AN INVESTIGATION OF DEVELOPMENT OF PROSPECTIVE ELEMENTARY TEACHERS' KNOWLEDGE TO TEACH ALGEBRA IN EARLY GRADES THROUGH CASE DISCUSSIONS

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ABSTRACT

AN INVESTIGATION OF DEVELOPMENT OF PROSPECTIVE ELEMENTARY TEACHERS' KNOWLEDGE TO TEACH ALGEBRA IN EARLY GRADES THROUGH CASE DISCUSSIONS

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The purpose of this study was to investigate the development of prospective elementary teachers' subject matter and pedagogical content knowledge for teaching algebra in elementary grades. Prospective elementary teachers attended a 5-week (10 hours) intervention as part of a method course, Teaching Mathematics I, that was designed based on case discussions. The participants of the study were nine third-year students who were enrolled in the course in the 2020-2021 fall semester as a must course of the undergraduate primary school education program in a private university. During the last 5 weeks of the 12-week course, the prospective elementary teachers were presented with the big ideas of equivalence and equations, generalized arithmetic, and functional thinking as the content of early algebra through text-based classroom cases. In these lessons, the prospective teachers were asked to discuss students' thinking, the teacher's instructions, and the tasks in the given classroom cases. Data were collected through one-hour individual interviews before and after the early algebra lessons. The interview questions were adapted from the related literature to examine prospective elementary teachers' subject matter and pedagogical content knowledge. Qualitative methods were used to analyze data. The analysis of the pre-interviews indicated that prospective elementary teachers did not have the required subject matter knowledge to guide algebraic thinking in elementary grades, such as the relational meaning of the equal sign, generalizing, representing, or justifying arithmetic or functional relationships and reasoning with them. Similarly, they were found not to have sufficient pedagogical content knowledge related to students' conception/misconceptions and appropriate instruction to foster early algebra. However, after the early algebra lessons based on case discussions, the prospective elementary teachers were found to progress in various aspects of teaching algebra in early grades, as both subject matter and pedagogical content knowledge related to teaching.

Keywords: Early Algebra, Prospective Elementary Teachers, Teacher Knowledge, Case Discussion

ÖZ

SINIF ÖĞRETMENİ ADAYLARININ DURUM TARTIŞMALARI YOLUYLA ERKEN CEBİR ÖĞRETİMİNE YÖNELİK BİLGİLERİNİN GELİŞİMİNİN İNCELENMESİ

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ÖZ

Bu çalışmanın amacı, durum tartışmaları temelli olarak tasarlanmış erken cebir derslerine katılan sınıf öğretmeni adaylarının, ilkokul seviyesinde cebir öğretimine yönelik alan ve pedagojik alan bilgilerinin gelişimini incelemektir. Çalışmanın katılımcıları, 2020-2021 güz döneminde sınıf öğretmenliği programının zorunlu dersi olan Matematik Öğretimi I dersini alan dokuz öğretmen adayıdır. Bu 12 haftalık dersin son 5 haftasında, sınıf öğretmeni adaylarına metin formundaki sınıf durumları aracılığıyla erken cebir içeriği olarak eşitlik ve denklem, genelleştirilmiş aritmetik ve fonksiyonel düşünme konuları sunulmuştur. Bu derslerde, öğretmen adaylarından verilen sınıf durumlarındaki öğrenci düşünceleri, öğretmen yönergeleri ve etkinlikler üzerine tartışmaları istenmiştir. Veriler, erken cebir derslerinden önce ve sonra yapılan birer saatlik bireysel görüşmelerle toplanmıştır. Görüşme soruları, sınıf öğretmeni adaylarının alan ve pedagojik alan bilgilerini incelemek için ilgili alan yazından uyarlanmıştır. Verileri analiz etmek için nitel yöntemler kullanılmıştır. Ön görüşmelerin analizi, öğretmen adaylarının, ilkokul seviyesinde cebirsel düşünmeye rehberlik edecek, eşit işaretinin ilişkisel anlamı, aritmetik veya fonksiyonel ilişkileri genelleme, temsil etme, gerekçelendirme ve bu ilişkiler üzerinde akıl yürütme gibi gerekli alan bilgisine sahip olmadıklarını göstermiştir. Benzer şekilde, erken cebir alanında öğrenci düşünmeleri/kavram yanılgıları ve uygun öğretim yolları ile ilgili yeterli pedagojik alan bilgisine sahip olmadıkları bulunmuştur. Ancak durum tartışmalarına dayalı erken cebir derslerinden sonra sınıf öğretmeni adaylarının ilkokul seviyesinde cebir öğretiminin çeşitli yönlerinde hem alan hem de pedagojik alan bilgisi olarak ilerleme kaydettikleri tespit edilmiştir.

Anahtar Kelimeler: Erken Cebir, Sınıf Öğretmeni Adayı, Öğretmen Bilgisi, Durum Tartışması

To my teachers who inspire me each and every day

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ABBREVIATIONS	
NCTM	National Council of Teachers of Mathematics
MoNE	Ministry of National Education
PSE	Primary School Education
PET	Prospective Elementary Teacher
SMK	Subject Matter Knowledge
CCK	Common Content Knowledge
PCK	Pedagogical Content Knowledge
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching

CHAPTER 1

INTRODUCTION

Twenty-six years ago, Schoenfeld (1995) stated that;

> Algebra today plays the role that reading and writing did in the industrial age. If one does not have algebra, one cannot understand much of science, statistics, business, or todays' technology. Thus, algebra has become an academic passport for passage into virtually every avenue of the job market and every street of schooling (p.11).

This fact is still accurate and might be more valid for our technology-wrapped era. To gain significant mathematics knowledge and reach future educational and employment opportunities, algebra is seen as a "gateway" (Kaput, 1998; National Mathematics Advisory Panel, 2008) and is located in the mathematics curriculum as a central learning domain.

Traditionally, mathematics education is on the grounds of arithmetic then algebra approach. More explicitly, students are first expected to gain procedural fluency for arithmetic in elementary grades; then, they face algebra, mostly based on a procedural approach in the middle grades (Blanton et al., 2007). Parallel with this approach, algebra does not appear in the Turkish mathematics curriculum as a learning domain for elementary grades (MoNE, 2018). However, the transition from concrete arithmetic thinking to increasingly abstract algebraic reasoning, which is required in secondary school and later grades, became a hurdle for students' mathematics learning (Bekdemir & Isik, 2007; Carpenter et al., 2000; Knuth et al., 2016). This problem led educators and mathematics education researchers to consider the "deep, long-term algebra reform" (Kaput, 1999, p. 134). Kaput (1999) described a route to that reform as "infusing algebra throughout the mathematics curriculum from the very beginning of the school" (p. 134). Teachers can provide students with a more sophisticated algebra background, which involves solid understandings and experiences for middle grades and high school, by placing algebra in the curriculum from kindergarten onward (NCTM, 2000). This new approach is currently known as "Early Algebra."

Early algebra does not mean serving common algebraic concepts and procedures addressed in the middle grades to the elementary students earlier (Carraher et al., 2008) Besides, early algebra is not an attempt to make the elementary curriculum bigger (Kaput et al., 2008). Early algebra is a way of thinking to provide students opportunities to generalize relationships and mathematical facts by delving into the concepts already in the curriculum to provide a deep and coherent mathematical understanding (Blanton et al., 2007).

Although integrating algebra in the elementary grades is a relatively new idea, recent research findings enable us to recognize the capability of elementary students (e.g., Blanton, Stephens, et al., 2015) and kindergarten students (e.g., Stephens et al., 2020) to think algebraically, gain insight into a classroom environment for early algebraic thinking (e.g., Bastable & Schifter, 2008) and teacher practices and necessary algebra knowledge (e.g., Blanton & Kaput, 2003). However, this situation is not valid in our national context. Although there are some studies related to elementary students' algebraic thinking process (e.g., Tanışlı, 2011; Turgut & Temur, 2017), we still need to know more about algebraic thinking in Turkish elementary grades. On the other hand, in both national and international contexts, the question: "How can prospective teachers be given a good start on developing essential knowledge of algebra for teaching?" (Fey et al., 2007, p. 27) has not been answered sufficiently. Prospective teachers' subject matter knowledge and knowledge to teach is at the center of the complex landscape of prospective teacher education (da Ponte & Chapman, 2008), and while investigating prospective teachers' preparation and their development of knowledge to teach is so critical, the number of studies that focused on prospective teachers' knowledge to teach algebra in early grades is quite limited.

The theory and practice gap, as a perennial issue related to prospective teacher education, has been discussed several times (Darling-Hammond, 2006; Gravett, Henning & Eiselen, 2011; Korthagen, 2001), and various solutions have been proposed to eliminate this gap. As one of them, since they take hold of the authenticity and complication of the instructional practice, classroom cases in teacher education are seen as an antidote to an overly theoretical approach (Smith & Friel, 2008). L. Shulman (1996) stated that “case-based teacher education offers safe contexts within which teachers can explore their alternatives and judge their consequences” (p. 214).

In the scope of these considerations, this study investigated the development of prospective elementary teachers’ knowledge to teach algebra in elementary grades through case discussions.

1.1 Purpose of The Study

The purpose of this study was to investigate the development of prospective elementary teachers' (PET) “subject matter knowledge” and “pedagogical content knowledge” (L. Shulman, 1987) to teach algebra in the elementary grades during their participation in the case discussions that focused on early algebra contents. More clearly, the study aimed to find out the change in PETs’ knowledge related to equivalence and equations, generalized arithmetic, and functional thinking as part of “subject matter knowledge” and their knowledge to teach these contents in terms of knowledge of students’ thinking, instructional strategies, and representations as part of “pedagogical content knowledge,” after participating in early algebra lessons including case discussions.

1.2 Research Questions

The specific research question that guided this study was the following:

How does the prospective elementary teachers' knowledge to teach early algebra develop after attending early algebra lessons including case discussions?

a. In what aspects does PETs' subject matter knowledge related to equivalence and equations, generalized arithmetic, functional thinking, and the concept of variable change after attending early algebra lessons including case discussions?
b. In what aspects does PETs' pedagogical content knowledge in terms of knowledge of students' thinking, instructional strategies, and representations for algebra in early grades change after attending early algebra lessons including case discussions?

1.3 Significance of The Study

"All students should learn algebra" (NCTM, 2000, p. 37) and early algebra studies conjecture that when young students have some sustained experiences related to algebraic reasoning, they develop "important habits of mind" and gain much deeper mathematical understanding, by comparison with the ones who have experiences focused on arithmetical competence and so they become better prepared for secondary algebra learning (Blanton et al., 2007, p. 8). Moreover, everyone accepts that one of the most critical factors on student learning is teacher knowledge (Fennema & Franke, 1992). Therefore, to provide better mathematics education, particularly algebra education, investigating prospective teachers' early algebra related knowledge development has significance. Exploring how prospective teachers make sense of early algebra and teaching algebra in early grades might help us gain insight into designing future teacher education courses and professional development programs.

Among early algebra researches, the ones focused on prospective teachers' knowledge are scarce, especially in the national context. Thus, the current study comes into prominence in terms of its contribution to the related literature. In addition, different from the existing research (e.g., Hohensee, 2017; McAuliffe & Vermeulen, 2018) which focus on the prospective teachers' subject matter knowledge, this study was designed to examine mainly the development of prospective teachers' pedagogical content knowledge. However, since subject matter knowledge is considered as a prerequisite for pedagogical content knowledge development (Agathangelou & Charalambous, 2020; Ball et al., 2005), prospective teachers' subject matter knowledge is also examined within the scope of the study. In other words, while previous studies focused on prospective teachers' comprehension of early algebra, this study also aimed to focus on examining their learning to teach early algebra. Moreover, similar to studies conducted by McAuliffe and Vermeulen; and Hohensee in which prospective teachers' knowledge is examined in a content course, the intervention in this study is designed as including case discussions. Case-based teaching, as compared to the conventional method, assists prospective teachers in comprehending the complexity of teaching and allows them to connect theory-based principles to practical impasses (Gravett et al., 2017). Therefore, the study is also important because of the scarcity of the studies using case discussions in this area. This feature of the study distinguishes it from the others.

1.4 Definition of Important Terms

Early Algebra

Early algebra refers to providing students with opportunities to make generalizations and examine mathematical relationships within the current curriculum to provide students with a deep and coherent understanding of mathematics and ultimately prepare them for learning advanced algebra (Blanton et al., 2007).

Prospective Elementary Teachers

Prospective Elementary Teachers are college students who were in their third year in a four-year primary school education (PSE) program at a private university in Ankara, Turkey. The graduates of the program are certified to teach 1st to 4th grades (elementary school).

Classroom Cases

Classroom cases are "an account of an experience in which our intentions have been unexpectedly obstructed, and the surprising event has triggered the need to examine alternative courses of action" (L. Shulman, 2004, p. 474). Cases attempt to provide a multidimensional representation of the situation's context, participants, and reality (Gravett et al., 2017). They are seen as "a way to bridge the abstract nature of principles and teaching standards to classroom practice" (J. Shulman, 2002, p. 2). In this study, classroom case refers to a classroom narrative that includes the descriptions of lessons' context, teacher' decisions, students' thinking and teacher-students interactions. In other words, a classroom case is a written account of what occurred during a lesson.

Case-Based Instruction

Case-based instruction entails utilizing real-world examples to assist teachers in gaining the knowledge and skills they need to respond to the complexities and authenticity of real-world classrooms (Merseth, 1996; Sykes & Bird, 1992). It is an instruction method in which students read, analyze, and reflect on the classroom cases (Ertmer & Stepich, 1999; Kowalski,1999). In the current study, case-based instruction refers to a way in which prospective elementary teachers discuss and reflect their ideas related to mathematical understanding, students' thinking, and teachers' moves on a classroom case.

Big Ideas

Big ideas are "key ideas that underlie numerous concepts and procedures across topics" (Baroody, Cibulskis, Lai, & Li, 2004, p. 24).

CHAPTER 2

LITERATURE REVIEW

This study investigates the development of prospective elementary teachers' knowledge to teach algebra in the early grades through case discussions. In this chapter, the related literature will be introduced in three parts. The first part will give information about the framework used for teacher knowledge. The second part will focus on the need for early algebra studies, the reconceptualization of algebra, and the nature and content of early algebra. Lastly, the use of case-based instruction in teacher education will be explained in the third part.

2.1 Teacher Knowledge

It is indeed beyond doubt that "to be a teacher requires extensive and highly organized bodies of knowledge" (L. Shulman, 1985, p. 447). Nevertheless, there is no agreement on what teachers need to know or what constitutes the teacher knowledge (Even & Tirosh, 2008), neither on pre-service teacher development and assessment (Hill et al., 2004) nor on what should be investigated in studies related to teacher knowledge (Petrou & Goulding, 2011). On the other hand, while knowledge and pedagogy were previously considered separately, Lee Shulman (1986, 1987) initiated a new wave by pointing out the content dimensions of teaching.

L. Shulman (1987) proposed seven dimensions of teacher knowledge by referring to content as a missing paradigm in teaching research. These dimensions are introduced in Table 2.1.

Table 2.1

L. Shulman's (1986,1987) Dimensions of Teacher Knowledge

General Dimensions	**Content Dimensions**
General pedagogical knowledge	Subject matter knowledge
Knowledge of learners	Pedagogical content knowledge
Knowledge of the educational context	Curricular knowledge
Knowledge of educational ends, purposes, and values	

According to L. Shulman's (1986) conceptualization, subject matter knowledge is "the amount and organization of the knowledge per se in mind of the teacher" (p. 9), and it involves not only knowing the subject related facts, principles, or rules and also knowing the reasons under the structures. Besides, L. Shulman identifies pedagogical content knowledge as knowing "ways of representing and formulating the subject that make it comprehensible to others" and "what makes the learning of specific topics easy or difficult" (p. 9). Moreover, understanding related to students' thinking or misconceptions is an essential component of pedagogical content knowledge. The last dimension of L. Shulman's conceptualization of teacher knowledge is curricular knowledge, and it is the knowledge of available instructional materials, scope, and sequence of the current curriculum.

