CASIMIR EFFECT IN A HALF EINSTEIN UNIVERSE

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ABSTRACT

CASIMIR EFFECT IN A HALF EINSTEIN UNIVERSE

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In this thesis, we considered the Casimir effect for the massless conformal scalar field in a half Einstein universe. First we calculated the renormalized quantum vacuum energy by using the mode sum method and studied the effects of different cutoff functions. Next we calculated the renormalized quantum vacuum energy momentum tensor by using the point-splitting method.
We constructed the Green's function by using the eigenfunctions, which are obtained by solving the wave equation with appropriate boundary conditions. We discussed the importance of this case as well as its relation to previous calculations.

Finally, we presented the details of the calculation of the renormalized quantum vacuum energy momentum tensor for the massless conformal scalar field in a closed Friedmann universe by using the covariant point-splitting regularization method.

Key Words : Casimir Effect, Vacuum Energy, Renormalization, Mode Sum, Cutoff, Point-Splitting, Regularization, Green's Function, Energy Momentum Tensor, Einstein Universe, Half Einstein Universe, Closed Friedmann Universe.

ÖZ

YARIM EINSTEIN EVRENİNDE CASIMIR ETKİ

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Bu çalışmada, yarım Einstein evrenindeki kütlesiz, konformal skalar alanın Casimir etkisini yeniden inceledik. Önce, yeniden normalleştirilmiş tanecik sıfır noktası enerjisini, kiplerin toplamı medotuyla hesapladık ve değişik kesilim fonksiyonlarının etkilerini araştırdık. Daha sonra, yeniden normalleştirilmiş tanecik sıfır noktası enerji momentum tansörünü şiddetli nokta medotu kullanarak hesapladık. Uygun sınır koşullarında, dalga denklemi çözülerek elde edilen öz fonksiyonları kullanıp, Green's fonksiyonunu oluşturduk. Bu sonuçların önemini ve önceki hesaplarla olan ilişkisini tartıştık. Son olarak, kapalı genişleyen Friedmann evreninde, kütlesiz konformal skalar alan için şiddetli nokta düzenleyicisi methodu kullanılarak, yeniden normalleştirilmiş tanecik sıfır noktası enerji momentum tansörü hesabının ayrıntılarını sunduk.

Anahtar Kelimeler : Casimir Etki, Sıfır Noktası Enerjisi, Yeniden normalleştirme, Kiplerin Toplamı, Kesilim, Şiddetli Nokta, Düzenleyici, Green's Fonksiyon, Enerji Momentum Tansörü, Einstein Evreni, Yarım Einstein Evreni, Kapalı Friedmann Evreni.

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CHAPTER I

INTRODUCTION

The measurable consequence of the quantized fields associated with the vacuum energy is Casimir effect. This effect is basically the polarization of the vacuum energy of the quantized fields on account of presence of boundaries and/or curvature [1,2].

The formal vacuum energy

$$E_0 = \sum_{\vec{k}} \frac{\hbar}{2} w_{\vec{k}} \tag{1.1}$$

is infinite, since there are infinite number of normal modes of increasingly high frequency. In flat space time the problem of infinite vacuum energy is usually bypassed by introducing normal ordering. The normal ordering process has the effect that the energy of the corresponding free vacuum state becomes zero [1]. However, presence of boundaries for the field induces a change in the energy spectrum and therefore modifies the vacuum energy. Thus, when the fields are confined to a finite volume, the vacuum energy cannot be removed by the simple normal ordering prescription used in conventional field theories.

The first zero point energy calculation in the presence of boundaries was done by Casimir in (1948) [3]. He evaluated the renormalized quantum vacuum energy for the electromagnetic field bounded by two parallel plates. Since the finite vacuum energy Casimir found was negative, he concluded that there must exist an attractive force between the plates. The attractive force:

$$F = -\frac{\pi^2 \hbar c}{240 a^4} \quad , \tag{1.2}$$

which is expressed in terms of the, Planck constant $\hbar$, the velocity of light c, and the distance between plates a, should act on unit area of two conducting plane parallel plates in vacuum.

This attractive force was later observed experimentally by Sparnaay [4] and Tabor [5]: For plates of $1cm^2$ in area and with $a = 0.5\mu m$, this force was found to be $\approx 0.2dyn$, in accordence with the theoretical prediction. Hence, it became clear that the Casimir effect is real.

Encouraged by the existence of this attractive force, Casimir hoped that one could construct an electron model, where the repulsive electric forces will be balanced by the attractive Casimir effect. However, the vacuum energy inside a spherical shell was later calculated by Boyer [6] and Davies [7] and they found it to be positive which means that the force on the shell is repulsive and depends only on the radius of the shell. This result did not confirm Casimir's suggestion. This being quite different from the case of two parallel plates shattered Casimir's hope to construct an electron model.

Boyer's result was intriguing but not completely convincing in that it depended on a particular choice of exponential cutoff function. In addition, he found it necessary to enclose the sphere in question in yet a larger sphere the radius of which is allowed to approach infinity. Boyer found this necessary to eliminate some of the divergences he encountered [6]. Later, Milton, DeRaad and Schwinger [8] confirmed Boyer's calculation by using Green's functions, where they did not have to introduce an artifically chosen cutoff function and also they did not have the need for an exterior sphere. Both papers calculated the integrated energy for the electromagnetic fields. In 1981, Olaussen and Ravndal [9] studied the distribution of this energy inside a sphere. In all these papers authors had to resort to numeric methods and approximate expressions at some point in their calculations.

We may take the Casimir's beautiful calculation of the quantum vacuum energy as a model for calculations in the case of a quantized field in a curved background spacetime. As an example the Casimir effect has also been calculated in Cosmology. It was first done by Ford [10,11]. Ford discussed the vacuum energy of a massless conformal scalar field in an Einstein universe by the mode sum method. Ford obtained the renormalized vacuum energy density in this case as

$$\rho = \frac{\hbar c}{480\pi^2 R_0^4} \quad , \tag{1.3}$$

where, R_0 is the radius of the universe and the pressure is given by $P = \frac{1}{3}\rho$.

Thus the energy momentum tensor is in the same form as that for the classical radition. In Ford's work on the Casimir effect in an Einstein universe, he showed that the energy is associated with the closed spatial topology. Ford used the mode sum method and employed an exponential cutoff in his calculations. The only divergence he encountered was quartic in the cutoff parameter, which could be eliminated easily by subtracting the divergent vacuum energy that one has in the Minkowski spacetime case, with no boundaries and for an equivalent volume. Later, Dowker and Critchley [12] verified Ford's result by using the covariant point-splitting method. Their calculation was not only covariant but also cutoff independent.

Even though Dowker and Critchley used the global topology of the universe to construct the Green's function, their result is also expected to be true locally i.e. the local renormalized energy density and pressure should be given by

$$(< T_0^0 >=)\rho = \frac{\hbar c}{480\pi^2 R_0^4} \quad , \qquad (- < T_1^1 >= - < T_2^2 >= - < T_3^3 >=)\, P = \frac{1}{3}\rho \quad . \tag{1.4}$$

In this thesis we discuss the quantum vacuum energy calculation for the massless conformal scalar field in a half Einstein universe by various methods [13,14]. We also present the details of some of the tedious calculations that are commonly used in this field.

The thesis is organized as follows. In section II we study the massless conformal scalar field with a spherical boundary located at $\chi_0 = \frac{\pi}{2}$ and with the interior geometry represented by the static closed Friedmann metric (This case is also called the half Einstein universe). We consider both the Dirichlet and the Neumann conditions. We first calculate the renormalized vacuum energy by using the mode sum method and compare the effects of different cutoff functions. Next, in section III we solve the problem by using the covariant point-splitting method. We construct the Green's function directly by using the wave functions satisfying the desired boundary conditions. Expressing $< T_{\mu\nu} >$ as the coincidence limit, we obtain the renormalized energy momentum tensor which turns out to be identical to the result given in Eq.(1.4) [13,14]. Half space case is very important in the sense that despite being in curved spacetime, and with a spherical boundary, it is exactly solvable by various techniques, thus allowing us to understand and compare the effects of different methods. Our results confirm the conclusions of Kennedy and Unwin [15], where they have written the Green's function for the full Einstein universe as an image sum and obtained the necessary Green's function for the "half Einstein universe" case by locating an image charge in the dual region.

Our results are obtained for static space time. In Appendix A, for future reference, we present the details of the calculation of the renormalized quantum vacuum energy momentum tensor for the massless conformal scalar field in a closed Friedmann universe by using the covariant point-splitting regularization method [16,17,18]. Finally, Appendix B includes some details of the Green's function calculations.

Our sign conventions are such that the metric signature is $(+,-,-,-)$, $R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\gamma,\delta} - ..$, and $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ [18]. The usual summation convention is in effect over Greek (spacetime) indices. Summations over latin (three-space) indices are indicated explicitly, but an index may be omitted from the summation sign when there is no chance of confusion. The units are such that $\hbar = c = 1$. We abbreviate the spacetime point $(t, \vec{x}) = (x^0, \vec{x})$ as x.

CHAPTER II

RENORMALIZED VACUUM ENERGY OF THE MASSLESS CONFORMAL SCALAR FIELD BY THE MODE SUM METHOD

This chapter consists of two parts. In the first part we will consider the quantum vacuum energy of the massless conformal scalar field in the full Einstein universe by the mode sum method. Later, we will consider the half Einstein universe by using the mode sum method.

2.1 Casimir Energy Calculation in an Einstein Universe

We begin with the basic properties of a massless conformal scalar field considered on curved background Einstein (static closed Friedmann universe) geometry, where the metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R_0^2[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \quad , \tag{2.1}$$

where, χ and θ run from 0 to π, while ϕ runs from 0 to 2π, and R_0 represents the radius of the universe and it is a constant.

The Lagrangian density is given as [18]

$$\mathcal{L}(\vec{x}) = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\Phi_{;\mu}(\vec{x})\Phi_{;\nu}(\vec{x}) - \frac{R}{6}\Phi^2(\vec{x})\right) \quad . \tag{2.2}$$

We construct the action

$$S = \int \mathcal{L}(\vec{x})d^4x \quad . \tag{2.3}$$

Varying S with respect to Φ, we obtain the scalar field equation,

$$0 = \frac{\delta S}{\delta \Phi} = \sqrt{-g}\left(\Phi_{;\mu}^{;\mu}(\vec{x}) + \frac{R}{6}\Phi(\vec{x})\right), \tag{2.4}$$

where $\frac{\delta}{\delta\Phi}$ indicates functional differentation, g is the determinant of the background metric $g_{\mu\nu}$, and $R(=\frac{6}{R_0^2})$ is the Ricci scalar corresponding to the metric Eq.(2.1). Hence, the conformally invariant Klein-Gordon equation for the massless conformal scalar field is given as [10,18]

$$\Box\Phi + \frac{1}{R_0^2}\Phi = 0 \quad . \tag{2.5}$$

Here, $\Box$ is the D'Alembertian operator given by

$$\Box\Phi = \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\mu\nu}\Phi_{;\mu})_{;\nu} \quad . \tag{2.6}$$

Solution of equation (2.5) could be easily found by using the method of separation of variables and is given as [10,19]

$$\Phi(x) = c_0\Pi_{w,l}(\chi)P_l^m(\cos\theta)e^{im\phi}e^{-iwt} \quad , \tag{2.7}$$

where $P_l^m(\cos\theta)$ is the associated Legendre function, c_0 is an appropriate normalization constant and

$$\begin{aligned} l &= 0,1,2,....,and \\ m &= -l,-l+1,...,0,1.2.....,(l-1),l \quad . \end{aligned} \tag{2.8}$$

$\Pi_{w,l}(\chi)$ satisfies the following differential equation:

$$\frac{d}{d\chi}\left(\sin^2\chi\frac{d\Pi_{w,l}}{d\chi}\right) + \left[(R_0^2w^2-1)\sin^2\chi - l(l+1)\right]\Pi_{w,l}(\chi) = 0 \quad . \tag{2.9}$$

Making the following substitution:

$$\Pi_{w,l}(\chi) \propto \sin^l(\chi)C(\cos\chi) \quad , \tag{2.10}$$

the equation to be solved for $C(\cos\chi)$ becomes

$$(1-x^2)\frac{d^2C}{dx^2} - x(2l+3)\frac{dC}{dx} + [(w^2-1) - l(l+2)]C = 0 \quad , \tag{2.11}$$

where $x = \cos\chi$, $x\epsilon[-1,1]$, and $N^2 = R_0^2w^2$.

This differential equation could be solved by the method of Frobenius. Hence, we express the function $C(x)$ by a power series as

$$C(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \quad . \tag{2.12}$$

Using the above expansion to solve Eq.(2.11), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \quad & \{(n+\alpha+1)(n+\alpha+2)a_{n+2} - [(n+\alpha)(n+\alpha-1) \\ & +(n+\alpha)(2l+3) - (N^2-1) + l(l+2)]a_n\}x^{n+\alpha} = 0 \quad , \end{aligned} \tag{2.13}$$

or

$$\sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+2} - [(n+l)^2 + 2(n+l) - (N^2-1)]a_n\}x^n = 0 \quad , \tag{2.14}$$

so that the coefficients a_n must satisfy the recursion relation:

$$\frac{a_{n+2}}{a_n} = \frac{(n+l+1)^2 - N^2}{(n+1)(n+2)} \quad . \tag{2.15}$$

The above recursion relation gives us the following series solution for $C(x)$:

$$C(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1} \quad , \tag{2.16}$$

$$\begin{aligned} C(x) \quad = \quad & a_0\{1 + \frac{(l+1)^2 - N^2}{2!}x^2 + ...\} \\ & +a_1\{x + \frac{(l+2)^2 - N^2}{3!}x^3 + ...\} \quad . \end{aligned} \tag{2.17}$$

This is in the form

$$C(x) = a_0 C_1(x) + a_1 C_2(x) \quad , \tag{2.18}$$

where $C_1(x)$ and $C_2(x)$ are linearly independent. Convergence of the series at the end points of our interval could be checked easily by using the Raabe test which says that if $\sum_{n=0}^{\infty} u_n$ is a series of positive terms and if the $\lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) = A$ the series is divergent for $A < 1$, convergent for $A > 1$ and the test fails for $A = 0$

[19]. At the end points we apply this test to the series with even powers. We write

$$\lim_{n\to\infty} n(\frac{a_{2n}}{a_{2n+2}} - 1) = A \quad , \tag{2.19}$$

and using Eq.(2.15), one obtains A as $\frac{1}{2} - l(l \geq 0)$. Hence, the infinite series $C_1(x)$ diverges at the end points. One could similarly check that $C_2(x)$ is also divergent at the end points of our interval. To obtain a regular solution in the entire interval $x\epsilon[-1,1]$ we terminate the infinite series after a finite number of terms by restricting N to have values given as

$$N = (n' + l) + 1 \quad , \tag{2.20}$$

where $n' = 0,1,2,3,4,....,$and $l = 0,1,2,3,4,......$ Thus, the integer values of w_N can be written as

$$w_{n',l} = \frac{N(n',l)}{R_0} \quad . \tag{2.21}$$

The polynomial solutions obtained this way are the well known Gegenbauer polynomials, hence we could write $C(x)$ in terms of the Gegenbauer polynomials as

$$C(x) = C^{l+1}_{N-l-1}(\cos\chi) \quad . \tag{2.22}$$

Values that $N(n',l)$ could take are presented in table (2.1).

Usually one rearranges the eigenvalues and defines a new index n such that the eigenvalues $N(n)$ and the degeneracy $g(n)$ are given as in table (2.2). These values are also given by

$$N(n) = n \qquad and \qquad g(n) = n^2 \qquad where \quad n = 1,2,3,4,5,6,7,... \quad . \tag{2.23}$$

Hence, $\Pi_{N,l}(\chi)$ could be given as

$$\Pi_{N,l}(\chi) = \sin^l \chi C^{l+1}_{N-l-1}(\cos\chi) \quad , \tag{2.24}$$

where

$$w_N = \frac{N}{R_0} \quad , \quad N = 1,2,3,4,5,.. \quad . \tag{2.25}$$

Table 2.1: For the Einstein universe this table gives $N(n', l)$ values, where $w = N(n', l)/R_0$.

l \ n'	0	1	2	3	4	5	6	.....
0	1	2	3	4	5	6	7	.....
1	2	3	4	5	6	7	8	.....
2	3	4	5	6	7	8	9	.....
3	4	5	6	7	8	9	10	....
4	5	6	7	8	9	10	11	....
.....	.....	.....	.....	.....	.....	.....	.....	......

Table 2.2: This table we have introduced a new index n and ordered the eigenvalues for the Einstein universe.

n	1	2	3	4	5	6	8	....
N(n)	1	2	3	4	5	6	7	....
g(n)	1	4	9	16	25	36	49	......

Appropriate normalization constant, along with the proportionality constant are absorbed into c_0 in Eq. (2.7).

Now, we discuss the Hamiltonian for the massless conformal scalar field. The Hamiltonian is obtained from the energy momentum tensor $T_{\mu\nu}$. This tensor for a scalar field in curved spacetime could be obtained from the variation of the action with respect to the metric tensor $g^{\mu\nu}$. The energy momentum tensor could be written as (explicitly calculation is given chapter III)

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} - \frac{1}{3}\Phi_{;\mu\nu}\Phi \\ &+\frac{1}{3}g_{\mu\nu}g^{\rho\sigma}\Phi\Phi_{;\rho\sigma} - \frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{12}g_{\mu\nu}R\Phi^2 \quad . \end{aligned} \tag{2.26}$$

Now setting $\mu = 0$ and $\nu = 0$ and also using $T_0^0 = g^{00}T_{00}$, we obtain

$$T_0^0 = \frac{1}{2}g^{00}\Phi_{;0}\Phi_{;0} + \frac{1}{6}g^{ij}\Phi_{;i}\Phi_{;j} - \frac{1}{3R_0^2}\Phi\Delta^{(3)}\Phi + \frac{1}{2R_0^2}\Phi^2 \quad , \tag{2.27}$$

where $\Delta^{(3)}$ is three space Laplacian. Thus, the Hamiltonian is

$$H = \int T_0^0\sqrt{-g}d^3x \quad . \tag{2.28}$$

We have set of solutions of the Eq.(2.5) which could be given as

$$F_\lambda = f_\lambda e^{-iw_N t} \quad . \tag{2.29}$$

Where λ stands for N, l and m. Here

$$f_{\lambda(N,l,m)}(\chi,\theta,\phi) = c\sin^l\chi C^{l+1}_{N-l-1}(\cos\chi)Y_{lm}(\theta\phi) \quad . \tag{2.30}$$

Where c is the normalization constant, and

$$w_N = \frac{N(n)}{R_0} \quad . \tag{2.31}$$

F_λ are the positive frequency mode functions and they form an orthonormal set with to the scalar product which is defined as [18]

$$(\Phi_1,\Phi_2) = -i\int\left(\Phi_1(\vec{x})\partial_t\Phi_2^*(\vec{x}) - \Phi_2(\vec{x})\partial_t\Phi_1^*(\vec{x})\right)\sqrt{-g}d^3x \quad . \tag{2.32}$$

Hence

$$(F_{\lambda_1}, F_{\lambda_2}) = \delta_{\lambda_1\lambda_2}. \tag{2.33}$$

In canonical quantization scheme the scalar field $\Phi(\vec{x})$ behaves like an operator and we impose the following equal-time commutation rules:

$$\begin{aligned}
&[\Phi(\vec{x}), \Phi(\vec{x'})] = 0 \quad , \\
&[\Pi(\vec{x}), \Pi(\vec{x'})] = 0 \quad , \\
&[\Phi(\vec{x}), \Pi(\vec{x'})] = i\delta(\vec{x} - \vec{x'}) \quad .
\end{aligned} \tag{2.34}$$

where Π is the canonically conjugate momentum which is defined by

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)} = \partial_t \Phi \quad . \tag{2.35}$$

The field modes given in Eq.(2.29) and their respective complex conjugate from a complete and orthonormal basis with the scalar product Eq.(2.32). Hence, $\Phi(\vec{x})$ may be expanded as

$$\Phi(\vec{x}) = \sum_\lambda (a_\lambda F_\lambda + a_\lambda^\dagger F_\lambda^*) \quad . \tag{2.36}$$

($\dagger$ denotes Hermitian conjugate, $*$ denotes complex conjugate) The equal-time commutation relations for Φ and Π are now equivalent to

$$\begin{aligned}
&[a_{\lambda_1}, a_{\lambda_2}] = 0 \quad , \\
&[a_{\lambda_1}^\dagger, a_{\lambda_2}^\dagger] = 0 \quad , \\
&[a_{\lambda_1}, a_{\lambda_2}^\dagger] = \delta_{\lambda_1\lambda_2} \quad .
\end{aligned} \tag{2.37}$$

Where, a_λ is referred to as an annihilation operator and $a_\lambda^\dagger$ as a creation operator for quanta in the mode λ.

