publications will be discussed in the subsequent chapters.

geometry is not only expressed as our ability to reason about them without running ourselves into any contradiction. He offered many thought experiments in which he depicted fictive worlds to show that different environmental conditions would compel us to reinterpret the primitive geometrical terms. This in turn makes possible the comprehension of different geometries for sentient beings who are equipped more or less with same kind of sense organs as ours. This, in effect, reflects the impact of Helmholtz's psycho-physiological arguments on Poincaré's philosophy of geometry. Helmholtz, before Poincaré, gave similar arguments, in his *On the Origin and Significance of Geometrical Axioms (1870),* in which he argued that the different environmental conditions would inevitably cause species like us to adopt a different geometry. Neither it was the case that the propositions of geometry were experimental facts, for "we do not make experiment on ideal lines or circles, we can only make them on material objects", (Poincaré, 2011) they were conventions; and our choice of one particular geometrical over another is carried out in the guidance of the nature. Nature does not dictate which particular geometry is to be chosen to account for the phenomena, nature can only be suggestive of which particular geometry is to be chosen.

Poincaré offered an exhaustive list of empirical and a-priori conditions to account for the constitutive factors in the genesis of space and geometry. The empirical conditions for a species to come up with the idea of space and geometry can be divided into two categories; *subjective* and *objective conditions*[25]. The subjective conditions *are* to have a *body* and *mobility,* and the objective condition is the *possibility of the*

[25] This division I announced here does not mean that subjective conditions are not objective; it only means that they are the conditions which is related to the experiencing subject, that is to say, the conditions must be satisfied by the subject.

motion of the invariable figure. Poincaré, first, begins by analyzing our sensations and how they contribute to the idea of space. He affirms, just like Kant did in his *Transcendental Aesthetic*, that “our sensations cannot give us the notion of space”, and by themselves “they have no spatial character” (Poincaré, 1898). But, unlike him, Poincaré thought that in order for an organism to have an idea of *space* and in turn be capable of doing *geometry*, the organism, first of all, must be capable of moving. He clearly acknowledges it when he says “For a being completely immovable, there would be neither space nor geometry.” (Poincaré, 1905, p. 48)

The origins of the idea of space depends upon the reciprocal relations formed between the subject and the object. There are some *external changes*, in which it is possible for the subject to restore the aggregate of primitive sensations through performing certain locomotor actions, and the idea of space is predicated upon our ability to compensate for the *external changes* through respective *internal changes.* It is through these compensatory acts and performed *displacements* that an organism gets to know the spatial relations and thus forms the idea of space. An organism incapable of performing certain displacements would not even know the very primitive spatial relations such as *contiguity* or *distance*. This is at odds with the Kantian thesis which claims that “space must already be presupposed in order for the appearances are represented as alongside one another” and ordered accordingly (Kant, 2007, B38/B39). This, however, would be totally meaningless to that organism according to Poincare. Poincaré, in his *Foundations of Geometry (1898),* wants to imagine us a hypothetical person “who possessed but a single immovable eye” (Poincaré, 1898). This man is completely paralyzed, and one of his eyes is blind. The other is not blind, but he is not able to move it at his will. For Poincaré, this man would not be able formulate these mentioned relations and he emphasizes that the origin of the idea of space as a framework of relations could be traced back to our capability of moving, and our capability of retaining the aggregate of primitive relations through performing certain *displacements.* Within this context, focusing on an object with our eyes and capable of following it continuously is an example of a displacement that we make with our eyes. Since our hypothetical man is incapable of performing these movements, the changes he spots upon his retina cannot be categorized spatially; because there remains no possibility for our man to retain his old impressions back.

The idea of the motion of an invariable figure plays an equally important role in the genesis of the Euclidean geometry and in the birth of the idea of space. Every change we observe in the nature is either a *change of position* or a *change of state*. The former category is comprised of changes that the solid bodies generally undergo. In order for compensation to be possible, Poincaré says "the external object in the first change must be displaced as an *invariable solid* would be displaced." (Poincaré, 1905, p. 60). He emphasized the importance of the possibility of free-mobility of objects, and said "if, then, there were no solid bodies in nature, there would be no geometry." (Poincaré, 1905, p. 61). *Changes of state*, on the other hand, can be exemplified by the chemical reactions of various sorts, or the displacements of fluids, which cannot be compensated for by making internal displacements. Since the motion of an invariable figure was not explicitly stated as an axiom by Euclid, Poincaré considered it to be an *implicit axiom* which was used by Euclid to provide a ground for the possibility of establishing *congruence relations.* In Euclid's system almost every proof is predicated upon the notion of congruence. Helmholtz himself said that "the foundation of all proofs in the Euclidean method is the proof of the congruence of the relevant lines, angles, plane figures, bodies, etc." (Helmholtz, 1977, p. 3) But this axiom, for Poincaré, is evidently disguised in the fourth axiom of *The Elements*, which is not explicitly stated by Euclid, but utilized nonetheless. The fourth axiom seeks to show that two figures are congruent if one can be superimposed on top of the other. But moving a figure in space in such a way requires that the figures conserve their shape while moving. This, for Poincaré, blends Euclidean geometry with an empirical element; *motion*. The alleged purity of the geometrical practice is thus stained by the introduction of a physical element. In contrast to Kant, Poincaré did not differentiate between *pure* and *empirical motion* as Kant did. For Poincaré, motion cannot be determined a-priorily, for we do not know *a-priori* that the motion of objects that we observe within our environment obey the *group of Euclidean displacements*; they could just as well obey to another *group of non-Euclidean displacements*.

Here, the summit of the thesis of Poincaré is reached, the ultimate empirical condition necessary for the genesis of geometry is nothing other than the reciprocal relation formed between the subject and the object. Subject contributes to it through performing certain *internal displacements* which are aimed to compensate for the change caused by the *external displacements* of an object. Together, they form a *displacement group*. This is why Poincaré held that alternative geometries are as equally *conceivable* as Euclidean geometries, for in a *possible world*, the solid bodies may obey to different laws of displacements than the ones to which have been long accustomed to observe. Perhaps the most famous thought experiment provided by Poincaré is his *Sphere-world* which he gave in his *Science and Hypothesis* (1903). This sphere world is in an imaginary world governed by different laws. The properties of the sphere-world are listed down by Poincaré as follows:

1. The world is enclosed within a sphere

2. Temperature is not uniform; it gradually decreases as we move away from the center of the sphere. The absolute temperature is proportional to $R^2 – r^2$

   R = Radius of the sphere

   r = Distance of the point considered from the origin

3. Each body has the same coefficient of dilatation, so that each body shrinks or expands in the same proportion as they move
4. The law of refraction is inversely proportional to $R^2 – r^2$
5. This means that the path of the ray of lights are not linear, they are circular.

The sentient beings, Poincaré argues, would cultivate a geometry different from ours; their geometry would be a non-Euclidean geometry. Imagine that two people; P_1 and P_2, are transferred into sphere-world from our world; P_1 and P_2, for so long, have been habituated to use Euclidean geometry, so the geometry that they have so long accustomed to is Euclidean geometry. The question is the following: would they notice the difference between those two worlds? If so, how? First of all, as Poincaré stated, the world would appear as infinite for those beings, whereas from our perspective, it appears as finite. The reason is that bodies shrink as they move away from the origin, and this makes the periphery not approachable for those beings, which would make them think that their world is infinite. P_1 and P_2 would not so easily be able to detect the effect of shrinking and expanding through measurement. Because every time they wanted to measure an object which moves far away from the origin, they would have to move away from the origin to reach that object and superimpose their measuring rod on top of that object, so their measuring rods, together with their bodies, would shrink in the same proportion as the body that they wanted to measure.

But is not there, then, a way to find out that the world in which P_1 and P_2 are embedded is different than their previous world? They would have to consider their *tactile* and *visual* impressions in the sphere-world and compare them with that in our Euclidean world. The realization of the differences between their tactile impressions in the sphere-world and those in the Euclidean world demands that they make certain experiments with their bodies or rigid rods. Suppose that P_1 and P_2 are located at the origin of the sphere. Let P_1 make a 90^o counter-clockwise rotation to the left and let P_2 make a 90^o rotation to the right. After each completes his rotation, let them walk ten steps in a straight line and stop and rotate their bodies back to their initial orientation. This indicates that they walked away from one another for about twenty steps, stopped, and reconfigured their bodies back to their initial orientation. Lastly, let each of them walk ten steps in a straight line again for the last time. This imply that they walk in parallel lines; for both are separated from one another for about twenty steps, and walking towards the same direction. Now if they want to measure the distance in between them, they have to walk towards one another and count their steps. They will notice that the number of steps that needs to be taken is far greater now, for they both moved away from the origin about ten steps and their bodies shrank as they moved, so their steps will be much smaller when they are away from the origin, and in turn the distance they measure when they move towards one another will be much greater. If those two people were transported to that world from a Euclidean world, they would be astonished, for they knew that in the Euclidean world, the parallel lines are equidistant to one another everywhere. But here, even though they moved parallel to one another, they see that the distance between them gets larger and larger as they move away from the origin. So, just like in hyperbolic geometry, the distance between two parallel lines in this world does indeed change; the more they move away from the origin, the more space there becomes in between them.

There are other ways to find out the differences between the sphere world and our Euclidean world. Through the measurement of the ratio of the circumference of a circle to its diameter with physical rods, those people would likely to discover that it is bigger than π; and as the diameter increases the ratio exceeds π more and more. The reason behind this is simple; as the physical rods move away from the origin, they shrink, so measuring the circumference with smaller rods means that more rods can placed on it compared to the number of the rods that could be placed on it if the diameter were smaller. This would in turn mean that the measurement of the circumference would yield exceedingly large values compared to the value of the diameter, and when the ratio is thus taken, this would imply an excess compared to the ordinary ratio between circumference and diameter.