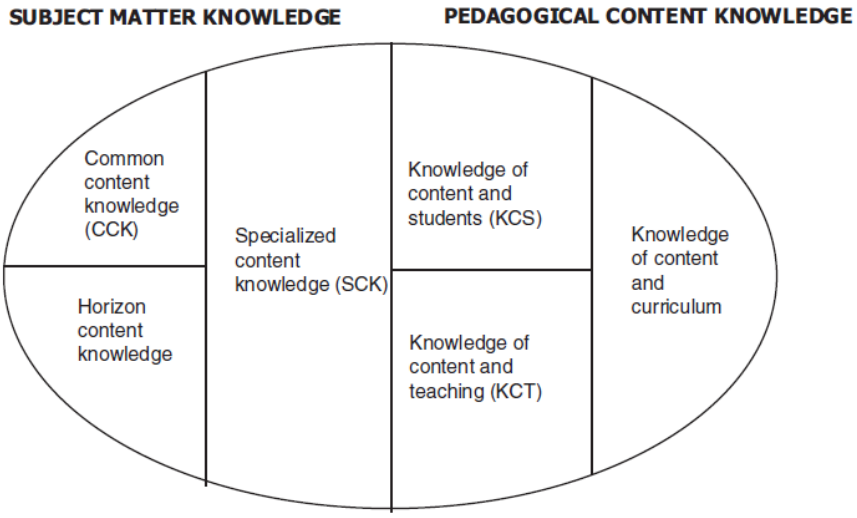
Although L. Shulman's conceptualization still maintains its influence on teacher education and research, it is criticized because of a lack of clear distinction between content and pedagogical content knowledge (Ball et al., 2008) and ignorance of the dynamic nature of teaching (Fennema & Franke, 1992). Then, based on his work, several conceptualizations or frameworks for teacher knowledge were put forward (e.g., Ball et al., 2008; Fennema & Franke, 1992; Grossman, 1995; Peterson, 1988). Among them, the model of "*Mathematical Knowledge for Teaching* (MKfT)," which was introduced by Ball, Thames, and Phelps (2008), guided this study and will be explained in the following section.

2.1.1 Mathematical Knowledge for Teaching Framework

Ball, Thames, and Phelps (2008) generated the Mathematical Knowledge for Teaching framework as a detailed categorization of L. Shulman's (1986,1987) conceptualization and the framework composed of two main domains, which are *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK). Then, while the domain of SMK is divided into three subdomains as *common content knowledge* (CCK), *specialized content knowledge* (SCK), and *horizon content knowledge*, the domain of PCK is also divided into three subdomains as *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT) and *knowledge of content and curriculum* (see Figure 2.1).

Figure 2.1

Domains of mathematical knowledge for teaching (Ball et al., 2008, p.403)



According to Ball et al. (2008), under subject matter knowledge, while common content knowledge refers to knowledge and skills that are not specific for teaching and used in other settings, specialized content knowledge refers to teaching specific knowledge and skills. For instance, to expect and respond to students' “why” questions such as why we need a common denominator to add two fractions, might be categorized under SCK. Furthermore, as the last component of SMK, horizon content knowledge is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Concerning the components of pedagogical content knowledge, firstly, knowledge of content and students includes the knowledge about students' conceptions and misconceptions by combining knowledge about content and students. Secondly, knowledge of content and teaching refers to the knowledge such as selecting and sequencing mathematical tasks with an awareness of decisions' advantages and disadvantages, orchestrating classroom discussion, or using students' strategies and responses to provide mathematical understanding. KCT is shortly a combination of knowledge about content and teaching. The third and last component of PCK, similar to L. Shulman's curricular knowledge, knowledge of content and curriculum refers to teachers' knowledge about what students have learned in previous years and what they will learn in the future related to their current learning area.

Hohensee (2017), asserted that in the learning and teaching process of early algebra, while elementary students need to transition from arithmetic to algebra, prospective elementary teachers need to transition from formal algebra back to early algebra. This is because early algebra is a relatively new idea and that most prospective teachers do not have early algebra related experience in their elementary education. Based on this, “if prospective elementary teachers are going through a similar process as elementary students (although in reverse), then it makes sense for prospective elementary teachers to first learn about early algebra as content before learning how to teach early algebra” (Hohensee, 2017, p. 233). Therefore, the current study, which aimed to investigate the development of prospective elementary teachers' knowledge to teach algebra in early grades, focused on KCS and KCT dimensions in terms of (a) students' conceptions and misconceptions and (b) instructional strategies and representations, respectively, as part of PCK, and as well as CCK as part of SMK. Assuming that PETs go through a similar learning process with elementary students, the knowledge and skills that elementary students should acquire, which early algebra studies have put forward, are regarded as common content knowledge related to the early algebra content, which PETs were expected to learn in this study.

2.2 Early Algebra

The primary purpose of elementary school mathematics is to provide young students with fundamental mathematical knowledge and skills and preparing them for higher mathematics. Algebra is one of the significant components of mathematics that students encounter in middle and high school (Kieran, 2004). Therefore, elementary school mathematics should offer young students content to develop necessary concepts and skills related to algebra. Nevertheless, because of the traditional "arithmetic-then-algebra" approach, students' studying for competence in arithmetic and procedural fluency in early grades is followed by learning algebra in middle grades as a "distinct subject matter standing in a particular order" (Schliemann et al., 2007, p. x). The distinction of arithmetic and algebra and the abrupt transition between them deprives students of developing significant mathematical schemes and makes learning algebra difficult in later grades (Kaput, 1998; Kieran, 2004). Hence, recently a consensus has emerged on the substantial role of algebra at all grades, and the necessity of reformulation of school algebra from kindergarten to higher grades has risen (e.g., NCTM, 2000). Supporting this idea, the curricula of countries such as Singapore and Korea, which have high success in mathematics in international exams (see results of TIMMS 2019 in Mullis et al., 2020), include content to support algebraic thinking at early grades. According to Ferrucci's (2004) overview, although it is not explicitly referenced, Singapore's curriculum provides activities to contribute to the algebraic thinking in early grades. Similarly, while Korean students begin to study formal algebra in grade 7, algebraic thinking is supported by several prerequisite activities at early elementary school levels.

As Carraher and his colleagues (2008) state, "early algebra is not the same as algebra early" (p. 235). In other words, early algebra means neither to down the traditional algebra curriculum into elementary grades nor to replace arithmetic with algebra. It means reforming our way of teaching arithmetic (Carpenter et al., 2005) to help students recognize and reason with underlying mathematical structures and properties and develops the ability to identify, describe, and analyze the relationships between varying quantities (Knuth et al., 2016). In a word, early algebra seeks an answer to the question, "What kinds of algebraic concepts can children learn in instructional settings that support algebraic thinking?" (Kaput et al., 2008, p. xviii), in order to prepare them for formal algebra in later grades.

The distinction between algebra in early grades and traditional school algebra brought the questions of what algebra is and what kind of thinking should be considered algebraic (e.g., Bell, 1996) and the need for the reconceptualization of those concepts (Kaput, 1998). To that end, various characterizations of algebra that highlight important aspects have been asserted (e.g., Bednarz et al., 1996; Kaput, 2008; Kieran, 1996; Usiskin, 1988). Among these characterizations, Kaput, who is the pioneer of the early algebra approach, introduced a useful framework, which guided this study.

This section in the literature review intends to present the rationale behind the early algebra studies. Next section will detail Kaput's (2008) framework, and it will be followed by the nature and content of early algebra. Lastly, the studies related to teaching early algebra will be overviewed.

2.2.1 Kaput’s Algebraic Reasoning Framework

According to Kaput (2008), algebraic reasoning has two core aspects (Core Aspect A & B), and these two aspects run through three content strands (Strand 1, 2 & 3). Figure 2.2 presents each core aspects and strands.

Figure 2.2

Kaput's core aspects and strands (Kaput, 2008, p.11)

The Two Core Aspects
(A) Algebra as systematically symbolizing generalizations of regularities and constraints.
(B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems.
Core Aspects A & B Are Embodied in Three Strands
1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning.
2. Algebra as the study of functions, relations, and joint variation.
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics.

As stated in Figure 2.2, while Core Aspect A focuses on regularities and relations to make generalizations and using symbols for generalizations, Core Aspect B focuses on symbol manipulations and following rules. Kaput (2008) explained that Core Aspect B is usually developed after Core Aspect A is developed. Stated another way, during the algebraic reasoning process, by using symbols as a tool, a relational understanding should be advanced first, then skills for acting on symbols should be considered. These two core algebraic thinking aspects are embodied in three content strands, and despite the presence of the strands as single entities in the figure, they overlap (Kaput, 2008). Strand 1 refers to generalized arithmetic, which is the "heart of algebra" (Kaput, 2008, p. 12). It could be explained as seeing how "algebra is inherent to arithmetic" (Carraher & Schliemann, 2007). It involves building generalization from structures of arithmetic (e.g., structures of arithmetic operations and properties) in terms of their form rather than their computed value. The next strand, Strand 2, refers to functions and focuses on reaching generalizations that describe systematic variation of samples in a domain and acting on the forms of these generalizations. Lastly, Strand 3 refers to modeling and is explained in three types. The first type is the number or quantity specific modeling, and it reflects the usage of the syntactic aspect of algebra to solve an arithmetic problem in which the variable regards as an unknown. The second type of modeling is expressing and generalizing regularities in situations as a form of function. The last one refers to generalizations as single-answer modeling, which builds on solutions to arithmetic problems by nature.

To investigate the development of prospective elementary teachers' knowledge to teach algebra in early grades, this study concentrates on using symbols for generalization (Core Aspect A) and acting on symbols (Core Aspect B) in the contents of generalized arithmetic (Strand 1) and functions (Strand 2).

2.2.2 The Content of Early Algebra

To put it merely, early algebra refers to "encompass algebraic reasoning and algebra-related instruction among young learners—from approximately 6 to 12 years of age" (Carraher & Schliemann, 2007, p. 670) and "immersing them in the *culture of algebra*" (Lins & Kaput, 2004, p. 47; italics in the original). It is believed that throughout the first six years of elementary school, developing arithmetic and algebraic thinking simultaneously provides the later algebra learning become a "natural and non-threatening extension of mathematics of elementary school curriculum" (Cai & Moyer, 2008, p. 3). Although there is no consensus about what early algebra contains, whatever content or activity helps students go beyond the arithmetic and computational fluency to understanding mathematical structures could be a part of early algebra (Cai & Knuth, 2005; Lins & Kaput, 2004). Algebraic thinking in early grades is defined by Kieran (2004) as:

> Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting (p. 149).

Later on, based on Kaput's (2008) core aspects, Blanton and her colleagues (2011) proposed the four core algebraic practices as *generalizing, representing, justifying, and reasoning with mathematical relationships.* These core algebraic practices were regarded as a skeleton for planning early algebra lessons and developing questions of data instrument. Besides, five big ideas for algebra in early grades were identified in the light of core content strands and previous early algebra studies. These five big ideas are (i) equivalence, expressions, equations, and inequalities; (ii) generalized arithmetic; (iii) variable; (iv) proportional reasoning; and (v) functional thinking (Blanton, Stephens, et al., 2015).

In the scope of this study, the development of prospective elementary teachers' SMK and PCK were investigated in three big ideas, which are (a) equivalence and equations, (b) generalized arithmetic, and (c) functional thinking. Each of these will be elaborated on in the following sections. The other big ideas, variable, proportional reasoning, and the inequalities and expressions as the component of a big idea were not investigated in the scope of this study. However, since variable notation is a powerful tool for expressing generalizations (Carraher & Schliemann, 2007), the prospective teachers were expected to use variables while representing the generalizations related to generalized arithmetic and functional thinking. Thus, PETs' conceptions of variables were also examined under the common content knowledge.

2.2.2.1 Equivalence and Equations

NCTM (2000) regards that "equality is an important algebraic concept that students must encounter and begin to understand in the lower grades" (p. 94). The equation is "a mathematical statement that uses an equal sign to show that two quantities are equivalent," and "using equations to reason about, represent, and communicate relationships between quantities is a cornerstone of algebra" (Blanton et al., 2011, p. 25). Studies suggest that the notion of equality and relational view of the equal sign (i.e., as a symbol showing the relation between quantities that are the same on both sides of the symbol) become significant for solving equations (e.g., $3x$-5 = $2x$ +1) (Knuth et al., 2005) and operating on the structure of equations (i.e., carry out the same operations both sides) (Kieran, 1992) and solving equations with an understanding rather than memorizing a series of rules (Falkner et al., 1999). Moreover, according to the national mathematics curriculum by MoNE (2018), students are expected to realize the meaning of the equal sign as an "equality" between the mathematical expressions at 2nd grade (see M.2.1.3.5. in MoNE, 2018). However, it is well documented that students do not view the equal sign as a symbol of equivalence; instead, they think that the equal sign is a signal to "do something" or an announcement of the result of an arithmetic operation (Falkner et al., 1999; Knuth et al., 2006; McNeil & Alibali, 2005). For example, in the study by Falkner et al. (1999), when the students at 1st to 6th grade were asked to find the missing value in the equation 8 + 4 = __ + 5, the majority of the students in each grade level thought that the missing value was either 12 or 17. Less than 10% of students at each grade level found the correct answer of 7. While the students' thinking who ignored 5 and answered 12 by adding 8 + 4 or answered 17 by adding all given numbers is defined as "operational thinking," the students' thinking who considered the equality on each side of the symbol and answered 7 is defined as "relational thinking." In the studies conducted in the national context, similarly, students were found to have an operational conception towards the equal sign (e.g., Bulut et al., 2018; Isler-Baykal et al., 2019; Yaman et al., 2003). For example, in the study of Isler-Baykal et al. (2019), about 60% of 3rd grade students, 30% of 4th grade students, and 23% of 5th grade students found the missing value in the equation 7 + 3 = __ + 4 as 10 or 14 with operational thinking.

Besides the students' conception of the equal sign, operational or relational, Carpenter et al. (2003) described students' strategies to find the missing value in the equation 8 + 4 = __ + 5. The students who thought operationally, as mentioned earlier, wrote 12 in the blank by considering "the answer comes next" to the equal sign and added 8 and 4, or they found the answer as 17 by ignoring where the equal sign appeared in the number sentence and "added all numbers," 8 + 4 + 5 = 17. On the other hand, the students who thought relationally found 7 as the correct answer either by computing or by recognizing the structure of the equation. The students who used the strategy of "computation" added 8 and 4 on the left side and found the number 7 thinking of what number would give 12 when added to 5 which was on the right side. As a more sophisticated and more flexible strategy, some students considered the "structure" of the equation and the relation between the numbers on both sides by recognizing that 5 is one more than 4 and that the missing value must be one less than 8.

One of the major stumbling blocks in learning algebra is students' poor understanding of the equal sign (Carpenter et al., 2003). Further, the misleading view of the equal sign does not improve with time or mathematical maturity (Freiman & Lee, 2004), and telling the meaning of the equal sign directly is not enough for students to develop a relational conception towards the equal sign (Falkner et al., 1999). However, studies have shown that student's conception of the equal sign can develop from operational to relational (e.g., Blanton, Stephens, et al., 2015; Warren et al., 2009). In order for it to happen, teachers themselves need to have a relational view of the equal sign and the required pedagogical content knowledge to support students to develop meaningful understanding and use of the equal sign.