We now return to Eq.(2.33). Using the spatial functions f_λ may be normalized so that

$$\int f_{\lambda_1} f_{\lambda_2} \sqrt{-g}\, d^3x = \frac{1}{2w} \delta_{\lambda_1\lambda_2} \quad . \tag{2.38}$$

The Hamiltonian now takes the familiar form

$$H = \frac{1}{2} \sum_{\lambda(N,l,m)} w_N (a_\lambda a_\lambda^\dagger + a_\lambda^\dagger a_\lambda) \quad . \tag{2.39}$$

The operators a_λ and $a_\lambda^\dagger$ satisfy the usual commutation relations and may be used to define a Fock space [10,18]. The existence of a global timelike killing vector leads us naturally to a unique choice for the vacuum state, which is defined as

$$a_\lambda \mid 0 >= 0 \quad , \qquad \text{for all } \lambda \text{ (N,l,m)} \; . \tag{2.40}$$

We take the expectation values of the Hamiltonian at the vacuum state, hence the vacuum energy is given as

$$E_0 = \sum_{n=1}^{\infty} \frac{1}{2} w_n g(n) \quad ,$$

$$\text{where} \quad w_n = \frac{n}{R_0} \qquad g(n) = n^2 \qquad n = 1, 2, 3, 4, ...$$

$$\text{becomes} \quad E_0 = \frac{1}{2R_0} \sum_{n=1}^{\infty} n^3 \quad . \tag{2.41}$$

This divergent expression must now be regularized by the insertion of a cutoff function. This regularization should be legitimized on physical grounds. We consider the Casimir attractive force for the electromagnetic field bounded by two uncharged parallel conducting plates at absolute zero temperature. This attractive force has appeared due to the fact that the electromagnetic field modes has discrete spectrum related to the plate separation. Clearly, this disturbance affects only the long wavelength modes. Since the conductivity of physical materials actually decreases at high frequencies and the conducting surfaces become transparent to electromagnetic waves of very short wavelength, the ultraviolet divergences are obviously irrelevant and may be regularized away. Under this legitimacy, the divergent vacuum energy Eq.(2.41) could be regularized by an exponential cutoff function. The convenient cutoff function which we choose is [10,11];

$$f(\alpha, w_n) = e^{-\alpha w_n} \quad , \tag{2.42}$$

where α is a cutoff parameter and w_n is given in Eq.(2.40). Hence, the cutoff vacuum energy is defined as

$$\begin{aligned} E_0 &= \sum_{\lambda} \frac{g_n}{2} w_n e^{-\alpha w_n} \quad , \\ &= \frac{1}{2R_0} \sum_{n=1}^{\infty} n^3 e^{-\alpha \frac{n}{R_0}} \quad . \end{aligned} \tag{2.43}$$

Here, α is a cutoff parameter which will be set to zero at the end.

Before we proceed, we recall the Euler-Maclaurin sum formula, which could be given as [19,20]

$$\begin{aligned} \sum_{j=0}^{n} F(j) &= \int_0^n F(x)dx + \frac{1}{2}F(0) + \frac{1}{2}F(n) \\ &+ \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!}(F^{(2s-1)}(n) - F^{(2s-1)}(0)) \\ &+ \int_0^n \frac{B_{2m} - B_{2m}(x-[x])}{(2m)!} F^{(2m)}(x)dx \\ &= F(0) + F(1) + \dots\dots\dots + F(n) \quad , \end{aligned} \tag{2.44}$$

where, m and n are integers such that $n > 0$ and $m > 0$ and

$$F^{(2m)}(x) = \frac{d^{2m}F(x)}{dx^{2m}} \tag{2.45}$$

is absolutely integrable over the interval $(0, n)$. Also $[x]$ denotes the integer in the interval $(x-1, x]$; in consequence, as a function of x, $B_{2m}(x-[x])$ is periodic and continuous, with period 1. B_{2m} are the Bernoulli numbers. Significance of the Euler-Maclaurin sum formula is that it could be used to evaluate a given sum in terms of the integral of a continuous and differentiable function, plus some correction terms.

We can rewrite the sum given in Eq.(2.43) using the Euler-Maclaurin sum formula, which leads to

$$
\begin{aligned}
E_0 &= \frac{1}{2R_0}\Bigg(\int_0^\infty F(n)dn + \frac{1}{2}F(0) \\
&+\frac{1}{2}F(\infty) + \frac{B_2}{2!}(F^{(1)}(\infty) - F^{(1)}(0)) \\
&+\frac{B_4}{4!}(F^{(3)}(\infty) - F^{(3)}(0)) \\
&+\frac{B_6}{6!}(F^{(5)}(\infty) - F^{(5)}(0)) + \ldots\ldots\ldots\Bigg) . \qquad (2.46)
\end{aligned}
$$

One could easily check that as m$\rightarrow \infty$ the remainder term $\rightarrow 0$. In equation (2.46) F(n) is given as

$$F(n) = n^3 e^{-\alpha \frac{n}{R_0}} \quad . \qquad (2.47)$$

The continuous and differentiable function F(n) and its derivatives up to the fifth order are given as

$$
\begin{aligned}
&F(0) = 0 && ; F(\infty) = 0 \\
&F^{(1)}(0) = 0 && ; F^{(1)}(\infty) = 0 \\
&F^{(2)}(0) = 0 && ; F^{(2)}(\infty) = 0 \\
&F^{(3)}(0) = 6 && ; F^{(3)}(\infty) = 0 \\
&F^{(4)}(0) = -24\frac{\alpha}{R_0} && ; F^{(4)}(\infty) = 0 \\
&F^{(5)}(0) = 60\frac{\alpha^2}{R_0^2} && ; F^{(5)}(\infty) = 0 \quad ,
\end{aligned}
\qquad (2.48)
$$

and

$$\int_0^\infty F(x)dx = \frac{6R_0^4}{\alpha^4} \quad . \qquad (2.49)$$

The vacuum energy given equation (2.43) becomes

$$E_0 = 3\frac{R_0^3}{\alpha^4} + \frac{1}{240R_0} + \frac{17\alpha^2}{360R_0^2} + \{ \text{ terms in positive powers of } \alpha \} \quad . \qquad (2.50)$$

If one divides E_0 by the spatial volume of the universe $2\pi^2 R_0^3$ one gets the divergent vacuum energy in an Einstein universe. It is appearent that the cutoff vacuum enegy density in an equal volume of Minkowski space is just

$$\bar{\rho}_0 = \frac{3}{2\pi^2 \alpha^4} \quad . \tag{2.51}$$

This quantity is independent of the physical properties of the system and diverges in the limit as $\alpha \to 0$. Hence, we define the physical vacuum energy density in an Einstein universe as

$$\rho = \lim_{\alpha \to 0} (\rho_0 - \bar{\rho}_0) \quad . \tag{2.52}$$

$$\rho = \frac{1}{480\pi^2 R_0^4} \quad . \tag{2.53}$$

The corresponding pressure may be obtained from the requirements that the conformally invariant energy momentum tensor be traceless. Since our renormalization procedure consists in substracting the Minkowski space energy momentum tensor from the Einstein universe energy momentum tensor, and since both are traceless, the renormalized energy momentum tensor is also traceless. Hence the pressure is [10]

$$P = -\frac{\rho}{3} \quad . \tag{2.54}$$

Thus, the only nonzero components of the renormalized energy momentum tensor of the vacuum state of the massless conformal scalar field on the background Einstein universe are given as [10,18]

$$< T_0^0 >_{ren} = \rho \quad , \qquad \text{and} \qquad < T_\chi^\chi > = < T_\theta^\theta > = < T_\phi^\phi > = -P \quad . \tag{2.55}$$

We will now investigate the massless conformal scalar field with a spherical boundary located at $\chi_0 = \frac{\pi}{2}$ and with the interior geometry represented by static closed Friedmann metric. The metric inside our boundary may be expressed as

$$ds^2 = dt^2 - R_0^2[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \quad , \tag{2.56}$$

where $\chi \epsilon\, [0, \pi/2]$, $\theta \epsilon\, [0, \pi]$, and $\phi \epsilon\, [0, 2\pi]$. We consider a massless conformal scalar field on this curved background geometry where the wave equation is given by

$$\Box\Phi + \frac{1}{R_0^2}\Phi = 0 \quad . \tag{2.57}$$

Solution of Eq.(2.57) could be found easily as [19,21]

$$\Phi_{N,l,m}(x) = c\Pi_{N,l}(\chi)Y_{lm}(\theta,\phi)e^{-iwt} \quad . \tag{2.58}$$

Where

$$\Pi_{N,l}(\chi) = \sin^l\chi \frac{d^{l+1} \quad \cos N\chi}{d(\cos\chi)^{l+1}} \quad , \qquad \text{and} \qquad w = \frac{N}{R_0} \quad . \tag{2.59}$$

For some specific values of l, $\Pi_{N,l}(\chi)$ and $\frac{d}{d\chi}\Pi_{N,l}(\chi)$ could be written as, respectively

$$\Pi_{N,0}(\chi) = \frac{N\sin N\chi}{\sin\chi} \quad ,$$

$$\Pi_{N,1}(\chi) = -\frac{N^2\cos N\chi}{\sin\chi} + \frac{N\sin N\chi\cos\chi}{\sin^2\chi} \quad ,$$

$$\Pi_{N,2}(\chi) = \frac{(N - N^3)\sin N\chi}{\sin\chi} - \frac{3N^2\cos N\chi\cos\chi}{\sin^2\chi} + \frac{3N\sin N\chi\cos^2\chi}{\sin^3\chi} \quad ,$$

$$\Pi_{N,3}(\chi) = \frac{(-4N^2 + N^4)\cos N\chi}{\sin\chi} + \frac{(9N - 6N^3)\sin N\chi\cos\chi}{\sin^2\chi}$$

$$-\frac{15N^2\cos N\chi\cos^2\chi}{\sin^3\chi}+\frac{15N\sin N\chi\cos^3\chi}{\sin^4\chi} \quad \text{and}$$

$$\begin{aligned}\Pi_{N,4}(\chi) &= \frac{(N^5-10N^3+9N)\sin N\chi}{\sin\chi}+\frac{(10N^4-55N^2)\cos N\chi\cos\chi}{\sin^2\chi}\\ &\quad -\frac{(45N^3+90N)\sin N\chi\cos^2\chi}{\sin^3\chi}-\frac{105N^2\cos N\chi\cos^3\chi}{\sin^4\chi}\\ &\quad +\frac{105N\sin N\chi\cos^4\chi}{\sin^5\chi}\ .\end{aligned} \tag{2.60}$$

$$\begin{aligned}\frac{d}{d\chi}\Pi_{N,0}(\chi) &= \frac{N^2\cos N\chi}{\sin\chi}-\frac{N\sin N\chi\cos\chi}{\sin^2\chi}\ ,\\ \frac{d}{d\chi}\Pi_{N,1}(\chi) &= \frac{(N^3-N)\sin N\chi}{\sin\chi}+\frac{2N^2\cos N\chi\cos\chi}{\sin^2\chi}-\frac{2N\sin N\chi\cos^2\chi}{\sin^3\chi}\ ,\\ \frac{d}{d\chi}\Pi_{N,2}(\chi) &= \frac{(-N^4+4N^2)\cos N\chi}{\sin\chi}+\frac{(4N^3-7N)\sin N\chi\cos\chi}{\sin^2\chi}\\ &\quad +\frac{90N^2\cos N\chi\cos^2\chi}{\sin^3\chi}-\frac{9N\sin N\chi\cos^3\chi}{\sin^4\chi}\ ,\\ \frac{d}{d\chi}\Pi_{N,3}(\chi) &= \frac{(-N^5+10N^3-9N)\sin N\chi}{\sin\chi}+\frac{(-7N^4+43N^2)\cos N\chi\cos\chi}{\sin^2\chi}\\ &\quad +\frac{(27N^3+27N)\sin N\chi\cos^2\chi}{\sin^3\chi}+\frac{60N^2\cos N\chi\cos^3\chi}{\sin^4\chi}\\ &\quad -\frac{60N\sin N\chi\cos^4\chi}{\sin^5\chi} \quad \text{and}\\ \frac{d}{d\chi}\Pi_{N,4}(\chi) &= \frac{(N^6-20N^4+44N^2)\cos N\chi}{\sin\chi}\end{aligned}$$

$$+\frac{(-11N^5+155N^3-9N^2+180N)\sin N\chi\cos\chi}{\sin^2\chi}$$

$$+\frac{(-65N^4+335N^2)\cos N\chi\cos^2\chi}{\sin^3\chi}+\frac{(150N^3-330N)\sin N\chi\cos^3\chi}{\sin^4\chi}$$

$$+\frac{525N^2\cos N\chi\cos^4\chi}{\sin^5\chi}-\frac{525N\sin N\chi\cos^5\chi}{\sin^6\chi} \quad . \tag{2.61}$$

We note that the boundary condition at $\chi = \chi_0$ has not been imposed on Eq.(2.58) yet. Hence, N still remains as a continuous parameter, while l and m take the values.

$$\begin{aligned} l &= 0,1,2,.... \; , \quad \text{and} \\ m &= -l,-l+1,...,0,1.2.....,(l-1),l \quad . \end{aligned} \tag{2.62}$$

We now impose the Dirichlet boundary condition for our problem i.e.

$$\Phi_{N,l,m}(\chi_0) = 0 \qquad \text{at} \qquad \chi_0 = \frac{\pi}{2} \quad . \tag{2.63}$$

Using the wave functions given in Eq.(2.60) we immediately get integer values for N, which could be tabulated as in table (2.3). We reorder the energy eigenvalues, and introduce a new index n. The new eigenvalues $N(n)$ and the degenaracy $g(n)$ are given in the table (2.4) and could be written as

$$N(n) = (n+2) \quad , \qquad g(n) = \frac{1}{2}(n+1)(n+2) \quad \text{where} \quad n = 0,1,2,3,... \quad . \tag{2.64}$$

The new eigenfunctions Eq.(2.58) are still the Gegenbauer polynomials, but only the ones that satisfy the boundary condition are picked. We compare the full Einstein universe case which is given in table (2.1) with the table (2.3). Table (2.3) could be formed by just taking the second, fourth, etc columns of the table (2.1).

Table 2.3: For the Dirichlet case this table gives $N(n', l)$ values, where $w = N(n', l)/R_0$.

l \ n'	0	1	2	3	4	5	6	.....
0	2	4	6	8	10	12	14	.....
1	3	5	7	9	11	13	15	.....
2	4	6	8	10	12	14	16	.....
3	5	7	9	11	13	15	17	....
4	6	8	10	12	14	16	18	....
.....	.....	.....	.....	.....	.....	.....	.....	......

Table 2.4: This table we have introduced a new index n and ordered the eigenvalues for the Dirichlet case.

n	0	1	2	3	4	5	6	....
N(n)	2	3	4	5	6	7	8	....
g(n)	1	3	6	10	15	21	28	......

The divergent vacuum energy, which could be written as

$$E_0 = \sum_n \frac{1}{2} w_n g(n) \quad ,$$

where $w_n = \frac{(n+2)}{R_0} \quad , \quad g(n) = \frac{1}{2}(n+1)(n+2) \qquad n = 0,1,2,3,4,\ldots \quad ,$

becomes

$$E_0 = \frac{1}{4R_0} \sum_{n=0}^{\infty} (n+2)^2 (n+1) \quad . \tag{2.65}$$

This divergent vacuum energy could be regularized by using the same exponential cutoff function used for the full Einstein universe case as [10]

$$E_0 = \sum_n \frac{1}{2} w_n(n) g(n) e^{-\alpha w_n} \quad ,$$

$$E_0 = \frac{1}{4R_0} \sum_{n=0}^{\infty} (n+2)^2 (n+1) e^{-\alpha(n+2)/R_0} \quad . \tag{2.66}$$

Redefining a new index $n' = n + 2$. Dropping primes one now obtains cutoff dependent vacuum energy for the Dirichlet boundary condition as

$$E_0 = \frac{1}{4R_0} \sum_{n=0}^{\infty} (n^3 - n^2) e^{-\alpha n/R_0} \quad , \tag{2.67}$$

where α is cutoff parameter which will be set to zero at the end. Using the Euler-Maclaurin sum formula Eq.(2.44) the vacuum energy given in Eq.(2.67) becomes

$$\begin{aligned} E_0 = \frac{1}{4R_0} \Bigg(& \int_0^{\infty} F(n) dn + \frac{1}{2} F(0) \\ & + \frac{1}{2} F(\infty) + \frac{B_2}{2!} (F^{(1)}(\infty) - F^{(1)}(0)) \\ & + \frac{B_4}{4!} (F^{(3)}(\infty) - F^{(3)}(0)) \\ & + \frac{B_6}{6!} (F^{(5)}(\infty) - F^{(5)}(0)) + \ldots\ldots\ldots \Bigg) . \end{aligned} \tag{2.68}$$

Here B_{2n} is a Bernoulli number and F(n) is defined by

$$F(n) = (n^3 - n^2) e^{-\alpha \frac{n}{R_0}} \quad . \tag{2.69}$$

The continuous and differentiable function F(n) and its derivatives up to the fifth order are given as

$$
\begin{aligned}
&F(0) = 0 && ; F(\infty) = 0 \\
&F^{(1)}(0) = 0 && ; F^{(1)}(\infty) = 0 \\
&F^{(2)}(0) = -2 && ; F^{(2)}(\infty) = 0 \\
&F^{(3)}(0) = 6\frac{\alpha}{R_0} + 6 && ; F^{(3)}(\infty) = 0 \\
&F^{(4)}(0) = -12\frac{\alpha^2}{R_0^2} && ; F^{(4)}(\infty) = 0 \\
&F^{(5)}(0) = 20\frac{\alpha^3}{R_0^3} + 60\frac{\alpha^2}{R_0^2} && ; F^{(5)}(\infty) = 0 \ .
\end{aligned}
\tag{2.70}
$$

And

$$\int_0^\infty F(n)dn = 6\frac{R_0^4}{\alpha^4} - 2\frac{R_0^3}{\alpha^3} \ . \tag{2.71}$$

Sustutiting Eq.(2.70) and Eq.(2.71) into the Eq.(2.68) the cutoff vacuum energy becomes

$$E_0 = \frac{3}{2}\frac{R_0^3}{\alpha^4} - \frac{R_0^2}{2\alpha^3} + \frac{1}{480\pi R_0} + \{\text{ terms in positive powers of } \alpha\} \ . \tag{2.72}$$

We see that this expression contains quartic and cubic divergences in α as $\alpha \to 0$. Renormalization procedure works as usual by subtracting the divergent vacuum energy in the Minkowski spacetime case (without a boundary and for an equivalent volume).

We now consider the Neumann boundary condition, which is given by

$$n^\alpha \nabla_\alpha \Phi(x) = 0 \ , \qquad \text{at} \qquad \chi_0 = \frac{\pi}{2} \ , \tag{2.73}$$

where n^α is the outward normal to the boundary. Hence, for the Neumann boundary condition

$$\frac{d}{d\chi}\Pi_{N,l}(\chi) = 0 \ , \qquad \text{at} \quad \chi_0 = \frac{\pi}{2} \ . \tag{2.74}$$

Using the equations in Eq.(2.61) we obtain the $N(n', l)$ values presented in table (2.5). As before, one rearranges the eigenvalues and defines a new index n such

Table 2.5: For the Neumann case this table gives $N(n', l)$ values, where $w = N(n', l)/R_0$.

l \ n'	1	2	3	4	5	6	7	.....
0	1	3	5	7	9	11	13	.....
1	2	4	6	8	10	12	14	.....
2	3	5	7	9	11	13	15	.....
3	4	6	8	10	12	14	16	....
4	5	7	9	11	13	15	17	....
.....	.....	.....	.....	.....	.....	.....	.....	......

Table 2.6: This table we have introduced a new index n and ordered the eigenvalues for the Neumann case.

n	1	2	3	4	5	6	7	....
N(n)	1	2	3	4	5	6	7	....
g(n)	1	3	6	10	15	21	28	......

that the eigenvalues $N(n)$ and the degeneracy $g(n)$ are given as in table (2.6). One could also use the following expressions for the eigenfrequencies w_n and the degeneracy $g(n)$:

$$w(n) = \frac{n}{R_0} \quad , \qquad g(n) = \frac{1}{2}n(n+1) \qquad \text{where} \quad n = 1, 2, 3, 4, \ldots \quad . \tag{2.75}$$

Notice that the $N(n', l)$ values in table (2.5) could be obtained from the full Einstein universe case in table(2.1) by taking the first, third, etc. columns. Once again, the eigenfunctions are the Gegenbauer polynomials, but only the ones that satisfy the Neumann boundary condition are allowed. Using the table (2.6) we obtain divergent vacuum energy for the Neumann boundary case with the same exponential cutoff:

$$\begin{aligned} E_0 &= \sum_n \frac{1}{2} w_n g(n) e^{-\alpha w_n} \quad , \\ E_0 &= \frac{1}{4R_0} \sum_{n=0}^{\infty} n^2(n+1) e^{-\alpha n/R_0} \quad . \end{aligned} \tag{2.76}$$

Now, using the Euler-Maclaurin sum formula Eq.(2.44) one obtains the regularized vacuum energy E_0:

$$\begin{aligned} E_0 &= \frac{1}{4R_0} \Bigg(\int_0^{\infty} F(n) dn + \frac{1}{2} F(0) \\ &+ \frac{1}{2} F(\infty) + \frac{B_2}{2!} (F^{(1)}(\infty) - F^{(1)}(0)) \\ &+ \frac{B_4}{4!} (F^{(3)}(\infty) - F^{(3)}(0)) \\ &+ \frac{B_6}{6!} (F^{(5)}(\infty) - F^{(5)}(0)) + \ldots\ldots.. \Bigg) , \end{aligned} \tag{2.77}$$

where

$$F(n) = (n^3 + n^2) e^{-\alpha \frac{n}{R_0}} \quad , \tag{2.78}$$

and $B_{2n}$is the Bernoulli number, α is a cutoff parameter which will be set to zero at the end. The continuous and differentiable function $F(n)$ and its derivatives up to the fifth order are given as

$$
\begin{aligned}
&F(0) = 0 && ; F(\infty) = 0 \\
&F^{(1)}(0) = 0 && ; F^{(1)}(\infty) = 0 \\
&F^{(2)}(0) = -2 && ; F^{(2)}(\infty) = 0 \\
&F^{(3)}(0) = -6\frac{\alpha}{R_0} + 6 && ; F^{(3)}(\infty) = 0 \\
&F^{(4)}(0) = 12\frac{\alpha^2}{R_0^2} - 24\frac{\alpha}{R_0} && ; F^{(4)}(\infty) = 0 \\
&F^{(5)}(0) = -20\frac{\alpha^3}{R_0^3} + 60\frac{\alpha^2}{R_0^2} && ; F^{(5)}(\infty) = 0 \quad . \qquad (2.79)
\end{aligned}
$$

And

$$\int_0^\infty F(n)dn = 6\frac{R_0^4}{\alpha^4} + 2\frac{R_0^3}{\alpha^3} \quad . \qquad (2.80)$$

Thus, the cutoff vacuum energy becomes

$$E_0 = \frac{3}{2}\frac{R_0^3}{\alpha^4} + \frac{R_0^2}{2\alpha^3} + \frac{1}{480\pi R_0} + \{\text{ terms in positive powers of } \alpha\} \quad . \qquad (2.81)$$