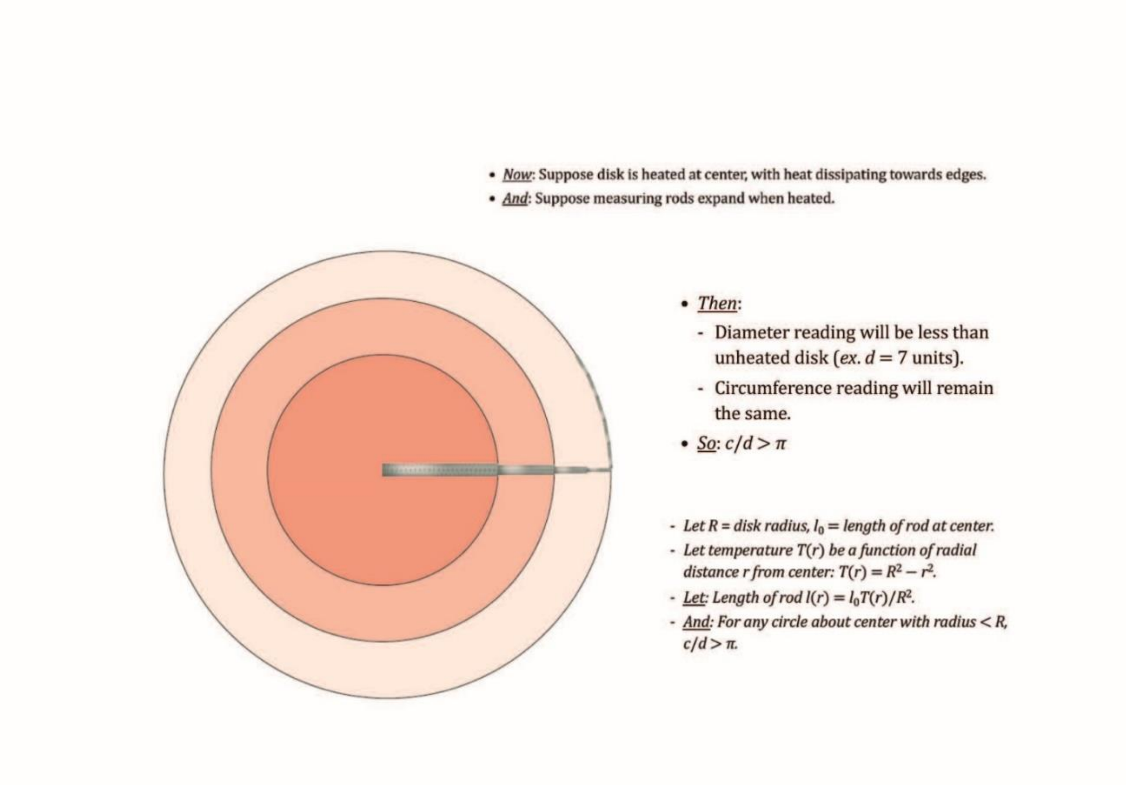

Figure 3

What about the visual impressions of those people in the sphere-world? How would the visual impressions of P_1 and P_2 be different from those they used to experience in their previous Euclidean world? Is the sphere-world been qualitatively identical with the Euclidean space? So far, considering only the tactile impressions of P_1 and P_2, the space in which they live display the characteristics of the hyperbolic space; in hyperbolic space the ratio of the circumference to its diameter is bigger than π and the parallel lines are not equidistant. So the question can be translated to the following: how different the visual impressions of beings like us in a hyperbolic space? Helmholtz, in his *The Origin of the Geometrical Axioms*, offered a detailed explanation about it. He said that the most distant objects of this space would be seen at a finite distance to the observer, but the distance between those objects and the observer appears to be dilated as the observer moves towards these objects. This means that two physical lines that are remotely placed relative to the standpoint of the observer appears to be parallel at first sight. But as the observer moves towards these physical lines, he would see that those lines bulge outwards, and the distance between them is increased. (Helmholtz, 1977). Considering what Helmholtz said about the visual estimations of the subject, it is evident in these passage that the visual impressions of P_1 and P_2 would also be different than that they had in their native world. Considering the difference both in their tactile and visual sensations in the sphere-world, can it be concluded that these people would necessarily adopt the hyperbolic geometry in explaining the succession of their impressions in that world? It is clear that the solid bodies that P_1 and P_2 encounters in the sphere world are totally different than the bodies that they experienced in their ordinary world. Despite the difference in their tactile and visual impressions, they would still regard the bodies that they encounter in the sphere-world as solid, for they are able compensate for the external changes through performing certain internal changes. But it would be a hasty conclusion to say that they would adopt hyperbolic geometry, because then “geometry would be only the study of the movements of solid bodies” (Poincaré, 1905, p. 70). Poincaré stated that experience could only guide us in our choice of a particular geometry, it could never dictate to us which geometry to choose among alternatives. This brings us to the role played by the a priori elements in choosing a particular geometry. Regarding this, Poincaré wrote:

> The object of geometry is the study of a particular "group"; but the general concept of group pre-exists in our minds, at least potentially. It is imposed on us not as a form of our sensitiveness, but as a form of our understanding; only, from among all possible groups, we must choose one that will be the *standard*, so to speak, to which we shall refer natural phenomena. (Poincaré, 1905, p. 70)

The a-priori element is the notion of *group*. Poincaré aimed to provide a group-theoretical foundations for both space and geometry. Among various groups, we are particularly interested in *displacement groups* to obtain the idea of space, and geometry is nothing other than a particular choice of a displacement group among the existing alternatives. Poincaré underscored that the notion of a group does not belong to our *form of sensibility*; it belongs to our *form of understanding*. This is important, for it separates Poincaré’s philosophical position from Kant. For Kant, space is a form of sensitiveness which precedes all the data provided to us by our senses. Poincaré thought that our capability of forming a network of relations is not due to our form of sensitiveness but due to our having the idea of group in the first place. “What mathematicians call a group is the ensemble of a certain number of operations and of all the combinations which can be made of them” said Poincaré (Poincaré, 1898, p. 13). Space and geometry owes its existence to these specific operations and the combinations of those operations that we are able to make. The very idea of the compensation of our aggregate of sensations, is grounded upon the idea of a displacement group. Our ability to compensate for an external change is taken as the fundamental group operation. The idea of compensation is not taken from experience, for experience *roughly* informs us that the sensations that we feel at t_1 are retained after making necessary displacements at t_2. But the very idea of making compensatory acts arises from within, and this alone is the condition of the possibility of classifying our sensations.

The set of operations Poincaré gives as examples can in fact be explained by using the language of group theory. To give an example, consider an aggregate of sensations that I receive through my thumb, A at t_1. Consider that an internal displacement S takes place at t_2, and this makes me feel the same set of sensations with my index finger at t_2. So at t_2, my index finger is now feeling the set of sensations A. Now consider that at t_3, an external displacement R takes place and makes my index finger feels an aggregate of sensations B with my index finger. Now I observe that, through making an *inverse displacement*, S' at t_4, I bring my thumb in the place of my index finger and my thumb feels now the aggregate of sensations B. Thus S' becomes the *inverse displacement* of S. *Inverse element* is one of the axioms of group theory and its utilization is nowhere limited to displacements; even in manipulating algebraic quantities that are totally devoid of spatial significance we use the same group structure. This is why the idea of group belongs to the form of understanding for Poincaré. Heinzmann declared it to be an *algebraic intuition*[26] which is applicable to our sensations and is useful in classifying them.

Even though it was first mentioned earlier as an empirical condition of the possibility of both space and doing geometry, the idea of the possibility of the motion

[26] See, Heinzmann (1999)

of an invariant figure is not something that we derive from experience directly. It is true that experience provides us with solid bodies that are invariant under any displacement, but solid bodies must not be confused with *rigid bodies*. Rigid bodies are idealized solid bodies. Solid bodies must be taken as approximations of rigid bodies. The difference between solid bodies and rigid bodies are made clear brilliantly by Hans Reichenbach in his *Philosophy of Space and Time*[27]. In brief, Reichenbach claimed that a rigid body is a solid body whose minute deformations can be ignored. This definition squares with Poincaré's intentions. Poincaré was aware of the fact that the mind intervenes and eliminates those minute deformations present in solid objects in creating *ideal objects*; and geometry does not study solid objects, it studies those ideal objects. This is where the opinions of Poincaré is departed from that of Helmholtz. Helmholtz thought that the idea of the motion of an invariable figure is obtained directly from experience. The following passage aims to exhibit the views of Helmholtz with respect to the origin of the possibility of the motion of an invariable figure:

> If, however, we want to build necessities of thought upon this assumption of free mobility of fixed spatial structures with unaltered form towards every part of space, then we must raise the question whether this assumption does not involve some logically undemonstrated presupposition. We shall presently see that it does in fact involve such a presupposition--and, indeed, one very rich in consequences. But if it does so, then every proof of congruence is based upon a fact taken only from experience. (Helmholtz, 1977, pp. 4-5)

Poincaré, in contradistinction Helmholtz's views, was aware of the fact that nature can only provide us with approximately rigid bodies. It is the mind which acts upon these approximate sensations and convert them into ideal ones. So the possibility of an

[27] The complete definition is given as follows: "Rigid bodies are solid bodies which are not affected by differential forces, or concerning which the influence of differential forces has been eliminated by corrections; universal forces are disregarded" (Reichenbach, 1958, p. 22) I refrained from providing the original definition in the text for I believed that certain terms that are used in it; such as 'differential forces' and 'universal forces', are apt to create more confusion than to clarify the point I tried to make. These terms will be expounded upon in the next section.

invariant figure is known a-priori in geometry, it is not attained from experience. This is one of the a-priori elements of geometry without which the practice of it becomes impossible.

Poincaré's conventionalism allowed him to ridicule the question whether the space is Euclidean or not. This question, for Poincaré, has no meaning, for one geometrical structure cannot be *true* or *false*, it can only be more *convenient* (Poincaré, 1905, p. 50). Experience can neither refute, nor verify the Euclidean geometry. Even if, as perhaps believed by the discoverers of non-Euclidean geometry such as Taurinus, Gauss and others, one day it will turn out that the *parallax* of a distant start is different than it currently is, the practitioners of geometry and science will be faced with two options; either they will give up on the Euclidean geometry and adopt non-Euclidean geometry, or they will give up on the law of optics which state that a ray of light propagates in a straight line and retain the Euclidean geometry. So his conventionality thesis is centered on the interdependence between physics and geometry. In the light of new experiments and observations, the current geometrical structure used in science may require a modification. But scientists will always be free to choose whether the geometrical structure is going to be modified or the laws of physics are going to be modified. The choice cannot be dictated to him by experience, experience can only guide the scientists to choose the *simplest* and the most *convenient* geometric structure to explain the relation between phenomena. Poincaré thought that the scientists will always favor the Euclidean geometry over the alternatives, for it is the *simplest* and the most *convenient* geometric structure to explain phenomena.

In conclusion, Poincaré renounced Kant's theory of space and geometry. Space could not be a priori form of sensitiveness, for a human being incapable of producing necessary movements would not be able to possess the idea of it. One of the constitutive a-priori elements of space[28] is the idea of *group*, which pre-exist in us as a form of understanding. Geometry, cannot be a body of knowledge comprised of synthetic a-priori truths; for there are alternative geometries which can also be used to describe the spatial behavior of objects. So there is no necessity in singling out the Euclidean geometry to describe the relation between phenomena.