The studies that focused on prospective or in-service teachers' knowledge for teaching equal sign, equivalence is quite limited. In one of the studies, Stephens (2006) assessed 30 prospective elementary teachers' preparedness to engage students in relational thinking and equivalence tasks. The findings of her study indicated that the majority of the prospective teachers showed awareness of relational thinking in identifying tasks' goals and sample students' work. Few participants, however, exhibited an awareness of the fact that many elementary school students had misconceptions regarding the meaning of the equal sign. Similar findings were recorded in Asquith et al. (2007) 's study. When they interviewed 20 middle school teachers, they found that teachers did not expect students to have misconceptions related to the equal sign, and they predicted that their students "have a stronger relational understanding of the equal sign than was actually demonstrated by student responses" (p. 262). Another study showed that teachers might not have the necessary knowledge for teaching equal sign. Vermeulen and Meyer (2017) interviewed three fifth and sixth grade teachers and concluded that teachers "lacked the knowledge and skills to identify, prevent, reduce, or correct students' misconceptions about the equal sign" (p. 136). In addition to the studies investigating the prospective or in-service teachers' initial knowledge for teaching equal sign, Santarone et al. (2020) assessed the development of prospective teachers' KCS and KCT related to the meaningful use of the equal sign by designing an intervention. Their study provided the prospective teachers with a research-based teacher intervention and assessed the development of knowledge for teaching equivalence through their practice-based experiences where they had opportunities to practice instructional strategies with students. They found that although the prospective teachers still demonstrated some difficulty to distinguish students' computational and relational views, and prompting the ways to entirely further students' relational thinking, the teacher intervention was found to help them to develop their KCS and KCT related to the equal sign.

In the current research, the development of the prospective elementary teachers' CCK, KCS, and KCT related to equivalence and equations were investigated in a learning environment including case discussions. The findings might contribute to the literature about preparing prospective teachers to teach equivalence and equations as a cornerstone for algebra learning.

2.2.2.2 Generalized Arithmetic

Traditionally, elementary school students spend most of their time performing computations, learning algorithms, and finding correct answers. Their experience with generalizations, studying the fundamental properties' structure, and searching regularities and patterns in numbers and operations are pretty limited. However, NCTM (2000) emphasizes the significance of generalization in arithmetic by stating that "analyzing the properties of the basic operations gives students opportunities to extend their thinking and to build a foundation for applying these understanding to other situations" (p. 161). Parallel with this idea, generalized arithmetic refers to "helping children see, describe, and justify patterns and regularities in operations and properties of numbers" in order to "move beyond arithmetic to algebraic thinking" (Blanton, 2008, p. 12). The development of algebraic thinking requires making and representing conjectures, as well as generalizing and justifying them (Kaput, 1999), and those activities "can bring a deeper purpose to arithmetic and children's arithmetic understanding" (Blanton, 2008, p. 12). In the national mathematics curriculum by MoNE (2018), several learning objectives under the domain of number and operations could be used to provide elementary students with opportunities to conjecture, represent, justify and generalize arithmetic relationships, and ultimately think algebraically. For example, the 2nd-grade learning objective that expects students to notice that changing the order of the multipliers would not change the product could be used to engage students in generalizing process.

The studies on generalized arithmetic mainly focuses on a) the fundamental properties of number and operations (e.g., commutative property of addition or zero is additive identity), b) the relationships among operations (e.g., inverse relationship between addition and subtraction), and c) the relationships in a class of numbers and outcomes of calculations (e.g., operations with odd and even numbers). Making conjectures, justifying, and generalizing mathematical reasoning about these arithmetic aspects is significant for the unification of arithmetic and algebra (Hunter, 2010). However, research showed that elementary students have limited experiences related to engaging in generalizations in arithmetic. For example, Anthony and Walshaw's (2002) study with the year 4 and year 8 students revealed that students struggle to reach correct generalizations related to commutativity. Although students were confident about the commutative property under addition and multiplication, they were not sure about the commutativity for subtraction. Moreover, they also found that students were not able to justify their conjectures with the models. Similar findings were recorded by Warren's (2001) study investigating elementary students' generalizations related to commutativity. Her study revealed that elementary school students were capable of generalizing, but because of incorrect sense-making, misleading teaching materials, and over generalizations of new learning, they had difficulties in reaching correct generalizations.

The studies so far also show us that thanks to the appropriate instruction, the difficulties that elementary students face while conjecturing, justifying, and generalizing can be eliminated, and the students can learn to construct and justify arithmetical generalizations. For example, in the study with students who were 9-11 years old, Hunter (2010) concluded that "opportunities to develop explanations with concrete material and use notation to represent conjectures led to students developing further generalizations" (p. 111). Likewise, in the study of Blanton, Stephens, et al. (2015), when the 3rd grade students were asked whether 39 + 121 = 121 + 39 was true or false, none of the students who thought that the statement was true could explain by relying on the structure of the equation in the pre-test. However, after the intervention including the activities that guide students to make arithmetic generalizations, in the post-test, 66% of the students provided such an explanation, including recognition of commutativity (e.g., "True, because 121 + 39 is just 39 + 121 in reverse"). In addition, Isler et al.'s (2013) study also showed that instruction could improve students' ways of justification. Their study noted that students were beginning to build representation-based arguments and provide generalizations using facts about the sum of even and odd numbers during the classroom intervention, despite their generally poor performance on the pre-assessment. Thus, it is teachers who can create instructional settings to encourage generalizations and "algebrafy" their resources (Blanton & Kaput, 2003, p. 76).

The studies that investigated teachers' knowledge related to teaching arithmetic generalizations is quite limited. The existing research focused on how prospective teachers make sense of fundamental properties and represent them. Monandi (2018) state that pre-service elementary teachers who took part in the study did not understand associative and distributive properties' use in simplifying numerical statements well by analyzing their performances on pre- and post-teaching algebra tests. Similarly, Ding et al. (2013), examining pre-service elementary teachers' knowledge to teach associative property of multiplication, found that only 14% of PETs defined the associative property accurately, about 30% could generate a correct algebraic formula, and 25% provided an arithmetic example correctly. Moreover, they noted that most participants could not use concrete contexts (e.g., pictorial representations and word problems) to represent the associativity of multiplication conceptually. Such results and the paucity of studies examining teacher subject matter knowledge and pedagogical content knowledge regarding arithmetic generalizations may indicate that there is still much needed to know to bridge the gap between teaching arithmetic and algebra.

2.2.2.3 Functional Thinking

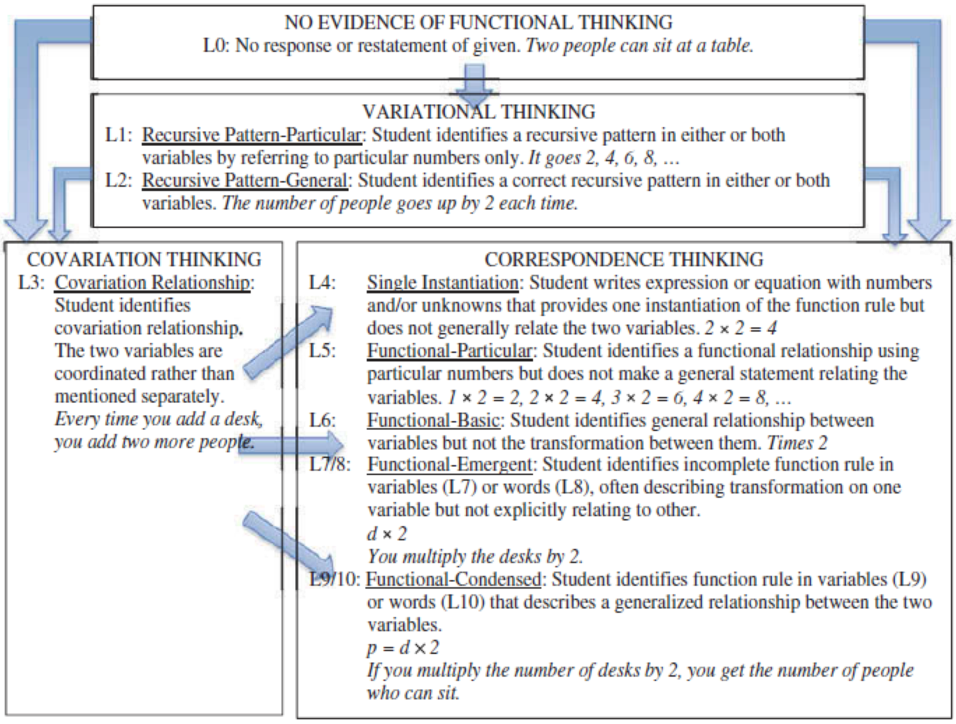
Functional thinking is closely related to the early algebraic thinking practices of generalizing, representing, justifying, and reasoning with mathematical relationships (Blanton et al., 2011; Kaput, 2008). Thus, functional thinking is seen as a critical route to learning and teaching early algebra (Carraher & Schliemann, 2007). According to Blanton, Brizuela, et al. (2015), functional thinking involves a) generalizing relationships between covarying quantities; b) representing and justifying these relationships in multiple ways such as using natural language, variable notation, tables, and graphs; and c) reasoning with these generalized representations (p. 512). Parallel with these practices, NCTM (2000) states that elementary students should learn to describe and extend generalizations about patterns; using words, symbols, tables, and graphs to represent patterns, and investigate the relationship between the variables change together. Similarly, in the national mathematics curriculum (MoNE, 2018), elementary students are expected to study patterns. Although the objectives do not directly ask students to focus on the relationship between the variables changing together, the elementary teachers could create an environment to guide students to think functionally. For example, the 3rd-grade learning objective that expects students to expand and generate the number patterns that have a constant difference could be used to encourage functional thinking.

Confrey and Smith (1991) propounded that students show three different modes of thinking when they generalize the functional relationships: recursive, covariation and correspondence. *Recursive patterns* represent variation in a single series of values, demonstrating how to get the following number in the sequence from the previous number (e.g., in the context of a varying number of dogs and the total number of the dogs' eyes; the number of dogs' eyes goes by 2); *covariational thinking* entails examining how two quantities vary in respect to one another and including that variation in the function's description (e.g., as the number of dog increases by 1, the total number of dogs' eyes increases by 2); and *correspondence relationship* is a function rule that expresses a coordination between two variables (e.g., the total number of dog eyes is twice the number of dogs) (Blanton et al., 2011).

Based on these three modes of generalizing functional relationships, researchers developed several frameworks related to the level of representing generalizations (e.g., Barbosa, 2010; Blanton, Brizuela, et al., 2015; Stephens et al., 2017). Stephens et al.'s framework of "Levels of sophistication describing generalization and representation of functional relationships" (2017, p. 153; see Figure 2.3) was taken as a reference in the functional thinking dimension of this study because it was more up-to-date and comprehensive.

Figure 2.3

Levels of sophistication describing generalization and representation of functional relationships (Stephens et al., 2017, p. 153)



There is plenty of research which indicated that elementary students could engage in functional thinking, even at the kindergarten level. For instance, research showed that elementary students could use tables to represent and reason with the relationships (e.g., Brizuela & Lara-Roth, 2002, grade 2; Tanışlı, 2011, grade 5); although students initially tended to focus on recursive patterns, not on the relationships between variables (e.g., Lannin et al., 2006, grade 6), they were found to be capable of correspondence thinking that went beyond what was typically taught in elementary school (Blanton & Kaput, 2004, grades K-5; Turkmen & Tanışlı, 2019; grades 3-5). Furthermore, elementary students could express the correspondence relationships in words (e.g., Martinez & Brizuela, 2006, grades 2–5; Moss & McNab, 2011, grade 2) and in variables (Blanton et al., 2017, grade 1; Isler et al., 2015, grades 3-5). Studies in both national and international contexts indicated that supportive instruction could help elementary students to engage in covariational and correspondence thinking (e.g., Akin, 2020, grade 5; Blanton, Stephens, et al., 2015, grade 3; Canadas et al., 2016, grade 2; Isler et al., 2017, grade 3-5; Ozturk et al., 2020, grade 3). Thus, to create a learning environment that supports functional thinking, teachers need to have subject matter knowledge and pedagogical content knowledge related to functional thinking.

Previous research showed that prospective teachers had difficulties in generalizing the functional relationships. Oliveria et al. (2021) examined Spanish and Portuguese prospective elementary teachers' functional thinking at the beginning of their teacher education program and found that successful strategies to generalize functional relationships were infrequent. When PETs were asked to find a distant term of a geometric pattern, %32 of Spanish PETs and %17 of Portuguese PETs were able to use correspondence strategies to find correct answers. Then, when they were asked to express the general term of the geometric pattern, only 30% of Portuguese PETs and 34% of Spanish PETs provided a correct general term. Strikingly, the majority of Portuguese PETs' strategies (47%) were recursive which did not lead to the generic term, as was the case with Spanish PETs' responses (35% of the strategies).

Similarly, Alajmi (2016) found that prospective elementary and middle school mathematics teachers were not confident in developing general rules for the tasks that involved linear, exponential, and quadratic situations. In his study, prospective elementary teachers preferred drawing or counting to support their thinking and mainly used recursive strategies rather than explicit rules. Likewise, Yesildere and Akkoc (2010) found that prospective elementary mathematics teachers used recursive strategies while generalizing the patterns that were non-linear (quadratic) in nature. Furthermore, the existing research showed that prospective teachers faced challenges when using symbolic notations for generalizing (Zazkis & Liljedahl, 2002) and had difficulties while providing justifications for their reasoning (Richardson et al., 2009). In the study conducted by Richardson et al. (2009) in which prospective elementary teachers were asked to complete pattern-finding tasks, the researchers concluded that "while the generalizations were valid in terms of a rule, attempts to explain the algebraic symbols of the rules were incomplete with respect to explaining the origins of the coefficient and/or the y-intercept of the rule" (p. 193). Besides those findings, we need to learn more about prospective teachers' algebraic thinking abilities, particularly their methods, misunderstandings, and challenges with a wide range of functional thinking topics (Yemen-Karpuzcu et al., 2017).

2.2.3 The Studies That Focus on Development of Teacher Knowledge Related to Early Algebra

As already noted, except for those focusing on the meaning of the equal sign, most of the studies investigating prospective teachers' early algebra-related knowledge focused on initial subject matter knowledge. This was also true in the limited studies that examined the development of prospective teachers' early algebra-related knowledge. Hohensee (2017) examined prospective teachers' thinking after participating in a method course that focused on generalized arithmetic, functional relationship, and the meaning of the equal sign as the content of early algebra. The purpose of his study was "to examine how PSTs [pre-service teachers] learn about early algebra rather than how they learn to teach early algebra" (Hohensee, 2017, p. 233). In the scope of the study, 13 prospective teachers firstly worked in groups on the activities that explored algebraic thinking, then they engaged in whole-class discussions during the 20 lessons (10 lessons for generalized arithmetic, 5 lessons for functional relationships, and 5 lessons for the meaning of the equal sign). Data of the study was collected through the meetings of the researcher and participants' groups of twos and threes. In those meetings, the prospective teachers worked on tasks that assessed their SCK related to the themes of the lessons. In the data analysis, participants' verbal responses were examined to figure out their meaningful insights and conceptional challenges after they attended the course. The findings indicated that the participants experienced the transition "from knowledge they had about formal algebra from high school to new knowledge about early algebra" (p. 242), they had new insights about identifying the operational and relational meaning of the equal sign and representing quantities which are unknown or variable in informal ways without using algebraic symbols. On the other hand, although the participants made significant progress in representing functional relationships, identifying functional relationships and conceiving variables as being different from unknowns remained challenging for them.