We see that the cubic divergence is still there even though its sign is opposite to the Dirichlet boundary condition case Eq.(2.72). In the flat spacetime case (with the spherical boundary) Boyer[6] avoided this problem by concentrating on the electromagnetic fields, where the cubic divergence cancels between the opposite parity modes (cancellation of the cubic divergence is also true for the fermion fields [22]). However, in the electromagnetic field case there is also a quadratic divergence. Boyer handled this by enclosing the sphere in question by another sphere, whose radius was later allowed to approach infinity [6]. Improved treatments of the Boyer problem have also concentrated on the electromagnetic fields, thus leaving the cubic divergence in the scalar field case unsolved [1,2,8,9]. Of course, one could always define an ad hoc renormalization scheme that disregards all the divergences(quartic,cubic and otherwise) and keeps only the cutoff independent term as the physically meaningful renormalized vacuum energy density in half Einstein universe, which for both boundary conditions (Dirichlet and

Neumann)leads to

$$\left(\frac{(E_0)_{ren}}{\pi^2 R_0^3} =\right)\rho = \frac{1}{480\pi^2 R_0^4} \quad . \tag{2.82}$$

We introduce an alternative approach in which we use a different class of cutoff function. We consider the vacuum energy density for the Dirichlet boundary condition case as:

$$\rho = \frac{1}{4\pi^2 R_0^4}\sum_{n=0}^{\infty}(n^3 - n^2) f(\alpha, w, R_0) \quad , \tag{2.83}$$

where $f(\alpha, w, R_0)$ is a suitable cutoff function and α is a cutoff parameter. We have some restriction on the allowable cutoff function [11] i.e.

$$\lim_{w_n \to \infty} f(\alpha, w_n, R_0) \to 0 \qquad \forall \alpha \neq 0 \quad , \tag{2.84}$$

rapidly enough to remove all divergences, while for $\alpha = 0$

$$f(0, w_n, R_0) = 1 \quad . \tag{2.85}$$

The other restriction on the form of f is that it should not affect the long wavelength modes so that

$$\lim_{w_n \to 0} f(\alpha, w_n, R_0) = 1 \quad . \tag{2.86}$$

We take $f(\alpha, w_n, R_0)$ to be defined for all real, positive values of w_n so that the limit is meaningful. It is often possible to express the quantity to be substracted as an integral over w_n. The remainder is then the difference between a sum and an integral of the same formal expression. To do this, we further restrict f to depend only on w_n only in certain combinations of w_n and R_0. Thus

$$f = f(\alpha, \Omega(w_n, R_0)) \quad , \tag{2.87}$$

where Ω is a function which is to be choosen suitably. For the Dirichlet boundary condition case we choose Ω as

$$\Omega(w, R_0) = \frac{1}{4R_0}(n^4 - \frac{4}{3}n^3) \qquad \text{and} \qquad w = \frac{n}{R_0} \quad . \tag{2.88}$$

Finally we take $f(\alpha,\Omega)$ as a sufficiently smooth function such that

$$\lim_{\alpha\to 0}\left[\frac{\partial^n f}{\partial\Omega^n}\right]\to 0 \qquad \text{for} \quad n=1,2,3,4.. \quad . \tag{2.89}$$

With this choice of cutoff function and using the Euler-Maclaurin sum formula Eq.(2.44) it is easy to show that the vacuum energy density inside our boundary is given as

$$\begin{aligned}\rho &= \frac{1}{4\pi^2}\Bigg(\int_0^\infty F(n)dn+\frac{1}{2}F(0)\\ &\quad +\frac{1}{2}F(\infty)+\frac{B_2}{2!}(F^{(1)}(\infty)-F^{(1)}(0))\\ &\quad +\frac{B_4}{4!}(F^{(3)}(\infty)-F^{(3)}(0))\\ &\quad +\frac{B_6}{6!}(F^{(5)}(\infty)-F^{(5)}(0))+.........\Bigg),\end{aligned} \tag{2.90}$$

where

$$F(n)=\frac{1}{R_0^4}(n^3-n^2)f(\alpha,\Omega) \quad , \tag{2.91}$$

and B_{2n} is the Bernoulli number, α is a cutoff parameter which will be set to zero at the end. The continuous and differentiable function F(n) and its derivatives up to the fifth order are given as

$$\begin{aligned}&F(0)=0 &&;F(\infty)=0\\ &F^{(1)}(0)=0 &&;F^{(1)}(\infty)=0\\ &F^{(2)}(0)=-\frac{2}{R_0^4} &&;F^{(2)}(\infty)=0\\ &F^{(3)}(0)=6\frac{1}{R_0^4} &&;F^{(3)}(\infty)=0\\ &F^{(4)}(0)=0 &&;F^{(4)}(\infty)=0\\ &F^{(5)}(0)=-40\frac{\alpha}{R_0^4} &&;F^{(5)}(\infty)=0 \quad .\end{aligned} \tag{2.92}$$

And

$$\int_0^\infty F(n)dn=\int_0^\infty f(\alpha,\Omega)d\Omega \quad . \tag{2.93}$$

Thus, the Eq.(2.89) becomes

$$\rho = \frac{1}{4\pi^2}\left\{\int_0^\infty f(\alpha,\Omega)d\Omega + \frac{1}{120\pi R_0^4} + \{\text{ terms in positive powers of } \alpha\}\right\}. \tag{2.94}$$

The integral on the right hand side contains the divergence and could be discarded in the usual way by substracting the energy density that one would get in Minkowski spacetime with no boundary, for an equivalent volume, and also calculated with the same cutoff function. Taking the limit as α goes to zero, we obtain the renormalized vacuum energy density in the half Einstein universe for the Dirichlet boundary condition case as

$$(\rho)_{ren} = \frac{1}{480\pi^2 R_0^4} \quad . \tag{2.95}$$

For the Neumann boundary condition we could use the same class of cutoff functions. However, we now take $\Omega(w, R_0)$ as

$$\Omega(w,R_0) = \frac{1}{4R_0^4}(n^4 + \frac{4}{3}n^3) \qquad \text{and} \quad w = \frac{n}{R_0} \quad . \tag{2.96}$$

We write the cutoff depended vacuum energy density as

$$\rho = \frac{1}{4\pi^2 R_0^4}\sum_{n=0}^{\infty}(n^3 + n^2) f(\alpha, \Omega(w_n, R_0)) \quad . \tag{2.97}$$

Following the same steps that leads us to, where

$$F(n) = \frac{1}{R_0^4}(n^3 + n^2) f(\alpha,\Omega) \quad , \tag{2.98}$$

and using

$$\begin{aligned} F(0) &= 0 \;, \\ F^{(1)}(0) &= 0 \;, \\ F^{(3)}(0) &= \frac{6}{R_0} \;, \qquad \text{and} \\ \lim_{\alpha\to 0} F^{(n)}(0) &\to 0 \qquad n > 3 \;, \end{aligned} \tag{2.99}$$

and the form of ρ for infinite R_0, which is

$$\bar{\rho} = \frac{1}{4\pi^2}\int_0^\infty f(\alpha,\Omega)d\Omega \quad , \tag{2.100}$$

we obtain $\rho_{ren.}$ as

$$\rho_{ren.} = \lim_{\alpha \to 0} (\rho - \bar{\rho}) = \frac{1}{480\pi^2 R_0^4} \quad . \tag{2.101}$$

CHAPTER III

CALCULATION OF THE RENORMALIZED QUANTUM VACUUM ENERGY MOMENTUM TENSOR BY THE POINT-SPLITTING METHOD

In this chapter, we evaluate the vacuum stress tensor for the massless conformal scalar field in a static closed Friedmann (Einstein universe) background space time and the renormalized vacuum energy momentum tensor in half Einstein universe by using the covariant point-splitting method. For the half Einstein universe we construct the Green's functions directly by using the eigenfunctions, which are obtained by solving the wave equation with the appropriate boundary conditions (Dirichlet and Neumann) [13,14].

3.1 Einstein Universe Case

The metric for the Einstein (Static closed Friedmann) universe could be given as

$$ds^2 = dt^2 - R_0^2[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \quad , \tag{3.1}$$

where χ and θ run from 0 to π, while ϕ runs from 0 to 2π, and R_0 is the radius of S^3.

The Lagrangian density and the action functional for the massless conformal scalar field in a curved background are given, respectively as

$$\mathcal{L}(\vec{x}) = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\Phi_{;\mu}(\vec{x})\Phi_{;\nu}(\vec{x}) - \frac{R}{6}\Phi^2(\vec{x})\right) \quad , \tag{3.2}$$

and

$$S = \int \mathcal{L}(\vec{x})d^4x \quad . \tag{3.3}$$

Where $\Phi(\vec{x})$ is the scalar field, g is the determinant of the background metric $g_{\mu\nu}$, and R $(= \frac{6}{R_0^2})$ is the Ricci scalar for this metric. Varying S with respect to Φ, we obtain the scalar field equation

$$0 = \frac{\delta S}{\delta \Phi} = \sqrt{-g}\left(\Phi_{;\mu}^{;\mu}(\vec{x}) + \frac{1}{R_0^2}\Phi(\vec{x})\right) \quad , \tag{3.4}$$

where $\frac{\delta}{\delta\Phi}$ indicates functional differentiation. Hence, the equation of motion is given as

$$\Box\Phi(\vec{x}) + \frac{1}{R_0^2}\Phi(\vec{x}) = 0 \quad . \tag{3.5}$$

The energy momentum tensor for a massless conformal scalar field in curved spacetime could be obtained from the variation of the action with respect to the metric tensor $g^{\mu\nu}$ as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \quad , \tag{3.6}$$

where

$$\delta S = \int \delta\mathcal{L} d^4x \quad . \tag{3.7}$$

Using the relation [18]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \quad . \tag{3.8}$$

We obtain $\delta\mathcal{L}$:(with respect to $g^{\mu\nu}$)as

$$\begin{aligned}\delta\mathcal{L} &= \frac{1}{2}\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\Phi_{;\mu}\Phi_{;\nu}\delta g^{\mu\nu} + \Phi_{;\mu}\Phi_{;\nu}\delta g^{\mu\nu}\right.\\ &\left.+\frac{1}{12}R\Phi^2\delta g^{\mu\nu} - \frac{1}{6}\delta R\Phi^2\right) .\end{aligned} \tag{3.9}$$

Using the following relations [18]

$$\begin{aligned}\delta R &= -R^{\mu\nu}\delta g_{\mu\nu} + g^{\rho\sigma}g^{\mu\nu}(\delta g_{\rho\sigma;\mu\nu} + \delta g_{\rho\mu;\sigma\nu}) \quad ,\\ \delta g^{\mu\nu} &= -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} \quad ,\\ g^{\mu\nu}\delta g_{\mu\nu} &= -g_{\mu\nu}\delta g^{\mu\nu} \quad ,\\ &\text{and}\\ R^{\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} \quad .\end{aligned} \tag{3.10}$$

We obtain

$$\delta S = \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}\left(\frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} - \frac{1}{3}\Phi_{;\mu\nu}\Phi + \frac{1}{3}g_{\mu\nu}g^{\rho\sigma}\Phi\Phi_{;\rho\sigma} - \frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{12}g_{\mu\nu}R\Phi^2\right). \quad (3.11)$$

Inserting Eq.(3.11) into Eq.(3.6) and rearranging, we obtain $T_{\mu\nu}$ as [18]

$$T_{\mu\nu} = \frac{1}{6}\left(4\Phi_{;\mu}\Phi_{;\nu} - g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} - 2\Phi_{;\mu\nu}\Phi + 2g_{\mu\nu}g^{\rho\sigma}\Phi\Phi_{;\rho\sigma} - R_{\mu\nu}\Phi^2 + \frac{1}{2}g_{\mu\nu}R\Phi^2\right). \quad (3.12)$$

Taking the vacuum expectation value of the above equation we obtain

$$< 0 \mid T_{\mu\nu} \mid 0 > = \frac{1}{6}\Big(4 < 0 \mid \Phi_{;\mu}(\vec{x})\Phi_{;\nu}(\vec{x}) \mid 0 > - g_{\mu\nu}g^{\rho\sigma} < 0 \mid \Phi_{;\rho}(\vec{x})\Phi_{;\sigma}(\vec{x}) \mid 0 > -2 < 0 \mid \Phi_{;\mu\nu}(\vec{x})\Phi(\vec{x}) \mid 0 > + 2g_{\mu\nu}g^{\rho\sigma} < 0 \mid \Phi(\vec{x})\Phi_{;\rho\sigma}(\vec{x}) \mid 0 > - R_{\mu\nu} < 0 \mid \Phi(\vec{x})\Phi(\vec{x}) \mid 0 > + \frac{1}{2}g_{\mu\nu}R < 0 \mid \Phi(\vec{x})\Phi(\vec{x}) \mid 0 >\Big). \quad (3.13)$$

In Eq.(3.12) the scalar field $\Phi(\vec{x})$ behaves like a field operator due to the canonical quantization scheme. We note that Eq.(3.13) is constructed from products of field operators and their covariant derivatives at the same spacetime point. It is this fact which causes the vacuum expectation value of the energy momentum tensor to diverge. To avoid these divergent quantities, one of the field operator $\Phi(\vec{x})$ in each product in Eq.(3.13) should be replaced by $\Phi(\vec{x'})$, where $\vec{x'}$ is some point near $\vec{x}$. Thus $< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >$ transforms like the product of two vectors, one at x, the other at x'. The point separated function $< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >$ is called the positive Wightmann function which is denoted by $G^{(+)}(x, x')$ $=< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >$. The vacuum expectation value of the anticommutator of the field could also be used, which is called the Hadamard Green's function

$$G^{(1)}(x,x') =< 0 \mid \left\{\Phi(\vec{x}), \Phi(\vec{x'})\right\} \mid 0 > \quad , \tag{3.14}$$

where $G^{(1)}(x,x')$ and $G^{(+)}(x,x')$ are related by

$$G^{(1)}(x,x') = 2ReG^{(+)}(x,x') \quad . \tag{3.15}$$

(Explicitly calculation is given in appendix B)

Using the point separated definition [18,20,23] we obtain

$$< 0 \mid \Phi_{;\mu}(\vec{x})\Phi_{;\nu}(\vec{x}) \mid 0 >= \lim_{x'\to x} \nabla_\mu \nabla_{\nu'} < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \quad , \quad \text{and} \tag{3.16}$$

$$g_{\mu\nu} g^{\rho\sigma} < 0 \mid \Phi_{;\rho}(\vec{x})\Phi_{;\sigma}(\vec{x}) \mid 0 >= \lim_{x'\to x} g_{\mu\nu'} g^{\rho\sigma'} \nabla_\rho \nabla_{\sigma'} < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \quad . \tag{3.17}$$

Here the prime on the indices indicates that the derivative is taken at the point x'. We note that $< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >$ and its derivatives are bitensors, i.e. they are functions which transform as tensors at two different points. We also have to consider

$$< 0 \mid \Phi(\vec{x})\Phi_{;\mu\nu}(\vec{x}) \mid 0 >= \lim_{x'\to x} \frac{1}{2}(\nabla_\mu \nabla_\nu + \nabla_{\mu'} \nabla_{\nu'}) < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \quad , \tag{3.18}$$

which is meaningless. Each term transforms differently as a bitensor, hence they cannot be added. We rewrite this term as

$$< 0 \mid \Phi(\vec{x})\Phi_{;\mu\nu}(\vec{x}) \mid 0 >= \lim_{x'\to x} \frac{1}{2}(g_{\nu'\lambda} \nabla^\lambda \nabla_\mu + g_{\mu\tau'} \nabla^{\tau'} \nabla_{\nu'}) < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \quad . \tag{3.19}$$

Similarly,

$$< 0 \mid g_{\mu\nu} \Phi(\vec{x})\Phi^{;\rho}{}_{;\rho}(\vec{x}) \mid 0 >= \lim_{x'\to x} \frac{1}{2} g_{\mu\nu'} (\nabla_\rho \nabla^\rho + \nabla_{\rho'} \nabla^{\rho'}) < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \quad . \tag{3.20}$$

$$< 0 \mid R_{\mu\nu} \Phi(\vec{x})\Phi(\vec{x}) \mid 0 >= \lim_{x'\to x} \frac{1}{2}(R^\sigma_\mu g_{\sigma\nu'} + R^{\rho'}_{\nu'} g_{\mu\rho'}) < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \quad . \tag{3.21}$$

Finally,

$$< 0 \mid g_{\mu\nu} R\Phi(\vec{x})\Phi(\vec{x}) \mid 0 >= \lim_{x'\to x} g_{\mu\nu'} R < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \quad . \tag{3.22}$$

All the equations (3.16)-(3.22) are inserted into the Eq.(3.13) to obtain the vacuum energy momentum tensor as the coincidence limit [12]

$$\begin{aligned} < 0 \mid T_{\mu\nu} \mid 0 >= \frac{1}{6} \lim_{x'\to x} \quad & \Big(4\nabla_\mu \nabla_{\nu'} + g_{\mu\nu'}(\nabla_\rho \nabla^\rho + \nabla_{\rho'} \nabla^{\rho'} - g^{\lambda\sigma'} \nabla_\lambda \nabla_{\sigma'}) \\ & - g_{\nu'\lambda} \nabla^\lambda \nabla_\mu - g_{\mu\tau'} \nabla_{\tau'} \nabla_{\nu'} \\ & - \frac{1}{2}[R^\sigma_\mu g_{\sigma\nu'} + R^{\rho'}_{\nu'} g_{\mu\rho'} - R g_{\mu\nu'}] \Big) < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > . \end{aligned} \tag{3.23}$$

We return to the equation of motion:

$$\Box\Phi(\vec{x}) + \frac{1}{R_0^2}\Phi(\vec{x}) = 0 \quad . \tag{3.24}$$

Here $\Box$ is the D'Alembertian operator, which is given by

$$\Box\Phi(\vec{x}) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi(\vec{x})) \quad . \tag{3.25}$$

The solution of Eq.(3.24) is given as

$$\Phi(\vec{x}) = c_0 \Pi_{w,l}(\chi) Y_{lm}(\theta,\phi) e^{-iwt} \quad . \tag{3.26}$$

Where Y_{lm} are the sperical harmonics with

$$l = 0,1,2,3,.... \quad , \quad and \quad m = -l, -l+1, ..., 0, 1, ..., l-1, l \quad . \tag{3.27}$$

The differential equation that $\Pi_{w,l}(\chi)$ satisfies is;

$$\frac{d}{d\chi}\left(\sin^2\chi \frac{d\Pi_{w,l}(\chi)}{d\chi}\right) + \left[\left(w^2 - \frac{1}{R_0^2}\right) R_0^2 \sin^2\chi - l(l+1)\right] \Pi_{w,l}(\chi) = 0 \quad . \tag{3.28}$$

For the full Einstein universe regularity of the solution at $\chi = \pi$ restricts N to integer values. In this case $\Pi_{w,l}$ are the Gegenbauer polynomials which could be written as

$$\Pi_{w,l}(\chi) = \sin^l(\chi) C^{l+1}_{N-l-1}(\cos\chi) \quad , \tag{3.29}$$

where

$$w_N = \frac{N(n', l)}{R_0} \quad . \tag{3.30}$$

Values that $N(n', l)$ could take are presented in table (2.1). Thus Eq.(3.26) becomes

$$\Phi(\vec{x}) = c_0 \sin^l \chi C^{l+1}_{N-l-1}(\cos\chi) Y_{lm}(\theta, \phi) e^{-iwt} \quad , \tag{3.31}$$

where $C^{l+1}_{N-l-1}(\cos\chi)$ are the Gegenbauer polynomials and c_0 is the normalization constant. To evaluate the normalization constant c_0 we write the scalar product which is generalized to

$$(\Phi_{\lambda_1}, \Phi_{\lambda_2}) = -i \int [\Phi_{\lambda_1}(\vec{x}) \partial_t \Phi^*_{\lambda_2}(\vec{x}) - \partial_t \Phi_{\lambda_1}(\vec{x}) \Phi^*_{\lambda_2}(\vec{x})] \sqrt{-g} d^3x \quad . \tag{3.32}$$

$\Phi(\vec{x})$ given Eq.(3.31) form a complete and an orthonormal set with respect to the product Eq.(3.32)

$$(\Phi_{\lambda_1}, \Phi_{\lambda_2}) = \delta_{\lambda_1 \lambda_2} \quad , \qquad (\Phi^*_{\lambda_1}, \Phi^*_{\lambda_2}) = -\delta_{\lambda_1 \lambda_2} \quad , \qquad (\Phi_{\lambda_1}, \Phi^*_{\lambda_2}) = 0 \quad . \tag{3.33}$$

Thus, the general mode solutions $\Phi(\vec{x})$ are normalized so that

$$\begin{aligned} 1 \;=\; & c_0^2 2w R_0^3 \int_0^{\pi} \sin\chi d\chi \sin^{2l}\chi \mid C^{l+1}_{N-l-1}(\cos\chi) \mid^2 \\ & \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \mid Y_{lm}(\theta, \phi) \mid^2 \quad , \end{aligned} \tag{3.34}$$

where

$$w = \frac{N}{R_0} \quad , \qquad N = 1, 2, 3, 4, 5, ... \quad . \tag{3.35}$$

One obtains c_0 as

$$c_0 = 2^l l! \left\{ \frac{(N-l-1)!}{\pi R_0^2 (N+l)!} \right\}^{1/2} \quad . \tag{3.36}$$

Thus, the complete set of mode solutions becomes

$$\Phi_\lambda(\vec{x}) = 2^l l! \left\{ \frac{(N-l-1)!}{\pi R_0^2 (N+l)!} \right\}^{1/2} \sin^l \chi C^{l+1}_{N-l-1}(\cos\chi) Y_{lm}(\theta, \phi) e^{-iwt} \quad . \tag{3.37}$$

Here λ stands for N, l and m values which are given in table (2.1).

The general solution of the field equation could be written as sum over these modes in the form

$$\Phi(\vec{x}) = \sum_{\lambda}(a_\lambda \Phi_\lambda(\vec{x}) + a_\lambda^\dagger \Phi_\lambda^*(\vec{x})) \quad . \tag{3.38}$$

Where, a_λ is referred to as an annihilation operator and $a_\lambda^\dagger$ as a creation operator for quanta in mode λ, while $\Phi_\lambda(\vec{x})$ is the solution of the wave equation given by Eq.(3.37).