Even though Poincaré rejected Kant's theory of space and geometry, he nonetheless tried to remain, loyal to the Kantian terminology throughout his works and sided with Kant on the issue of the *content* of mathematics. Very similar to Kant, for Poincaré mathematics is not devoid of an intuitive content. He, too, believed that mathematics harbored extra-logical elements in it, and thought that it cannot be reduced to logic. He associated these extra-logical elements with our intuitions just like Kant. However, the term 'intuition' received very different connotations with Poincaré. The geometrical intuition, in the sense it was used by Kant, is likened to a sort of intuition which is fallible and unable to provide any certainty. Poincaré shared in his book *The Value of Science* the following quote taken from Royce's article, *Kant's Doctrine of the Basis of Mathematics*: "That very use of intuition which Kant regarded as geometrically ideal, the modern geometer regards as scientifically defective, because surreptitious. No mathematical exactness without explicit proof from assumed principles-such is the motto of the modern geometer." (Poincaré, 1958. p. 2) The Kantian style of construction of a spatial magnitude resulting from a successive spatio-temporal synthesis carried out in the transcendental imagination

[28] The notion of group is not the only mental capacity that plays a constitutive role in the genesis of space. Poincaré lists other powers of the mind which equally contributes to the genesis of space. The remaining capacities are the power of an indefinite repetition (principle of mathematical induction) and the idea of continuum. These are often mentioned as *intuitions*, the idea of group is expressed as the *algebraic intuition*, mathematical induction as *arithmetic intuition,* and continuum as *topological intuition* by Heinzmann in his article*, Poincare on Understanding Mathematics (1999)*

cannot rigorously account for the *continuity* of the produced magnitude in the sense that modern mathematics today demand it. Michael Friedman, in his *Kant's Theory of Geometry*, discussed that the existence of certain points that are used in the diagrammatic representations of certain propositions becomes problematic if the Euclid-style constructive procedure is chosen to generate them. The existence of such points can only be established by using *polyadic quantification theory*, to which Kant simply had no access. Friedman also stresses the fact that had Kant known the polyadic quantification theory, he would not have tried to base the origin of space in our pure intuition. If it is to be remembered from the second chapter of this thesis, Kant offered an argument to show that our space is a non-conceptual representation, for the mereological structure of it does not obey that of concepts, and the representation of a concept which contains *within it* infinitely many concepts could not be achieved with the tools of the logic. His argument was indeed brilliant, for Friedman, in showing the inadequacy of the *monadic logic*[29] in representing the *infinity*. Friedman wrote:

> We can now begin to see what Kant is getting at in his doctrine of construction in pure intuition. For Kant logic is of course syllogistic logic or (a fragment of) what we call monadic logic. So, for Kant, one cannot represent or capture the idea of infinity formally or conceptually: one cannot represent the infinity of points on a line by a formal theory […] If logic is monadic, one can only represent such infinity intuitively: by an iterative process of spatial construction (Friedman, 1985, p. 466)

But the discovery of *polyadic quantification theory*[30] availed logicians the opportunity to represent the infinite logically. What is at stake here is actually the representation of the *infinite divisibility*, for Kant seemed to have concerned himself with the

[29] Monadic logic is a branch of 1st-order logic that involves well-formed formulas constructed from a one-place predicate. Every well-formed formula involves a single argument about a single object in monadic logic.

[30] Polyadic logic, in contrast to monadic logic, involves well-formed formulas constructed from a many-place predicate. Every well-formed formula involves an argument about multiple objects, so the predicates in polyadic logic are essentially relational predicates, and the quantifiers denote the essential order relations among the variables that enter into the well-formed formula.

impossibility of the representation the constituents of a geometric line logically. He thought that such representation could only be achieved through an indefinite number of bisection, which is a synthetic activity of the geometer that takes place in our pure intuition. But modern quantification theory showed that the representation of *infinite divisibility* and continuity are in fact possible. Friedman said "what makes this representation itself possible is precisely the quantifier-dependence of modern polyadic logic." (Friedman, 1985, p. 474) Geometric intuition, when it comes to the representation of infinite divisibility, continuum and differentiability, has the potential to mislead and to err for Poincaré. Poincaré thus treats the *geometric intuition* of Kant as a *sensible intuition* by affiliating it to the productivity of our imagination, and accuses it for not being able to provide the rigor the *pure intuition,* such as the *pure intuition of number*[31], can give. Despite all that, he did not seek refuge in *formalism* either; he thought that formalizability of infinite divisibility and continuum does not tell anything about the true character of what continuum is and where its origin lies. Poincaré, in his *Last Essays*, stated that we have a direct *intuition of continuum*[32]. The idea of continuum is already pre-supposed by the logician and expressed as an axiom

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[31] *Pure intuition of number*, for Poincaré, is essentially a synthetic a-priori intuition. It is the awareness of our ability to iterate indefinitely. We use this intuition more than often in arithmetic and geometry when we want to generalize over particulars and prove a certain theorem by using mathematical induction. This ability, for Poincaré, cannot be reduced to logic, for it represents an infinite number of syllogisms link together in a series. This intuition is not reducible to logic, for logic cannot provide us how to pass infinitely many numbers of syllogisms to reach a general conclusion without recurring to this power of our minds

[32] Poincaré accuses Hilbert of using this intuition and treating as if it is an axiom of logic in his axiomatic treatment of geometry. *The axiom of order*, which was used as an axiom by Hilbert in his *Foundations of Geometry,* has its root in our *topological intuitions* for Poincaré. He wrote the following:

As to the axioms of order, […] they are true intuitive propositions relating to *analysis situs*. We see that the fact that the point C is *between* two other points on a line relates to the method of *cutting up* one-dimensional continuum with the aids of *cuts* formed by impassable points. (Poincaré, 1963, p. 43) of logic. But the truth is that the axiom is made possible only through the availability of that intuition.

5.3. Reichenbach and the Relativity of Geometry

Poincaré and Helmholtz strived vigilantly to establish the possibility of conceiving a different geometry in different environmental conditions. They meticulously strived for explaining how, on the basis of the data provided to us by our senses, we generate the web of relations called space whose geometrical character depends solely on the observed character of those relations. The role that our sensations play in the adoption of a particular geometry even became more noticeable and gained a pedagogical importance in the *possible worlds* that they have created. Within these possible worlds there were different set of spatial relations observed among bodies which are completely alien to us. It was first seen with Albert Einstein that that our actual world, turned out to be as bewildering as those possible worlds that Poincaré and Helmholtz generated in their thought experiments. Unfortunately, both Poincaré and Helmholtz could not live long enough to see Einstein's remarkable achievements. Einstein successfully implemented Riemannian geometry to our actual world and overthrown the old Newtonian conception which was built upon the system of Euclid. According to Einstein's general relativity theory (GRT), the light is bent when it travels close to a gravitational region, and the bending of light becomes more noticeable as the strength of the gravitational field is increased. This bending of light was something that replaced the old conception of straight line; the straight line in Einstein's universe was no more defined as was defined by the Euclidean geometry, and to give the name straight to the curvilinear path traced by a ray of light would mean to adopt non-Euclidean geometry.

Having witnessed the success of physics in describing accurately the spatial relations between objects by way of non-Euclidean geometry, logical positivists raised concerns similar to that of Poincaré and Helmholtz about the orthodox conception of the nature of space and geometry. The general complaint raised by the logical positivists, such as Schlick, Carnap and later Reichenbach, to Kant's philosophy of geometry is its failure to distinguish between *pure geometry* and *applied geometry*. The subject of pure geometry is the study of the logical relations between un-interpreted primitive terms. So it is a science which is concerned solely with derivability in accordance with the laws of formal logic. Every term that is used in pure geometry is devoid of any content; only the relations between these terms are concerned. Applied geometry, on the other hand aims to select a particular structure which best fits the data acquired by means of observations and experiments. To achieve that explanatory success, un-interpreted terms find their meaning in applied geometry. The terms "point", "straight line", etc. are no more devoid of meaning; each of them is successfully coordinated to a physical object. The distinction between pure and applied geometry can be boiled down to the distinction between *mathematical space* and *physical space*. Mathematical space is that in which the mathematicians work with possible spatial structures. They, as it were, deal with hypothetical spaces and hand them on to physicists whose job is to select among those hypothetical spaces the one which truly describes the physical space, that is to say, the space described by physics.

The general concept of space seems to be bifurcated into two distinct conceptions of space with the logical positivists after Poincaré. The idea of a *mathematical space* is nowhere spotted in Poincaré's philosophy, and it cannot be, for Poincaré did not give any credence to the possibility of the conception of the axioms of geometry independent of our sensations and the relations between sensations. For him, there is also no such thing as *physical space* in his philosophy, for there is no *true metric* which can we select to depict the spatial relations. Any metric will do the job, for the choice is always conventional. The introduction of this new dichotomy between mathematical space and physical is partly due to the success of Einstein's GRT in describing and predicting the phenomena and partly due to the work done by Hilbert in his *Foundations of Geometry*[33]. Einstein's impact, both as a scientist and a philosopher, on these philosophers cannot be underestimated. After all, in the era of positivists, Einstein's famous dictum; "as far as the laws of mathematics refer to reality, they are not certain; as far as they are certain they do not refer to reality" (Einstein, 1921) echoed and taken as a maxim of an utmost value.

Perhaps the most outstanding work in the philosophy of space and time was carried out by Reichenbach shortly after the reign of positivism over philosophy of science. Reichenbach shared the tenets of conventionalism and positivism, and provided a successful mixture of them. Unlike Poincaré, he did not believe that the choice of a particular geometry is purely conventional to describe spatial relations. However, similar to Poincaré, he thought that there is also a conventional ingredient in geometry, and it is the way congruence is defined. The geometry of the physical

[33] Hilbert thought that geometry is devoid of a particular content; according to his view, geometry is nothing but a system of relations between primitives that are not yet defined. In his *Foundations of Geometry*, showed that the constructive procedures deemed as necessary by Kant are just auxiliary tools and therewith not a necessary condition for proving any result in Euclidean geometry. Hilbert argued that it is because of the deficiency of the axiomatic structure of the Euclidean geometry that the geometers had to recur to diagrams and visible figures. In a rigorously established axiomatic system, there would be no need for any figure for Hilbert.

space can be determined only after the conventional definition of congruence is given; once the congruence is defined, the problem of the geometric character of space becomes an empirical problem. To define congruence, a physical object must be coordinated to the concept of unit length, this is called a *metrical coordinative definition* (Reichenbach, 1958, p. 15). Definitions in physics are different than that of mathematics, for in the former, the *definiens* is a physical object that do the job of defining the corresponding concept whereas in the latter, the *definiens* is generally another set of concepts that aim to define the target concept. The standard meter in Paris is coordinated to the concept unit length. This is a great example of a metrical coordinative definition. The completion of our coordinative definition of congruence requires the comparison of two unit lengths at different locations. Once the unit length is physically defined, what remains to be done is to define how the rod should behave when it is transported from one region to another. The definition of a *rigid body* is then predicated on the definition of the behavior of our measuring rod during its transport. The question that needs to be asked at this point is this: would not it suffice to consider our *factual observations* made distinctly at different places to conclude that the same rod is congruent to itself in different places? Reichenbach answers this question negatively; we cannot conclude from *observed facts* that two rods are congruent to one another at different places; to assume that they are always equal in length would only be an additional convention. But he also states that this conventional definition can be *empirically verified* through comparing the length of the rods measured at different places. This is why, Reichenbach stated that "one can say that the factual relations holding for a local comparisons of rods, though they do not require the definition of congruence in terms of transported rods, make this definition admissible". (Reichenbach, 1958. P. 17) In Poincaré’s sphere-world *gedanken*, it was shown that the same rod turned out to be self-congruent across transportation, for there were no noticeable change in its shape during its transport. However, the comparison of the measurement of certain ratios (such as π) has shown that the shape of the rod must have been altered during its transport. This is why we have to make, prior to the observations, a metrical coordinative definition; and this is why, for Reichenbach, the definition of congruence is “not a matter of cognition, but a matter of definition.” (Reichenbach, 1958. P. 17).