Another study that investigated the development of prospective teachers' early algebra-related knowledge in a course, like the current study, was conducted by McAuliffe and Vermeulen (2018), focusing on functional thinking. Their study aimed to investigate the prospective teachers' knowledge for teaching functional thinking during their teaching practicum. In the study, 26 prospective teachers were enrolled in an Early Algebra course lasting 24 weeks, 8 of which were teaching practicum in schools. Early Algebra course involved reading and discussing early algebra-related journal articles and planning an early algebra lesson for their teaching practicum. The study's data comprised participants' written lesson reflections, and video-recorded lessons were analyzed by considering four aspects of their SCK: representations, working with students' responses, restructuring tasks, and questioning. The findings revealed that the Early Algebra course helped prospective teachers develop their SCK for using different representations of functions, but they still needed support to encourage students to generalize and describe functional relationships.

The findings of these studies provide insight into the development of prospective teachers' knowledge related to early algebra and inform the teacher education courses. However, we still need to know further about this issue. The current study aimed to build on the existing research findings and, different from them, investigated the prospective elementary teachers' early algebra-related knowledge more comprehensively by focusing on SMK and PCK together.

2.3 Case-Based Teacher Education

Lee Shulman (2004) voiced that teaching has an uncertain and unpredictable nature. Therefore, to prepare teachers for the dynamic teaching work, teacher education programs must provide them such skills to analyze the situations and make quick decisions. However, teacher education programs are criticized because of the distinction between theory and practice (Ball & Cohen, 1999). Due to the theory and practice gap, novice teachers face practice shock (Stokking et al., 2003) and complain that the overly theoretical courses of their professional education do not provide necessary practical knowledge for real-life situations (Lambert, 2010). As part of the endeavor to eliminate that problem, using an instructional method in teacher education is asserted: case-based instruction, which is also referred to as case-study pedagogy (Heitzmann, 2008).

The use of cases in the education of professions such as law, medicine, and business goes a long way back. Nevertheless, an alternative to traditional teaching methods (e.g., lecture-based instruction) was offered to be case-based instruction in teacher education (Merseth, 1996), and it started to be used after the 1980s. With the emphasis on the complex nature of teaching, Hutchings (1993) provided a rationale for using cases in teacher education as:

> Cases have the ability to situate the conversation about teaching on this middle ground between process and content (technique and substance) where a particular teacher, with particular goals, teaches a particular piece of literature (in this instance) to a particular student (p. 10).

Judith Shulman (2002) considered the use of cases as "a way to bridge the abstract nature of principles and teaching standards to classroom practice" (p. 2) and defined a classroom case as "… a piece of controllable reality, more vivid and contextual than textbook discussion, yet more disciplined and manageable than observing or doing work in the world itself" (1992, p. xiv). In case-based instruction, on the other hand, students read, analyze, and reflect on the cases (Ertmer & Stepich, 1999; Kowalski, 1999). Mostly this process starts with individually reading or observing the case and taking notes, and then it is followed by small group discussion and eventually finishes with whole group discussion (Morris, 2008).

It is well documented that cases can be an effective tool in teacher education (e.g., Lundeberg et al., 2000; J. Shulman, 1992; Smith & Friel, 2008; Sudzina, 1999). Since cases can reflect the multidimensional representation of real-life situations with the context and participants (Gravett et al., 2017) and have the power to integrate theory and practice (Mitchell, 2001), cases help teachers to create a repertoire of solutions for everyday problems in teaching (Kleinfeld, 1992), encourage critical thinking and decision-making abilities (Butler et al., 2006), assist to reason about dilemmas in instruction (Markovits & Even, 1999) and provide opportunities to enhance subject matter and pedagogical content knowledge (Henningsen, 2008). In a word, case-based instruction enables teacher candidates to "think like a teacher" (Kleinfeld, 1992, p. 33).

Merseth (1996) divided case purpose into three categories: (a) cases as exemplars, (b) cases as opportunities to practice analysis and contemplate actions, and (c) cases as stimulants to personal reflection. The exemplar cases are generic examples of practice, theory, instructional method, or principle supplied to students' discussion. Moreover, exemplar cases can be used "to honor 'best practice' or to make effective teaching more public" (p. 728). The second category of cases provides opportunities to practice actions rather than being confirmed or specified practice. In the light of their own experiences, emotions, and prior knowledge (Merseth & Lancey, 1993), during the analysis of cases, students are expected to practice decision making and problem-solving. Lastly, cases in the third category involve multiple perspectives and aims to foster the interpretation of students' reflectivity.

Besides the purposes and uses of cases, the forms of cases are also various in the literature, such as text-based cases, video-based cases, and multi-media cases. In this study, prospective teachers' knowledge development was investigated in a learning environment that involves text-based cases and case discussions. The cases were selected as content-specific and used as exemplars and opportunities to practice analysis and consider actions, according to Merseth's (1996) categorization. Although some studies asserted that video-based cases could be a more powerful tool than text-based cases in teacher education (e.g., Moreno & Valdez, 2007), text-based cases are also used as an effective tool in teacher education (e.g., Henningsen, 2008) since the nature of human thought is narrative and narrative knowledge is linked to memorable occasions in a person's life (Bruner, 1987). Based on this idea and because it was not feasible to produce video-based or multimedia cases during this study, the case discussions were carried out through text-based cases.

CHAPTER 3

METHODOLOGY

This study investigated the development of prospective elementary teachers' subject matter and pedagogical content knowledge to teach algebra in elementary grades through case discussion. The methodology will be presented in this chapter. Firstly, the design of the study will be introduced, and then the participants and the context of the study will be described. These sections will be followed by an explanation of the data collection tool, the data collection procedure, and data analysis. After that, issues related to the trustworthiness of the study will be presented. Lastly, the limitations of the study will be adverted.

3.1 Design of The Study

The current study aimed to investigate the development of PETs' knowledge to teach algebra in early grades by analyzing their individual interviews before and after their participation in the early algebra lessons in a method course. The research process was performed to understand PETs' knowledge development and examine their experiences. The design of the study was qualitative since "qualitative researchers are interested in understanding how people interpret their experiences, how they construct their world and what meaning they attribute to their experiences" (Merriam & Tisdell, 2016, p. 6).

There are several qualitative research approaches. Among all, the case study approach was performed in this study since the investigation was "an in-depth description and analysis of a bounded system" (Merriam & Tisdell, 2016, p. 37). In parallel with the purpose and design of this study, Yin (2003) defined the case study as " an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context may not be clearly evident" (p. 13). The current study used individual interviews as the data source to investigate the case of PETs' early algebra-related knowledge development in a learning environment, including case discussions.

While this study can be categorized as an instrumental case study, according to Stake (2005), it is categorized as a descriptive case study, according to Yin (2003). A case study that a researcher aims to "describe an intervention or phenomenon and real-life context in which it occurred" (Yin, 2003, p. 15) is called a descriptive case study. On the other hand, an instrumental case study examines a particular case "mainly to provide insight into an issue or to redraw a generalization. The case is of secondary interest; it plays a supportive role, and it facilitates our understanding of something else" (Stake, 2005, p. 437). Both aspects fit the nature of the current study.

3.2 Context of The Sudy

The data for answering the research questions of this study was collected during the particular five weeks of an undergraduate course in the primary school education program. Therefore, the study's context comprises of the teacher education program, the undergraduate course, and intervention implemented during the course, the early algebra lessons. Each of them will be explained in the following sections.

3.2.1 The Teacher Education Program

The participants of the study were enrolled in an undergraduate primary school education (PSE) program in a private university in Ankara, Turkey. The program of PSE is a 4-year undergraduate teacher education program. In this Turkish-medium program, the students are offered an option to take English language preparation education. The PSE program trains prospective teachers working with primary school students (grades 1-4).

In general terms, the primary school education program offers subject matter knowledge, the teaching profession, practice, and general culture courses. The subject matter knowledge courses include theoretical courses such as basic mathematics and basic science for primary school, drama, games, and physical activities and several teaching methods courses for all disciplines at the primary school (e.g., mathematics, science, social studies, Turkish language, foreign language, music, visual arts). These subject matter knowledge-related courses are spread to four-year education. Besides, the courses of educational sciences (Psychology of Education, Educational Philosophy, Sociology of Education), Educational Technology, Instructional Principles and Methods, Classroom Management, Measurement and Assessment, and Guidance are offered as teaching profession courses, and most of those are completed by the end of the third year. In the last year, the students are expected to complete the courses of teaching practice during the two semesters. The teaching practice courses are carried out in the training schools with the supervision of course instructors and teachers in training schools. Lastly, besides the subject matter knowledge and teaching profession courses, the primary school education (PSE) program offers some general culture courses related to Ataturk's Principles and History of Turkish Revolution, Turkish Language, Foreign Language, Information Technologies, Community Service, and some elective courses which are depended on students' interest.

Among the courses in the PSE program, Basic Mathematics in Primary School and Teaching Mathematics courses are the ones related to prospective teachers' knowledge to teach mathematics, which is the focus of this study. Basic Mathematics in Primary School course is offered in the fall semester of the program's 1st year. Within the scope of this course, content such as basic operations, various number systems, functions, sets, and data analysis are focused on. In the Teaching Mathematics course, methods, and techniques for teaching mathematics at the primary school level are presented. Since the development of PETs' knowledge to teach algebra in early grades was investigated through the last five weeks of the

Teaching Mathematics I course, in the following section, detailed information about the course will be provided.

3.2.2 Teaching Mathematics Course

The Teaching Mathematics course is offered to 3rd year prospective elementary teachers in both fall and spring semesters as Teaching Mathematics I and Teaching Mathematics II. These courses are compulsory and do not have any prerequisite courses. In a general manner, the purpose of the courses is to provide prospective teachers with insight and knowledge related to the principles of mathematics teaching, the teaching and learning strategies, fundamental learning theories, basic mathematical skills, measurement, and assessment in mathematics teaching. Moreover, as one of the main objectives, the PETs are expected to gain the necessary knowledge to teach the contents and objectives in the primary school mathematics curriculum at the end of Teaching Mathematics courses. The detailed learning outcomes for these courses are presented in Table 3.1.

Table 3.1

The learning outcomes of Teaching Mathematics courses

Course	Learning Outcomes
Teaching Mathematics I	Know the goals and basic principles of mathematics education Know the teaching and learning principles in mathematics education Apply mathematics teaching and learning strategies Know the goals and philosophy of elementary education Use the basic abilities of relation, communication, reasoning Use information technologies in mathematics teaching Know the steps of number concept development Develop activities on 1., 2. and 3. grades mathematics program
Teaching Mathematics II	Take precautions for the misconceptions of fractions Prepare activities related to fractions Know the steps of the development of geometric thinking Prepare activities for geometry concepts Develop activities for measurement concepts Develop activities for data analysis Know the assessment strategies in mathematics education

Note: This information was taken from the related official webpage of the university.

As mentioned earlier, according to the Turkish mathematics curriculum, algebra is not a learning domain for elementary grades, and there is no emphasis on algebraic thinking. Similarly, in the Primary School Education Program and particularly in Teaching Mathematics courses, the prospective elementary teachers have not been presented any algebra and teaching algebra-related content. Teaching Mathematics I course would be offered in the fall semester when the researcher contacted the course instructor, and its content was found appropriate to conduct this study.

The five-week early algebra content was added to the Teaching Mathematics I course schedule in the 2020-2021 fall semester. The details of the early algebra lessons will be given in the following section.

3.2.3 Early Algebra Lessons

The last five weeks of the 12-week Teaching Mathematics I course were devoted to introducing early algebra content to prospective elementary teachers, which comprised the intervention for this study. Each week, there were 2 lesson hours. The researcher guided these lessons on Microsoft Teams during distance education. Throughout the 10 hours of the intervention, the prospective elementary teachers were expected to develop their subject matter knowledge and pedagogical content knowledge for teaching algebra in elementary grades. The learning goals were adapted for each big idea in two groups: subject matter knowledge (particularly CCK) and pedagogical content knowledge (particularly KCT and KCS). Moreover, three more learning goals were adapted for the concept of variable, which was significant to represent the relationships related to the big ideas. These three learning goals were also categorized under the common content knowledge. The learning goals for CCK were adapted from the LEAP project (see Blanton, Stephens, et al., 2015) with permission. The LEAP project focused on understanding the impact of a systematic, multi-year approach to teaching and learning algebra in the elementary grades. In the different phases of the project, the researchers developed a curricular framework for early algebra, an instructional sequence, and grade-level assessment tools. They investigated the effectiveness of instructional sequence in grades 3-5 overtime in terms of students' algebra understanding. Hence, the order and the learning goals were appropriate for providing prospective elementary teachers with early algebra content in this study. On the other hand, the learning goals for KCT and KCS were adapted from Ball et al. (2008)'s descriptions of teacher knowledge categories. The learning goals that guided the early algebra lessons are presented in Table 3.2.

Table 3.2

The learning goals for the Early algebra lessons

Big Ideas / Concept	Learning Goals
Equivalence and Equations	**CCK1.** Develop a relational understanding of the equal sign by identifying and reasoning with structural relationships in the equation or by using arithmetic strategies **KCS1.** Analyze students' possible conceptions and misconceptions. Identify students' conceptions of the equal sign as relational or operational. **KCS2.** Expect students to use different strategies to find a missing value in a number sentence. **KCT.** Design a lesson with appropriate strategies and representations to guide students to gain a relational understanding of the equal sign
Variable	**CCK1.** Understand that a variable represents the measure or amount of the object, not the object itself **CCK2.** Understand that the role of a variable in a functional relationship is that of varying quantity **CCK3.** Understand how to use variables to stand for unknowns in equations

Table 3.2 (continued)

Generalized Arithmetic	**CCK1.** Identify and generalize arithmetic relationships • fundamental properties in use in computation • the relationships among operations and their inverse relationships • arithmetic relationships in a context such as classes of numbers or outcomes of calculations **CCK2.** Describe arithmetic generalizations in words and variables **CCK3.** Understand that arithmetic generalizations are true for all values of variables in a specified number domain. **CCK4.** Justify arithmetic generalizations by empirically, by using representation-based reasoning, by using general arguments, or by using algebraic arguments **CCK5.** Reason with the arithmetic generalization **KCS.** Analyze students' possible conceptions and misconceptions in the processes of conjecturing, representing, justifying, and generalizing arithmetic relationships **KCT.** Design a lesson with appropriate strategies and representations to provide students to engage in conjecturing and generalizing and to experience core algebraic thinking practices
Functional Thinking	**CCK1.** Identify and describe recursive, covariational, and correspondence relationships in words **CCK2.** Describe correspondence relationships, or function rules using variables **CCK3.** Justify relationships represented in words and variables using function rules, tables, or problem context **CCK5.** Reason with generalization by interpreting representations (graph, table, function rule) and linking representations **KCS.** Analyze students' functional thinking approaches while generalizing and representing the relationships between two variables that change together. **KCT.** Design a lesson with appropriate strategies and representations to provide students to think functionally by coordinating the relations between the variables, instead of focusing on the change in one variable

Note: The CCK learning goals were adapted from the LEAP project (see http://algebra.wceruw.org)

In line with these learning goals, in the early algebra lessons, the prospective elementary teachers were involved in case discussions on the learning and teaching early algebra. Besides, they worked on the activities that can be used with elementary school students to support their algebraic thinking. Each week, usually, the first lesson focused on PETs' common content knowledge and activities were implemented to enable them to experience core algebraic thinking practices of generalizing, representing, justifying, and reasoning. Then, in the second lesson, through case discussions, PETs were provided to enhance their pedagogical content knowledge in terms of possible student thinking, instructional strategies, and representations to be used. More specifically, PETs were asked to read, analyze, and discuss the cases from the point of mathematical understanding, students' thinking, and teachers' moves (Schifter & Bastable, 2008) to improve their knowledge for teaching the core algebraic thinking practices in the particular content areas. For example, while reading the given classroom cases, they were asked questions including “How do you interpret those students' answers?” “Why does the teacher need to ask this question?” “What do you think about the teacher's move?” “As a teacher, would you prefer a different strategy?” “Besides this student's explanation, what kind of student response might be expected for that question?” The activities and the classroom cases in which the case discussions were carried out were chosen from the literature according to the learning goals. In determining the classroom cases, the presence or absence of a knowledge or skill specified in the learning goal in the classroom vignette was taken into consideration. For example, in the classroom case named Two of Everything (Wickett et al., 2008, pp. 4-12), the students examine a growth pattern, record the data on a table, describe the relationships they see in the table, in words and in variables. This classroom vignette was considered as appropriate exemplar case for discussing how patterns could be used to encourage functional thinking. In the case of Two of Everything, not only the appearance of the knowledge and skills specified in the learning goal (e.g., describing a recursive pattern, creating a table, justifying the correspondence relationship) but also their absence was considered important for the case discussions. For example, none of the students in that case described the relationship between the variables using covariational thinking. This situation was used as an opportunity to engage prospective elementary teachers in a discussion about other possible student thinking and the teacher strategies to guide students think covariationally. The other classroom cases were determined in a similar way and presented to the participants to have reflection and discussion related to the content specified for that lesson. Table 3.3 shows the schedule and contents of the early algebra lessons.