We could construct the two point function $G^{(+)}(x,x')$ from these mode function as [16,18,20]

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >= \sum_{\lambda} \Phi_\lambda(\vec{x})\Phi_\lambda^*(\vec{x'}) \quad , \tag{3.39}$$

which leads to

$$\begin{aligned} < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \; &= \; \frac{1}{\pi R_0^2} \sum_{N=1}^{\infty} e^{-iN\frac{\Delta t}{R_0}} \sum_{l=0}^{N-1} 2^{2l} \frac{(N-l-1)!(l!)^2}{(N+l)!} \\ &\quad . \sin^l \chi \sin^l \chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(\cos\chi') \\ &\quad . \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) \quad . \end{aligned} \tag{3.40}$$

Here the time separation $t-t'$ has been written as Δt. Using the addition formula for the spherical harmonics, which is given as

$$\frac{2l+1}{4\pi} P_l(\cos\gamma) = \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) \quad , \tag{3.41}$$

where

$$\cos\gamma = cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') \quad , \tag{3.42}$$

we obtain

$$\begin{aligned} < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \; &= \; \frac{1}{\pi R_0^2} \sum_{N=1}^{\infty} e^{-iN\frac{\Delta t}{R_0}} \sum_{l=0}^{N-1} 2^{2l} \frac{(l!)^2 (N-l-1)!(2l+1)}{(N+l)!} \\ &\quad \sin^l \chi \sin^l \chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(\cos\chi') P_l(\cos\gamma) \quad . \end{aligned} \tag{3.43}$$

Now, using addition theorem for the Gegenbauer polynomials which is given [24]

$$C^1_{N-1}(\cos\alpha) = \sum_{l=0}^{N-1} 2^{2l}(l!)^2 \frac{(N-l-1)!(2l+1)}{(N+l)!}$$

$$\sin^l\chi\sin^l\chi' C^{l+1}_{N-l-1}(\cos\chi)C^{l+1}_{N-l-1}(\cos\chi')P_l(\cos\gamma) \quad , \tag{3.44}$$

where

$$\cos\alpha = \cos\chi\cos\chi' + \sin\chi\sin\chi'\cos\gamma \quad , \quad \text{and} \tag{3.45}$$

also using [24]

$$C^1_{N-1}(\cos\alpha) = \frac{\sin N\alpha}{\sin\alpha} \tag{3.46}$$

we obtain

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x}') \mid 0 >= \frac{1}{4\pi^2 R_0^2}\frac{1}{\sin\alpha}\sum_{N=1}^{\infty}\left(e^{-i\Delta t/R_0}\right)^N \sin N\alpha \quad . \tag{3.47}$$

Substituting $\Delta t - i\epsilon$ for Δt with the understanding that we will let $\epsilon \to 0$ at the end we get

$$\mid e^{-i\Delta t/R_0} \mid < 1 \quad . \tag{3.48}$$

Now we could use the following relation [24]

$$\sum_{N=1}^{\infty} p^N \sin N\alpha = \frac{p\sin\alpha}{1-2p\cos\alpha+p^2} \qquad \text{if} \quad \mid p \mid < 1 \quad , \tag{3.49}$$

to get

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x}') \mid 0 >= \frac{1}{8\pi^2 R_0^2}\left(\frac{1}{\cos\frac{\Delta t}{R_0} - \cos\alpha}\right) . \tag{3.50}$$

Where α is the angle between the vectors $\vec{r}$ and $\vec{r'}$ in the directions (χ, θ, ϕ) and (χ', θ', ϕ'). The geodesic distance on S^3 is denoted by $(\alpha R_0 =)\Delta s(q, q')$ where $q = q(\chi, \theta, \phi)$: q and $q' \epsilon\ S^3$ and $x = x(t, q)$. The spacelike separation Δs is given in terms of the coordinates (χ, θ, ϕ) and (χ', θ', ϕ') as

$$(\cos\alpha =) \cos\frac{\Delta s}{R_0} = \cos\chi\cos\chi' + \sin\chi\sin\chi'$$

$$(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')) \quad . \tag{3.51}$$

Thus, we have the following two point function

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >= \frac{1}{8\pi^2 R_0^2}\left(\frac{1}{\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta s}{R_0}}\right). \tag{3.52}$$

The two point function Eq.(3.52) may be expanded as an infinite sum [12,25]. For this reason, we define a new variable

$$u = \ln\left(\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta s}{R_0}\right). \tag{3.53}$$

Differentiating this equation with respect to $\Delta s/R_0$ and using it in Eq.(3.52), one obtains

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >= \frac{1}{8\pi^2 R_0^2 \sin\frac{\Delta s}{R_0}}\frac{du}{d(\frac{\Delta s}{R_0})} \quad . \tag{3.54}$$

Now we use the following infinite product expansion [24]

$$\begin{aligned}\left(\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta s}{R_0}\right) &= 2\left(1-\frac{\Delta t^2}{\Delta s^2}\right)\sin^2\frac{\Delta s}{2R_0} \\ &\quad \prod_{n=1}^{\infty}\left(1-\frac{\Delta t^2}{(2\pi n R_0+\Delta s)^2}\right)\left(1-\frac{\Delta t^2}{(2\pi n R_0-\Delta s)^2}\right).\end{aligned} \tag{3.55}$$

Substituting Eqs.(3.55) and (3.53) into Eq.(3.54), we obtain the Green's function $G^{(+)}$ for the massless conformal scalar field in an Einstein universe as an expression in terms of a sum over classical paths on the spatial section S^3 of spacetime as

$$(< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >=)D(x,x') = -\frac{1}{4\pi^2 R_0 \sin\frac{\Delta s}{R_0}}\sum_{n=-\infty}^{\infty}\frac{(2\pi n R_0+\Delta s)}{[\Delta t^2-(2\pi n R_0+\Delta s)^2]} \quad . \tag{3.56}$$

(Strictly speaking, in our $D(x,x')$ Δt should be repleaced by $\Delta t - i\epsilon$, where $\epsilon \to 0$). Here, n expresses the multiply connected nature of S^3, being a sum over the indirect geodesics which connect (χ,θ,ϕ) and (χ',θ',ϕ') by encircling S^3 $\mid n \mid$ times [12,25]. The only term in the expression for the Green's function which diverges at the coincidence limit as $\Delta t, \Delta s \to 0$ is the $n=0$ term. It is easy to see that only the $n = 0$ term in Eq.(3.56) will lead to divergences in the evaluation of Eq.(3.23). Furthermore this is the only term ($n = 0$) that is infinite at the coincidence limit, and is the only term to survive as the radius R_0, tends to infinity to give Minkowski expression (in appendix B this quantity is calculated explicitly). Once again, only the $n = 0$ term diverges, consequently, the remaining terms contribute a finite amount to vacuum stresses. Thus a natural procedure to remove the infinities in $< T_{\mu\nu} >$, and other expressions involving in $D(x, x')$ is to substract the $n = 0$ term in Eq.(3.56). This process is called the renormalization procedure. We define a renormalized Green's function, $D_{ren}(x, x')$, as the series Eq.(3.56) omitting the n=0 term. Then the renormalized $< T_{\mu\nu} >$ is given by Eq.(3.23) with $D(x, x')$ replaced by $D_{ren}(x, x')$. We shall calculate the renormalized Green's function at the coincidence limit i.e. $(t' \to t)\Delta t \to 0$ and $\Delta s \to 0$. We take the Eq.(3.56) with the $n = 0$ term omitted as

$$D_{ren}(x, x') = \frac{-1}{4\pi^2 R_0 \sin\frac{\Delta s}{R_0}} \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{(2\pi n R_0 + \Delta s)}{[\Delta t^2 - (2\pi n R_0 + \Delta s)^2]} . \tag{3.57}$$

Taking the coincidence limit, one obtains

$$\lim_{x' \to x} D_{ren}(x, x') = \frac{1}{4\pi^2 R_0} \lim_{\Delta s \to 0} \frac{1}{\sin\frac{\Delta s}{R_0}} \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{1}{2\pi n R_0 (1 + \frac{\Delta s}{2\pi n R_0})} . \tag{3.58}$$

We treat the last term in Eq.(3.58) by using the binomial expansion to get

$$\lim_{x' \to x} D_{ren}(x, x') = \frac{1}{4\pi^2 R_0} \lim_{\Delta s \to 0} \frac{1}{\sin\frac{\Delta s}{R_0}} \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \left(\frac{1}{2\pi n R_0} - \frac{\Delta s}{(2\pi n R_0)^2} + \frac{\Delta s^2}{(2\pi n R_0)^3} - \frac{\Delta s^3}{(2\pi n R_0)^4} + \frac{\Delta s^4}{(2\pi n R_0)^5} - ...... \right) . \tag{3.59}$$

Using the following relations:

$$\lim_{\Delta s \to 0} \frac{\sin \frac{\Delta s}{R_0}}{\frac{\Delta s}{R_0}} = 1 \quad ,$$

$$\left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{1}{n^{2p-1}} = 0 \quad \text{and}$$

$$\left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{1}{n^{2p}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \quad \text{where} \quad p = 1,2,3,4... \qquad (3.60)$$

the renormalized Green's function at the coincidence limit becomes

$$\lim_{x \to x'} D_{ren}(x,x') = \frac{-1}{8\pi^4 R_0^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad . \qquad (3.61)$$

Differentiating expression Eq.(3.57) twice with respect to t one obtains

$$\begin{aligned} \nabla_0 \nabla_0 D_{ren}(x,x') &= -\nabla_0 \nabla_{0'} D_{ren}(x,x') \\ &= \frac{-1}{4\pi^2 R_0} \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{(2\pi n R_0 + \Delta s)}{\sin \frac{\Delta s}{R_0}} \\ &\quad \left\{ -\frac{2}{[\Delta t^2 - (2\pi n R_0 + \Delta s)^2]^2} + \frac{8\Delta t^2}{[\Delta t^2 - (2\pi n R_0 + \Delta s)^2]^3} \right\} . \end{aligned} \qquad (3.62)$$

(Here $\nabla_{0'}$ indicates differentiating with respect to t') Taking the coincidence limit ($\Delta t \to 0, \Delta s \to 0$) one obtains

$$\begin{aligned} \lim_{x' \to x} \nabla_0 \nabla_0 D_{ren}(x,x') &= -\lim_{x' \to x} \nabla_0 \nabla_{0'} D_{ren}(x,x') \\ &= \frac{1}{2\pi^2 R_0} \lim_{\Delta s \to 0} \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{1}{\sin \frac{\Delta s}{R_0}} \\ &\quad \frac{1}{(2\pi n R_0)^3 (1 + \frac{\Delta s}{2\pi n R_0})^3} \quad . \end{aligned} \qquad (3.63)$$

Using the binomial expansion for the last term of the above equation we get

$$\frac{1}{(2\pi n R_0)^3 (1 + \frac{\Delta s}{2\pi n R_0})^3} = \frac{1}{(2\pi n R_0)^3} - \frac{3\Delta s}{(2\pi n R_0)^4} + \frac{6\Delta s^2}{(2\pi n R_0)^5} - ...... \quad . \qquad (3.64)$$

Subtituting Eq.(3.64) into Eq.(3.63) and using the Eq.(3.60), we obtain

$$\begin{aligned}\lim_{x'\to x}\nabla_0\nabla_0 D_{ren}(x,x') &= -\lim_{x'\to x}\nabla_0\nabla_{0'} D_{ren}(x,x') \\ &= -\frac{3}{16\pi^6 R_0^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \quad . \end{aligned} \tag{3.65}$$

Now, we calculate the covariant derivatives at the coincidence limit of the geodesic distance on S^3 which is given by

$$\begin{aligned}\Delta s &= R_0 \arccos[\cos\chi\cos\chi' + \sin\chi\sin\chi' \\ &\quad (\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'))] \quad . \end{aligned} \tag{3.66}$$

We have

$$\lim_{x'\to x}\nabla_i\Delta s = 0 \qquad ; \qquad \lim_{x'\to x}\nabla_{i'}\Delta s = 0 \quad . \tag{3.67}$$

Where i refers to χ , θ and ϕ. We will use the properties of $g^{\mu}_{\nu'}$ by studying the action of $g^{\mu}_{\nu'}$ on $\nabla^{\nu'}\Delta s$ which is tangent to the geodesics at x', has length equal to the geodesic distance between x and x', and is oriented in the $x' \to x$ direction [18,20,23]. We have

$$\nabla^{\mu}\Delta s = -g^{\mu}_{\nu'}\nabla^{\nu'}\Delta s \quad . \tag{3.68}$$

When a tangent vector is parellel transported, it remains tangent and keeps the same length. So the action of $g^{\mu}_{\nu'}$ on $\nabla^{\nu'}\Delta s$ must give $-\nabla^{\mu}\Delta s$, which is tangent to the geodesic at x and has the same length as $\nabla^{\nu'}\Delta s$. The minus sign comes from the fact that $\nabla^{\mu}\Delta s$ is oriented in the $x' \to x$ direction [23]. Thus, we have

$$\nabla_{\mu}\nabla^{\mu}\Delta s = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\Delta s \quad , \tag{3.69}$$

and

$$\nabla_{\mu'}\nabla^{\mu'}\Delta s = -g^{\mu'\nu}\nabla_{\mu'}\nabla_{\nu}\Delta s \quad . \tag{3.70}$$

Taking the covariant derivative of the renormalized Green's function at the coincidence limit.

$$\lim_{x'\to x} \nabla_i\nabla_j D_{ren}(x,x') = -\frac{1}{4\pi^2 R_0}\lim_{x'\to x} \left(\left(\sum_{n=-\infty}^{-1}+\sum_{n=1}^{\infty}\right)\csc(\frac{\Delta s}{R_0})\right.$$

$$\nabla_i\nabla_j\Delta s\left[\frac{1}{[\Delta t^2-(2\pi nR_0+\Delta s)^2]}\right.$$

$$-\frac{1}{R_0}\cot(\frac{\Delta s}{R_0})\frac{(2\pi nR_0+\Delta s)}{[\Delta t^2-(2\pi nR_0+\Delta s)^2]}$$

$$\left.\left.+\frac{2(2\pi nR_0+\Delta s)^2}{[\Delta t^2-(2\pi nR_0+\Delta s)^2]^2}\right]\right), \quad (3.71)$$

and taking the limit $\Delta t \to 0$ one obtains

$$\lim_{x'\to x} \nabla_i\nabla_j D_{ren}(x,x') = -\frac{1}{4\pi^2 R_0}\lim_{\Delta s\to 0} \left(\left(\sum_{n=-\infty}^{-1}+\sum_{n=1}^{\infty}\right)\csc(\frac{\Delta s}{R_0})\right.$$

$$\nabla_i\nabla_j\Delta s\left[\frac{1}{(2\pi nR_0)^2}\frac{1}{(1+\frac{\Delta s}{2\pi nR_0})^2}\right.$$

$$\left.\left.+\frac{1}{2\pi nR_0^2}\cot(\frac{\Delta s}{R_0})\frac{1}{(1+\frac{\Delta s}{2\pi nR_0})}\right]\right). \quad (3.72)$$

Consider the last parenthesis of the above equation using the binomial expansion and the following formula

$$\cot(\frac{\Delta s}{R_0}) = \frac{R_0}{\Delta s}-\frac{\Delta s}{3R_0}-\frac{\Delta s^3}{45R_0^3} -\frac{2\Delta s^5}{945R_0^5}+\{\text{Terms in positive powers of } \tfrac{\Delta s}{R_0}\}\ , \quad (3.73)$$

we obtain

$$\lim_{x' \to x} \nabla_i \nabla_j D_{ren}(x,x') = -\frac{1}{4\pi^2 R_0} \lim_{\Delta s \to 0} \left(\left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \csc(\frac{\Delta s}{R_0}) \right.$$

$$\nabla_i \nabla_j \Delta s \left[\frac{1}{(2\pi n R_0 \Delta s} - \frac{\Delta s}{6\pi^2 n R_0^3} \right.$$

$$-\frac{\Delta s}{8\pi^3 n^3 R_0^3} + \frac{\Delta s^2}{12\pi^2 n^2 R_0^4} + \frac{\Delta s^2}{8\pi^4 n^4 R_0^4}$$

$$+\frac{\Delta s^3}{90\pi n R_0^5} - \frac{\Delta s^3}{24\pi^3 n^3 R_0^5} - \frac{\Delta s^3}{8\pi^5 n^5 R_0^5}$$

$$\left. \left. +\{\text{Terms in positive powers of } \tfrac{\Delta s}{R_0}\} \right] \right) . \tag{3.74}$$

We also have the following covariant derivatives and relations:

$$\begin{aligned} \nabla_\chi \nabla_\chi \Delta s &= -\nabla_\chi \nabla_{\chi'} \Delta s \\ &= -R_0^2 \{ -\frac{1}{\Delta s} + \frac{\Delta s}{3R_0^2} - \frac{\Delta s^3}{45 R_0^4} \\ &\quad -\frac{\Delta s^5}{945 R_0^6} + \{\text{Terms in positive powers of } \tfrac{\Delta s}{R_0}\}\} \quad . \end{aligned} \tag{3.75}$$

$$\begin{aligned} \nabla_\theta \nabla_\theta \Delta s &= -\nabla_\theta \nabla_{\theta'} \Delta s \\ &= -R_0^2 \sin\chi \sin\chi' \cos\gamma \{ -\frac{1}{\Delta s} - \frac{\Delta s}{6R_0^2} + \frac{7\Delta s^3}{360 R_0^4} \\ &\quad +\{\text{Terms in positive powers of } \tfrac{\Delta s}{R_0}\}\} \quad . \end{aligned} \tag{3.76}$$

$$\begin{aligned} \nabla_\phi \nabla_\phi \Delta s &= -\nabla_\phi \nabla_{\phi'} \Delta s \\ &= -R_0^2 \sin\chi \sin\chi' \sin\theta \sin\theta' \cos(\phi - \phi') \{ -\frac{1}{\Delta s} - \frac{\Delta s}{6R_0^2} + \frac{7\Delta s^3}{360 R_0^4} \\ &\quad +\{\text{Terms in positive powers of } \tfrac{\Delta s}{R_0}\}\}. \end{aligned} \tag{3.77}$$

And

$$\csc(\frac{\Delta s}{R_0}) = \frac{R_0}{\Delta s} + \frac{\Delta s}{6R_0} + \frac{7\Delta s^3}{360R_0^3}$$

$$+\frac{31\Delta s^5}{15120R_0^5} + \{\text{Terms in positive powers of } \tfrac{\Delta s}{R_0}\}. \qquad (3.78)$$

Substituting Eqs.(3.75-3.78) into Eq.(3.74) one obtains

$$\begin{aligned}\lim_{x'\to x} \nabla_i \nabla_j D_{ren}(x,x') &= -\lim_{x'\to x} \nabla_i \nabla_{j'} D_{ren}(x,x') \\ &= -\frac{1}{4\pi^2 R_0} g_{ij} \lim_{\Delta s\to 0} \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \\ &\quad \left[-\frac{1}{n2\pi\Delta s^3} + \frac{1}{n^3(2\pi R_0)^3\Delta s} + \frac{1}{n6\pi R_0^2\Delta s} \right. \\ &\quad -\frac{1}{n^4 8\pi^4 R_0^3} - \frac{1}{n^2 12\pi^2 R_0^3} + \frac{1}{n^5}\frac{\Delta s}{8\pi^5 R_0^4} \\ &\quad \left. +\{\text{Terms in positive powers of } \tfrac{\Delta s}{R_0}\} \right]. \end{aligned} \qquad (3.79)$$

The odd powers of n is equal to zero since

$$\left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{1}{n^{2p-1}} = 0 \quad , \quad \text{where} \quad p = 1,2,3,4.. \quad . \qquad (3.80)$$

After using Eq.(3.80) and taking the coincidence limit, we may use the following sum properties

$$\left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty} \right) \frac{1}{n^{2p}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{2p}}, \quad \textit{where} \quad p = 1,2,3,4... \qquad (3.81)$$

Thus, we find

$$\begin{aligned}\lim_{x'\to x} \nabla_i \nabla_j D_{ren}(x,x') &= -\lim_{x'\to x} \nabla_i \nabla_{j'} D_{ren}(x,x') \\ &= \frac{1}{8\pi^4 R_0^4} g_{ij} \sum_{n=1}^{\infty} \left\{ \frac{1}{3n^2} + \frac{1}{2\pi^2 n^4} \right\}. \end{aligned} \qquad (3.82)$$

We now proceed to evaluate the renormalized $< 0 \mid T_{\mu\nu} \mid 0 >=< T_{\mu\nu} >$. From the Eq.(3.23), we have

$$< 0 \mid T_{00} \mid 0 >_{ren} = \frac{1}{6} \lim_{x' \to x} \Big(4\nabla_0 \nabla_{0'} + g_{00'}(\nabla_\rho \nabla^\rho + \nabla_{\rho'} \nabla^{\rho'} - g^{\lambda\sigma'} \nabla_\lambda \nabla_{\lambda'})$$

$$- g_{0'\lambda} \nabla^\lambda \nabla_0 - g_{0\tau'} \nabla_{\tau'} \nabla_{0'}$$

$$- \frac{1}{2} [R_0^\sigma g_{\sigma' 0} + R_{0'}^{\rho'} g_{0\rho'} - R g_{00'}] \Big) D_{ren}(x, x') \quad . \tag{3.83}$$

This could be simplified as

$$< 0 \mid T_{00} \mid 0 >_{ren} = \frac{1}{6} \lim_{x' \to x} \left(4\nabla_0 \nabla_{0'} + g_{00'}(\nabla_i \nabla^i + \nabla_{i'} \nabla^{i'}) + g_{00'} \frac{2}{R_0} \right) D_{ren}(x, x') \quad . \tag{3.84}$$

Using the covariant derivatives, one obtains

$$< 0 \mid T_0^0 \mid 0 >_{ren} = \frac{3}{16\pi^6 R_0^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \quad . \tag{3.85}$$

Since,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad . \tag{3.86}$$

We find

$$< 0 \mid T_0^0 \mid 0 >_{ren} = \frac{1}{480\pi^2 R_0^4} \quad . \tag{3.87}$$

Similarly we have

$$< 0 \mid T_i^i \mid 0 >_{ren} = \frac{1}{6} \lim_{x' \to x} \Big(2g^{ij} \nabla_i \nabla_{j'} + \nabla_\rho \nabla^\rho + \nabla_{\rho'} \nabla^{\rho'}$$

$$- 2(\nabla_i \nabla^i + \nabla_{i'} \nabla^{i'}) \Big) D_{ren}(x, x') \quad . \tag{3.88}$$

We note that in $< T_i^i >_{ren}$ no summation is implied. Using the covariant derivatives given in Eqs.(3.75-3.78) we obtain

$$< 0 \mid T_i^j \mid 0 >_{ren} = -\frac{1}{1440\pi^2 R_0^4} g_{ij} \quad . \tag{3.89}$$

We easily see that the renormalized vacuum expectation value of the stress tensor turns out to be identical to the result given first by Ford [10]. However, this result is not only obtained by covariant method but is also cutoff independent [12].