The aim of a coordinative definition of congruence is to eliminate *universal* and *differential forces*, and establish the possibility of empirically determining the geometry of the physical space. Universal forces are forces which affect all materials in the same way. Going back to Poincaré’s sphere-world, the uniform increase in the temperature is an effect produced by a universal force. Each body, in that sphere-world, is affected by the temperature equally, and this was expressed by each body having the same *coefficient of dilatation*. The local comparison of the lengths of the transported rods in sphere-world were not noticeable, this is why Reichenbach stated that it is “fundamentally impossible to detect changes that were caused by universal forces.” (Reichenbach, 1958, p. 16) A coordinative definition of congruence aims to eliminate universal forces, this is called, by Rudolf Carnap, *the principle of the elimination of the universal forces.* (Reichenbach, 1958, p. vii) This is the exact place where Reichenbach criticizes Poincaré's conventionalism; for Reichenbach, there is a disturbing element of arbitrariness in our choice of a particular geometrical structure in Poincaré's conventionalism, and he wanted to eliminate that point of arbitrariness by introducing his principle of the elimination of the universal forces. Once the universal forces are not admitted, a unique geometrical system can be chosen to describe our observations. Carnap says: “if this principle is accepted, the arbitrariness in the choice of o a measuring procedure is avoided and the question of the geometrical structure of physical space has a unique answer." (Reichenbach, 1958, p. vii) Differential forces, on the other hand, are forces which does not affect every material in the same way; different materials respond differently to differential forces. These forces also must be eliminated to reach the idea of *rigid body*. For it corrects the minute differences in each body produced by various internal and external forces. Through the elimination of the differential forces, we no longer consider those minute deformations in bodies as a change in the geometrical structure of geometry. If we do not eliminate differential forces, then we would have as many geometries as there are bodies which reacts differently to same forces (such as heat). This would unnecessarily overcomplicate the task of the physicists, so by definition, all differential forces are set to zero.

In conclusion, the determination of the geometry of the physical world depends on the coordinative definition of congruence, until then, the physical geometry is *indeterminate*. "The geometry of the physical space is not an immediate result of experience, but depends on the choice of coordinative definition of congruence." (Reichenbach, 1958, p.19) The criteria for selecting the most adequate definition of congruence is the same criteria that Poincaré embraced; *simplicity*, and *convenience*. However, Reichenbach argues that the scientist will not always select the theory which involves the simplest geometry, but which involves overall the simplest structure. This is what Einstein did in his GRT, he chose the simplest coordinative definition of congruence, not the simplest geometry to describe the relations between phenomena.

Reichenbach's own unique *conventionality thesis* implies that one is free to choose whatever geometrical structure one wishes to describe the physical space if universal forces are admitted. This analysis directly lead us to the *relativity of* *geometry*. Going back to sphere-world experiment again, the same set of observed relations can be explained in two different ways Let G_0 = Euclidean Geometry, G_1 = Non-Euclidean Geometry, F = Universal Forces that causes materials to shrink or expand. We can either say that the geometry of the sphere-world is Euclidean and there are universal forces which affects all the materials in it (G_0 + F), or we can say that the geometry of the sphere-world is non-Euclidean and there are no universal forces in it (G_1).

The relativity of geometry made Reichenbach renounce the Kantian thesis that the Euclidean geometry is *synthetic a-priori*. He did not believe that the Euclidean geometry is *epistemologically prior* to other geometries. However, it is possible to retain Euclidean geometry in every scenario, all we have to do is to choose between the set of possible coordinative definitions of congruence, the one which includes the Euclidean geometry. He listed the reasons which predisposes us to cling onto the Euclidean geometry in every possible scenario. I entitle these reasons as *visual preferability* and *local soundness*[34].

Notwithstanding the success of these criticisms of Poincaré and Reichenbach, and how they rendered Kant's philosophy of geometry obsolete, there were other group of philosophers; P. F. Strawson and Gottlob Frege being important representatives, who tried to rescue Kant's philosophy from these death-blows. Strawson's thesis was centered on the view that even though Kant's philosophy of geometry cannot truly describe the space described by physics, it still necessarily and universally holds for the *space of human visualization* and for *local space*. In essence,

[34] By *local soundness*, I mean the soundness and the validity of the geometry within a confined region of the entire universe.

he narrowed down the scope of Kant’s theory of geometry to encompass only the visual and local space.

To begin with, it must be noted that Strawson seemed to have endorsed the existence of the *physical space*. He clearly stated that we have a conception of physical space in his *Bounds of Sense*. He also went further and claimed that the space talked about by Kant in *The Metaphysical Exposition of Space* is actually the physical space. He wrote:

> I have already remarked that the space declared to be "essentially one" can only be understood to be physical space, the space in which there stand, mutually related, public physical bodies conceived of by us as objects distinct from our perceptions of them.

He also affirmed that the geometry studied by the astronomers and physicists were different than the Euclidean geometry. His following words suggests that he was aware of the discrepancy between the *local* and *global* properties of space:

> The testing of Euclidean geometry by observation and measurement shows its theorems to be verified with an acceptable degree of accuracy for extents of space less than those which astrophysics is concerned; but for astrophysics itself, a different physical geometry, inconsistent with Euclidean, is found to accommodate observation and measurement (Strawson, 1966, p. 286)

The Euclidean geometry holds *true* in small areas. The curvature of the space cannot be detected within these small areas, therefore the deviation from Euclidean space cannot be detected in small areas. The necessary corrections that must be made to make possible the transition from the Euclidean and non-Euclidean geometry also lie within the errors of observation, thus they are not realizable.

In addition to the postulated dichotomy between mathematical geometry and physical geometry by positivists, he postulated the existence of another kind of geometry, which he calls *phenomenal geometry,* which is distinct from *physical geometry*, and known a-priori. The phenomenal geometry is the geometry of the visual images. "The visual imagination cannot supply us with physical figures, but it can supply us with *phenomenal figures*" said Strawson. (Strawson, 1966, p. 282). Strawson said that this third option was completely overlooked by positivists. He wrote:

> What we have had to notice is that there is a third way, different from either of these, which is also possible and which the positivist view neglects […] Euclidean geometry may also be interpreted as a body of unfalsifiable propositions about phenomenal straight lines, circles, etc. (Strawson, 1966, p. 286)

Strawson said that we can never "see" or "picture" two straight lines between two points. If there are two lines between two points, at least one of them has to be curved. Since in non-Euclidean geometry, two straight lines can be drawn between two points (specifically, in Riemannian geometry), then it seems that we can form Euclidean but not non-Euclidean pictures. In short, Strawson's *phenomenal geometry* strived for accommodating Kant's theory of geometry with the advancements in physics and mathematics. A geometry is phenomenally true only insofar as it can be interpreted by virtue of phenomenal figures. This is why the Euclidean geometry is necessarily and universally true; for every geometric concept is interpreted according to the phenomenal items that corresponds to those concepts. The postulates of the phenomenal geometry are *phenomenally analytic* (Strawson, 1966, p. 286), that is to say, they are true in virtue of the meanings attached to the concepts that they contain, and those meanings are themselves phenomenal, or visual. To give an example, whenever I think of the concept of straight line, the phenomenal item, the picture of a straight line, is analytically contained in it. So there is a necessary identity relation between the concept and the picture which makes my phenomenal interpretation of the given concept necessarily true.

To what extent Strawson's modification of Kant's theory of geometry can said to be successful? There are many gaps that needs to be filled in Strawson's account. It must not go unnoticed that Strawson did not said anything about the tri-partite relation between the phenomenal geometry, the physical geometry, and the mathematical geometry. Reichenbach tackles the issue of *visual a-priori* in his *The Philosophy of Space and Time (1958)*. The following words brilliantly summarizes Reichenbach's take on the issue of *visual a-priori*:

> The theory contends that an innate property of the human mind, the ability of visualization, demands that we adhere to Euclidean geometry. In the same way as a certain self-evidence compels us to believe the laws of arithmetic, a visual self-evidence compels us to believe in the validity of Euclidean geometry. It can be shown that this self-evidence is not based on logical grounds. Since mathematics furnishes a proof that the construction of non-Euclidean geometries does not lead to contradictions, no *logical* self-evidence can be claimed for Euclidean geometry, This is the reason why the self-evidence of Euclidean geometry has sometimes been derived, in Kantian fashion, from the human ability of visualization conceived as a source of knowledge. (Reichenbach, 1958, p. 32)

Reichenbach states that our subjective preference for Euclidean geometry stems from the *epistemological function of visualization* (Reichenbach, 1958, p. 34), which is a function of utmost importance in terms of the psychological and pedagogical utility that it brings. But this, for Reichenbach, does not violate the principle of the relativity of geometry, for every geometry which can be mapped onto one another must be treated epistemologically on par with each other[35]. Because of the epistemological

[35] Reichenbach states that as long as two spaces are *topologically equivalent*, the mapping can be done. One cannot, however, map a toroidal space or a spherical space to Euclidean geometry without modifying *the law of causality* accordingly. In that scenario, an observer who is actually moving on the surface of a torus would periodically confront the same set of impressions after covering certain amount of distance. This happens because the observer goes through the same regions over and over again due to the fact that the constant positive curvature of toroidal and spherical spaces forms loops. If the observer wants to retain the Euclidean geometry, he must change, along with the laws of physics, the law of causality. For that space seems to display a *causal anomaly* which is completely at odds with the classical (Kantian) conception of causality according to Reichenbach. So, a Kantian would be having a very hard time explaining the causal relations on that space, for certain regions which are separated by a certain distance would display identical events when the classical conception of causality is preferred. A Kantian's overall system would be in the form of $G_0 + F + A$, where 'A' refers to a newly introduced principle which goes by the name of *the pre-established harmony*. This pre-established harmony aims to explain "the instantaneous coupling of distant events." (Reichenbach, 1958, p. 65)

function of visualization, we generally prefer the Euclidean geometry by setting the geometry to Euclidean, and then, according to our observations and experiments, introduce the existence of universal forces. So our overall system, if we stick with the Euclidean geometry, will always be in the form of $G_0 + F$; in which $F = 0$ or $F \neq 0$, depending on the results we obtain from experiments and observation.