Table 3.3

The Schedule and Content of the Early algebra lessons

Week	Big Idea	Lesson	Topic	Task / Case
1	Equivalence and Equations	1	What is early algebra?	Lecture, classroom discussion (Blanton, 2008; Kaput, 2008)
		2	The meaning of the equal sign	Case Discussion – Students' responses to the missing value item 8 + 4 = __ + 5 (Carpenter et al., 2003, pp. 10-13)
2	Generalized Arithmetic	3	Arithmetic relations of the fundamental properties of number and operation	Commutative Property Task
		4	Arithmetic relationships in a context such as classes of numbers or outcomes of calculations	Case Discussion – Defining Even Numbers (Schifter et al., 2018, pp. 15-19)

Table 3.3 (continued)

3	Generalized Arithmetic	5	The concept of variable	The adapted version of the Candy Problem (Blanton, 2008, p. 161)
		6	Justification of arithmetic relations	Discussing the justification of $a + b - b = a$ (Carpenter et al., 2003, pp. 98-101)
4	Functional Thinking	7	Identifying and describing recursive, covariational, and correspondence relationships in words and variables. Justifying relationships represented in words or variables using function rules, tables, or problem context	The adapted version of the Outfit Problem (Blanton, 2008, p. 177)
		8		
5		9		Case Discussion – Two of Everything (Wickett et al., 2008, pp. 4-12)
		10		

The early algebra lessons started with the ones focused on the big idea of equivalence and equations. Before discussing the meaning of the equal sign and the structures of equations, firstly, prospective elementary teachers were asked to think and discuss whether teaching algebra in elementary school is necessary and how algebra teaching should be at that level. After that, they were presented with the core algebraic practices and content strands for algebra learning in elementary grades. In the second lesson of the first week, PETs were asked to engage in a case discussion. They read the classroom cases in which students showed different conceptions of the equal sign, relational or operational, while finding the missing value in the 8 + 4 = __ + 5 (see Carpenter et al., 2003, pp. 10-13). Then PETs discussed the students' ways of thinking, what the equal sign means to them, questions to be asked to students who show operational thinking, and teaching methods that can be used to lead students to relational thinking.

The second week of early algebra lessons focused on the big idea of generalized arithmetic. In the first lesson, they worked on the commutative property task, which involves some multiplication problems that includes an opposite different number of groups and group size (e.g., 7 soccer teams with 5 players vs. 5 soccer teams with 7 players). While working on that task, the PETs were asked to come up with a conjecture from the result of multiplications, generalize, represent in word and variable and justify why these conjectures are true. In this way, the PETs were provided to engage in core algebraic practices. Then, in the second lesson, they were asked to participate in the case discussion by analyzing a classroom case in which elementary students define the even numbers, justify why their definitions are true for each even number, and lastly discuss the sum of even numbers (see Schifter et al., 2018, pp. 15-19). In a word, in the classroom case, the students were going through a conjecturing process. While reading that classroom case, PETs discussed the students' conjectures related to even numbers and other possible conjectures, evaluated the questions that asked by the teacher in the case, explained what they would do if they were the teacher of this class and discussed how this kind of lesson could lead students to think algebraically.

The third week of the lessons was also revolved around generalized arithmetic. In the first lesson of this week, PETs worked on the adapted version of the Candy Problem (see Blanton, 2008, p. 161) and discussed the roles of variables (varying quantity or unknown) in an equation. After that, in the second lesson, they read a case in which a student tries to justify why the conjecture of $a + b - b = a$ is true (see Carpenter et al., 2003, pp. 98-101). During the discussion of that student and her teacher in the case, the student first uses some empirical strategies for justification. Then she justifies the truthiness of $a + b - b = a$ with general arguments thanks to her teacher's guiding questions. While discussing the case, PETs were asked to think about the student's ways of justification, how the teacher's questions challenge the student, how they would respond to the student's different thinking, and how elementary teachers can guide students to use general arguments while justifying conjectures.

The last two weeks of the early algebra lessons concentrated on the big idea of functional thinking. In the 4th week, during the two lessons, PETs worked on the adapted version of the Outfit Problem (see Blanton, 2008, p. 177). They were asked to examine a functional relationship, generalize, and represent in words and variables. Then they were also asked to justify the truthiness of that functional generalization and reason with it. By this task, PETs were provided to engage in core algebraic practices with a functional relationship and compare their different level generalizations (variational, covariational, and correspondence). In the 5th week, during the two lessons, PETs were asked to participate in the case discussion through the case of Two of Everything (see Wickett et al., 2008, pp. 4-12). As mentioned earlier, in this classroom case, the students examine a growth pattern, which could be represented as a linear function rule, $a = 2b$, record the data on a table, describe the relationships they see in the table in words and variables. While analyzing that classroom case, PETs were led to think and discuss the possible purposes of advantages of constructing a table, students' ways of examining patterns in the table, students' ways of generalizing and describing functional relationships, students' other possible answers, the purposes and effectiveness of the questions that were asked by the teacher and how to use the number patterns to guide elementary students to think functionally.

3.2.4 Role of the Researcher

The researcher designed the interview protocol and the plans of early algebra lessons. Then, she implemented the early algebra lessons and took the role of facilitator while carrying out the case discussions. Moreover, the role of the interviewer was also taken on by the researcher for the pre-and post-interviews. In the data analysis process, she was the main coder.

3.3 Participants of the Study

The participants of the study were selected among the 3^{rd}-year students who attended the primary school program and took the Mathematics Teaching I course in the fall semester of 2020-2021. Forty-three prospective elementary teachers were enrolled in the Teaching Mathematics I course, and all of them attended the early algebra lessons. None of the prospective teachers have an experience related to early algebra in both their elementary school education and teacher education. Eighteen of these 43 students volunteered to participate in the pre-and post-interviews. Among the volunteer prospective teachers, nine people who had completed the Basic Mathematics course were selected as the research participants. The selection of the participants was performed by a purposive sampling method by considering the PETs' GPA and Basic Mathematics course grades. A final score out of 100 was formed by calculating the mean of each participant's GPA and passing grades of the Basic Mathematics course. Except for one, the final scores of the participants were over 65. According to the final scores, the participants were divided into four groups as below 65 points (group 1), between 65-75 points (group 2), between 75-85 (group 3), and above 85 points (group 4). There were 1, 3, 7, and 7 participants in the groups, respectively. Then, the research participants were selected as nine people, one person from group 1, 2 people from group 2, and three people from both groups 3 and 4. In this way, a mixed group was aimed to be created in terms of achievement. All participants were female. The details about the participants of the study are presented in Table 3.4.

Table 3.4

Participants' Information

ID	Gender	GPA (out of 100)	Basic Mathematics course grade
PET1	F	40	45
PET2	F	60	85
PET3	F	60	85
PET4	F	80	77
PET5	F	80	84
PET6	F	80	85
PET7	F	80	92
PET8	F	80	95
PET9	F	80	100

3.4 Data Collection Tool and Procedure

The data of the current study was collected through individual interviews before and after the early algebra lessons. The purpose of the data collection was to examine the change in the prospective elementary teachers' subject matter knowledge and pedagogical content knowledge related to big ideas of equivalence and equations, generalized arithmetic, and functional thinking as the content of early algebra, and their conception of variable. As Patton (2002) stated we need to ask questions for the thing we cannot directly observe, and interviewing provides to "find out what is in and on someone else's mind" (p. 341); thus, the individual interviews was used as the data collection tool. The individual interviews lasted about one hour and were conducted on Zoom by the researcher.

The identical interview protocol was used in pre- and post-interviews. The protocol was prepared as semi-structured and consisted of three parts. In each part there were questions about the subject matter knowledge and pedagogical content knowledge related to a big idea (see Appendix A). More specifically, part A focused on equivalence and equation in terms of the meaning of the equal sign and the structure of equations; part B focused on generalized arithmetic, and part C on functional thinking. In part A of the protocol there was also a question for the participants' conception of variable.

While developing the interview protocol, the questions for the subject matter knowledge, particularly common content knowledge, were taken or adapted from the literature by considering the learning goals that guided the early algebra lessons. For the big idea of equivalence, prospective elementary teachers were expected to develop a relational understanding of the equal sign by identifying and reasoning with structural relationships in the equation. Correspondingly, the questions asked for the same purpose in the literature, Interpreting Equal Sign Item (Knuth et al., 2005, p. 70; see Table 4.1), Missing Value Item (Carpenter et al., 2003; see Table 4.2), and True/False Items (Stephens, 2006; see Table 4.4) were asked to examine participants' common content knowledge related to the meaning of the equal sign and the structure of equations . For the concept of variable, the PETs were expected to understand that a variable represents the measure or amount of the object and may stand for a varying quantity or an unknown in an equation. In line with that objective, Which Is Larger Item (Knuth et al., 2005, p. 70; see Table 4.5) took part in the interview protocol to investigate participants' conception of the variable. For the big idea of generalized arithmetic, PETs were asked to develop their knowledge of identifying and generalizing arithmetic relationships, defining arithmetic generalizations in words and variables, verifying generalizations' validity, and reasoning with them. In order to investigate their knowledge and ways of thinking on these topics, as suggested by Blanton (2008), the prospective elementary teachers were presented with a set of computations (see Figure 4.1) to lead them to make arithmetic generalizations. Then, they were asked to make a conjecture from this set of computations, describe their conjecture in variables, explain why they think their conjecture is true and reason about whether their conjecture is valid for all numbers or not. Finally, for the big idea of functional thinking, PETs were expected to identify and describe recursive, covariational, and correspondence relationships in words, describe a function rule in variables, justify the correctness of the relationships and reason with these relationships. Therefore, similar to the studies of Stephens et al. (2017) and Blanton et al. (2015), in the interview, PETs were provided a problem, Saving for a Bicycle (adapted from Blanton, 2008, p. 179; see Figure 4.2), which includes the quantities that change together, and they were asked to respond to the questions focused on the relationship between the quantities in the problem.

Besides the common content knowledge, the questions to investigate prospective elementary teachers' pedagogical content knowledge, namely knowledge of content and students and knowledge of content and teaching, were developed by the researcher based on the literature. The current study aimed to investigate the PETs' knowledge of content and students in terms of students' conceptions and misconceptions. Following this purpose, in the interviews, the participants were asked to explain their expected students' responses for a given problem, and then they were provided with some possible students' responses and were asked to explain students' ways of thinking. In the similar way, Tanisli and Kose (2013) and Asquith et al. (2007) asked prospective teachers to reflect on the sample students' response in their studies which focus on knowledge of content and students. In more detail, in part A, The Missing Value Item, 8 + 4 = [ ] + 5, was presented to participants again and asked how elementary students could respond to the question. After that, as has been reported in the literature (e.g., Carpenter et al., 2003; Knuth et al., 2005), student answers to the same question that can be given through operational and relational thinking were presented (see Table 4.24) to the participants to observe how they interpret students' thinking. In part B, PETs were asked to develop a conjecture that elementary students might make from the set of computations (Figure 4.3). Then, based on the justification approaches proposed by Carpenter et al. (2003, p. 87), the student responses justifying an arithmetic relationship empirically or using general arguments were presented to the participants, and they were asked to interpret them. Lastly, in part C, Saving for a Bicycle Problem, was provided to participants again, and they were asked what patterns the elementary students would notice after completing the table and how they could establish relationships between the number of weeks and the total amount of money. After that, based on Stephens et al.'s framework of "Levels of sophistication describing generalization and representation of functional relationships" (2017, p. 153), the students' possible responses that are in the variational, covariational, and correspondence thinking levels were presented to participants to interpret.

The second component of pedagogical content knowledge that the study focused on was knowledge of content and teaching. The prospective elementary teachers' KCT was investigated in terms of instructional strategies and representations. To investigate PETs' knowledge, they were given an objective from the curriculum developed by the Ministry of National Education (MoNE, 2018) related to each big idea and asked to explain what kind of lesson they would plan to address those objectives and what they would consider. In this way, the aim was to observe what kind of opportunities PETs create for algebraic thinking in the lessons they plan.

In the process of the development of interview questions, after receiving expert opinion from a mathematics education researcher focusing on early algebra, the pilot study of the interview was conducted. A 3rd-year student from a Primary School Education program in a different university volunteered for the pilot interview. The pilot interview gave the researcher a chance to test the interview questions and see whether they were understood as intended. After the pilot study, some changes were made to the order of the questions in the interview protocol. For example, the question that asked the participants to describe a lesson for a given learning goal was ordered before the ones that included activities and problems that can be used for this purpose. In this way, it was ensured that the prospective teachers were not affected by other questions while describing their lesson.

3.5 Data Analysis

The current study aimed to investigate the development of prospective elementary teachers' knowledge for teaching early algebra, and the data of the study was the participants' verbal responses to the interview questions. Therefore, the study was qualitative in nature. According to Merriam (2009), all qualitative data analysis is content analysis in the sense that it analyzes the content of the data source. Thus, the content analysis was utilized for the individual interviews.

Firstly, all video recorded individual interviews were transcribed. Then, the data analysis process was carried out in two parts: subject matter knowledge and pedagogical content knowledge. The prospective teachers' responses to the interview questions in these two parts were read, categorized, and coded several times. During this process, while some codes were taken or adapted from the literature, the analysis of some questions were performed by emerging codes. Since most of the interview questions to investigate the PETs' common content knowledge were adapted from the related literature, these questions were mostly analyzed with the existing codes. On the other hand, the analysis of the questions for pedagogical content knowledge was carried with emerging codes guided by the stated framework. For example, the question, "which is the larger problem" (Knuth et al., 2005, p.70) was adapted from the literature and the approaches for answering that question was already determined with the existing codes as "variable explanation," "single value explanation" and "operation." On the other hand, in the questions such as asking participants to come up with a conjecture from a set of computations, participant responses were examined, grouped, and coded with the emerging codes. All the codes and explanations are provided in Chapter 4.