3.2 Half Einstein Universe Case

In this section, we will now consider the renormalized energy momentum tensor for the massless conformal scalar field with a spherical boundary located at $\chi_0 = \frac{\pi}{2}$. We solve the problem by using the covariant point-splitting method. We aim to construct the Green's function directly by using the wave equation satisfying both the Dirichlet and the Neumann boundary conditions.

We have the solution of the wave equation (3.31) which is given as

$$\Phi(\vec{x}) = c_0 \sin^l \chi C_{N-l-1}^{l+1}(\cos\chi) Y_{lm}(\theta,\phi) e^{-iwt} \quad , \tag{3.90}$$

where $C_{N-l-1}^{l+1}(\cos\chi)$ are the new eigenfunctions which are still the Gegenbauer polynomials but only the ones that satisfy the Dirichlet or the Neumann boundary conditions in the half Einstein universe, and c_0 is the normalization constant. To evaluate the normalization constant c_0, we can use the scalar product given in Eq.(3.32). Thus the general mode solutions $\Phi(\vec{x})$ may be normalized so that

$$\begin{aligned} 1 &= c_0^2 2wR_0^3 \int_0^{\frac{\pi}{2}} \sin\chi d\chi \sin^{2l}\chi \mid C_{N-l-1}^{l+1}(\cos\chi) \mid^2 \\ & \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \mid Y_{lm}(\theta,\phi) \mid^2 \quad , \end{aligned} \tag{3.91}$$

where

$$w = \frac{N(n',l)}{R_0} \quad . \tag{3.92}$$

$N(n',l)$ are integer numbers. Specific values of $N(n',l)$ for the Dirichlet and Neumann boundary conditions are presented in tables (2.3) and (2.5). We find the normalization constant in the following form

$$c_0 = 2^{l+1} l! \left\{ \frac{(N-l-1)!}{2\pi R_0^2 (N+l)!} \right\}^{1/2} . \tag{3.93}$$

Thus, the mode solution becomes

$$\Phi_\lambda(\vec{x}) = 2^{l+1} l! \left\{ \frac{(N-l-1)!}{2\pi R_0^2 (N+l)!} \right\}^{1/2} \sin^l \chi C_{N-l-1}^{l+1}(\cos\chi) Y_{lm}(\theta,\phi) e^{-i\omega t} \quad . \tag{3.94}$$

The Green's function that we aim to calculate is defined by [16,18,20]

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >= \sum_\lambda \Phi_\lambda(\vec{x})\Phi_\lambda^*(\vec{x'}) \quad . \tag{3.95}$$

Where $\Phi_\lambda(\vec{x})$ are the mode functions satisfying the appropriate boundary conditions. Using the mode solutions given in Eq.(3.94), one obtains

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > \; = \; \frac{1}{\pi R_0^2} \sum_{\lambda(N,l,m)} e^{-i\omega\Delta t} \frac{2^{2l+1}(N-l-1)!(l!)^2}{(N+l)!}$$

$$\sin^l \chi \sin^l \chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(\cos\chi') Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi). \tag{3.96}$$

We define the Green's function for the Dirichlet and Neumann boundary conditions as

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >= D_D(x,x') \quad , \tag{3.97}$$

and

$$< 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 >= D_N(x,x') \quad . \tag{3.98}$$

We easily see that

$$D_D(x,x') + D_N(x,x') = 2D(x,x') \quad . \tag{3.99}$$

Where $D(x,x')$ is the Green's function for the full Einstein universe. Since the sum of the Dirichlet and Neumann eigenvalues for half Einstein universe which is depicted in tables (2.3) and (2.5) is equal to full Einstein universe eigenvalues given in table (2.1) we find

$$\begin{aligned} D_D(x,x') + D_N(x,x') &= 2D(x,x') \\ &= \frac{2}{\pi R_0^2}\sum_{N=1}^{\infty} e^{-iN\frac{\Delta t}{R_0}} \sum_{l=0}^{N-1} \frac{2^{2l}(N-l-1)!(l!)^2}{(N+l)!} \\ & \sin^l\chi \sin^l\chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(\cos\chi') \\ & \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \quad . \end{aligned} \tag{3.100}$$

The factor of 2 comes from the fact that the mode functions used in $D(x,x')$ are normalized with respect to the full Einstein universe.

To evaluate $D_D(x,x')$ and $D_N(x,x')$ explicitly we find it convenient to write them as

$$D_D(x,x') = D(x,x') - \frac{1}{2}[D_N(x,x') - D_D(x,x')] \quad , \tag{3.101}$$

and

$$D_N(x,x') = D(x,x') + \frac{1}{2}[D_N(x,x') - D_D(x,x')] \quad . \tag{3.102}$$

The second term on the right hand side of Eqs.(3.101) and (3.102) could be written as $D_B(x,x') = \frac{1}{2}[D_N(x,x') - D_D(x,x')]$. A closed expression for $D(x,x')$ is already known which given in Eq.(3.50) and could be written as

$$D(x,x') = \frac{1}{8\pi^2 R_0^2}\frac{1}{(\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta s}{R_0})} \quad . \tag{3.103}$$

Where $\Delta t = t - t'$, and

$$\begin{aligned} \cos\frac{\Delta s}{R_0} &= \cos\chi\cos\chi' + \sin\chi\sin\chi' \\ & (\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')) \,. \end{aligned} \tag{3.104}$$

Using the information given in tables (2.3) and (2.5), and definitions of $D_D(x,x')$ and $D_N(x,x')$ one could write as

$$\begin{aligned} D_B(x,x') &= \frac{1}{2}[D_N(x,x') - D_D(x,x')] \\ &= \frac{1}{2\pi R_0^2}\sum_{N=1}^{\infty} e^{-iN\frac{\Delta t}{R_0}} \sum_{l=0}^{N-1} \frac{2^{2l+1}(N-l-1)!(l!)^2(-1)^{N+l-1}}{(N+l)!} \\ & \sin^l\chi \sin^l\chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(\cos\chi') \\ & \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) \quad . \end{aligned} \tag{3.105}$$

Using the addition formula for the spherical harmonics, which is given as

$$\frac{2l+1}{4\pi}P_l(\cos\gamma) = \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \quad . \tag{3.106}$$

Where

$$\cos\gamma = cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \tag{3.107}$$

And $P_l(\cos\gamma)$ are the Legendre polynomials. We rewrite the above equation as

$$\begin{aligned} D_B(x,x') &= \frac{1}{4\pi^2 R_0^2}\sum_{N=1}^{\infty} e^{-iN\frac{\Delta t}{R_0}} \sum_{l=0}^{N-1} \frac{2^{2l}(l!)^2(N-l-1)!(-1)^{N+l-1}(2l+1)}{(N+l)!} \\ & \sin^l\chi \sin^l\chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(\cos\chi') P_l(\cos\gamma) \quad . \end{aligned} \tag{3.108}$$

Using the following property of the Gegenbauer polynomials [21]:

$$\begin{aligned} C_{N-l-1}^{l+1}(\cos(\pi-\chi')) &= C_{N-l-1}^{l+1}(-\cos\chi') \\ &= (-1)^{N+l-1} C_{N-l-1}^{l+1}(\cos\chi') \quad , \end{aligned} \tag{3.109}$$

we rewrite Eq.(3.108) as

$$D_B(x,x') = \frac{1}{4\pi^2 R_0^2}\sum_{N=1}^{\infty} e^{-iN\frac{\Delta t}{R_0}} \sum_{l=0}^{N-1} \frac{2^{2l}(l!)^2(N-l-1)!(2l+1)}{(N+l)!}$$

$$\sin^l\chi \sin^l\chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(-\cos\chi') P_l(\cos\gamma) \quad , \tag{3.110}$$

we have used $\sin(\pi - \chi') = \sin\chi'$. Now, using the addition theorem for the Gegenbauer polynomials, which is given as

$$C_{N-1}^1(-\cos\chi\cos\chi' + \sin\chi\sin\chi'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')])$$

$$= \sum_{l=0}^{N-1} \frac{2^{2l}(l!)^2(N-l-1)!(2l+1)}{(N+l)!}$$

$$\sin^l\chi \sin^l\chi' C_{N-l-1}^{l+1}(\cos\chi) C_{N-l-1}^{l+1}(-\cos\chi') P_l(\cos\gamma) \quad , \tag{3.111}$$

we could write Eq.(3.110) as

$$D_B(x,x') = \frac{1}{4\pi^2 R_0^2}\sum_{N=1}^{\infty} e^{-iN\frac{\Delta t}{r_0}} C_{N-1}^1(\cos\Delta\sigma) \quad . \tag{3.112}$$

Where

$$\cos\Delta\sigma = -\cos\chi\cos\chi' + \sin\chi\sin\chi'\cos\gamma \quad . \tag{3.113}$$

Writing $C_{N-1}^1(\cos\Delta\sigma)$ explicitly,

$$C_{N-1}^1(\cos\Delta\sigma) = \frac{\sin N\Delta\sigma}{\sin\Delta\sigma} \quad . \tag{3.114}$$

$D_B(x,x')$ becomes

$$\begin{aligned} D_B(x,x') &= \frac{1}{2}[D_N(x,x') - D_D(x,x')] \\ &= \frac{1}{4\pi^2 R_0^2 \sin\Delta\sigma}\sum_{N=1}^{\infty}\left(e^{-i\Delta t/R_0}\right)^N \sin N\Delta\sigma \quad . \end{aligned} \tag{3.115}$$

Letting $\Delta t \to \Delta t - i\epsilon$ where $\epsilon \to 0$, then

$$| e^{-i\Delta t/R_0} | < 1 \quad . \tag{3.116}$$

Now the sum in Eq.(3.115) could be evaluated using the following summation formula:

$$\sum_{N=1}^{\infty} p^N \sin N\Delta\sigma = \frac{p \sin \Delta\sigma}{1 - 2p\cos\Delta\sigma + p^2} \qquad \text{if} \quad | p | < 1 \quad , \tag{3.117}$$

and leads to

$$D_B(x, x') = \frac{1}{8\pi^2 R_0^2} \left(\frac{1}{\cos\frac{\Delta t}{R_0} - \cos\Delta\sigma} \right) . \tag{3.118}$$

Where, $\Delta\sigma$ is defined in Eq.(3.113).

Substituting Eq.(3.118) into Eq.(3.101) and into Eq.(3.102), we find the Green's function for the half Einstein universe for the Dirichlet and Neumann boundary conditions, respectively

$$D_D(x, x') = \frac{1}{8\pi^2 R_0^2} \left(\frac{1}{\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta s}{R_0}} - \frac{1}{\cos\frac{\Delta t}{R_0} - \cos\Delta\sigma} \right) , \tag{3.119}$$

and

$$D_N(x, x') = \frac{1}{8\pi^2 R_0^2} \left(\frac{1}{\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta s}{R_0}} + \frac{1}{\cos\frac{\Delta t}{R_0} - \cos\Delta\sigma} \right) . \tag{3.120}$$

Where $\cos\frac{\Delta s}{R_0}$ and $\cos\Delta\sigma$ are defined as in Eqs.(3.104) and (3.113).

In these Green's functions the first terms contain all the divergences, as we have discussed in the previous section.

Now, we can rewrite the vacuum energy momentum tensor as the coincidence limit, as given in Eq.(3.23)

$$\begin{aligned} < 0 \mid T_{\mu\nu} \mid 0 >= \frac{1}{6} \lim_{x' \to x} \quad & \Big(4\nabla_\mu \nabla_{\nu'} + g_{\mu\nu'} (\nabla_\rho \nabla^\rho + \nabla_{\rho'} \nabla^{\rho'} - g^{\lambda\sigma'} \nabla_\lambda \nabla_{\sigma'}) \\ & - g_{\nu'\lambda} \nabla^\lambda \nabla_\mu - g_{\mu\tau'} \nabla_{\tau'} \nabla_{\nu'} \\ & - \frac{1}{2} [R^\sigma_\mu g_{\sigma\nu'} + R^{\rho'}_{\nu'} g_{\mu\rho'} - R g_{\mu\nu'}] \Big) \bar{G}(x, x') \quad . \end{aligned} \tag{3.121}$$

Where $\bar{G}(x,x')$ stands either for $D_D(x,x')$ or $D_N(x,x')$. We see that the first term in Eqs.(3.119) and (3.120) gives the finite contribution to the renormalized energy momentum tensor, $< T_{\mu\nu}D(x,x') >_{ren}$, which is identical to the complete Einstein universe case, which is given in the previous section.

Next, we take into account the second term in Eqs.(3.119) and (3.120), which are identical arise from their sign. To evaluate the energy momentum tensor, we write $< T_0^0 >$ and $< T_i^i >$ explicitly as

$$< 0 \mid T_0^0 D_B(x,x') \mid 0 >= \frac{1}{6}\lim_{x'\to x} \left(4\nabla_0\nabla_{0'} + g_{00'}(\nabla_i\nabla^i + \nabla_{i'}\nabla^{i'}) + g_{00'}\frac{2}{R_0}\right) D_B(x,x') \quad , \tag{3.122}$$

and

$$< 0 \mid T_i^i D_B(x,x') \mid 0 >= \frac{1}{6}\lim_{x'\to x} \left(2g^{ij}\nabla_i\nabla_{j'} + \nabla_\rho\nabla^\rho + \nabla_{\rho'}\nabla^{\rho'} - 2(\nabla_i\nabla^i + \nabla_{i'}\nabla^{i'})\right) D_B(x,x') \quad . \tag{3.123}$$

Where $D_B(x,x')$ is given in Eq.(3.118) and we again note that in $< T_i^i >$ no summation is implied over i.

For the energy momentum tensor calculations we will start by finding the coincidence limits of the Green's function, $D_B(x,x')$ and its covariant derivatives. First we consider

$$\lim_{x'\to x} D_B(x,x') = \frac{1}{8\pi^2 R_0^2}\lim_{x'\to x}\left(\frac{1}{\cos\frac{\Delta t}{R_0} - \cos\Delta\sigma}\right), \tag{3.124}$$

where $x' \to x$ means that $t' \to t$, $\chi' \to \chi$, $\theta' \to \theta$ and $\phi' \to \phi$. Using

$$\lim_{x'\to x} \cos\frac{\Delta t}{R_0} = 1 \quad , \tag{3.125}$$

and

$$
\begin{aligned}
\lim_{x'\to x}\cos\Delta\sigma &= \lim_{x'\to x}\Big[-\cos\chi\cos\chi' + \sin\chi\sin\chi' \\
&\qquad (\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'))\Big] \\
&= -\cos 2\chi
\end{aligned}
\tag{3.126}
$$

we obtain

$$
\lim_{x'\to x} D_B(x,x') = \frac{1}{16\pi^2 R_0^2\cos^2\chi} \quad . \tag{3.127}
$$

Turning to Eq.(3.118) and differentiating twice with respect to t, one obtains

$$
\begin{aligned}
\nabla_0\nabla_0 D_B(x,x') &= -\nabla_0\nabla_{0'} D_B(x,x') \\
&= \frac{1}{8\pi^2 R_0^4}\left(\frac{\cos\frac{\Delta t}{R_0}}{\left(\cos\frac{\Delta t}{R_0}-\cos\Delta\sigma\right)^2} + \frac{2\sin^2\frac{\Delta t}{R_0}}{\left(\cos\frac{\Delta t}{R_0}-\cos\Delta\sigma\right)^3}\right).
\end{aligned}
\tag{3.128}
$$

Taking the coincidence limit of this equation we find

$$
\begin{aligned}
\lim_{x'\to x}\nabla_0\nabla_0 D_B(x,x') &= -\lim_{x'\to x}\nabla_0\nabla_{0'} D_B(x,x') \\
&= \frac{1}{32\pi^2 R_0^4\cos^4\chi} \quad .
\end{aligned}
\tag{3.129}
$$

Since

$$
\lim_{x'\to x}\begin{cases}
\sin\frac{\Delta t}{R_0} = 0 \\
\cos\Delta\sigma = -\cos 2\chi \\
\cos\frac{\Delta t}{R_0} = 1 \quad .
\end{cases}
\tag{3.130}
$$

Now, we take the covariant derivatives (defining $\Delta\sigma = \frac{\Delta\bar{s}}{R_0}$)

$$\left(\nabla_i\nabla^i + \nabla_{i'}\nabla^{i'}\right) D_B(x,x') = -\frac{1}{8\pi^2 R_0^3}\Bigg[\frac{\sin\frac{\Delta\tilde{s}}{R_0}}{(\cos\frac{\Delta t}{R_0} - \sin\frac{\Delta\tilde{s}}{R_0})}$$

$$\left(\nabla_i\nabla^i + \nabla_{i'}\nabla^{i'}\right)\Delta\tilde{s}$$

$$+\left(\nabla_i\Delta\tilde{s}\nabla^i\Delta\tilde{s} + \nabla_{i'}\Delta\tilde{s}\nabla^{i'}\Delta\tilde{s}\right)$$

$$\left(\frac{\cos\frac{\Delta\tilde{s}}{R_0}}{R_0\left(\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta\tilde{s}}{R_0}\right)^2} - \frac{2\sin^2\frac{\Delta\tilde{s}}{R_0}}{R_0\left(\cos\frac{\Delta t}{R_0} - \cos\frac{\Delta\tilde{s}}{R_0}\right)^3}\right)\Bigg] . \tag{3.131}$$

We find

$$\lim_{x'\to x}\left(\nabla_i\nabla^i + \nabla_{i'}\nabla^{i'}\right) D_B(x,x') = \frac{1}{8\pi^2 R_0^4}\frac{\sin^2\chi}{\cos^4\chi} . \tag{3.132}$$

In these calculations the following covariant derivatives and equations are used:

$$\lim_{x'\to x}\begin{cases} \sin\Delta\sigma = \sin 2\chi \quad , \\ \nabla_i\nabla^i\Delta\tilde{s} = \frac{2}{R_0}\cot 2\chi \quad , \\ \nabla_{i'}\nabla^{i'}\Delta\tilde{s} = -\frac{2}{R_0}\csc 2\chi \quad , \\ (\nabla_i\nabla^i + \nabla_{i'}\nabla^{i'})\Delta\tilde{s} = -\frac{2}{R_0}\tan 2\chi \quad , \\ \nabla_\chi\Delta\tilde{s} = -R_0 \quad , \\ \nabla_{\chi'}\Delta\tilde{s} = -R_0 \quad , \\ \nabla_\theta\Delta\tilde{s} = 0 \quad , \\ \nabla_{\theta'}\Delta\tilde{s} = 0 \quad , \\ \nabla_\phi\Delta\tilde{s} = 0 \quad , \\ \nabla_{\phi'}\Delta\tilde{s} = 0 \qquad \text{and} \\ (\nabla_i\Delta\tilde{s}\nabla^i\Delta\tilde{s} + \nabla_{i'}\Delta\tilde{s}\nabla^{i'}\Delta\tilde{s}) = 0 \quad . \end{cases} \tag{3.133}$$

Substituting Eqs.(3.127), (3.129) and (3.132) into the Eqs.(3.122) and (3.123), one obtains

$$< T_0^0 D_B(x.x') >= 0 \quad ,$$

$$< T_i^i D_B(x,x') >= 0 \quad . \tag{3.134}$$

We see that the second terms in Eqs.(3.119) and (3.120) gives the zero contribution to the renormalized energy momentum tensor $< T_{\mu\nu} >_{ren}$ thus, we find the following expressions for the complete renormalized energy momentum tensor for both the Dirichlet and the Neumann boundary conditions as

$$< T_0^0 >_{ren}= \frac{1}{480\pi^2 R_0^4} \quad . \tag{3.135}$$

$$< T_j^i >_{ren}= -\frac{1}{1440\pi^2 R_0^4} g_j^i \quad . \tag{3.136}$$

Which is identical to the full Einstein universe case [13,14,15].

CHAPTER IV

CONCLUSION

In this thesis we considered quantum vacuum energy of the massless conformal scalar field in curved background half Einstein universe. This case is interesting in that it could be solved analtically by several different methods. These calculations that we have presented are based on the approximate methods of the quantum field theory on curved background geometry i.e. gravity is still considered as a classical field [18]. However, in the early days of the development of quantum electrodynamics we have experienced that some of the results that emerge from such an approximate theory may still survive and remain to be valid even in the exact theory. We preferred to work with the massless conformal scalar field since it is the simplest yet closest analogy we have to electromagnetic theory [26]. We concentrated on the renormalization of the vacuum expectation value of the energy momentum tensor $< 0 \mid T_{\mu\nu} \mid 0 >$. This expression contains divergences and we discussed how to extract finite meaningfull quantities from these divergent expressions.

In sections II and III we found the quantum vacuum energy for the massless conformal scalar field in half Einstein universe by considering mode sums and renormalized the divergent vacuum energy by introducing different cutoff functions. We calculated the renormalized vacuum energy momentum tensor by using the covariant point-splitting method. We constructed the Green's functions by using the eigenfunctions, which are obtained by solving the wave equation with the appropriate boundary conditions (Dirichlet and Neumann). Our result is in agreement with the results obtained by the method of images [15,25]. Our result is interesting on various accounts. First of all the Casimir energy is calculated for an isolated sphere, where the interior geometry is given by Eq.(2.56). The global topology of the universe as well as the outside geometry are not important and have no effect on our result. This is very important for applications of the Casimir effect to the bag models [22,27,28]. Second, its flat spacetime counterpart (massless conformal scalar field with a spherical boundary) cannot be solved analytically for any value of the boundary. In addition, it also has the problem of a cubic divergence, which cannot be eliminated by the conventional definition of renormalization. So far the general attitude towards this problem has been either to concentrate on the electromagnetic or fermion fields, where the cubic divergence cancels between the opposite parity modes [1,2,6,8,9,27,28] or the simply to declare the vacuum energy of the massless conformal scalar field with a spherical boundary in flat spacetime to be divergent [29].