Is it true that we can only visualize the Euclidean geometry? If the answer provided to this question is negative, then Strawson's attempts to rescue Kant's philosophy of geometry inevitably fails. Can human beings visualize non-Euclidean geometries? To recall what was written in the chapter where the views of Helmholtz and Poincaré are discussed, the answer that they have provided to this question seems to be positive; both of them thought that our visual impressions would change in different environments where bodies succeed one another according to different laws. This, in a sense, would compel us to adopt a different geometry, which would in turn compel us to associate different images with different geometric concepts. But they did not tackle the issue of visualization in a great detail. Reichenbach attempts to provide more satisfactory answers than his predecessors with regard to the possibility of visualizing non-Euclidean geometries. To do this, he begins by determining the properties of the visual space.

Visualization is "the reproduction of the particular object in the form of image" (Reichenbach, 1958, p. 38). The attainment of the precision of the image requires more effort on part of the subject. So Reichenbach seems to divide the ability to produce an image into distinct levels. When, for example, we attempt to visualize a particular triangle, or any other geometric object, a blurred image somehow emerges in our mind. This image lack the vividness and particular details. Reichenbach calls these images *schematic images* (Reichenbach, 1958, p. 38). The schematic images lack particular details and exact metrical properties, but have general properties that belong to the object. The exact length of the sides of a triangle, or the angle between its vertices, cannot apprehended precisely in the imagination. Nonetheless, we never fail to apprehend the number of its sides. The former stage is capable of representing the *topological properties*[36] but not the *metrical properties* of the figures, that is to say, it is able to provide us with a rough sketch of the object but it fails to provide us with the exact quantitative relations among the parts of the objects. Reichenbach calls this particular function of the imagination which is able to produce schematic images *image-producing function* (Reichenbach, 1958, p. 39).

The second stage of the visualization is called *the normative function of visualization* (Reichenbach, 1958, p. 39), and for him, it is the stage which is philosophically important. The normative function of visualization is used to make clearer the relations between the objects that I imagined in the former stage. Compared to the images provided to us by the image-producing function, the normative function is able to provide clearer images and is able to correct the drawings we performed in our imagination in the first stage. The rough sketch that is generated in the former process is transformed into an exact diagram which is capable of representing the relations between the images more accurately. When I am asked, for example, to count all the diagonals in a pentagon, I need to pay considerable attention to the figure I am constructing in my head, since it is not the same thing as counting the sides of a

[36] Topological properties of a figure are the properties which do not involve any quantitative measure. Poincaré tends to call these properties *qualitative properties*, and the area of mathematics which studies these qualitative relations is *analysis situs.* (Poincaré, 1963, p. 25) The topological relations between objects include, *adjacency, in-betweenness, connectivity, etc.* Poincaré states that topology precedes geometry epistemologically, for it is possible to disregard the metric properties of a figure and study those qualitative relations, but it is impossible to disregard those qualitative relations and study the metrical relations. (Poincaré, 1963, p. 26)

triangle. In the rough sketch, which is nothing but the output of the image-producing function, certain properties of the image may be wrong, that is to say, I may find the number of diagonals that can be drawn inside the pentagon less or more than its true value. But when the normative function takes over, the total number of diagonals in a pentagon can be apprehended clearly. Even the normative function does able to provide us with more exact images, it does not have the power to represent accurately the exact metrical relations between objects since measurement has nothing to do with our sense of sight. Every measurement is carried out with measuring rods in the physical space.

Reichenbach claims that “Kant’s synthetic a-priori intuition springs from the normative function of visualization” (Reichenbach, 1958, p. 39), and that this function alone singles out the Euclidean geometry from other geometries. The a-priority of visualization is explained as the conformity of the imagination to certain *tacit conditions* when producing an image. These presumed tacit conditions are the norms imposed upon the figures that we draw, therefore the normative function, according to these presumed tacit conditions, directs and restrains our imagination so as to provide images that obey certain visual characteristics. As a consequence, *visual impossibility/possibility* is related to these tacit conditions, and these tacit conditions, in turn, are related to the generally preferred *topological structure* in accordance with which our imagination produces images. When we are asked, for example, whether there exists a surface with one side, we hastily say “no”. But “every student of a lecture on topology has taken a strip of paper, twisted around itself, pasted it together in the form of a ring” (Reichenbach, 1958, p. 41) to form a one sided surface. So if we modify the underlying topological conditions which we impose upon our scenery of imagination as norms, we can turn impossible into possible.

The Euclidean geometry is singled out among alternative geometries in a similar fashion. The particular Euclidean objects that we construct in our imagination are constructed in a space which has a determinate topological structure; and this topological structure act as a norm in producing images. Consider the following question: “Do parallel lines diverge?” Offhand, we must say “no”, if we consider the surface on which these lines are constructed as flat. But if we change the surface on which these two straight lines are constructed, the answer to this question can in fact turn out to be positive. There seems to be no necessary relation between the image and the concept in Reichenbach’s treatment in contradistinction to the treatment of Kant and Strawson. Kant thought that the image is necessarily connected to the concept through a *schema*, and Strawson thought that images are necessarily contained under the concepts. But for Reichenbach, the connection between the image and the concept is flexible and guided by certain tacit conditions which can be modified. These tacit topological conditions, which act as a norm in producing an image, are implicitly presumed in the conceptual elements of the particular geometrical structure. These conceptual elements are the postulates, definitions and axioms. What we are doing is developing a function which is habituated to associate certain images and rules of construction with the conceptual skeleton of the geometry that we are practicing. In support of this view, Reichenbach wrote:

> The merit of visualization consists only in the fact that it translates the logical compulsion of Euclidean geometry into a visual compulsion. The normative function of visualization is revealed as a correlate of the logical compulsion and achieves the same results by means of the elements furnished by the image-producing function as the logical inference does by means of the conceptual elements of thought. (Reichenbach, 1958, p. 42)

Geometrical practice seems to be essentially logical for Reichenbach, a view which is not shared by Poincaré. The diagrams are useful in aiding us to carry out the proof which is essentially logical in nature. The normative function of the visualization is a function whereby we associate images with logical concepts; this in turn help us to complete the proof through using diagrams.

Even though Poincaré and Reichenbach did not agree on the content of geometry, they agreed on the possibility of the association of different images with different concepts, which go against the central doctrine of Kant's theory of geometry. For them, non-Euclidean geometries are as equally plausible as the Euclidean geometry. For Poincaré, different adaptive conditions would compel the organism to adopt a different geometry. Similarly, for Reichenbach, visualization of the Euclidean geometry is a result of a *biological habit* (Reichenbach, 1958, p. 82), and he believed that we can gradually break this habit. This habit of ours resulted from our everyday experience of the behavior of solid bodies. If the solid bodies behaved differently, we would be able to strain the normative function of visualization to be able to adopt a new way of imagining and visualizing geometric relations. In brief, what is actually needed is an emancipation from the congruence relations that belong to our *native geometry*. If we were immersed in a non-Euclidean environment, we would at first resisted to redefine the coordinative definition of congruence and interpret those changes as an actual change in the shape of an object. But after a while, we would no longer perceive those changes as a change in the shape of an object, but rather as a change in our perspective. Reichenbach states that "the moment we no longer see any change in the transported object, we have accomplished a visual adjustment. (Reichenbach, 1958, p. 54)

In the light of these discussions, Reichenbach rejects the view that there exists a visualization which is static and not changing according to different environmental conditions that produces different visual sensations in an organism. It cannot be the case that there existed a pure form of visualization which is necessarily Euclidean as claimed by Kant. The visualization of the Euclidean space was cultivated as a result our observations of rigid rods and light rays. It was cultivated over the course of the biological history of our species as a developmental adaptation. This is why he called it a biological habit and implicitly stressed the role played by *evolution*. In fact, similar views were shared by Poincaré in his discussions about the possibility of the adaptation of different geometries. One of the dominant force that is likely to have shaped Poincaré's conventionalism is the theory of evolution. One of the examples provided by Poincaré is centered around the role of *adaptation* and *inheritance* in the acquisition of the idea of space. In this example, Poincaré raises important questions as to whether the origination of the idea of space truly happen on an individual level, or it is a fruit which is a result of a long chain of continuation of habitual movements of the members of a race and inherited throughout the biological history of the race. The example given by Poincaré is displayed as follows:

> It will be seen that though geometry is not an experimental science, it is a science born in connexion with experience; that we have created the space it studies, but adapting it to the world in which we live. We have chosen the most convenient space, but experience guided our choice. As the choice was unconscious, it appears to be imposed upon us. Some say that it is imposed by experience, and others that we are born with our space ready-made. After the preceding considerations, it will be seen what proportion of truth and of error there is in these two opinions. In this progressive education which has resulted in the construction of space, it is very difficult to determine what the share of the individual is and what of the race. To what extent could one of us, transported from his birth into an entirely different world, where, for instance, there existed bodies displaced in accordance with the laws of motion of non-Euclidian solids-to what extent, I say, would he be able to give up the ancestral space in order to build up an entirely new space? (Poincaré, 1914, pp. 115-116)

Reichenbach rejects the idea that there exists a faculty in us, given completely prior to any experience and is the condition of the possibility of generating images. He rejects that there is a separation between the *form* of the image and the *content* of it. The form, for Reichenbach, is not over and above the content, which is nothing but the displayed visual qualities of an object, such as its color or brightness. In support of this, Reichenbach wrote "visual forms are not perceived differently from color or brightness. They are sense qualities, and the visual character of geometry consists in these sense qualities." (Reichenbach, 1958, p. 84)

CHAPTER 6

CONCLUSION

My thesis aimed to provide an answer whether it is possible to reconcile Kant's theory of geometry with non-Euclidean geometries in the light of the criticisms and modifications, a sufficient portion of which was displayed above. And the answer that thesis gives to that question is negative; it seems not possible to reconcile Kant's theory of geometry with non-Euclidean geometries. Kant's theory of geometry cannot embody non-Euclidean geometries even if undergoes appropriate modifications. Strawson, Frege and others have tried to rescue Kant's theory of geometry by reducing the scope of its validity. They have tried to show that Kant's theory that geometry is synthetic a-priori is still tenable in the face of non-Euclidean geometries, for they thought that the Euclidean geometry is still necessarily applicable to our visual space, even if it does not explain the structure of the world studied by physicists and scientists. This modification, however, did not stand a chance against the criticisms of Helmholtz, Poincaré and Reichenbach, for both of them thought that it is possible to visualize other geometries in different environments. As a philosopher who did not witness the revolutionary turns in logic, mathematics, physics and biology, Kant's current philosophical stance towards the nature of geometry must not be accused of its ignorance as to these matters. Had he known the theory of evolution, he might have contemplated the possibility of a dynamic and evolving intuition, which is capable of adapting itself to the environment. Had he known, similarly, the new logics discovered in the 19th and 20th Century, the discovery of non-Euclidean geometries, etc., he would have reconsidered his philosophical stance towards the nature of geometrical construction. He simply lack all the valuable information that would have helped him to revise his own philosophical position and his transcendental idealism.