3.6 The Trustworthiness of The Study

Both qualitative and quantitative studies should persuade the readers about the study's reliability and validity. According to Merriam (2009), regardless of the type of research, this can be achieved by paying attention to the conceptualization of the study, how data is collected, analyzed, and interpreted, and how findings are presented. However, since qualitative and quantitative studies are different in nature, while the quantitative study could convince the readers with some procedures and short descriptions, the qualitative study should provide enough detail "to show that the author's conclusion 'makes sense'" (Firestone, 1987, p. 19). Therefore, instead of *validity* and *reliability,* it is widely accepted to use the different terminology suggested by Lincoln and Guba (1985) for qualitative research. They proposed replacing the terms internal validity with credibility, external validity with transferability, and reliability with consistency or dependability. In order to ensure the trustworthiness of the study, credibility, transferability, and consistency or dependability of the study will be explained in the following sections.

3.6.1 Credibility and Transferability

Credibility refers to the internal validity. According to Merriam (2009), internal validity in all research is dependent on the meaning of reality, and internal validity deals with "how research findings match reality" and if "the findings capture what is really there" (p. 213). In this study, peer examination as a proposed strategy by Merriam (2009) was performed to enhance credibility. Peer examination or peer debriefing involves the researcher discussing the research process and findings with a colleague who have used qualitative methods before (Krefting, 1990). Lincoln and Guba (1985) stated that this process helps the researcher be 'honest', and the researcher's biases be probed, providing the basis for the interpretation to be clarified.

To perform peer examination in this study, a researcher who was experienced in qualitative research was involved in all the research process. At the beginning of the study, she shares her opinions and critics about the study's significance and feasibility. Related to methodology, particularly the development of the data collection tool, the questions' quality and purposes were debriefed. Then during the data collection, each week, the research process was discussed together. The strategy codes used were discussed in terms of their convenience and comprehensibility in the data analysis process. Moreover, at the end of the study, the correctness of the presentation and the interpretation of findings were examined by that experienced researcher.

Transferability refers to the external validity. Merriam (2009) describes the extent of the external validity as "the findings of one study can be applied to other situations" (p. 223). Although the current study is qualitative and aimed to investigate the development of the prospective elementary teachers' knowledge to teach early algebra in a particular learning environment rather than making a generalization, it is accepted that researchers need to provide "sufficient descriptive data to make transferability possible" (Lincoln & Guba, 1985, p. 298). Thus, the current study's thick description, which involves the theoretical and methodological approaches may enable the readers to understand and compare with other studies.

3.6.2 Consistency or Dependability

Consistency or dependability refers to the reliability. According to Merriam (2009), reliability deals with "whether the results are consistent with the data collected" (p. 221) and the "research findings can be replicated" (p. 220). To that end, the study's procedures should be described in-depth and allow the reader to evaluate the degree to which acceptable research methods were followed (Shenton, 2004). Peer review is suggested by Merriam (2009) as one of the strategies that can be performed to provide consistency in qualitative studies. As detailed in the previous section, peer review helped the researcher to ensure the consistency of the findings and the data collected in this study.

Furthermore, in the data analysis process, while coding participants' responses, reaching an intercoder agreement helps interpreting the participants' responses correctly. A second coder who was a mathematics educator researcher coded the randomly selected 20% of the data independently to assess the reliability of coding. When the agreement between the two coders was less than 80%, the codes were debated, and adjustments were recorded in the analysis until the two coders reached an agreement of 80%.

3.7 Ethical Issues

While conducting the current study, the ethical issues were considered, and it was ensured that the participants were not physically or psychologically harmed in any part of the study. Firstly, before starting the research, the required permission was received from Human Subjects Ethics Committee (HSEC) (see Appendix B). In addition, the permission was obtained from the dean of the faculty of education, where the PSE program is offered, and from the instructor who delivered the Teaching Mathematics course. At the beginning of the study, the participants were informed about the purpose of the study and the process. All prospective teachers who attended the early algebra lessons and to those who participated in the individual interviews were explained that the participation was entirely voluntary and that there would be no evaluation or grading after the lessons or interviews. A consent form explaining this information in detail was presented to the participants, and they signed it before the study.

In the data collection process, with the permission of the participants, the interviews were video recorded. Before starting the interviews, the participants were reminded that they could end the interview at any time and that they did not have to answer the questions they did not want. The researcher took care not to create a judgmental environment both in the lessons and in the individual interviews.

Participants were told that their responses to the questions would be kept confidential and that only the researcher and her supervisor would have access to the data. Additionally, while presenting the study's findings, the names of the participants were coded as pseudonyms, such as PET1, to ensure that the data and the participants' identities were not linked directly.

3.8 Limitations

This study has some limitations. The first one is the way lessons are delivered. Due to the pandemic conditions in the 2020-2021 academic year, the lessons were implemented through distance education. The five-week early algebra lessons offered to prospective teachers within the scope of the study were also held online, contrary to what was planned before the study. In this case, the inexperience of both the prospective teachers and the researcher in online courses may have affected the learning and teaching processes.

Another limitation of the study is related to the data collection tool. In this study, only individual interviews were used as a data collection tool. Not using more than one data tool may limit the study's findings.

CHAPTER 4

FINDINGS

This study was conducted to investigate the development of prospective elementary teachers' knowledge for teaching early algebra through case discussion. The subject matter knowledge and pedagogical content knowledge of participants were aimed to be examined in the study. In this chapter, findings related to the change in prospective elementary teachers' knowledge before and after early algebra lessons will be presented. The chapter consists of two main sections. In the first section, the findings related to prospective elementary teachers' common content knowledge as the subject matter knowledge will be introduced. The findings related to prospective teachers' pedagogical content knowledge will be presented in the second section. Moreover, findings related to each category of knowledge will be shared according to the big ideas: equivalence and equations, variable, generalized arithmetic, and functional thinking (Blanton, Stephens, et al., 2015). While presenting the findings, the codes used in the data analysis will also be shared. During the data analysis, different categories were used for each interview question but there were two common categories for all questions: Other and No response. No response (NR) code was used when the participant did not respond or replied as "I have no idea." Other (O) code was used when the participant answered with a strategy other than the determined ones or when the strategy was not discerned. Throughout this chapter, the findings will be presented by detailing the major categories.

4.1 Development of Prospective Elementary Teachers' Subject Matter Knowledge

This study aimed to investigate the development of prospective elementary teachers' knowledge of early algebra, and that development was examined through pre-and post-interviews before and after the early algebra lessons. During the early algebra lessons, as part of the common content knowledge, PETs were expected to a) comprehend the relational meaning of equal sign and concept of variable, b) identify, express in words and variables, justify and reason with arithmetic generalization and c) identify, express in words and variables, justify and reason with recursive, covariational and correspondence relationships (see Table 3.2 for all learning goals). The change in their knowledge related to these big ideas was examined with corresponding interview questions. The codes used to analyze PETs' verbal responses to those questions will be presented first in each section.

4.1.1 Equivalence and Equations

There were three interview questions related to the meaning of the equal sign and one question related to the concept of variable. These questions and the codes used in the analysis were taken or adapted from the related literature.

For the meaning of the equal sign, the first item was "interpreting the equal sign" (Knuth et al., 2005, p. 70). The item and the codes are shown in Table 4.1.

Table 4.1

Interpreting the Equal Sign Item and Codes

3 + 4 = 7
↑
A2. What is the name of the symbol indicated by the arrow?
What does this symbol mean?

Strategy Codes	Definition	Example
Relational – RL	Participant expresses the general idea that the equal sign means "the same as"	*The values of 3 + 4 and 7 are the same/equal.* *Both sides of the equal sign have the same value*
Operational – OP	Participants express the general idea that the equal sign means "add the numbers" or "the answer."	*It expresses the solution. It means the result of 3 + 4 is 7. It shows the result of the addition.*

Note: The problem and the codes were taken from Knuth et al. (2005, p. 70).

In the pre-interviews before the early algebra lessons, when PETs were asked what equal sign means to them, 5 out of 9 participants showed an operational understanding of the equal sign (see Table 4.2). For example, PET4 said that "*At the end of the computation, we use the equal sign to show the result,*" and PET9 explained the meaning of the equal sign as "*It says that the sum of 3 plus 4 is 7, so as a result, it reaches the sum of 7."* After the early algebra lessons, when the same question is asked to prospective teachers in the post-interviews, except one PET, 8 out of 9 of them showed a relational understanding of the equal sign. While PET4 still thought that "*The equal sign is to show the result here, 3 plus 4 equals 7*", the others described the equal sign as a relational symbol. For example, PET1 said that "*It means the two sides are equal to each other,*" and PET5 said that "*There are values on the right side of the equal sign and the left side of the equals sign. It shows that these are equal to each other*" Besides this kind of expression, some preservice teachers emphasized the "sameness." For example, PET7 explained the equal sign's meaning as "*It shows they have the same value."*

Table 4.2

The Frequencies of the Strategies in Interpreting the Equal Sign Item

	Pre (n=9)	Post (n=9)
Operational	5	1
Relational	4	8

The second interview question related to the relational meaning of the equal sign was the missing value problem that was taken (A3.1) and adapted (A3.2) from Carpenter et al. (2003). The items and the codes are shown in Table 4.3. The codes for this item are taken from Blanton et al. (2015).

Table 4.3

Missing Value Item and Codes

A3. Find the values of [ ] that make each number sentence true. Explain your answer. a) 8 + 4 = [ ] + 5 b) 67 + 83 = [ ] + 82		
Strategy Codes	**Definition**	**Example**
Operational – OP	The participant adds the two numbers to on the left and stops or adds all the numbers.	*8 + 4 = 12* *8 + 4 + 5 = 17* *67 + 83 = 150* *67 + 83 + 82 = 232*
Computational – C	To balance the two sides, the participant adds the two numbers on the left side and subtracts the number on the right side.	*8 + 5 = 12, 12 – 5 = 7* *67 + 83 = 150, 150 – 82 = 68*
Structural – S	The participant recognizes the structure in the equation and solves it without the need for a calculation.	*5 is one more than 4, so the number in the blank must be one less than 8* *82 is one less than 83, so the number in the blank must be one more than 67*

Note: The codes were taken from Blanton, Stephens, et al., 2015, p. 51.

The frequencies of PETs' usage of these strategies for missing value problem are presented in Table 4.4.

Table 4.4

The Frequencies of the Strategies in The Missing Value Question

	A3.1 8 + 4 = [ ] + 5		A3.2 67 + 83 = [ ] + 82	
	Pre (n=9)	Post (n=9)	Pre (n=9)	Post (n=9)
Operational	1	0	1	0
Computational	6	4	5	4
Structural	2	5	3	5

As shown in Table 4.4, in the pre-interviews, one prospective teacher showed an operational understanding and thought that missing values should be 12 and 150, respectively. Moreover, before the early algebra lessons, most of the PETs found the missing values by computation. For example, PET2 explained how she found the missing value for item A3.1: "*First I added eight to four, it is twelve. When we subtract five from twelve, it is seven.*" On the other hand, in the post-interviews, the operational understanding was not observed in PETs' responses, and this time, the most commonly used strategy was structural. In both items, 5 out of 9 PETs found the missing value without computation. For example, PET6, who used the computational strategy for item A3.2 in the pre-interview, explained her strategy in the post-interview as "*There is 83 here, on the opposite side there is 82, that is one less. Then there is 67, so that [the missing value] must be one more than the one on the opposite side*".

True/False Problem was the last question for the meaning of the equal sign. The participants were asked to evaluate whether the given statements were true or false. The items A4.1 and A4.2 were adapted from Stephens (2006), and item A4.3 was developed by the researcher. Moreover, the codes used for the analysis of those items were generated by drawing on the articles and emergent codes from the data. The items and the codes are shown in Table 4.5.

Table 4.5

True/False Item and Codes

A4.1 If 16 + 15 = 31, the expression of 16 + 15 – 9 = 31– 9 is also true.
The statement is TRUE / FALSE. Because...

Strategy Codes	Definition	Example
Compute – C	Participant makes the operations on both sides of the equation to show that they have the same value.	*It is true because both of the operations 16 + 15 – 9 and 31 – 9 results in 22.*

Table 4.5 (continued)

Same Expression – SE	To justify the truth of the statement, participant asserts that 16 + 15 and 31 are the same value.	*Both 16 + 15 and 31 are the same. It does not matter which one you write. They are the same value.*
Structure – S	Participant recognizes the equivalence of the two equations and states that the same operation is performed on both sides of the equal sign in the second one, and the balance is preserved.	*The same number is subtracted from both sides of the equation.*

A4.2 The equations 3x – 12 = 51 and 3x – 12 + 3 = 51 + 3 have the same solution. The statement is TRUE / FALSE. Because...

Strategy Codes	Definition	Example
Structure – S	Participant recognizes the equivalence of the two equations and states that the same operation is performed on both sides of the equal sign in the second one, and the balance is preserved.	*The same number is added to both sides of the equation.*
Variable Misconception – VM	Participant thinks that these two equations are different and that different values of the variable *x* results in different solutions.	*The letter x is a variable, and if we assign different values to them, they will not be the same.*
Solving Equations – SOL	Participant solves both of the equations correctly or incorrectly to see whether the values of *x* in these equations are the same or not.	*When we solve both of these equations, the value of x is found to be 21 each time.*

Table 4.5 (continued)

A4.3 The expression of 17 = 17 is mathematically meaningful. The statement is TRUE / FALSE. Because...		
Strategy Codes	**Definition**	**Example**
Expression of Equality – EOE	Participant thinks that the statement is meaningful because it is an expression of equality.	*It is true since they are equal.* *17 is equal to 17, so it is equality.*
The need of the operation – NOP	Participant thinks that the expression is meaningless in this form; it must be rewritten as including an operation.	*It is not meaningful; if written as 15 + 2 = 17 for example, it could be.*
Trivial – T	Participant thinks that it is already known; we do not need to write such an expression. So, it is meaningless.	*17 is already equal to 17, it is known. Writing such an expression is meaningless.*

In both the pre-and post-interviews, all prospective elementary teachers thought that if 16 +1 5 = 31, the expression of 16 + 15 – 9 = 31 – 9 is also true, but their reasonings varied. In the pre-interviews, 5 out of 9 participants thought this statement was true because 16 + 15 and 31 are the same expression (see Table 4.6). Three of them made computations to see the equality. Only one participant, PET6, could recognize the structure of the equation and stated that "*the same numbers subtracted from both sides of the equation*" in the pre-interviews. Similarly, in the post-interviews, most participants, 5 out of 9, explained their reasoning based on “the same expression”. Also, the number of prospective teachers who could see the structure of the equality increased to 3.

Table 4.6

The Frequencies of the Strategies in The True/ False Item – A4.1

	Pre (n=9)	Post (n=9)
Compute	3	1
Same Expression	5	5
Structure	1	3

As for the item A4.2 of the true/false problem, in the pre-interviews 5 PETs stated that the solutions of the equations $3x - 12 = 51$ and $3x - 12 + 3 = 51 + 3$ are not the same. Three of these participants' incorrect answers were due to the variable misconception (see Table 4.7). For example, PET9 said that "*The value we substitute to x in the first expression and the value we substitute to x in the second expression may be different. That is why they do not have the same solution*". Before the early algebra lessons, only PET2 explained that these two equations have the same solution by considering the structure of the equation. On the other hand, in the post-interviews, all participants except 1 thought that the statement was true. The participant who thought that the statement was false used the strategy of solving the equations but found different solutions for both equations because she made an error while solving them. Different from the pre-interviews, no variable misconceptions were observed in the post-interviews, and 4 PETs explained the correctness of the statement by noticing the structure of the equation. For instance, PET9 stated that "*It is true because the same number is added to both sides of the equation.*" Besides, the other 3 PETs detected the equivalence of the equations by solving them.