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APPENDIX A

RENORMALIZED QUANTUM VACUUM ENERGY MOMENTUM TENSOR OF THE MASSLESS CONFORMAL SCALAR FIELD IN A CLOSED FRIEDMANN UNIVERSE VIA THE POINT SPLITTING METHOD

In this section we will write the closed Friedmann metric as

$$ds^2 = a^2(\eta)[d\eta^2 - d\chi^2 - sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \quad , \qquad (A.1)$$

where χ and θ run from 0 to π, and ϕ runs from 0 to 2π, while $a(\eta)$ is the scale factor.

Defining $C(\eta) = a^2(\eta)$ and $D = \frac{\dot{C}}{C}$, where a dot denotes differentation with respect to η, the nonzero Christoffel symbols for the metric Eq.(A.1) are (indices 0,1,2,3 corresponding to η, χ, θ, ϕ respectively)

$$\begin{aligned}
&\Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{1}{2}D \quad , \\
&\Gamma^0_{22} = \frac{1}{2}D\sin^2\chi \quad , \qquad \Gamma^0_{33} = \frac{1}{2}D\sin^2\chi\sin^2\theta \quad , \\
&\Gamma^1_{22} = -\sin\chi\cos\chi \quad , \qquad \Gamma^1_{33} = -\sin\chi\cos\chi\sin^2\theta \quad , \\
&\Gamma^3_{13} = \Gamma^2_{12} = \cot\chi \quad , \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \quad , \qquad \Gamma^3_{23} = \cot\theta \qquad (A.2)
\end{aligned}$$

From Eq.(A.2) the nonzero compenents of the Ricci tensor and hence the Ricci scalar could be computed as

$$\begin{aligned}
&R_{00} = \frac{3}{2}\dot{D} \quad , \qquad R_{11} = -\frac{1}{2}\dot{D} - \frac{1}{2}D^2 - 2 \quad , \\
&R_{22} = -\frac{1}{2}\dot{D}\sin^2\chi - \frac{1}{2}D^2\sin^2\chi - 2\sin^2\chi \quad , \\
&R_{33} = -\frac{1}{2}\dot{D}\sin^2\chi\sin^2\theta - \frac{1}{2}D^2\sin^2\chi\sin^2\theta + \cos^2\chi\sin^2\theta - \sin^2\chi\sin^2\theta - \sin^2\theta \quad ,
\end{aligned}$$

$$(A.3)$$

$$R = C^{-1}[3\dot{D} + \frac{3}{2}D^2 + 6] \quad . \tag{A.4}$$

The Lagrangian density is given as [18]

$$\mathcal{L}(\vec{x}) = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\Phi_{;\mu}(\vec{x})\Phi_{;\nu}(\vec{x}) - m^2\Phi^2(\vec{x}) - \frac{R}{6}\Phi^2(\vec{x})\right) \quad . \tag{A.5}$$

We construct the action

$$S = \int \mathcal{L}(\vec{x})d^4x \quad . \tag{A.6}$$

Varying S with respect to Φ, we obtain the scalar field equation

$$0 = \frac{\delta S}{\delta \Phi} = \sqrt{-g}\left(\Phi^{;\mu}_{;\mu} + (\frac{1}{6}R + m^2)\Phi\right). \tag{A.7}$$

where $\frac{\delta}{\delta\Phi}$ indicates functional differentiation, g is the determinant of the background metric $g_{\mu\nu}$. Hence, the equation of motion is given as

$$\Box\Phi(\vec{x}) + m^2\Phi(\vec{x}) + \frac{R}{6}\Phi(\vec{x}) = 0 \quad , \tag{A.8}$$

where $\Phi(\vec{x})$ is the scalar field, m is the scalar field's mass, and R is the Ricci scalar. We will let $m \to 0$ in equation (A.8) for the massless conformal scalar field, hence the equation of motion is given as

$$\Box\Phi(\vec{x}) + \frac{R}{6}\Phi(\vec{x}) = 0 \quad . \tag{A.9}$$

The solution of equation (A.9) could be obtained by separation of variables as

$$\Phi_{\vec{k}}(\eta, \chi, \theta, \phi) = \frac{1}{C^{1/2}(\eta)}\Psi_{\vec{k}}(\eta)\mathcal{Y}_{\vec{k}}(\chi, \theta, \phi) \quad . \tag{A.10}$$

The function $\mathcal{Y}_{\vec{k}}$ could be written as

$$\mathcal{Y}_{\vec{k}}(\chi, \theta, \phi) = \Pi^{(+)}_{kl}(\chi)Y_{lM}(\theta, \phi) \quad , \tag{A.11}$$

where

$$k = 1, 2, 3, \ldots \quad ,$$
$$l = 1, 2, 3, \ldots, (k-1) \quad , \quad and$$
$$M = -l, -l+1, \ldots\ldots, 0, \ldots., l \quad . \tag{A.12}$$

Here, the $Y_{lM}(\theta,\phi)$'s are the standart spherical harmonics and $\Pi_{kl}^{(+)}(\chi)$ satisfies the following differential equation:

$$\frac{d}{d\chi}\left(sin^2\chi\frac{d\Pi_{kl}^{(+)}}{d\chi}\right)+\left(sin^2\chi(k^2-1)-l(l+1)\right)\Pi_{kl}^{(+)}(\chi)=0 \quad . \tag{A.13}$$

We have seen that the solution of the above differential equation is given as [21]

$$\Pi_{kl}^{(+)}(\chi)=C_0 sin^l(\chi)C_{k-l-1}^{l+1}(cos\chi) \quad , \tag{A.14}$$

where C_0 is a constant and $C_{k-l-1}^{l+1}(cos\chi)$ are the Gegenbauer polynomials:

$$C_{k-l-1}^{l+1}(cos\chi)\propto\frac{d^{l+1}\cos k\chi}{d(cos\chi)^{l+1}} \quad . \tag{A.15}$$

The functions $\mathcal{Y}_{\vec{k}}(\chi,\theta,\phi)$ are the eigenfunctions of the three-space Laplacian $\Delta^{(3)}$ such that

$$\Delta^{(3)}\mathcal{Y}_{\vec{k}}(\chi,\theta,\phi)=-(k^2-1)\mathcal{Y}_{\vec{k}}(\chi,\theta,\phi) \quad . \tag{A.16}$$

The time-dependent mode functions $\Psi_k(\eta)$ satisfy

$$\frac{d^2\Psi_{\vec{k}}(\eta)}{d\eta^2}+k^2\Psi_{\vec{k}}(\eta)=0 \quad . \tag{A.17}$$

The solution of this equation could be written as

$$\Psi_{\vec{k}}(\eta)=e^{-ik\eta} \quad . \tag{A.18}$$

Thus Eq.(A.9) becomes

$$\Phi_{\vec{k}}(\eta,\chi,\theta,\phi)=\frac{c_0}{C^{1/2}(\eta)}\sin^l\chi C_{k-l-1}^{l+1}(\cos\chi)Y_{lM}(\theta,\phi)e^{-ik\eta} \quad . \tag{A.19}$$

To evaluate the normalization constant c_0 we use the generalized scalar product which is given as

$$(\Phi_{\vec{k_1}},\Phi_{\vec{k_2}})=-i\int\left[\Phi_{\vec{k_1}}(\vec{x})\partial_\eta\Phi^*_{\vec{k_2}}(\vec{x})-\partial_\eta\Phi_{\vec{k_1}}(\vec{x})\Phi^*_{\vec{k_2}}(\vec{x})\right]\sqrt{-g}d^3x \quad , \tag{A.20}$$

and obtain

$$c_0=2^l l!\left\{\frac{(k-l-1)!}{\pi(k+l)!}\right\}^{\frac{1}{2}} . \tag{A.21}$$

Thus, the complete set of mode solutions become

$$\Phi_{\vec{k}}(\eta,\chi,\theta,\phi) = \frac{1}{C^{1/2}(\eta)} 2^l l! \left\{ \frac{(k-l-1)!}{\pi(k+l)!} \right\}^{\frac{1}{2}} \sin^l \chi C^{l+1}_{k-l-1}(\cos\chi) Y_{lM}(\theta,\phi) e^{-ik\eta}. \tag{A.22}$$

We could write the two point function by using the above mode solutions $\Phi_{\vec{k}}$ as

$$< 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 >= \sum_k \Phi_k(\vec{x''})\Phi^*_k(\vec{x'}) \quad , \tag{A.23}$$

where k is a set of mode indices.

$$\begin{aligned} < 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 > \; = \; & \frac{1}{C^{1/2}(\eta'')C^{1/2}(\eta')} \sum_{k=1}^{\infty} e^{-ik\Delta\eta} \\ & \sum_{l=0}^{k-1} \frac{2^{2l}(l!)^2(k-l-1)!}{\pi(k+l)!} \sin^l \chi'' \sin^l \chi' \\ & C^{l+1}_{k-l-1}(\cos\chi'')C^{l+1}_{k-l-1}(\cos\chi') \sum_{M=-l}^{l} Y_{lM}(\theta'',\phi'')Y^*_{lM}(\theta',\phi') \quad , \end{aligned} \tag{A.24}$$

where the temporal separation $\eta'' - \eta'$ has been written as $\Delta\eta$.

Using the addition formula for the spherical harmonics, which is given as

$$\frac{2l+1}{4\pi} P_l(\cos\gamma) = \sum_{M=-l}^{l} Y_{lM}(\theta'',\phi'')Y^*_{lM}(\theta',\phi') \quad , \tag{A.25}$$

where

$$\cos\gamma = \cos\theta'' \cos\theta' + \sin\theta'' \sin\theta' \cos(\phi'' - \phi') \quad . \tag{A.26}$$

We obtain

$$\begin{aligned} < 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 > \; = \; & \frac{1}{4\pi^2 C^{1/2}(\eta'')C^{1/2}(\eta')} \sum_{k=1}^{\infty} e^{-ik\Delta\eta} \\ & \sum_{l=0}^{k-1} \frac{2^{2l}(l!)^2(k-l-1)!(2l+1)}{(k+l)!} \sin^l \chi'' \sin^l \chi' \\ & C^{l+1}_{k-l-1}(\cos\chi'')C^{l+1}_{k-l-1}(\cos\chi')P_l(\cos\gamma) \quad . \end{aligned} \tag{A.27}$$

Now using addition theorem for the Gegenbauer polynomials which is given as

$$C^1_{k-1}(\cos\Delta s) = \sum_{l=0}^{k-1} \frac{2^{2l}(l!)^2(k-l-1)!}{\pi(k+l)!}\sin^l\chi''\sin^l\chi' \\ C^{l+1}_{k-l-1}(\cos\chi'')C^{l+1}_{k-l-1}(\cos\chi')P_l(\cos\gamma) \quad , \qquad (A.28)$$

and

$$C^1_{k-1}(\cos\Delta s) = \frac{\sin k\Delta s}{\sin\Delta s} \quad , \qquad (A.29)$$

we obtain

$$< 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 >= \frac{1}{4\pi^2 C^{1/2}(\eta'')C^{1/2}(\eta')}\sum_{k=1}^{\infty} e^{-ik\Delta\eta}\frac{\sin k\Delta s}{\sin\Delta s} \quad . \qquad (A.30)$$

Using the following relation

$$\sum_{k=1}^{\infty} p^k \sin k\Delta s = \frac{p\sin\Delta s}{1-2p\cos\Delta s+p^2} \quad \text{if} \quad |\, p \,|< 1 \quad , \qquad (A.31)$$

the two point function becomes

$$< 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 >= \frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{8\pi^2(\cos\Delta\eta - \cos\Delta s)} \quad . \qquad (A.32)$$

The spacelike separation Δs is given in terms of the coordinates $(\eta, \chi, \theta, \phi)$ as

$$\cos\Delta s = \cos\chi''\cos\chi' + \sin\chi''\sin\chi' \\ [\cos\theta''\cos\theta' + \sin\theta''\sin\theta'\cos(\phi''-\phi')] \, . \qquad (A.33)$$

However, because the space is isotropic we may choose, $\theta'' = \theta'$, $\phi'' = \phi'$ so that $\Delta s = \chi'' - \chi'$.

We now assume that the separations $\Delta\eta$ and Δs are infinitesimally small, the points (η'', χ'') and (η', χ') being separated along a non-null geodesic by a proper distance 2ϵ. The mid points of the geodesic coincide with the point of interest at which $< 0 \mid T_{\mu\nu} \mid 0 >$ is to be calculated. At the end the calculation we let both x'' and x' approach x. The small distance from x to x' is denoted ϵ and the direction of the geodesic is parametrized by the tangent vector t^μ at x. If the separated point x'', x' remain in a normal neighbourhood of x, then this geodesic will be unique. To make use of this formalism the first step is to convert the two point function into a function of ϵ and t^μ rather than x'' and x'. For this reason it is necessary to evaluate the following expansions up to fifth order ϵ.

$$\begin{aligned} \chi''(\epsilon) &= \chi_0 + \epsilon t_1^1 + \frac{1}{2}\epsilon^2 t_2^1 + \frac{1}{6}\epsilon^3 t_3^1 + \frac{1}{24}\epsilon^4 t_4^1 + \frac{1}{120}\epsilon^5 t_5^1 + \ldots.. \\ \chi'(-\epsilon) &= \chi_0 - \epsilon t_1^1 + \frac{1}{2}\epsilon^2 t_2^1 - \frac{1}{6}\epsilon^3 t_3^1 + \frac{1}{24}\epsilon^4 t_4^1 - \frac{1}{120}\epsilon^5 t_5^1 + \ldots.. \quad , \end{aligned} \tag{A.34}$$

$$\begin{aligned} \eta''(\epsilon) &= \eta_0 + \epsilon t_1^0 + \frac{1}{2}\epsilon^2 t_2^0 + \frac{1}{6}\epsilon^3 t_3^0 + \frac{1}{24}\epsilon^4 t_4^0 + \frac{1}{120}\epsilon^5 t_5^0 + \ldots.. \\ \eta'(-\epsilon) &= \eta_0 - \epsilon t_1^0 + \frac{1}{2}\epsilon^2 t_2^0 - \frac{1}{6}\epsilon^3 t_3^0 + \frac{1}{24}\epsilon^4 t_4^0 - \frac{1}{120}\epsilon^5 t_5^0 + \ldots.. \quad , \end{aligned} \tag{A.35}$$

where t_α^0 and t_α^1 ($\alpha = 1, 2, 3, 4, 5, ., ., .$) represent the tangent vectors timelike and spacelike, respectively.

The expansion coefficients in Eq.(A.34) and (A.35) are found by solving the geodesic equation. Consider a family of geodesics, where each geodesic satisfies the following differential equation

$$\frac{dt^\mu}{d\epsilon} + \Gamma^\mu_{\alpha\beta} t^\alpha t^\beta = 0 \quad , \tag{A.36}$$

where

$$t^\mu = \frac{dx^\mu}{d\epsilon} \quad , \qquad \frac{dt^\mu}{d\epsilon} = \frac{d^2 x^\mu}{d\epsilon^2} \quad . \tag{A.37}$$

Solving the geodesic equation in power series

$$t^\mu = t_1^\mu + \epsilon t_2^\mu + \frac{1}{2}\epsilon^2 t_3^\mu + \frac{1}{6}\epsilon^3 t_4^\mu + \frac{1}{24}\epsilon^4 t_5^\mu + \frac{1}{120}\epsilon^5 t_6^\mu + \ldots.. \quad , \tag{A.38}$$

where

$$t_2^\mu = \frac{dt_1^\mu}{d\epsilon} = -\Gamma^\mu_{\alpha\beta} t^\alpha t^\beta \quad , \quad t_3^\mu = \frac{dt_2^\mu}{d\epsilon} \quad , \quad t_4^\mu = \frac{dt_3^\mu}{d\epsilon} \quad . \tag{A.39}$$

Only the $\mu = 0, 1$ (η, χ) components are non zero. The results up to the required adiabatic order are

$$\begin{aligned} t_2^0 &= -\frac{1}{2} D (t_1^0)^2 - \frac{1}{2} D (t_1^1)^2 \quad , \\ t_2^1 &= -D t_1^0 t_1^1 \quad , \end{aligned}$$

$$
\begin{aligned}
t_3^0 &= \left(-\frac{1}{2}\dot{D}+\frac{1}{2}D^2\right)(t_1^0)^3+\left(\frac{3}{2}D^2-\frac{1}{2}\dot{D}\right)t_1^0(t_1^1)^2 \quad , \\
t_3^1 &= \left(-\dot{D}+\frac{3}{2}D^2\right)(t_1^0)^2+\frac{1}{2}D^2(t_1^1)^3 \quad , \\
t_4^0 &= \left(-\frac{1}{2}\ddot{D}+\frac{7}{4}\dot{D}D-\frac{3}{4}D^3\right)(t_1^0)^4+\left(5\dot{D}D-\frac{1}{2}\ddot{D}-\frac{9}{2}D^3\right)(t_1^0t_1^1)^2 \\
&\quad \left(\frac{1}{4}\dot{D}D-\frac{3}{4}D^3\right)(t_1^1)^4 \quad , \\
t_4^1 &= \left(-\ddot{D}+5\dot{D}D-3D^3\right)(t_1^0)^3t_1^1+\left(2\dot{D}D-3D^3\right)(t_1^1)^3t_1^0 \quad , \\
t_5^0 &= \left(-\frac{1}{2}\dddot{D}+\frac{11}{4}\ddot{D}D-\frac{23}{4}\dot{D}D^2+\frac{7}{4}\dot{D}^2+\frac{3}{2}D^4\right)(t_1^0)^5 \\
&\quad +\left(-\frac{1}{2}\dddot{D}+\frac{15}{2}\ddot{D}D+5\dot{D}^2-32\dot{D}D^2+15D^4\right)(t_1^0)^3(t_1^1)^2 \\
&\quad +\left(\frac{3}{4}\ddot{D}D+\frac{1}{4}\dot{D}^2-\frac{33}{4}\dot{D}D^2+\frac{15}{2}D^4\right)t_1^0(t_1^1)^4 \quad , \\
t_5^1 &= \left(-\dddot{D}+\frac{15}{2}\ddot{D}D+5\dot{D}^2-\frac{43}{2}\dot{D}D^2+\frac{15}{2}D^4\right)(t_1^0)^4t_1^1 \\
&\quad +\left(\frac{7}{2}\ddot{D}D+2\dot{D}^2-\frac{47}{2}\dot{D}D^2+15D^4\right)(t_1^0)^2(t_1^1)^3 \\
&\quad +\left(-\dot{D}D^2+\frac{3}{2}D^4\right)(t_1^1)^5 \quad .
\end{aligned}
\tag{A.40}
$$

We now calculate the cosines in the denominator of Eq.(A.32). Using the expansion of cosines

$$
\cos\Delta s = \cos\Delta\chi = 1-\frac{\Delta\chi^2}{2}+\frac{\Delta\chi^4}{24}-\ldots.. \quad ,
$$

$$\cos\Delta\eta \;=\; 1-\frac{\Delta\eta^2}{2}+\frac{\Delta\eta^4}{24}+..... \quad . \qquad \text{(A.41)}$$

Thus

$$\cos\Delta\eta-\cos\Delta\chi=\frac{1}{2}[\Delta\chi^2-\Delta\eta^2]+\frac{1}{24}[\Delta\eta^4-\Delta\chi^4]+.. \quad . \qquad \text{(A.42)}$$

Using the Eqs.(A.34) and (A.35) one obtains

$$\begin{aligned}
\Delta\eta &= \eta''-\eta'=2\epsilon t_1^0+\frac{1}{3}\epsilon^3t_3^0+\frac{1}{60}\epsilon^5t_5^0+.. \;, \\
\Delta\chi &= \chi''-\chi'=2\epsilon t_1^1+\frac{1}{3}\epsilon^3t_3^1+\frac{1}{60}\epsilon^5t_5^1+.. \;, \\
\Delta\eta^2 &= 4\epsilon^2(t_1^0)^2+\frac{4}{3}\epsilon^4t_1^0t_3^0+\frac{1}{9}\epsilon^6(t_3^0)^2+\frac{1}{15}\epsilon^6t_5^0t_1^0+... \;, \\
\Delta\chi^2 &= 4\epsilon^2(t_1^1)^2+\frac{4}{3}\epsilon^4t_1^1t_3^1+\frac{1}{9}\epsilon^6(t_3^1)^2+\frac{1}{15}\epsilon^6t_5^1t_1^1+... \;, \\
\Delta\eta^4 &= 16\epsilon^4(t_1^0)^4+\frac{32}{3}\epsilon^6(t_1^0)^3t_3^0+... \;, \\
\Delta\chi^4 &= 16\epsilon^4(t_1^1)^4+\frac{32}{3}\epsilon^6(t_1^1)^3t_3^1+... \quad .
\end{aligned} \qquad \text{(A.43)}$$

Thus, Eq.(A.42) becomes

$$\begin{aligned}
\cos\Delta\eta-\cos\Delta\chi \;=\; & 2\epsilon^2[(t_1^1)^2-(t_1^0)^2] \\
& +\frac{2}{3}\epsilon^4[t_1^1t_3^1-t_1^0t_3^0+(t_1^0)^4-(t_1^1)^4] \\
& +\frac{\epsilon^6}{90}[5((t_3^1)^2-(t_3^0)^2)+3(t_5^1t_1^1-t_5^0t_1^0) \\
& +40((t_3^1)^2t_3^0-(t_1^1)^3t_3^1)] \quad .
\end{aligned} \qquad \text{(A.44)}$$