Helmholtz, Poincaré and Reichenbach tried to show that the synthetic a-priori nature of geometry is not tenable under these new developments mentioned above. They both stressed the importance of the role played by the empirical elements in the formation of a geometry; the possibility for a species to develop new biological habits in new environments is a great example of it. According to Kant, this cannot be possible, for the determination of space cannot be a function of the environment. The character of the space is invariant under any different environmental context according to Kant. Even though Poincaré and Reichenbach differed in their views as to role played our minds in the formation of geometry, they agreed that it nevertheless is one of the conditions of the possibility of geometry as a science proper, but not in the sense that Kant had thought. Poincaré sought the role played by our minds in the formation of geometry in other mental powers that belong to our form of understanding; Reichenbach in logic, but what is common in both is that they both reduced the normative mental operations carried out in the alleged *transcendental imagination* to psychological operations. This reduction in turn rid the normative constructive procedure that takes place in the imagination of its epistemological import. So the *transcendental idealism* of Kant is reduced to mere *psychologism*, and the normativity found in the construction of a geometric entity to a habit developed over time. This is why the term ‘intuition’ received very different connotations after Kant in the light of these advancements in both pure mathematics, physics and biology. *Poincaré,* in Janet Folina’s words, “attempted to reconceive, or reconfigure intuition” (Folina, 2018, p.

165). For Poincaré, intuition become something *psychological*, it became a faculty which enabled us to "to see the end from a far" *(*Poincaré, 1958, p. 22). It does not and cannot provide us with the expected rigor that Kant thought that it provided, it could only be seen as a fallible tool of discovery. For Reichenbach, on the other hand, the term *pure intuition* simply means '*pure visualization'*, which is some sort of a '*biological habit'*, developed as a "result of an adaptation of a psychological capacity to the environment." (Reichenbach, 1958, p.82). The appropriate modifications of the term 'intuition' is a way of renouncing the thesis that our pure intuition of space provides the ground of the necessity and universality of the propositions of geometry. This is due to the fact that Kant's theory of geometry is modally connected to his theory of space, and, as was clearly argued in the conclusion part of the second chapter, as the pillar geometry falls, so must the pillar of space. I thereby conclude that Kant's overall theory of space and geometry is rendered obsolete in the light of these new advancements in sciences and their philosophical consequences.

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APPENDICES

A. TURKISH SUMMARY / TÜRKÇE ÖZET

Bu tez Öklid-dışı geometrilerin Kant'ın matematik felsefesi ile uzlaştırılabilirliğinin mümkünatını konu almıştır. Tezimde Kant'ın geometri kuramının Öklid-dışı geometrileri içerecek bir kuram olamadığı ve Kant'ın sistemi üzerinde yapılan hiçbir modifikasyonun Öklid-dışı geometriler ile Kant'ın sistemini uzlaştırmaya muktedir olmadığı gösterilmeye çalışılmıştır. Uzlaştırma sözcüğünün kapsamı, Öklid-dışı geometrilerin varlığının Kant felsefesi için bir tehdit teşkil etmemesi ve hem Öklid-dışı geometrilerin hem de Kant'ın geometri kuramının birbirlerine sorun teşkil etmeden aynı anda var olabilmelerini içerir. Varılan sonuçları temellendirmek amacıyla tezin ilk dört bölümünde Kant'ın geometri kuramı ve bu kuramı oluşturmasını gerektirmiş olan tarihsel ve felsefi problemlere detaylı bir şekilde değinilmiştir. Dördüncü bölümde Öklid-dışı geometrilerin keşfi ve son bölümde Helmholtz, Poincaré ve Reichenbach gibi düşünürlerin eleştrileri ışığında Öklid-dışı geometrilerin Kant'ın geometri kuramı üzerindeki etkileri araştırılmıştır.

Kant için geometri uzayın özelliklerinin sentetik ve a-priori belirlenimidir. Geometrik bilgimiz a-priori'dir çünkü deneyimden türetilmiş bir bilgi türü değildir. Geometrik bilgimiz sentetiktir çünkü hiçbir kavramsal analiz bize analize tabi olan kavrama ilişkin tüm özellikleri veremez. Kant' tan önce geometrik bilgimizin içeriğine ait yaygun görüş onun kavramsal bir etkinlik olduğuna dairdi. Rasyonalist gelenekten gelen Leibniz ve Wolff, geometrik önermelerin, o önermeleri oluşturan kavramların analizinin yapılarak gösterildiğini düşünüyorlardı. Bu filozoflar için benim üçgenin özelliklerine ilişkin her bilgim üçgen kavramını analiz etmemle ortaya çıkıyordu. Kant Öklid'in Elemanlar'ında geçen kanıt prosedürlerini baz alarak geometrik bilginin bu tarz salt kavramsal analizle mümkün olamayacağı görüşünü ileri sürmüştür. Sözgelimi bir üçgenin iç açılarının toplamının iki dik açıya eşit olduğunu bulmak istersek bize verilen üçgen kavramının ötesine gitmemiz gerekir. Biz üçgenin iç açılarını bulmak için öncellikle bir üçgen çizerek başlarız. Fakat asla bu figürle sınırlı kalmayız. Bize verilen üçgenin tabanını uzatır ve o taban üzerinde toplamları iki dik açının toplamını veren bir iç ve bir dış açı yaratırız. Sonra bu dış açıyı kesecek ve dış açının komşusu olan iç açının gördüğü kenara paralel olacak şekilde bir düz çizgi daha çizeriz. Bu sayede elde ettiğimiz yeni dış açının iç açılardan birine eşit olduğunu buluruz. Bu sayede, bir kağıdın üzerinde ya da imgelemimizde oluşturduğumuz ve genişlettiğimiz figürler üzerinden akıl yürüterek göstermek istediğimiz önermeyi, yani bir üçgenin iç açılarının toplamının iki dik açıya eşit olduğunu göstermiş oluruz. Bu bakımdan geometrik bilgimiz sentetik bir bilgidir. Çünkü asla bize ilk başta verile üçgen figürü ile sınırlı kalmaz tasımlamamız; biz yeni figürler ve yeni bağıntılar inşa ederek göstermek istediğimiz şeyi göstermeye girişiriz. Geometri bu şekilde farklı şekillerin uzay görümüz içindeki sentezi ile olanaklı olan bir bilimdir.

Geometriye ilişkin sententik a-priori bilgimizin olanaklılığının koşulu uzayın saf bir görü formu olmasından kaynaklanır. Uzay ve zaman Kant için arı birer sezgidir ve deneyimimizin olanaklılığının koşuludurlar; objeler bize bu arı görü formları olmadan verilemez ve bu arı görü formlarımız objelerin birbirleri ile uzamsal olarak ilişkilendirilebilmelerinin ve sıralanabilmelerinin zeminini oluştururlar. Kant öncesi düşünürler uzayın zihinden bağımsız olduğu görüşünü benimsemişlerdir. Kant öncesi uzayın kökenine ilişkin yapılan felsefi tartışmalar iki başlık altında toplanabilir; uzayın ilişkisel olduğuna dair görüşler ve uzayın mutlak olduğuna dair görüşler. Leibniz uzayın aslen şeyler arasındaki ilişkilerden olduğu görüşünü savunmuştur. Newton ise uzayın mutlak olduğunu ve şeylerden bağımsız olarak var olduğunu savunmuştur. Her iki görüşte Kant için yanlıştı çünkü her iki görüşte uzayın zihinden bağımsız bir gerçeklik olduğu kanısına dayanıyordu. Kant ile beraber uzayın ideal ve zihne bağlı birer çerçeve olduğu görüşü ortaya çıkmıştır. Kant'a göre uzay düşünen bir bireyden bağımsız olarak kendi içinde var olabilen bir nen olamaz. Uzay bizim kendi tasarımımızdır. Uzay bir nevi bir gözlüğe benzetilebilir. Biz bu gözlükler olmadan ne görüngüler ile temas halinde olamayız; bizi görüngülerle doğrudan ve dolaysız bir şekilde temas haline sokan şey bu gözlüklerdir. Biz asla bu gözlükleri çıkarıp gerçekliğin kendi içinde nasıl olduğunu da bilemeyeceğiz; çevremizde olan biten herşeyi bu gözlüklerde bakarak algılamak zorundayızdır. Kısacası uzay a-priori bir çerçevedir ve görüngülerin bize verilebilmesinin olanaklılığının koşulunu oluşturur.

Geometri bu saf uzay görümüzün içerisinde belli birtakım inşalar yaparak sürdürdüğümüz bir etkinliktir. Saf görü formumuzu çeşitli bir takım inşalarla tahdit ederek belli geometrik objeler oluştururuz. Geometrik bilgimizin içeriği bu görü formunda inşa edilmiş objeler ve bu objeler arasındaki ilişkileridir. Geometrik bilmin önermelerin zorunluluğu ve evrenselliğinin mümkünatının koşulu bu bize a priori olarak verilen uzay görümüzdür, eğer bu içinde bir takım inşalar yaptığımız çerçevenin kendisi a priori olarak bizlere verilmeseydi geometrik edimimizin kendisi de asla a priori olamayacaktı.