Table 4.7

The Frequencies of the Strategies in The True/ False Item – A4.2

	Pre (n=9)	Post (n=9)
Variable Misconception	3	0
Solving Equations	3	4
Structure	1	4
Other	2	1

When participants were asked whether "the expression of 17 = 17 is mathematically meaningful" true or not, 3 prospective teachers found that expression is meaningless in pre-interviews (see Table 4.8). Two PETs thought that it was a trivial expression and one of them asserted that it needs to be written with an operation by saying "*It did not really make much sense to me. For example, it would make more sense to say 17 equals 15 plus 2 or say 10 plus 7.*" However, after early algebra lessons, all participants affirmed the correctness of the statement by recognizing it as an expression of equality. For example, PET9 said that "*I think it makes sense. It has the same number on both sides, showing that they are both equal*."

Table 4.8

The Frequencies of the Strategies in The True/ False Item – A4.3

	Pre (n=9)	Post (n=9)
T	2	0
NOP	1	0
EOE	6	9

4.1.2 Variable

In the interviews, one question, "which is the larger problem" (Knuth et al., 2005, p.70) was asked to examine prospective elementary teachers' conceptions of the variable. The items and the codes are shown in Table 4.9.

Table 4.9

Which Is Larger Item and Codes

A7. Can you tell which is larger, 3*n* or *n* + 6? Please explain your answer.

Codes	Definition	Example
Variable explanation – VEX	Participant expresses the general idea that one cannot determine which quantity is larger because the variable can take on multiple values.	*We cannot decide because it depends on the value of n.* *When the value of n changes, the big one also changes.*
Single Value Explanation – SVEX	Participant tests a single value and draws a conclusion on that basis; thus, the conclusions vary depending on the value tested.	*If the value of n is one, then 3n is equal to 3, and n + 6 is equal to 7. Thus n + 6 is the bigger one.*
Operation – OP	Participant expresses the general idea that one type of operation leads to larger values than the other (for example, multiplication produces larger values than addition).	*3n is bigger since it is multiplication, and multiplication gives us bigger results than the addition does.*

Note: The item and the codes were taken from Knuth et al., 2005, pp. 70-71.

Before the early algebra lessons, when prospective elementary teachers were asked if they can decide which of 3*n* or *n*+6 is larger, 4 out of 9 PETs stated that 3*n* was the bigger one because of the idea that multiplication produces larger values than addition (see Table 10). For instance, PET5 said that "*we can say that while addition has fewer results, multiplication results in larger numbers.*" Moreover, 2 participants stated that n + 6 was the bigger one by testing a single value. For example, PET8 stated that "*If we substitute one for n, it becomes 3 (for 3n), the other becomes 7 (for n + 6), so n+ 6 is larger.*" On the other hand, in the post-interviews, 8 out of 9 participants asserted that we could not determine which quantity is larger because the variable can take on multiple values using VEX. For example, PET6 said that;

> We can decide by substituting numbers. For example, when we substitute 1 for n, the value of 3n is 3 and the value of n +6 is 7. However, when we substitute 4 for n, the value of 3n is 12, and the value of the expression n + 6 becomes 10. I think it depends on the values we give. That's why we can't use an exact expression.

Table 4.10

The Frequencies of the Strategies in Which is Larger Item

	Pre (n=9)	Post (n=9)
SVEX	2	1
OP	4	0
VEX	3	8

4.1.3 Generalized Arithmetic

In the area of generalized arithmetic, prospective elementary teachers were expected to identify and generalize arithmetic relationships. Moreover, during that generalizing process, they were expected to describe those relationships in words and variables, justify and reason with them. In this direction, the prospective teachers were asked to make a conjecture from a set of computations (see Figure 4.1).

Figure 4.1

Set of Computation for Conjecturing

Computation Task

Do the following computations

17 – 8 + 8 =	**98 – 29 + 29 =**
12 – 12 + 71 =	**13 – 13 + 72 =**

After completing the computations, the prospective elementary teachers were asked, "What do you notice in computations? Describe your conjecture in words" (item B2). The conjectures made by the participants were grouped into four different categories, one of which (OOP) was mathematically incorrect. These categories are shown in Table 4.11.

Table 4.11

Participants' Generalizations from Computation Task

B2. What do you notice in computations? Describe your conjecture in words.		
Codes	**Definition**	**Example**
Getting Zero – GEZ	After completing computations, participant recognizes that if a number is subtracted from itself, we get zero **OR** opposite signs of the same number cancel each other out.	*The values with opposite signs cancel each other out.* *Subtracting a number from itself results in zero*
Getting the same number that was started with – GSN	After completing computations, participant recognizes that adding and then subtracting the same number in a computation does not change the result.	*If we subtract a number from another number and add it, the result will be the first number.*
Order of Operations – OOP	Participant states some incorrect conjectures related to the order of operation.	*In such a computation, the addition should be done first.*

As a result of the analysis made in this way, it was noted that three of the participants made mathematically incorrect conjecture in the pre-interviews by considering the order of the operations (see Table 4.12). For example, PET2 stated her conjecture as "*first, addition operations should be performed, not subtraction operations*." While one of the remaining participants could not come up with a conjecture, two of them made the conjecture of getting zero, and three of them made the conjecture of getting the same number that was started with. As for the post-interviews, 7 out of 9 prospective teachers stated mathematically correct conjectures. Only one PET's conjecture was incorrect. Five of the correct conjectures PETs stated related to the computations were "getting zero." For example, PET7 stated her conjecture as "*when we subtract a number from itself, we get zero*." The remaining three correct conjectures were about recognizing that adding and then subtracting the same number does not change the result. For instance, PET9 said that "*If we add the same number to a number and then subtract it, the result will be our first number*." In short, 5 out of 9 prospective teachers made mathematically correct conjectures in the pre-interviews, while this number was recorded as 7 in the post-interviews.

Table 4.12

The Frequencies of the Conjectures from Computation Task

	Pre (n=9)	Post (n=9)
GEZ	2	5
GSN	3	2
OPP	3	1
O	0	1
NR	1	0

In the interviews, after prospective teachers asserted their conjectures, they were asked to reason with them, describe in variables, and justify. Hence, the analysis process was continued for the correct conjectures (for 5 PETs in the pre-interviews, for 7 PETs in the post-interviews). The first question the participants were asked to answer about their conjecture was: For what numbers is your conjecture true? Is it true for all numbers? (item B3). The participants' strategies for this question are presented in Table 4.13.

Table 4.13

Participants' Strategies for Reasoning with The Conjectures

B3. For what numbers is your conjecture true? Is it true for all numbers?

Codes	Definition	Example
Structure – S	Participant makes a statement showing the recognition of the underlying structure and provides a general argument which shows that the conjecture is true for all numbers.	*It is true for all numbers because adding a number and then subtracting the same number means adding zero. And adding zero always gives the starting number.* *It is true for all numbers because adding a number with its opposite sign means subtracting a number from itself and equals zero.*
Compute – C	Participant computes or uses specific examples in response without referring to the structure using some numbers from a number set to test the correctness of her conjecture.	*I think it is true. It can be a rational number; -2/5 and +2/5 will cancel out again.*

Note: The codes were adapted from Blanton, Stephens, et al., 2015, p. 64.

Two of the five prospective elementary teachers who asserted correct conjectures in the pre-interviews thought that their conjectures, "getting the same number that was started with," was not true for all numbers (see Table 4.14). One of the remaining PETs who thought that her conjecture was true for all numbers, PET6, explained her reasoning by showing the recognition of the underlying structure stating, "*adding and subtracting the same number means adding zero for me because they cancel each other out*." The other participant who thought that her conjecture, GSN, was true for all numbers, PET9, explained her reasoning by using some numbers from a number set to test the correctness of her conjecture. The remaining participant's justification, who also stated that the conjecture was true for all numbers, was not coded as using one of the strategies above and was coded as Other.

In the post-interviews, all 7 participants who made correct conjectures stated that their conjectures were correct for all numbers. While making this decision, 3 of the 7 participants drew attention to the mathematical structure, and 2 made computations with numbers from different number sets. For example, PET5, who asserted her conjecture as getting zero, explained her reasoning by recognizing the structure and said, "*Because when we add a number, we subtract it again. So, it will be zero*". On the other hand, PET6, one of the participants, who explained her reasoning with computations, stated, "*Natural numbers, rational numbers, irrational numbers... I was satisfied to see that when I added and subtracted the same number as the numbers representing those numbers, it gave that expression. So, I tried one example from each set of numbers*".

Table 4.14

The Frequencies of the Strategies for Reasoning with The Conjectures

	Pre (n=5)	Post (n=7)
Structure	1	3
Compute	1	2
Other	3	2

The second question the participants were asked to answer about their conjecture was: "How do you write your conjecture using variables?" (item B4). The prospective elementary teachers who made a correct conjecture in item B2 were expected to write a complete equation corresponding to their conjectures. However, the participants were observed to create equations which were not corresponding to their conjecture they stated or was not complete (they were expressions instead of equations and/or included numbers instead of variables). Therefore, in the strategy codes, distinctions have been made for these situations. The codes and details are presented in Table 4.15.

Table 4.15

Participants' Strategies for Representing Conjectures in Variables

B4. How do you write your conjecture using variables?		
Codes	**Definition**	**Example**
Corresponding – Complete – CC	Participant states a complete equation which is corresponding to the conjecture generated in Item B2.	For the conjecture GEZ: $a – a = 0$ For the conjecture GSN: $a – b + b = a$
Corresponding – Incomplete – CIC	Participant states an incomplete equation which is corresponding to the conjecture generated in Item B2.	For the conjecture GEZ: $a – a$ For the conjecture GSN: $a – b + b$, $18 – b +b = 18$, $18 – b +b$
Not Corresponding – complete – NCC	Participant states a complete equation which is not corresponding to the conjecture generated in Item B2.	For the conjecture GEZ: $a – b + b = a$ For the conjecture GSN: $a – a = 0$
Not Corresponding – incomplete – NCIC	Participant states an incomplete equation which is not corresponding to the conjecture generated in Item B2.	For the conjecture GEZ: $a – b + b$, $18 – a + a$, $18 – a + a = 18$ For the conjecture GSN: $a – a$

In the pre-interviews, none of the 5 participants who made correct conjectures in item B2 could describe their conjecture in variables as "corresponding – complete" (see Table 4.16). While the responses of 2 of these participants were coded as corresponding – incomplete, the responses of the remaining 3 participants were coded as not corresponding – incomplete. For example, PET7 described the conjecture of getting the same number you started with, GSN, in variables as $18 + x – x = 18$. This expression corresponded to her conjecture but was not complete because of the using numbers instead of variables. PET8, whose conjecture was getting zero, GEZ, described her conjecture in variables as -a + a + 12 = 12. However, since this expression corresponded to the conjecture GSN, not GEZ, and included numbers, it was coded as "not corresponding – incomplete."

On the other hand, in the post-interviews, 2 of the 7 participants whose conjectures were mathematically correct could describe them in variables as "corresponding – complete." For example, PET9 stated "$x + y - y = x$" to get the same number conjecture. That expression was a complete equation and corresponded to her stated conjecture in item B2. Four of the remaining 5 participants' responses were coded as "corresponding – incomplete," and one response was coded as "not corresponding - complete."

Table 4.16

The Frequencies of the Strategies for Representing Conjectures in Variables

Strategies	**Pre (n=5)**	**Post (n=7)**
CC	0	2
CIC	2	4
NCC	0	1
NCIC	3	0

Related to the generalized arithmetic big idea, lastly, prospective elementary teachers were asked to justify their conjectures. Question B5 was: "Explain why your conjecture is correct." The codes used for the analysis of this item were adapted from the related literature, which focused on the justification of conjectures (e.g., Blanton, Stephens, et al., 2015; Carpenter et al., 2003). The codes and details are presented in Table 4.17.

Table 4.17

Participants' Strategies for Justifying Conjectures

B5. Explain why your conjecture is correct.		
Codes	**Definition**	**Example**
Authority Information – AUI	Participant justifies the conjecture by relying on information from an authority (i.e., teacher or books).	*This is how we were taught* *This is how we learned in lessons*
Empirical – E	Participant justifies the conjecture by showing it works for one or more examples.	*I have tried on some numbers and seen it works.*
Generic Example – GE	Participant justifies the conjecture by showing the structure or relationship using a particular example as a generic example.	*Subtracting a number from itself results in zero. Since let's say I have 17 pens. I'll throw those pens later. I have nothing left. So, it is zero.*
Using a General Argument – GA	Participant justifies the conjecture by using accepted mathematical arguments or definitions.	*If we subtract another number from one number and then add it, the result will be the first number. Because that also means adding zeros and adding zeros to any number gives the number itself.*

The frequencies of PETs' usage of these strategies for justifying their conjectures are presented in Table 4.18.

Table 4.18

The Frequencies of the Strategies for Justifying Conjectures

Strategies	**Pre (n=5)**	**Post (n=7)**
AUI	0	1
E	2	3
GE	2	2
GA	0	1
Other	1	0

As seen in Table 4.18, different from the pre-interviews, in the post-interviews, one participant, PET6, justified her conjecture using a general argument, GA, by stating that "*Because when we add and subtract the same number, they create a zero. That's why it gives the first number*."

4.1.4 Functional Thinking

The last content that prospective elementary teachers were expected to develop related to algebraic thinking was functional thinking. PETs were expected to identify, describe, justify, and reason with recursive, covariational, and correspondence relationships as common content knowledge. In this direction, to evaluate their knowledge related to quantities that change together, they were asked to reason on the "Saving for a Bicycle" problem (see Figure 4.2; adapted from Blanton, 2008, p. 179).

Figure 4.2

Saving for a Bicycle Problem

Saving for a Bicycle Problem

Every week Mert's dad gives him 3₺ for helping with chores around the house. Mert is saving his money to buy a bicycle. How much money has he saved after two weeks? Three weeks? Fill in the table below.

Week	Total Money
1	
2	
3	
4	
5	

After the participants filled the given table, they were asked to check over the data and to answer the following two questions, respectively: "Describe the patterns that you see in the table," (item C2), and "How do you describe the relationship between the number of weeks and the total amount of money?" (item C3). In the analysis of these items, Stephens et al.'s framework of "Levels of sophistication describing generalization and representation of functional relationships" (2017, p. 153) was taken as reference. The strategy codes are introduced in Table 4.19.

Table 4.19

Strategies for Describing Generalization and Representing Functional Relationships in Words

C2. Describe the patterns that you see in the table. C3. How do you describe the relationship between the number of weeks and the total amount of money in words?			
Codes		**Definition**	**Example**
Variational Thinking	**Recursive Pattern Particular – RP-P**	The participant defines the recursive pattern only with particular numbers.	*It goes as 3, 6, 9, 12…*
	Recursive Pattern General – RP-G	Participant identifies a correct general recursive pattern.	*The amount of total money goes up by 3*
Covariational Thinking – CR		Participant identifies a correct covariational relationship. The two variables (number of weeks and amount of total money) need to be coordinated rather than mentioned separately.	*When the number of weeks goes up 1, the amount of money goes up by 3.*
Correspondence Thinking	**Functional Particular – FR-P**	Participant identifies a functional relationship using particular numbers but does not make a general statement relating to the variables.	*1x3, 2x3, 3x3…*
	Functional Basic – FR-B	Participant identifies a general relationship between the two variables but does not identify the transformation between them.	*x3* *multiply by 3*

Table 4.19 (continued)

	Functional Emergent – FR- E	Participant identifies an incomplete function rule in words, often describing a transformation on one variable but not explicitly relating it to the other or not clearly identifying one of the variables.	*It is three times the number of weeks.* *They are just multiplying the number of weeks by 3.*
	Functional Condensed – FR-C	Participant identifies a function rule in words that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.	*The amount of total money is three times the number of weeks.* *If you multiply the number of weeks by three, you get the amount of total money.*

Note: The codes were adapted from Stephens et al., 2017, p. 153.