Using the result presented in Eq.(A.40) and taking the common paranthesis $2\epsilon^2[(t_1^1)^2-(t_1^0)^2]$, one obtains

$$\cos\Delta\eta - \cos\Delta\chi = 2\epsilon^2[(t_1^1)^2 - (t_1^0)^2]\Bigg[1 + \frac{\epsilon^2}{6}\left\{(D^2-2)(t_1^1)^2 + (D^2 - \dot{D} - 2)(t_1^0)^2\right\}$$

$$+\frac{\epsilon^4}{180}\Bigg[(\frac{23}{4}D^4 - 3\dot{D}D^2 - 20D^2)(t_1^1)^4$$

$$+(-\frac{185}{4}\dot{D}D^2 + \frac{33}{4}\ddot{D}D + 4\dot{D}^2 + 40\dot{D} - 80D^2 + \frac{49}{2}D^4)(t_1^0t_1^1)^2$$

$$-(\frac{3}{2}\dddot{D} - \frac{33}{4}\ddot{D}D + \frac{79}{4}\dot{D}D^2 - \frac{13}{2}\dot{D}^2 - 20\dot{D} - \frac{23}{4}D^4 + 20D^2)(t_1^0)^4\Bigg]\Bigg]. \tag{A.45}$$

In these calculations we neglect the $(t_5^0)^2$ and $t_5^0t_3^0$ terms because they are bigger than the adiabatic order four. We rewrite Eq.(A.32) as

$$< 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 >= \frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{16\pi^2\epsilon^2[(t_1^1)^2 - (t_1^0)^2]}C(1+x)^{-1} \quad , \tag{A.46}$$

where

$$x = \frac{\epsilon^2}{6}\left\{(D^2-2)(t_1^1)^2 + (D^2 - \dot{D} - 2)(t_1^0)^2\right\}$$

$$+\frac{\epsilon^4}{180}\Bigg[\left(\frac{23}{4}D^4 - 3\dot{D}D^2 - 20D^2\right)(t_1^1)^4$$
$$+\left(-\frac{185}{4}\dot{D}D^2 + \frac{33}{4}\ddot{D}D + 4\dot{D}^2 + 40\dot{D} - 80D^2 + \frac{49}{2}D^4\right)(t_1^0t_1^1)^2$$

$$-\left(\frac{3}{2}\dddot{D} - \frac{33}{4}\ddot{D}D + \frac{79}{4}\dot{D}D^2 - \frac{13}{2}\dot{D}^2 - 20\dot{D} - \frac{23}{4}D^4 + 20D^2\right)(t_1^0)^4\Bigg]. \tag{A.47}$$

Using the normalization condition

$$g_{\mu\nu}t^\mu t^\nu = [(t_1^0)^2 - (t_1^1)^2]C(\eta)$$
$$= \Sigma \quad , \tag{A.48}$$

where $\Sigma = +1$ if t^μ is timelike and $\Sigma = -1$ if t^μ is spacelike. Thus Eq.(A.46) becomes

$$< 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 >= -\frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{16\pi^2\epsilon^2\Sigma}C(1+x)^{-1} \quad , \tag{A.49}$$

Using the binomial expansion we get

$$(1+x)^{-1} = 1 - x + x^2 - .... \quad , \tag{A.50}$$

where in Eq.(A.49), x is small compared to 1. Thus, Eq.(A.49) becomes

$$< 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 > = -CC^{-1/2}(\eta'')C^{-1/2}(\eta')(18\pi^2\epsilon^2\Sigma)^{-1}\Big[1 + \frac{\epsilon^2}{6}\Big((-D^2+2)(t_1^1)^2$$
$$+(-D^2+\dot{D}+2)(t_1^0)^2\Big) + \frac{\epsilon^4}{120}\Big[\Big(2\dot{D}D^2 - \frac{1}{2}D^4 + \frac{40}{3}\Big)(t_1^1)^4$$
$$+\Big(\dddot{D} - \frac{11}{2}\ddot{D}D + \frac{13}{2}\dot{D}D^2 - \dot{D}^2 - \frac{1}{2}D^4 + \frac{40}{3}\Big)(t_1^0)^4$$
$$+\Big(\frac{145}{6}\dot{D}D^2 - \frac{11}{2}\ddot{D}D - \frac{8}{3}\dot{D}^2 + \frac{80}{3}D^2$$
$$-\frac{40}{3}\dot{D} - \frac{29}{3}D^4 + \frac{80}{3}\Big)(t_1^0t_1^1)^2\Big]\Big] . \tag{A.51}$$

The expansion of $C^{-1/2}(\epsilon)C^{-1/2}(-\epsilon)$ in the numerator of Eq.(A.32) is once again straightforward, the result being

$$C^{-1/2}(\epsilon)C^{-1/2}(-\epsilon) = C^{-1}\Big[1+\frac{\epsilon^2}{2}\Big((-\dot{D}+\frac{1}{2}D^2)(t_1^0)^2+\frac{1}{2}D^2(t_1^1)^2\Big)$$

$$+\frac{\epsilon^4}{24}\Big[\Big(-\dddot{D}+\frac{7}{2}\ddot{D}D+5\dot{D}^2-\frac{15}{2}\dot{D}D^2+\frac{3}{2}D^4\Big)(t_1^0)^4$$

$$+\Big(\frac{7}{2}\ddot{D}D+2\dot{D}^2-\frac{31}{2}\dot{D}D^2+6D^4\Big)(t_1^0t_1^1)^2$$

$$+\Big(-D^2\dot{D}+\frac{3}{2}D^4\Big)(t_1^1)^4\Big]\Big]. \qquad (A.52)$$

Multiplying the two series (A.51) and (A.52) together gives

$$\lim_{\epsilon\to 0} < 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 > = -\frac{1}{16\pi^2\epsilon^2\Sigma}+\frac{1}{48\pi^2\Sigma}\Big[(\dot{D}-\frac{1}{4}D^2-1)(t_1^0)^2$$

$$+(-\frac{1}{4}D^2-1)(t_1^1)^2\Big] + \frac{\epsilon^2}{2880\pi^2\Sigma}\Big[\Big(6\dddot{D}-18\ddot{D}D$$

$$-21\dot{D}^2+24\dot{D}D^2-3D^4+30\dot{D}-15D^2-20\Big)(t_1^0)^4$$

$$+\Big(-24\ddot{D}D-11\dot{D}^2+\frac{115}{2}\dot{D}D^2-\frac{31}{2}D^4+50\dot{D}$$

$$-70D^2-40\Big)(t_1^0t_1^1)^2+\Big(\frac{9}{2}\dot{D}D^2-3D^4-15D^2+20\Big)(t_1^1)^4\Big]. \qquad (A.53)$$

It is possible to express Eq.(A.53) purely in terms of local geometric quantities at the point of interest (η,χ). For this reason we calculate the components of all the geometrical tensors of the appropriate adiabatic order. Using the following geometrical quantities (; represents the covariant derivative)

$$R^2 = C^{-2}[9\dot{D}^2+9\dot{D}D^2+\frac{9}{4}D^4+36\dot{D}+18D^2+36] \quad ,$$

$$
\begin{aligned}
R_{00;00} &= \frac{3}{2}\dddot{D} - \frac{15}{4}\ddot{D}D - \frac{3}{2}\dot{D}^2 + \frac{9}{4}\dot{D}D^2 \quad , \\
R_{00;11} &= \frac{5}{4}\dot{D}D^2 - \frac{3}{4}\ddot{D}D - \frac{1}{4}\dot{D}^4 - D^2 \quad , \\
R_{01;01} &= -\frac{1}{2}\ddot{D}D + \frac{5}{4}\dot{D}D^2 - \frac{3}{8}D^4 - \frac{3}{2}D^2 \quad , \\
R_{01;10} &= -\frac{1}{2}\ddot{D}D + \frac{3}{2}\dot{D}D^2 - \frac{1}{2}\dot{D}^2 - \frac{3}{8}D^4 + \dot{D} - \frac{3}{4}D^2 \quad , \\
R_{11;00} &= -\frac{1}{2}\dddot{D} + \frac{1}{4}\ddot{D}D - \frac{1}{2}\dot{D}^2 + \frac{9}{4}\dot{D}D^2 - \frac{3}{4}D^4 + 2\dot{D} - 3D^2 \quad , \\
R_{11;11} &= \frac{1}{4}\ddot{D}D + \frac{3}{4}\dot{D}D^2 - \frac{1}{2}D^4 - 2D^2 \quad , \\
R_{;00} &= C^{-1}[3\dddot{D} - \frac{9}{2}\ddot{D}D^2 + \frac{9}{2}\dot{D}D^2 + \frac{9}{4}D^4 - 6\dot{D} + 9D^2] \quad , \\
R_{;11} &= C^{-1}[-\frac{3}{2}\ddot{D}D + \frac{3}{4}D^4 + 3D^2] \quad , \\
R^0_\lambda R_{\lambda 0} &= C^{-1}\frac{9}{4}\dot{D}^2 \quad , \\
R^\lambda_1 R_{\lambda 1} &= C^{-1}[-\frac{1}{4}\dot{D}^2 - \frac{1}{2}\dot{D}D^2 - \frac{1}{4}D^4 - 2\dot{D} - 2D^2 - 4] \quad , \\
RR_{00} &= C^{-1}[\frac{9}{2}\dot{D}^2 + \frac{9}{4}\dot{D}D^2 + 9\dot{D}] \quad , \\
RR_{11} &= C^{-1}[-\frac{3}{2}\dot{D}^2 - \frac{9}{4}\dot{D}D^2 - \frac{3}{4}D^4 - 9\dot{D} - 6D^2 - 12] \quad , \\
R_{\alpha\beta}R^{\alpha\beta} &= C^{-2}[3\dot{D}^2 + \frac{3}{2}\dot{D}D^2 + \frac{3}{4}D^4 + 6\dot{D} + 6D^2 + 12] \quad , \\
R^2_{00} &= \frac{9}{4}\dot{D}^2 \quad , \\
R^2_{11} &= \frac{1}{4}\dot{D}^2 + \frac{1}{4}D^4 + 4 + \frac{1}{2}\dot{D}D^2 + 2\dot{D} + 2D^2 \quad , \\
R_{00}R_{11} &= -\frac{3}{4}\dot{D}^2 - \frac{3}{4}\dot{D}D^2 - 3\dot{D} \quad ,
\end{aligned}
\tag{A.54}
$$

our two point function becomes

$$
\begin{aligned}
\lim_{\epsilon\to 0} < 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 > \;=\; & -\frac{1}{16\pi^2\epsilon^2\Sigma} + \frac{1}{48\pi^2}[R_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1} - \frac{1}{6}R] \\
& + \frac{\epsilon^2\Sigma}{2880\pi^2}\Big[2R^\lambda_\alpha R_{\lambda\beta}t^\alpha t^\beta\Sigma^{-1} + 4RR_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1} \\
& -\frac{1}{3}R^2 - R_{;\alpha\beta}t^\alpha t^\beta\Sigma^{-1} - 14R_{\alpha\beta}R_{\gamma\delta}t^\alpha t^\beta t^\gamma t^\delta\Sigma^{-2} \\
& +6R_{\alpha\beta;\gamma\delta}t^\alpha t^\beta t^\gamma t^\delta\Sigma^{-2}\Big] .
\end{aligned}
\tag{A.55}
$$

The Hadamard's Green's function is defined by [16,18]

$$G^{(1)}(x,x') = 2Re < 0 \mid \Phi(\vec{x''})\Phi(\vec{x'}) \mid 0 > \quad . \tag{A.56}$$

Thus, the Hadamard's Green's function becomes

$$\begin{aligned} G^{(1)}(x,x') &= -\frac{1}{8\pi^2\epsilon^2\Sigma} + \frac{1}{24\pi^2}[R_{\alpha\beta}t^\alpha t^\beta \Sigma^{-1} - \frac{1}{6}R] \\ &+ \frac{\epsilon^2\Sigma}{1440\pi^2}\Big[2R^\lambda_\alpha R_{\lambda\beta}t^\alpha t^\beta\Sigma^{-1} + 4RR_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1} \\ &- \frac{1}{3}R^2 - R_{;\alpha\beta}t^\alpha t^\beta\Sigma^{-1} - 14R_{\alpha\beta}R_{\gamma\delta}t^\alpha t^\beta t^\gamma t^\delta\Sigma^{-2} \\ &+ 6R_{\alpha\beta;\gamma\delta}t^\alpha t^\beta t^\gamma t^\delta\Sigma^{-2}\Big] \ . \end{aligned} \tag{A.57}$$

To renormalize, we now subtract terms up to adiabatic order four in the DeWitt-Schwinger expansion. The DeWitt-Schwinger Green's function could be given as [18]

$$\begin{aligned} G^{(1)}_{DS}(\epsilon,t^\mu) &= -\frac{1}{8\pi^2\epsilon^2\Sigma} + \frac{m^2}{4\pi^2}\left(\frac{1}{2}\ln\mid m^2\epsilon^2\mid + \gamma\right) \\ &+ \frac{1}{24\pi^2}R_{\alpha\beta}\frac{t^\alpha t^\beta}{\Sigma} - \frac{m^2}{8\pi^2} + \frac{\epsilon^2\Sigma}{24\pi^2}\left(\frac{1}{2}\ln\mid m^2\epsilon^2\mid + \gamma\right)\Big[-3m^4 \\ &- 2m^2R_{\alpha\beta}\frac{t^\alpha t^\beta}{\Sigma}\Big] + \frac{\epsilon^2\Sigma}{1440\pi^2}\Big[225m^4 + 60m^2R_{\alpha\beta}\frac{t^\alpha t^\beta}{\Sigma} \\ &+ 6R_{\alpha\beta;\gamma\delta}\frac{t^\alpha t^\beta t^\gamma t^\delta}{\Sigma^2} - 14R_{\alpha\beta}R_{\gamma\delta}\frac{t^\alpha t^\beta t^\gamma t^\delta}{\Sigma^2} \\ &+ 4R_{\alpha\rho}R^\rho_\beta\frac{t^\alpha t^\beta}{\Sigma} - \frac{4}{3}RR_{\alpha\beta}\frac{t^\alpha t^\beta}{\Sigma} - \Box R - R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R\Big] \\ &+ \frac{1}{1440\pi^2 m^2}\Big[R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{3}R^2 - \Box R\Big] . \end{aligned} \tag{A.58}$$

Defining the renormalized Green's function

$$D^{(1)}_{ren.}(x,x') = G^{(1)}(x,x') - G^{(1)}_{DS}(\epsilon,t^\mu) \quad , \tag{A.59}$$

we find the renormalized Green's function as

$$
\begin{aligned}
D^{(1)}_{ren.}(x,x') &= -\frac{R}{144\pi^2} - \frac{m^2}{4\pi^2}\left(\frac{1}{2}\ln \mid m^2\epsilon^2 \mid +\gamma\right) \\
&+\frac{m^2}{8\pi^2} - \frac{\epsilon^2\Sigma}{24\pi^2}\left(\frac{1}{2}\ln \mid m^2\epsilon^2 \mid +\gamma\right)\left[-3m^4 - 2m^2 R_{\alpha\beta}t^\alpha t^\beta \Sigma^{-1}\right] \\
&+\frac{\epsilon^2\Sigma}{1440\pi^2}\Bigg[-225m^4 - 60m^2 R_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1} \\
&-R_{;\alpha\beta}t^\alpha t^\beta\Sigma^{-1} + \frac{4}{3}RR_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1} \\
&+2R_{\lambda\beta}R^\lambda_\alpha t^\alpha t^\beta\Sigma^{-1} - \frac{2}{3}R^2 + \Box R + R_{\alpha\beta}R^{\alpha\beta}\Bigg] \\
&-\frac{1}{1440\pi^2 m^2}\left[R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{3}R^2 - \Box R\right].
\end{aligned}
\tag{A.60}
$$

We return to the energy momentum tensor which is defined by the functional differentiation of the action with respect to the metric tensor $g^{\mu\nu}$ as

$$
T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \quad , \tag{A.61}
$$

where

$$
\delta S = \int \delta\mathcal{L} d^4x \quad . \tag{A.62}
$$

Using the relation [18]

$$
\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \quad , \tag{A.63}
$$

we obtain $\delta\mathcal{L}$ (with respect to $g^{\mu\nu}$)as

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{1}{2}\Bigg(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu}\delta g^{\mu\nu} + \sqrt{-g}\Phi_{;\mu}\Phi_{;\nu}\delta g^{\mu\nu} \\
&+ \frac{1}{12}\sqrt{-g}g_{\mu\nu}R\Phi^2\delta g^{\mu\nu} - \frac{1}{6}\sqrt{-g}\delta R\Phi^2 + \frac{1}{2}m^2\sqrt{-g}g_{\mu\nu}\Phi^2\delta g^{\mu\nu}\Bigg).
\end{aligned}
\tag{A.64}
$$

Using the following relations [18]

$$\begin{aligned}
\delta R &= -R^{\mu\nu}\delta g_{\mu\nu} + g^{\rho\sigma}g^{\mu\nu}(\delta g_{\rho\sigma;\mu\nu} + \delta g_{\rho\mu;\sigma\nu}) \ , \\
\delta g^{\mu\nu} &= -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} \ , \\
g^{\mu\nu}\delta g_{\mu\nu} &= -g_{\mu\nu}\delta g^{\mu\nu} \ , \\
&\text{and} \\
g^{\rho\sigma}\Phi_{;\rho\sigma} &= \Box\Phi \ ,
\end{aligned} \tag{A.65}$$

we obtain

$$\begin{aligned}
\delta S = \ & \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}\Big(\frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} \\
& -\frac{1}{3}\Phi_{;\mu\nu}\Phi - \frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{36}g_{\mu\nu}R\Phi^2 + \frac{1}{6}m^2 g_{\mu\nu}\Phi^2\Big) \ .
\end{aligned} \tag{A.66}$$

Inserting (A.66) into (A.61) and rearranging, we obtain $T_{\mu\nu}$ [18] as

$$\begin{aligned}
T_{\mu\nu} = \ & \frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} - \frac{1}{3}\Phi_{;\mu\nu}\Phi \\
& -\frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{36}g_{\mu\nu}R\Phi^2 + \frac{1}{6}m^2 g_{\mu\nu}\Phi^2 \ .
\end{aligned} \tag{A.67}$$

Using the equation of motion (A.8) one could write this as

$$\begin{aligned}
T_{\mu\nu} = \ & \frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} - \frac{1}{3}\Phi_{;\mu\nu}\Phi \\
+ \ & \frac{1}{12}g_{\mu\nu}\Phi\Box\Phi - \frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{24}Rg_{\mu\nu}\Phi^2 + \frac{1}{4}m^2 g_{\mu\nu}\Phi^2 \ .
\end{aligned} \tag{A.68}$$

The trace of $T_{\mu\nu}$ is defined as

$$(T =)T^{\mu}_{\mu} = g^{\mu\nu}T_{\mu\nu} \ . \tag{A.69}$$

Also using

$$R = g^{\mu\nu}R_{\mu\nu} \ , \tag{A.70}$$

we obtain

$$\begin{aligned}
T = \ & \frac{2}{3}g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} \\
& -\frac{1}{3}g^{\mu\nu}\Phi_{;\mu\nu}\Phi + \frac{1}{3}\Phi\Box\Phi - \frac{1}{12}R\Phi^2 + \frac{1}{2}m\Phi^2 \ .
\end{aligned} \tag{A.71}$$

Using the following relations;

$$\begin{aligned} -\frac{1}{12}R\Phi^2 &= \frac{1}{2}\Phi\Box\Phi + \frac{1}{2}m^2\Phi^2 \ , \\ g^{\mu\nu}\Phi_{;\mu\nu}\Phi &= \Phi\Box\Phi \ , \end{aligned}$$

and

$$g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} = (g^{\mu\nu}\Phi_{;\mu}\Phi)_{;\nu} - g^{\mu\nu}\Phi_{;\mu\nu}\Phi \ , \tag{A.72}$$

we find that

$$T = m^2\Phi^2 \ . \tag{A.73}$$

Taking the vacuum expectation value $< T >$ and using the fact that the renormalized energy-momentum tensor could be written as

$$< 0 \mid T \mid 0 >_{ren} = \frac{1}{2}m^2 D^{(1)}_{ren.}(x, x') \ , \tag{A.74}$$

and taking $m \to 0$ we obtain

$$< 0 \mid T \mid 0 >_{ren} = \frac{1}{2880\pi^2}[\Box R - (R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{3}R^2)] \ . \tag{A.75}$$

Using Eq.(A.54) the trace could be written as

$$< 0 \mid T \mid 0 >_{ren} = \frac{C^{-2}}{2880\pi^2}[3\dddot{D} - 3\dot{D}D^2] \ . \tag{A.76}$$

The nonzero components of the renormalized energy-momentum tensor could be obtained from the trace anomoly by using the fact that the renormalized energy-momentum tensor is conserved [18]

$$\left(< 0 \mid T^{\nu}_{\mu} \mid 0 >\text{ren}\right)_{;\nu} = 0 \ , \tag{A.77}$$

The conservation condition could be written as

$$T^0_{0,0} + \frac{D}{2}T^0_0 - \frac{3}{2}DT^1_1 = 0 \ . \tag{A.78}$$

using

$$T^{\alpha}_{\alpha} = T^0_0 + 3T^1_1 \ . \tag{A.79}$$

Eq.(A.79) in Eq.(A.78) we obtain

$$T^0_{0,0} + DT^0_0 = \frac{1}{2} DT^\alpha_\alpha \quad . \tag{A.80}$$

Also using

$$\begin{aligned} T^0_0 &= g^{00} T_{00} \quad , \\ T^0_{0,0} &= g^{00} T_{00,0} \quad , \end{aligned} \tag{A.81}$$

we find

$$T_{00,0} + DT_{00} = \frac{1}{2} CDT^\alpha_\alpha \quad , \tag{A.82}$$

or

$$\frac{1}{C}\frac{d}{d\eta}(C(\eta)T_{00}) = C\frac{D}{2}T^\alpha_\alpha \quad . \tag{A.83}$$

This also means

$$\frac{d}{d\eta}(C(\eta) < 0 \mid T_{00} \mid 0 >\text{ren.}) = \frac{1}{2} C^2 D < 0 \mid T \mid 0 >\text{ren.} \quad . \tag{A.84}$$

Inserting (A.76) into (A.84) and after rearranging we obtain

$$\frac{d}{d\eta}(C < 0 \mid T_{00} \mid 0 >\text{ren.}) = \frac{1}{2880\pi^2}\frac{d}{d\eta}(\frac{3}{2}\ddot{D}D - \frac{3}{8}D^4 - \frac{3}{4}\dot{D}^2) \quad . \tag{A.85}$$

Now we could determine the complete renormalized energy-momentum tensor for the massless conformal scalar field in an expanding closed Friedmann model (valid up to adiabatic order four). Equation (A.85) could be integrated to yield [18]

$$< 0 \mid T^0_0 \mid 0 >\text{ren} = \frac{1}{2880\pi^2 C^2}(\frac{3}{2}\ddot{D}D - \frac{3}{8}D^4 - \frac{3}{4}\dot{D}^2 + A) \quad . \tag{A.86}$$

A is an integration constant to be determined from the static case. Using the result of section Il (2.53) we see that for closed Friedmann models A is actually 1. These expressions (A.76) and (A.86) are in complete agreement with the results obtained by using dimensional regularization and point splitting [18]. Thus one obtains both the trace anomaly and the Casimir energy. As we have mentioned the trace anamoly vanishes for the static models. As far as the other components of the renormalized energy momentum tensor is concerned, in general, the renormalized energy momentum tensor will have the following components

$$< T_1^1 >_{ren.} = < T_2^2 >_{ren.} = < T_3^3 >_{ren.} \quad , \qquad < T_\nu^\mu >_{ren.} = 0 \quad \text{where} \quad \mu \neq \nu \ . \tag{A.87}$$

Because of the symmetry of the closed Friedmann universe, the energy momentum tensor will have only two independent components, the 00 component T_0^0 and 11 component T_1^1. The nonzero components of the renormalized energy momentum tensor could be obtained from the trace anamoly i.e.

$$< 0 \mid T_1^1 \mid 0 >_{ren.} = \frac{1}{3}(< 0 \mid T \mid 0 >_{ren.} - < 0 \mid T_0^0 \mid 0 >_{ren.}) \quad , \tag{A.88}$$

hence

$$< 0 \mid T_1^1 \mid 0 >_{ren.} = \frac{1}{2880\pi^2 C^2}\left[\dddot{D} - \dot{D}D^2 - \frac{1}{2}\ddot{D}D + \frac{1}{8}D^4 + \frac{1}{4}\dot{D}^2 - \frac{1}{3}\right] . \tag{A.89}$$

APPENDIX B

GREEN'S FUNCTIONS

We consider a scalar field $\Phi(t, \vec{x})$ in a four dimensional Minkowski spacetime which satisfies the field equation [18]

$$(\Box + m^2)\Phi = 0 \quad , \tag{B.1}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\eta^{\mu\nu}$ is the Minkowski metric tensor (we take the signature as $+, -, -, -$). The quantity m is defined as the mass of the field quanta when the theory is quantized.