Kant uzay görümüzün belirlenimin zorunlu ve evrensel olarak Öklidyen olduğunu savunmuştur. Öklit geometrisinin önermeleri kendilerini akla bir zorunlulukla dayatır ve Öklid-dışı geometrilerin kavranabilmesi Kant'ın felsefesi içinde mümkün değildir. Kant Öklid-dışı geometrilerin imkansızlığının bu geometrilerin kavramlarının saf görüde inşasının mümkün olmamasından ötürü olduğunu tartışır. Söz gelimi birbirlerine paralel ve belirsizce uzatılmış düz birer çizgi bir alan kapar önermesi bir imkansızlığa işaret eder. Fakat bu önermenin imkansızlığı mantıksal bir imkansızlık değildir; çünkü biz bu önermeyi değili ile birlikte ele aldığımızda asla bir çelişki yarattığını göremeyiz. Önermenin değili göz önüne alındığında, değilinin de orjinali kadar mantıksal olarak imkanlı olabileceği göz önünde bulundurulmalıdır. Hem orjinalinin hem de değilin mantıksal açıdan eş düzeyde olanaklı olmalarının sebebi düz çizgi kavramı ne kadar analiz edilirse edilsin, birbirine paralel ve belirsizce uzatılan iki düz çizginin kapayabileceği bir figürün imkansız olduğunu bize göstermez. Önermede dile getirilen figür, ancak diğer bir takım daha primitif olan figürlerin bir araya getirilmesi ile bir sentez sonucu meydana getirilebilir. Bu önermede bahsi geçen kavramın imkansızlığı Kant'a göre bu figürü oluşturabilecek sentezin imkansızlığı ile alakalıdır. Bu bahsi geçen kavrama duyumda (ve ya görüde) bir obje verememememizden kaynaklanır. Sonuç olarak Öklid-dışı geometrilerin imkansızlığı mantıksal bir imkansızlığa işaret etmez, görüsel/sezgisel bir imkansızlığa işaret eder.

Helmholtz, Poincaré ve Reichenbach gibi filozoflar, Öklid-dışı geometrilerin duyumsal bir içeriğe sahip olabilmesinin imkansız olduğuna dair görüşü reddederler. Bu filozoflar farklı fiziksel koşulların içinde bulunduğumuzda, içerisinde bulunduğumuz dünyayı ve bu dünyanın içindeki objelerin arasındaki ilişkileri betimlemek için farklı geometrik yapıları benimseyeceğimizi söylerler. Yani bu ilişkilerin betimi için bir sürü aday geometrik yapı arasından seçilen Öklidyen geometrinin asla ve asla diğer geometrik yapılara karşı epistemolojik olarak bir üstünküğü ve önceliği var sayılamaz. Özellikle Helmholtz ve Poincaré, oluşturmuş oldukları varsayımsal olanaklı dünyaların içerisinde yer alan ve bizimle aynı biyolojik donanıma sahip organizmaların değişen çeşitli görsel ve taktil duyumları doğrultusunda benimseyecekleri farklı geometrik yapılar olduğunu tartışırlar. Poincaré'nin meşhur küre-uzay deneyi bunun en iyi örneklerinden biridir. Poincaré bizden farklı bir dünya tasvir etmemizi ister ve bu dünyanın özelliklerini sıralar. Bu dünya bir kürenin içinde hapsolmuştur. Bu dünyada sıcaklık da yeknesak değildir; bu kürenin merkezinden uzaklaşmaya başladığımızda sıcaklık düşer. Bu dünyada var olan tüm maddelerin sıcaklığa bağlı genleşme katsayısı da bizim dünyamızdaki maddelerin aksine aynıdır. Yani her cismin sıcaklık yüzünden boyutlarında meydana gelen değişmeler bu kürenin içinde nerede olduklarına göre belirlenecektir. Poincaré ayrıca bu dünyada ışığın kırılma endeksinin de bizim dünyamızdan farklı olduğunu, ve ışığın düz bir yol değil eğimli bir yol izlediğini ekler. Bu dünya Poincaré için düşünülmesi imkansız olan bir uzay değildir. Bu uzay düşünülebilir, çünkü herhangi bir mantıksal çelişmeden muhaftır. Bu uzayı oluşturan bahsi geçmiş özelliklerin hiçbiri birbiri ile çelişir değildir. Poincaré' yi Kant'tan ayıran düşüncesi onun bu dünyanın algılanmasının da mümkün olduğunu söylemesinde yatar. Poincaré' ye göre bu dünyada yaşayan bize benzer canlıların da geometrik bilgisinin olacağını, ve bu geometrik bilginin bizimkinden farklı olacağını savunmuştur. Bu küre-dünyada bu canlıların gözlemleyeceği ilişkiler Öklit geometrisi ile açıklanmaktansa hiperbolik geometri kullanılarak açıklanacaktır Poincaré'ye göre. Eğer bi Bu dünyayı bir gün biz ziyaret etseydik, başta herşeyi Öklit geometrisi kullanarak açıklamay çalışacaktık. Fakat zamanla bu küre-dünyada edindiğimiz yeni taktil ve görsel izlenimler ışığında geometrik sistemimizi değiştirecek ve gözlemlediğimiz ilişkileri farklı geometrik yapılar kullanarak açıklamaya girişecektik. Özet olarak bu üç filozof, Kant'ın aksine, geometrik önermelerin sentetik a-priori olduğu fikrini reddeder, çünkü geometrik kavramlarla eşleştireceğimiz imgeler veya objeler tamamen deneyimlerimiz ışığında belirlenir. Biz düz çizgi kavramını bugün Öklit geometrisin buyurduğu şekilde tanımlıyorsak bu deneyimi açıklayıcılığı bakımdan en uygun geometrik yapının Öklidyen geometri olmasından kaynaklanır. Farklı fiziksel dış koşullar düz çizgi kavramını nasıl tanımlayacağımız konusunda bize farklı şekilde kılavuzluk edebilirler.

Her ne kadar Helmholtz ve Poincaré'nin geometrinin kaynağına ilişkin görüşleri birbirlerinden farklı olsa da, mütabık oldukları görüş geometrinin önermelerinin asla a priori bilinemeyeceği ve deneyimin bu önermelerin bilinebilmesinde bir payının olduğudur. Helmholtz geometrik bilgimizin tamamen empirik olduğunu savunmuştur. Fakat Poincaré geometrik bilgimizi mümkün kılan ön koşulların uzlaşımsal karakteri üzerinde durmuştur. Poincaré için geometrik önermeler uzlaşımsaldır; çünkü biz geometri yapmaya başlamadan bir takım uzlaşımlar hakkında mütabık oluruz. Uzlaşımlar gizlenmiş tanımlardır ve Öklit geometrisi bu uzlaşımlarla doludur. Bir örnek verecek olursak, Öklit Geometrisi'nde önermelerin neredeyse hepsi kongrüans ilkesi üzerinde temellenir. Kongrüans ilkesi iki cismin birbirlerine eşit olması için uzayda üst üste denk getirilebilmesi gerektiğini söyler. Poincaré için kongrüans ilkesi uzlaşımsal bir ilkedir ve geometrinin temelinde bu ilke vardır. Bu ilke bir ölçüde deneyimden türetilir. Bunun sebebi doğada herhangi iki nokta arasında hareket ederken şekil değiştirmeyen ve izlenimlerini vücudumuzun karşılıklı bir hareketiyle düzeltebildiğimiz katı cisimlerin var olmasıdır. Öklit geometrisinde kongrüans ilkesi örtük bir belite işaret eder ve bu belit bu yukarıda bahsedilen hareketin mümkünatıdır. Sözgelimi Öklit iki cisim arasındaki denkliği kanıtlamak istediğinde bu cisimler uzayda hareket ettirerek üst-üste getirmeye çalışır. Bu kanıtın başarılı olması için belli geometrik cisimlerin şekil değiştirmeden hareket edebildiği varsayılmalıdır. Fakat bu Öklit geometrisinde açıkça bir belit olarak belirtilmez. Buna ancak dolaylı yoldan varılır. Öte yandan her ne kadar doğada bu tarz cisimler gözlemlesekte doğa bize ancak ve ancak aşağı yukarı ilkeler sunabilir. Gerçekten şekil değiştirmeden hareket eden cismi aklın bir ürünüdür. Fakat bu düşünce Kant'ın düşündüğü gibi saf sezgimizde tasarlayabildiğimiz bir düşünce değildir, anlama yetisinin bir ürünüdür.

Bu bağlamda Poincaré için katı cisimlerin ve ışığın hareketleri konusunda önden vermiş olduğumuz bir ön tanım uzlaşımsal olan bir elementtir. Katı cisimlerin ve ışık süzmelerinin hareketlerini tanımlayışımız bize hangi geometrik yapının gözlemlediğimiz ilişkileri açıklamak için kullanacağını belirlerler. Bu tanım gözlemlenen katı cisimler ve ışığın hareketi doğrultusunda değişebilir. Durmadan şekil değiştiren cisimlerin içinde ve de yörüngesi düzlemsel olmayan ışık süzmelerinin gözlemlenebildiği bölgelerde uzunca süreler yaşamış olsaydık Öklidyen bir düz çizgi tanımına ulaşmamız mümkün olmayabilirdi. Deneyim bize hangi tanımı kullanacağımız konusunda yardımcı olabilir, ama asla hangi tanımın kesinlikle seçileceğine deyin bir şey söyleyemez. Her tanım iş görebilir, fakat bazı tanımlar diğer tanımlardan daha kullanışlıdır. Bunun sebebi bazı tanımlar ışığında gözlemlenen ilişkileri betimlemek diğer tanımlara kıyasla çok daha kolay ve elverişlidir. Bunun sonucunda deneyimin asla ve asla bir geometrik yapının doğru ya da yanlış olduğuna dair bir yargıda bulunmamıza bir olanak tanımayacağında şahit oluruz. Poincaré için geometrinin önermelerinin uzlaşımsal olması tam da bu demektir; bir geometrik yapı diğerinden daha doğru ya da daha yanlış olamaz, ancak daha kullanışlı ve uygun olabilir. Bu durumda Öklit geometrisi asla deneyim ışığında yanlışlanamaz. Bir bilim adamı gözlemlediği ışık süzmelerinin ve katı cisimlerin düz bir doğrultuda hiç bir zaman ilerlemediğini saptarsa yapması gereken şey katı cisimlerin hareketine ilişkin yasalar ile optikte ışığın hareketine ilişkin yasaları değiştirip Öklit geometrisini tutmak olacaktır. Geometri ve fizik yasaları bir noktada birbirleri ile karşılıklı sınanma ilişkisi içerisindedir.

Öklit geometrilerinin diğer geometrik yapılara tercih edilmesinin arkasında yatan sebepler konusunda Reichenbach kapsamlı bir çalışma yapmıştır. Reichenbach Poincaré gibi geometrinin uzlaşımsal bir karakteri olduğunu savunmuştur. Ona göre kongrüans ilkesine ilişkin verilen koordinatif tanım, bütün geometrik egzersizimden önce gelir. Reichenbach bütün geometrik egzersizin öncelikle bir birimin uzunluğunun tanımlanması ile başladığını söyler. İkincil elzem olan tanım bu ilk aşamada tanımlanmış birim uzunluğun hareket ederken şekil değiştirmediğini öne süren tanımdır. Bu iki tanım yapıldığı anda bir geometrik yapıya işaret eder. Reichenbach, Poincaré'ye benzer bir biçimde gözlemlediğimiz ham olguları dilediğimiz geometrik yapı ile belirleyebileceğimizi vurgular. Dikkat edilmesi gereken şey deneyimde gözlenen olgularla tam örtüşmesi bakımından fiziksel yasaların kullanılan geometrik dile bağlı olarak modifiye edilip edilmeyeceğidir. Örnek verecek olursak biz bir takım olguları Öklidyen geometri kullanarak modelleyebiliriz, fakat eğer olguları açıklayıcılığı bakımından Öklit geometrisi yeterli değil ise, biz fizik yasalarını da kurduğumuz sistemin olgularla örtüşmesi bakımından modifiye etmeliyiz. Eğer gözlemlediğimiz iki düz çizgi arasındaki mesafe açılıyorsa ve bu olguyu açıklamak için Öklit geometrisinde ısrarcı oluyorsak, bu iki düz çizgi arasındaki açılmayı evrensel bir kuvvetin varlığından söz ederek açıklamaya girişmeliyiz.