In this section we will write a formal solution to Eq.(B.1) using the Green's function technique. We define Green's function as the solution of

$$(\Box + m^2)\mathcal{G}(x, x') = -\delta(x - x') \quad , \tag{B.2}$$

where

$$\delta(x - x') = \frac{1}{(2\pi)^4}\int e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')}dk^0 d^3k \quad . \tag{B.3}$$

We now want to solve for $\mathcal{G}(x, x')$. We will do this first in the momentum space. Considering

$$\mathcal{G}(x, x') = \int dk^0 d^3k e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')}g(k^0, \vec{k}) \quad , \tag{B.4}$$

we write

$$\begin{aligned}(\Box + m^2)\mathcal{G}(x, x') &= \int dk^0 d^3k \left[(-ik^0)^2 - (i\vec{k})^2 + m^2\right] e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')}g(k^0, \vec{k}) \\ &= -\frac{1}{(2\pi)^4}\int e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')}dk^0 d^3k \quad .\end{aligned} \tag{B.5}$$

This gives us

$$g(k^0, \vec{k}) = \frac{1}{(2\pi)^4}\frac{1}{\left((k^0)^2 - |\vec{k}|^2 - m^2\right)} \quad , \tag{B.6}$$

so that

$$\mathcal{G}(x,x') = \frac{1}{(2\pi)^4}\int dk^0 d^3k \frac{e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')}}{(k^0)^2 - |\vec{k}|^2 - m^2} \quad . \tag{B.7}$$

This result is only a formal solution since the integral is over all of k^0 space including poles at $(k^0)^2 - |\vec{k}|^2 - m^2 = 0$. We have different results for the Green's function depending on how we treat the poles.

We calculate the explicit solutions for $\mathcal{G}(x,x')$. We start out by rewriting Eq.(B.7)

$$\mathcal{G}(x,x') = \frac{1}{(2\pi)^4}\int d^3k e^{i\vec{k}.(\vec{x}-\vec{x}')}\mathcal{I}(w_{\vec{k}},(t-t')) \quad , \tag{B.8}$$

where

$$\mathcal{I}(w_{\vec{k}},(t-t')) = \int dk^0 \frac{e^{-ik^0(t-t')}}{(k^0)^2 - w_{\vec{k}}^2} \quad \text{and} \quad w_{\vec{k}}^2 = |\vec{k}|^2 + m^2 \quad . \tag{B.9}$$

Using the following identity

$$\frac{1}{(k^0)^2 - w_{\vec{k}}} = \frac{1}{2w_{\vec{k}}}\left[\frac{1}{k^0 - w_{\vec{k}}} - \frac{1}{k^0 + w_{\vec{k}}}\right], \tag{B.10}$$

we have

$$\mathcal{I}(w_{\vec{k}},(t-t')) = \int_C \frac{dk^0}{2w_{\vec{k}}} e^{-ik^0(t-t')}\left[\frac{1}{k^0 - w_{\vec{k}}} - \frac{1}{k^0 + w_{\vec{k}}}\right]. \tag{B.11}$$

The most important contours are shown in Figure.1. From these contours the following Green's functions are easily calculated.

1. We work with the contour that runs above the poles. We calculate $\mathcal{I}(w_{\vec{k}},(t-t'))$ for this contour. We will do this by closing the contour and applying the residue theorem, and subtracting off any contribution that we may get from the curve used to close the contour. For $t < t'$, the contour could be closed in the upper half plane, and this gives $\mathcal{G} = 0$. For $t > t'$, the contour must be closed in the lower half plane, giving rise to the retarded Green's function:

$$\begin{aligned} G_R(x,x') &= \frac{i}{(2\pi)^3}\int \frac{d^3k}{2w_{\vec{k}}}\left[e^{-iw_{\vec{k}}(t-t')} - e^{iw_{\vec{k}}(t-t')}\right] \quad \text{if} \ t > t' \ , \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{w_{\vec{k}}} \sin w_{\vec{k}}(t-t')\theta(t-t') \ , \qquad \text{where} \end{aligned} \tag{B.12}$$

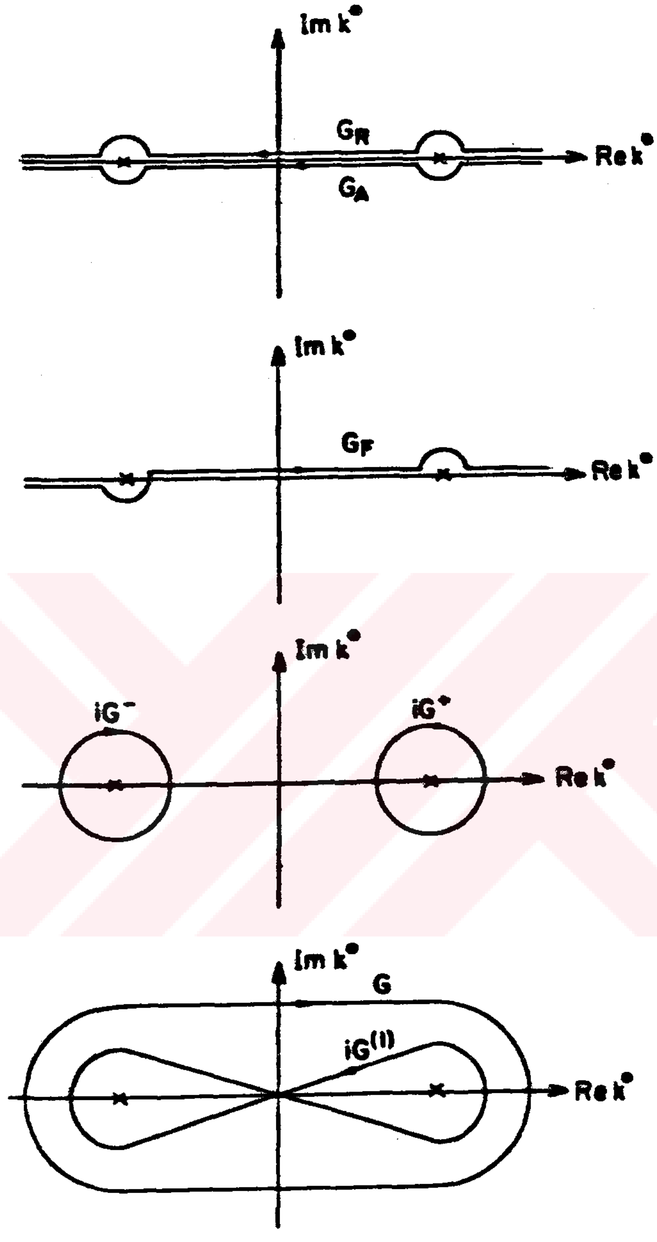

Figure 1. The various Green functions are each associated with the above contours in the complex k^0 plane.
The poles on the real axis at $k^0 = \pm(|\vec{k}|^2 + m^2)^{1/2}$ are marked with crosses.

$$\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{B.13}$$

2. Similarly, putting the contour below the poles yields the advanced Green's function, which equals 0 if $t > t'$ and in general

$$G_A(x,x') = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{w_{\vec{k}}} \sin w_{\vec{k}}(t-t')\theta(t'-t) \quad . \tag{B.14}$$

3. Contour plows right through the poles, G_R and G_A average is denoted as

$$\bar{G}(x,x') = \frac{1}{2}\left[G_R(x,x') + G_A(x,x')\right]. \tag{B.15}$$

4. The other Green's function G_F is called the Feymann propagator which is obtained by going under the left pole but over the right one. We displace the poles (taking the limit $\epsilon \to 0$ later). We have

$$G_F(x,x') = \frac{1}{(2\pi)^4}\int dk^0 d^3k \frac{e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')}}{(k^0)^2 - \mid \vec{k} \mid^2 - m^2 + i\epsilon} \quad . \tag{B.16}$$

5. The right-hand pole contributes $G^{(+)}(x,x')$ is called the Wightmann function

$$G^{(+)}(x,x') = \frac{i}{(2\pi)^3}\int \frac{d^3k}{2w_{\vec{k}}} e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')} \quad . \tag{B.17}$$

6. Similarly, the integral around the left-hand pole is denoted $G^{(-)}(x,x')$. We find

$$G^{(-)}(x,x') = \frac{-i}{(2\pi)^3}\int \frac{d^3k}{2w_{\vec{k}}} e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')} \quad . \tag{B.18}$$

Note that $G^{(+)}(x,x') = G^{(-)^*}(x,x')$.

7. We go around both poles counterclockwise:

$$G(x,x') = G_A(x,x') - G_R(x,x') \quad . \tag{B.19}$$

Using Eqs.(B.12) and (B.14) we obtain

$$G(x,x') = \frac{1}{(2\pi)^3}\int \frac{d^3k}{w_{\vec{k}}} e^{i\vec{k}.(\vec{x}-\vec{x}')} \sin w_{\vec{k}}(t-t') \quad ,$$

$$G(x,x') = 2ImG^{(+)}(x,x') \quad . \tag{B.20}$$

$G(x,x')$ is known as the Pauli-Jordan or Schwinger function.

8. A contour that goes clokwise around the left pole and counterclockwise around the right one produces the Hadamard function $G^{(1)}(x,x')$ defined as

$$G^{(1)}(x,x') = G^{(+)}(x,x') + G^{(-)}(x,x') \quad ,$$

$$G^{(1)}(x,x') = \frac{i}{(2\pi)^3}\int \frac{d^3k}{2w_{\vec{k}}} e^{i\vec{k}.(\vec{x}-\vec{x}')} \cos w_{\vec{k}}(t-t') \quad . \tag{B.21}$$

$G^{(+)}$ is related to $G^{(1)}$ as

$$G^{(1)}(x,x') = 2ReG^{(+)}(x,x') \quad . \tag{B.22}$$

$G^{(1)}$ is related to G_F by

$$G_F(x,x') = -\bar{G}(x,x') - \frac{1}{2}iG^{(1)}(x,x') \quad . \tag{B.23}$$

It is clear that G, $G^{(1)}$, $G^{(+)}$ and $G^{(-)}$ all satisfy the homogeneous equation

$$(\Box + m^2)\mathcal{G}(x,x') = 0 \quad , \tag{B.24}$$

while $G_F(x,x')$ satisfy

$$(\Box + m^2)\mathcal{G}(x,x') = -\delta(x-x') \quad . \tag{B.25}$$

We easily see that G_R, G_A, $\bar{G}$ and $G^{(1)}$ are all real valued but G_F, $G^{(+)}$ and $G^{(-)}$ are complex.

Let us formally calculate the vacuum expectation values of various products of free field operator. We return to Eq.(B.1), where we have a set of solutions of the equation (B.1) which could be given as

$$U_{\vec{k}} = A_0 e^{i\vec{k}.\vec{x}-iwt} \quad , \tag{B.26}$$

where A_0 is a normalization constant, and

$$w = (k^2+m^2)^{\frac{1}{2}} \quad . \tag{B.27}$$

$U_{\vec{k}}$ are the positive frequency mode functions and they form an orthonormal set with respect to the scalar product which is defined as [18]

$$(\Phi_1, \Phi_2) = -i \int [\Phi_1(x)\partial_t\Phi_2^*(x) - [\partial_t\Phi_1(x)]\Phi_2^*(x)]\, d^3x \ , \qquad \text{(B.28)}$$

hence

$$(U_{\vec{k}'}, U_{\vec{k}}) = \delta(\vec{k} - \vec{k}') \ . \qquad \text{(B.29)}$$

In canonical quantization scheme the scalar field Φ behaves like an operator and we impose the following equal-time commutation rules:

$$[\Phi(t, \vec{x}), \Phi(t, \vec{x}')] = 0 \ , \qquad \text{(B.30)}$$

$$[\Pi(t, \vec{x}), \Pi(t, \vec{x}')] = 0 \ , \qquad \text{(B.31)}$$

$$[\Phi(t, \vec{x}), \Pi(t, \vec{x}')] = i\delta^3(\vec{x} - \vec{x}') \ , \qquad \text{(B.32)}$$

where Π is the canonically conjugate momentum which is defined by

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)} = \partial_t \Phi \ . \qquad \text{(B.33)}$$

The field modes given in equation (B.24) and their respective complex conjugate form a complete and orthonormal basis with the scalar product (B.26). Hence, Φ may be expanded as

$$\Phi(t, \vec{x}) = \int d\vec{k}[a_{\vec{k}}U_{\vec{k}} + a_{\vec{k}}^{\dagger}U_{\vec{k}}^*]. \qquad \text{(B.34)}$$

($\dagger$ denotes Hermitian conjugate , * denotes complex conjugate.) The equal time commutation relations for Φ and Π are now equivalent to

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 \ , \qquad \text{(B.35)}$$

$$[a_{\vec{k}}^{\dagger}, a_{\vec{k}'}^{\dagger}] = 0 \ , \qquad \text{(B.36)}$$

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'} \quad . \tag{B.37}$$

Where, $a_{\vec{k}}$ is referred to as an annihilation operator and $a^{\dagger}_{\vec{k}}$ as a creation operator for quanta in the mode $\vec{k}$.

We now write the two point function of a quantized field

$$\begin{aligned} < 0 \mid \Phi(\vec{x})\Phi(\vec{x'}) \mid 0 > &= < 0 \mid \int d^3k a_{\vec{k}} U_{\vec{k}} \int d^3k' a^{\dagger}_{\vec{k}'} U^{*}_{\vec{k}'} \mid 0 > \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3k}{2w_{\vec{k}}} e^{i\vec{k}.(\vec{x}-\vec{x}')-iw_{\vec{k}}(t-t')} \\ &= -iG^{(+)}(x,x') \quad . \end{aligned} \tag{B.38}$$

Similarly,

$$iG^{(-)}(x,x') =< 0 \mid \Phi(\vec{x'})\Phi(\vec{x}) \mid 0 > \quad . \tag{B.39}$$

Therefore

$$iG(x,x') =< 0 \mid \left[\Phi(\vec{x'}), \Phi(\vec{x})\right] \mid 0 > \quad , \tag{B.40}$$

and

$$G^{(1)}(x,x') =< 0 \mid \left\{\Phi(\vec{x}), \Phi(\vec{x'})\right\} \mid 0 > \quad , \tag{B.41}$$

where the braces indicate the anticommutator of the operators:

$$\left\{\Phi(\vec{x}), \Phi(\vec{x'})\right\} = \Phi(\vec{x})\Phi(\vec{x'}) + \Phi(\vec{x'})\Phi(\vec{x}) \quad . \tag{B.42}$$

We now return to Eq.(B.7). We have written the Feymann Green's function with the understanding that all integrations are to be performed along the real axis. Alternatively, we could have displaced the contour to the imaginary axis. To evaluate G_F we will use the analytic continuation method.

$$G_F(x,x') = \frac{1}{(2\pi)^4} \int dk^0 d^3k \frac{e^{i\vec{k}.(\vec{x}-\vec{x}')-ik^0(t-t')}}{(k^0)^2 - \mid \vec{k} \mid^2 - m^2 + i\epsilon} \quad . \tag{B.43}$$

Now making use of the integral identity

$$\int_0^{\infty} d\tau e^{i\tau\left((k^0)^2 - |\vec{k}|^2 - m^2 + i\epsilon\right)} = \frac{-i}{(k^0)^2 - \mid \vec{k} \mid^2 - m^2 + i\epsilon} \quad . \tag{B.44}$$

We obtain

$$G_F(x,x') = \frac{-i}{(2\pi)^4}\int_0^\infty d\tau \int dk^0 d^3k e^{i\vec{k}.(\vec{x}-\vec{x'})-ik^0(t-t')+i\tau((k^0)^2-|\vec{k}|^2-m^2+i\epsilon)}. \quad (B.45)$$

After straightforward calculations, we obtain

$$G_F(x,x') = \frac{-i}{(2\pi)^4}\int_0^\infty d\tau \int dk^0 d^3k e^{-i\tau(\vec{k}-\frac{(\vec{x}-\vec{x'})}{2\tau})^2+i\tau(k^0-\frac{(t-t')}{2\tau})^2} e^{-\frac{i}{2\tau}\sigma - i\tau m^2} \quad , \quad (B.46)$$

where $\sigma = \frac{1}{2}[(t-t')^2 - (\vec{x}-\vec{x'})^2]$. Using the following integral

$$\int_{-\infty}^{\infty} e^{iax^2}dx = \sqrt{\frac{\pi}{|\,a\,|}}e^{a/|a|}e^{i\pi/4} \quad , \quad (B.47)$$

we obtain

$$G_F(x,x') = -\frac{\pi^2}{(2\pi)^4}\int_0^\infty \frac{d\tau}{\tau^2}e^{-\frac{i}{2\tau}\sigma - i\tau m^2} \quad . \quad (B.48)$$

In the final form the negative imaginary part $(i\epsilon)$ is attached to σ. G_F is really the boundary value of a function which is analytic in the lower half plane. Defining new variables

$$\begin{aligned} z &= (2m^2\sigma)^{\frac{1}{2}} \\ u &= -im\sqrt{2}\frac{\tau}{\sigma^{\frac{1}{2}}} \quad . \end{aligned} \quad (B.49)$$

Which convert Eq.(B.48) to

$$G_F(x,x') = -\frac{im\sqrt{2}}{16\pi^2\sigma^{1/2}}\int_{-i\infty}^{0}\frac{du}{u^2}e^{\frac{1}{2}z[u-\frac{1}{u}]} \quad . \quad (B.50)$$

Using the well known integral representation

$$H_1^{(2)}(z) = -\frac{1}{i\pi}\int_{-i\infty}^{0}\frac{du}{u^2}e^{\frac{1}{2}z[u-\frac{1}{u}]} \quad (B.51)$$

of the Hankel function of second kind of order 1. We finally have

$$G_F(x,x') = \frac{m\sqrt{2}}{16\pi\sigma^{1/2}}H_1^{(2)}([2m^2\sigma]^{1/2}) \quad . \quad (B.52)$$

For small values of z it is convenient to use the power series expansions

$$H_1^{(2)}(z) = J_1(z) - iY_1(z) \quad , \quad (B.53)$$

$$H_1^{(2)}(z) = \frac{z}{2} - \frac{z^3}{2^2 4} + \frac{z^5}{2^2 4^2 6} - \cdots\cdots\cdots$$

$$+i\frac{2}{\pi z} - i\frac{2z^2}{2^2\pi} + i\frac{2z^4}{\pi 2^2 4^2} + \cdots\cdots$$

$$\text{where} \quad z = (2m^2\sigma)^{1/2} \quad . \tag{B.54}$$

We find

$$G_F(x,x') = \frac{m\sqrt{2}}{16\pi\sigma^{1/2}}\left[\frac{z}{2} - \frac{z^3}{2^2 4} + \frac{z^5}{2^2 4^2 6} - \cdots\cdots\cdots \right.$$

$$\left. +i\frac{2}{\pi z} - i\frac{2z^2}{2^2\pi} + i\frac{2z^4}{\pi 2^2 4^2} + \cdots\cdots\right],$$

$$\text{where} \quad z = (2m^2\sigma)^{1/2} \quad . \tag{B.55}$$

In the massless limit (taking $m \to 0$)the Green's functions are denoted by D rather than G. In this limit the Feymann propagator in four dimensions reduces to

$$D_F(x,x') = \frac{i}{8\pi^2\sigma} - \frac{1}{8\pi}\delta(\sigma) \tag{B.56}$$

while the Hadamard's elementary function becomes

$$D^{(1)}(x,x') = -\frac{1}{4\pi^2\sigma} \quad . \tag{B.57}$$

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