Reichenbach' ın kendine mahsus uzlaşımsalcılığı uzayı açıklamada seçilecek geometrinin göreli olduğunu vurgular. O da Poincaré ile bu hususta taraf olarak geometrik bilgimizin Kant'ın düşündüğü gibi sentetik a-priori olmadığını savunmuştur. O da tıpkı Poincaré gibi bir geometrik yapının bir diğerinden epistemolojik anlamda daha üstün olamayacağını söyler. Fakat Poincaré' nin aksine bizim neden Öklidyen geometriyi diğer geometrik sistemlere tercih ettiğimiz üzerinde kapsamlı açıklamalar yapmaya çalışır. Öklit geometrisi lokal ve deneyimleyebildiğimiz uzayı açıklayıcılığı bakımından en makul geometri olduğundan ve görselleştirilebilirliği bakımından özel bir epistemolojik fonksiyonu olduğundan diğer geometrilere tercih edilir. Poincaré' nin aksine Reichenbach, Einstein'in Öklid-dışı geometrileri başarılı bir şekilde aktüel deneyimimizdeki ham olguları açıklamak için kullandığına tanık olmuş bir bilim filozofudur. Bu bir nevi ona uzayın geometrik karakterinin tam anlamı ile uzlaşımsal olmadığını ve birim uzunluğa ve onun hareketlerine ilişkin uzlaşımsal tanımlar yapıldıktan sonra uzayın geometrik karakterinin ampirik olarak belirlenebileceği görüşünü kazandırmıştır. Reichenbach, Einstein'in kuramının astronomik ölçekte zuhur eden ilişkilerin açıklanmasında kullanıldığını biliyordu. Fakat Öklit geometrisinin Öklid-dışı geometrilerle olan farkının insanların gündelik hayatta gözlemlediği objelerin ölçeğinde saptanamayacağının da farkındaydı. Bu bakımdan fiziksel uzamın lokal ve global özelliklerini betimlemede seçilecek geometrik yapılar birbirinden farklı olabilirdi. Bu bakımdan Reichenbach Öklit geometrisinin bu gündelik hayat ölçeğinde tercih edilebileceğini savunmuştur.

Reichenbach ikincil olarak Öklit geometrisinin görselleştirilebilirliği bakımdan özel bir epistemolojik fonksiyonu olduğunu söyler. Sözgelimi biz imgelemimizde düz bir çizgi tasarlamaya çalıştığımızda genellikle Öklidyen niteliklere sahip bir düz çizgi tasarlarız. Çoğu Kant sonrası filozof Kant'ın geometri kuramını kurtarmak için Öklit geometrisinin zorunlu ve evrensel olarak görsel uzayımız için geçerli olduğunu savunmuştur. Bu filozoflardan biri olan Strawson'a göre bizim görsel uzayımız, yani imgelemimizde tasvir ettiğimiz objeler ve onların ilişkileri, zorunlu ve evrensel olarak Öklidyen niteliklere sahiptir. Strawson görsel uzayımızı konu alan geometriye fenomenal geometri ismini verir. Fenomenal geometrinin objeleri fenomenal imgelerdir. Bu imajların geometrik karakteri zorunlu ve evrensel olarak Öklidyen geometri ile betimlenebilir. Strawson Öklidyen geometrisinde bahsi geçen kavramlara tekabül eden imajların zorunlu olarak o kavramlarla birlikte düşünüldüğünü söylemiştir. Biz düz çizgi kavramını düşündüğümüzde o kavramın altında zorunlu olarak Öklidyen niteliklere sahip bir düz çizgi tasvir ederiz. İmaj zorunlu olarak kavramda içerilir. Bu bağldamda Strawson bizim asla ve asla Öklid-dışı bir düz çizgiyi imgelemimizde tasvir edemeyeceğimizden bahseder. Bu ona göre imkansızdır çünkü biz Öklid-dışı bir düz çizgi düşündüğümüzde ister istemez bir eğri düşünürüz. Yani Öklid-dışı geometrilerde var olan kavramların imgelemdeki tasviri daimi olarak Öklid geometrisinin birtakım başka kavramları ile eşleştirdiğimiz imgeler vasıtası ile mümkün olabilir.

Reichenbach için görsel uzayımız asla ve asla a-priori belirlenemez. O görsel uzayımızın Öklidyen olmasının epistemolojik bir fonksiyonu olduğundan bahseder fakat Strawson gibi ileri giderek görsel uzayımızın geometrik belirleniminin zorunlu ve evrensel olarak Öklidyen olduğunu ileri sürmez. Reichenbach imgelemimizde bir kavrama tekabül eden imajı canlandırdığımızda ya da çizdiğimizde bir takım düzgüsel ve örtük koşulların etkisi altında kaldığımızdan bahseder. Biz bir düz çizgi düşündüğümüzde ister istemez o düz çizginin çizildiği yüzeyin düz olduğunu hayal ederiz. O yüzey çeşitli deformasyonlara uğradığında artık üzerine çizilen düz çizgilerin birbirleri ile girdikleri çeşitli geometrik ilişkilerin aynı kalması beklenemez. Bu bir bakıma şunu ifade eder: biz geometrik edimimizi imgelemimizde gerçekleştirirken daima bir takım topolojik bir yapıyı örtük olarak benimseriz. Bu topolojik yapı bir değişime uğradığında ister istemez geometrik edimimizin içeriğini oluşturan imgeler ve onların arasındaki ilişkiler de değişime uğrar. Örnek vermek gerekirse topoloji dersi almış bir matematik öğrencisine tek yüzü olan bir yüzeyin mümkün olup olmadığı sorulsa öğrenci buna olumlu bir cevap verecektir. O bir yüzeyi alacak ve onun bir ucunu 180 derece bükerek diğer ucu ile birleştirecek ve tek yüzeyi olan bir yüzey inşa edecektir. İki paralel çizginin kesişip kesişmediğine ilişkin soru da bu türden bir sorudur. Bu soruya verilecek cevap önceden benimsenmiş topolojik bir yapıya göre şekillenir. Yüzeyleri düzlemsel olarak tasvir eden bir kimse bu soruya olumsuz yanıt verecektir. Yüzeyleri küresel olarak tasvir eden bir kimse de bu soruya olumlu yanıt verecektir.

Reichenbach'a göre bu düzgüsel ve örtük koşulların kökeni biyolojik bir alışkanlığa dayanır. Bu bir nevi psikolojik, fizyolojik ve evrimsel bir sürecin nihai sonucudur. Bu sebeple bizim gibi organizmalar, kendi biyolojik ve evrimsel tarihleri hesaba katıldığında, bu tarz düzgüsel ve örtük koşulların içinde evrilmiş ve belirli bir topolojik yapıyı görselleştirme edimi için örtük olarak benimsemiş olsa da, bu süregelen alışkanlığı bozabilirler. Bu benimsenmiş ve düzgüsel olarak kendini bize dayatan topolojik yapı, biz farklı bir kongrüans ilişkisine adapte olmaya başladığımızda değişime uğrar. Biz kendi biyolojik ve evrimsel tarihimiz boyunca belirli bir kongrüans tanımına adapte olan ve bu tanım üzerinden düzgüsel bir topolojik yapıyı örtük olarak benimsemiş bir organizmayızdır. Farklı çevresel koşullar bize farklı kongrüans ilişkilerini seçmeye, ve bununla beraber farklı topolojik yapıları imgelemimizdeki imajları tasvir etmek için benimsememize sebep olur. Poincaré' de Reichenbach gibi benzer bir evrimsel ve biyolojik argüman sunmuştur. O da geometrik bilgimizin kökenlerine ilişkin yaptığı sorgulamada atasal deneyimlerimizin önemi üzerinde durur. Poincaré gerçek anlamda uzaya ilişkin tasarımlarımızın sadece bireyi deneyimlerine indirgenip indirgenemeyeceğini sorgulamıştır. Uzay fikrinin oluşumu ve bunun sonucu olarak sürdürdüğümüz geometrik edimlerimiz ya bireyin sadece kendi hayatında deneyimledikleri ile doğrudan ilişkilidir ya da o bireyin bir üyesi olduğu ırka mensup bir bilgi çeşididir. Sonuç olarak Poincaré belli bir geometrik sistemin diğer sistemlere tercih edilmesinde rol oynayabilecek faktörlerin başında adaptasyon ve inheritans olabileceği üzerinde durur, fakat bu argümanları için eksiksiz bir temel aramaya girişmez.

Özetle bahsi Helmholtz, Poincaré ve Reichenbach geometrik bilgimizin asla ve asla Kant'ın öne sürdüğü gibi sentetik a priori olmadığını savunmuştur. Bunu göstermek için fizyolojimizin, çevresel koşullarımızın, biyolojik tarihimizin ve psikolojimizin oynadığı rolün altını çizmişlerdir. Bu filozoflar Kant'ın aksine Darwin'in evrim kuramına, Öklid-dışı geometrilerin keşfine ve bu Öklid-dışı geometrilerin Einstein tarafından başarılı bir şekilde aktüel dünyamızdaki uzamsal ilişkileri betimlemede kullanılmasına tanık olmuşlardır. Bütün bu farklı alanlardaki gelişmeler hesaba katıldığında Kant'ın geometriye ilişkin kuramı kaçınılmaz olarak bu filozoflar tarafından yoğunca eleştrilmiş ve geometrik bilgimizin kökenine ilişkin alternatif olasılıklar ortaya sürülmüştür. Kant'ın geometrik bilgimizin zorunluluğunu ve evrenselliğinin olanaklılığının koşulunu açıklamak için ortaya sürdüğü arı görü formları bu filozoflarca artık kabul edilmemiştir. Uzay'ın bize içkin olduğu görüşü zaman içinde geometrik bilgimizin zorunluluğu ve evrenselliği sorgulanmaya başlandığında reddedilmeye başlanmıştır

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TEZİN TÜRÜ / DEGREE: **Yüksek Lisans** / Master ☒ **Doktora** / PhD ☐

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