In the pre-interviews, 6 out of 9 prospective teachers described the pattern they saw in the table as a recursive pattern (see table 4.20). For example, PET2 answered item C2 by stating that "*here their difference is the same, increasing by three*." The remaining 3 participants' responses were coded at the correspondence thinking level (one as FR-B, two as FR-C). On the other hand, in the post-interviews, all prospective teachers described patterns at the correspondence level, and 5 of these nine responses were recorded as FR-C. For instance, PET1 expressed "*three times the number of weeks equals the total money each week*." Moreover, while one of the remaining responses was FR-P, three of them were recorded as FR-B. For example, the answer of PST8, "*it always goes as multiple of three,*" at the level of FR-B.

As for item C3, when participants were asked to describe the relationship between the number of weeks and the total amount of money in words, two participants described the relationship as FR-C in the pre-interviews. The number of participants who responded at this level was recorded as 3 in the post-interviews. Moreover, the number of participants who defined the relationship as CR increased from 1 in the pre-interviews to 4 in the post-interviews.

Table 4.20

The Frequencies of the Strategies for Describing Generalization and Representing Functional Relationships in Words

		C2		**C3**	
Levels		**Pre (n=9)**	**Post (n=9)**	**Pre (n=9)**	**Post (n=9)**
Variational	RP-P				
	RP-G	**6**		1	
Covariational	CR			1	**4**
Correspondence	FR-P		1		
	FR-B	1	2	2	
	FR-E				
	FR-C	2	**5**	2	3
Other			1	3	2

After describing the relationship between the number of weeks and the total amount of money in words, the participants were asked to describe those relationships in variables. Item C4 was: “How do you describe the relationship between the number of weeks and the total amount of money by using variables?” In the analysis of this item, similar to the previous one, Stephens et al. (2017, p. 153)’s framework was taken as reference. The strategy codes are shown in Table 4.21.

Table 4.21

Participants' Strategies for Representing the Relationship in Variables

C4. How do you describe the relationship between the number of weeks and the total amount of money by using variables?		
Codes	**Definition**	**Example**
Incorrect Function Rule – INFR	Participant identifies an incorrect function rule using variable by identifying the relationship as an additive pattern.	$n + 3$
Functional – Emergent – FR – E	Participant identifies an incomplete function rule using variables, often describing a transformation on one variable but not explicitly relating it to the other.	$3n$ $3 \times x$
Functional – Condensed – FR – C	Participant identifies a function rule using variables in an equation that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.	$y = 3x$ $a = b \times 3$

Note: The codes were adapted from Stephens et al., (2017), p. 153.

In the pre-interviews, 7 out of 9 prospective teachers' responses were recorded as incorrect function rule-INFR or Other (see Table 4.22). Only 2 participants could express a correct function rule in variables, but they were emergent–function rules. In other words, the variables were not related to each other directly. For example, PET7 described the relationship between the number of weeks and the total amount of money as n × 3, where *n* referred to the number of weeks. On the other hand, 3 participants' expressions were coded as INFR or Other in the post-interviews. Moreover, while 4 of the remaining 6 participants' responses were coded as FR-E, two participants could describe the relationship as FR-C in the post-interviews. For instance, PET7 described the relationship as "$3x = y$," where *x* referred to the number of weeks, *y* referred to the amount of total money.

Table 4.22

The Frequencies of the Strategies for Representing the Relationship in Variables

Strategies	**Pre (n=9)**	**Post (n=9)**
INFR	4	1
FR-E	2	4
FR-C	0	2
Other	3	2

The next item, item C5, was asked to examine how the prospective teachers justified the functional relationships they formed. The question was: "How do you know your relationship works?" Based on the learning goal (see Table 3.2 for all learning goals), prospective elementary teachers were expected to use the function rules, tables, or problem context while justifying relationships. Therefore, the codes used in the analysis of item C5 were generated by the guidance of that learning goal. The codes are presented in Table 4.23.

Table 4.23

Participants' Strategies for Justifying the Functional Relationship

C5. How do you know your relationship works?		
Codes	**Definition**	**Example**
Using Problem Context – PC	Participant uses the information or conditions in the problem to show that the relationship/rule is true.	*It goes by three because his father gives him 3 liras each week.*
Using Table – T	Participant extends the table by adding more data to show that the relationship/rule works for more data OR uses the data on the given table to show the stated relationship/rule is true.	*We can continue to generate the table with more weeks and see the rule works.*
Using Function Rule – FR	Participant substitutes a chosen number of weeks and amount of total money on function rule (algebraic expression/equation of the relationship) to show that the inferred relationship/rule is true.	*We know that in the fifth week, he has 15 liras. When we substitute these numbers to the rule, which is 3x=y where x refers to the number of weeks, y refers to the total amount of money, and we can see it works.*

The frequencies of PETs' usage of these strategies to justify why their relationships work are presented in Table 4.24.

Table 4.24

The Frequencies of the Strategies for Justifying Conjectures

Strategies	Pre (n=9)	Post (n=9)
PC	2	0
T	3	3
FR	3	5
Other	1	1

As seen in Table 4.24, there was an increase in the number of participants who used the function rule in the post-interviews to show that the relationship is correct and working. For example, PET3, who used the table to justify her relationship in the pre-interview, explained her thinking by using the function rule in the post-interview and stated that "*I'll substitute it. I said x refers to total money. Our total money is equal to 3 at first… We said y refers to the number of weeks; I substitute 1 in y, then three equals three. This way, we can see that it is correct when we try all the weeks.*"

Lastly, prospective elementary teachers were expected to reason with the relationships they formed. In the last question, which was a multiple-choice item, related to functional thinking, participants were provided prices and asked whether they could decide the price of the bicycle that Mert bought at the end of any week. The question and the strategy codes used in the analysis are presented in Table 4.25.

Table 4.25

Participants' Strategies for Reasoning with the Relationships

C6. If it is known that at the end of any week Mert spent all his money to buy a bicycle, which of the following might be the price of that bicycle? In what week he bought his bicycle? Explain your answer. a)110 TL b)120 TL c)130 TL d)140 TL		
Codes	**Definition**	**Example**
Need of Information – NI	The participant thinks that the information given is not sufficient to decide on the price of the bicycle and states that the number of weeks that Mert bought the bicycle should be known.	*If we knew what week he bought the bicycle, we could find the price of it.*
Multiple of Three – M3	The participant thinks that regardless of the week he bought the bicycle, the price must be a multiple of three.	*It is 120 because the amount of total money goes up by three and each week it is multiple of three.*

In the pre-interviews, while 3 out of 9 prospective teachers thought that the price should be multiple of three and said that the price might be 120 TL, the other 3 of them thought that to decide the price of the bicycle, we need more information. For example, PET4 thought that we could not decide the price because "*the number of weeks should be given in the question*." On the other hand, in the post-interviews, no participant thought that the information provided was insufficient. Six out of 9 prospective teachers could reason that the price should be a multiple of three, no matter what week it is. For example, PET7 stated that "*It was going up as multiple of three. So, it must be a number that is divisible by three without a remainder. Since 120 is a number that is divisible by three without a remainder, it is 120.*"

4.2 Development of Prospective Elementary Teachers' Pedagogical Content Knowledge

In the early algebra lessons through case discussion, besides the common content knowledge related to early algebra, the prospective elementary teachers were also expected to develop their knowledge for teaching algebra in the elementary grades. Therefore, prospective elementary teachers' pedagogical content knowledge was also examined in the pre-and post-interviews before and after the early algebra lessons. The change in participants' PCK as knowledge of content and students (KCS) and knowledge of content and teaching (KCT) will be presented in two subsections. Each section will start with the codes used to analyze the PETs' verbal responses, and then the corresponding findings will be shared.

4.2.1 Knowledge of Content and Teaching (KCT)

To integrate algebraic thinking into the elementary curriculum, teachers need to develop their knowledge related to instructional strategies and representations as a component of pedagogical content knowledge, namely knowledge of content and teaching. In this study, to examine the prospective elementary teachers' knowledge of content and teaching, they were provided one objective related to each big idea from the mathematics curriculum. Then they were asked to explain what kind of lesson they would plan to meet those objectives and what they would consider.

First, the participants were provided an objective related to the meaning of the equal sign and asked how they would construct this lesson and what activities they would do. In the analysis process, strategy codes were developed by the researcher according to the meanings of the equal sign the participants focused on in the lessons they designed. The objective and the strategy codes are presented in Table 4.26.

Table 4.26

Participants' Strategies for Planning a Lesson Regarding the Meaning of the Equal Sign

A1. How would you design a lesson for the second-grade objective below? What would be your strategies and representations?

M.2.1.3.5. Students notice the meaning of equal sign that refers to the equality between mathematical expressions.

Codes	Definition	Example
Referring to the Result – R	The participant emphasizes that the equal sign indicates the result of an operation in the activity she developed for the objective.	*Ali had three pens. Ayşe gave Ali 5 more. We ask how many pencils Ali has. We can teach it by showing the result.*
Referring to the Same Objects – SO	The participant emphasizes the "same objects" in the activity she developed for the objective. She shows students that objects placed on two different sides are the same.	*We put bananas and apples on one side. We put bananas and apples on the other side in the same way and show that they are equal.*
Referring to the Same Amount – SA	The participant emphasizes the "same amount" (number, weight, etc.) in the activity she developed for the objective. She shows that the objects placed on two different sides have the same number/weight.	*I would put 10 counting beans on one side and 10 on the other side. I would have the students count it. They would count. They would say there were ten and the other had 10.*
Referring to the Balance – BAL	The participant emphasizes "balance" in the activity she developed for the objective. She shows the students the state of being in balance between the numbers/quantities of the objects placed on two different sides.	*I would put the same number of objects on both arms of the pan balance so that the students could see the equality between those objects. I would let them see it was equal when I put the same number objects, and it was equal when it was in balance.*

The frequencies of the prospective elementary teachers' usage of these strategies to teach the meaning of the equal sign are presented in Table 4.27.

Table 4.27

The Frequencies of The Strategies for Planning a Lesson Regarding the Meaning of the Equal Sign

Strategies	Pre (n=9)	Post (n=9)
R	2	0
SO	1	1
SA	3	2
BAL	1	6
Other	2	0

As shown in Table 4.27, 2 participants designed a task in the pre-interviews that have students think that an equal sign is a command to write a result. For example, PET8 explained that she could set up a lesson for this objective as:

> "I get the students on the board. Ali has three pens. Ayşe gave Ali 5 more. How many pens did Ali have? […] Then, when you go to mathematical notation, you prepare the numbers with magnets, you show them by sticking them on the board. You write both 3 and 5. As a result of these, I think we can teach equals as the result is the following."

Besides, 3 participants describe a lesson in which the equal sign referred to the same amount. For instance, PET7 stated that:

> "For example, I would put 10 counting beans here on one side and 10 on the other. I would have the students count it. They also counted. They would say there were 10, and the other would say there were 10. In this way, they would be made to realize that they were equal in the same amount."

While one of the remaining participants' response was coded as "referring to the balance", another one's response referred to same objects in the pre-interviews. On the other hand, in the post-interviews, 6 out of 9 prospective elementary teachers created activities to show the balance meaning of the equal sign in the lessons they described. The number of participants who created a lesson in this way was recorded as 1 in the pre-interviews. As an example, PET5 explained her lesson design in the post-interview as:

> "I'd bring a pan balance to class because it's an expression of equality. For example, when we put a mass on one side, let's say the simplest example is 10 kilos; when 2 kilos are put on the other side, equality is not achieved, one side is outweighed. I could show it as an equation that the scales balance when we put in a value that weighs an equal amount of 10 kilograms."

Furthermore, while only 1 prospective teacher considered using a pan balance for that objective in the pre-interviews, 6 participants thought to use the pan balance in their lessons in the post interviews.

To investigate the change in the prospective elementary teachers' knowledge of content and teaching, the second objective that was provided to them was related to the generalized arithmetic big idea. The objective and the codes of strategies that participants used in their lessons are provided in Table 4.28.

Table 4.28

Participants' Strategies for Planning a Lesson Regarding the Arithmetic Generalization

B1. How would you design a lesson for the fourth-grade objective below? What would be your strategies and representations?

> M.4.1.4.2. Students show that changing the order of the multipliers in multiplication with three natural numbers does not change the result.

Codes	Definition	Example
Showing through Example(s) – EX	Participant uses one or more examples to show students this fact works. She may use manipulatives.	*For example, let's say 5 times 2, it equals 10; when I switch 2 and 5, the result does not change. I can show it like this.*
Leading students to make a conjecture – CON	Rather than showing that this fact works, participant leads students to consider whether such a fact exists. She creates an environment to make conjectures. For this purpose, she may use problem contexts or ask leading questions such as "Why did not the result change?", "How do you know this is true?", "Is it always true?" etc.	*I would create a discussion environment and let students to think about why the result did not change when we switch the numbers. I would ask them to think about whether this fact is always true or not.*

When participants' verbal responses were analyzed, it was detected that in the pre-interviews, all prospective teachers tended to use one or more examples to show that this relationship works, rather than enabling students to make an arithmetic generalization. For instance, PET6 reported her lesson design as:

> "I would write it on the board and show it. 2 × 3 × 5. I used to have them multiply one by one, actually. I'd want them to multiply 2 by 3 and then multiply by 5. Then I would have them multiply 2 by 5 and then multiply by 3. This is a somewhat classical method, but they would see that the results were the same."

However, in the post-interviews that were conducted after the early algebra lessons, 6 prospective teachers designed a lesson to provide students an environment to make conjectures by thinking that "*I wouldn't give the rule of multiplication directly. I want them to reach it at the end*" (PET8). They considered asking questions to lead students to generalize the arithmetic relationship. For example, "*Will this be the same for every number*?" (PET4), "*Why did you do that? How do you think? How do you know it is? Is it true for all numbers?*" (PET8). The remaining 3 prospective teachers used examples to show students that this fact works.

The last objective was related to the functional thinking big idea. To understand how prospective elementary teachers present varying quantities and whether they consider guiding students to investigate relationships between quantities, they were given an objective related to number patterns and asked to plan a lesson. Table 4.29 shows the objective of the lesson and also the codes of strategies.

Table 4.29

Participants' Strategies for Planning a Lesson Regarding Functional Thinking

C1. How would you design a lesson for the third-grade objective below? What would be your strategies and representations? M.3.1.1.7. Students expand and generate the number patterns that have a constant rate.		
Codes	**Definition**	**Example**
Expanding the Pattern – EXP	The participant asks students to find the next term of the given pattern and expand the pattern in the activity she developed for the objective.	*We can write 2, 4, 6, and put three dots, and ask them to continue.*
Finding the Missing Term – MIS	The participant expects the students to find the missing terms of the given pattern and to place the appropriate terms in the blank spaces in the pattern in the activity she developed for the objective.	*I would write 2, 4, 6, 8…, 12. I would ask the student to find the number to fill in the blank there. Then I would repeat. I would like them to fill in the pattern by putting spaces in between.*
Attempting to Guide to Functional Thinking – FT	The participant not only creates and expands the number pattern in the activity she develops for the objective but also attempts to create a functional thinking environment. She may have used a problem situation or a table with quantities that change together.	*There was the question of the chair. If a chair has 4 legs, how many legs do 2 chairs have? We then convert this into a table. We let them see through the table.*

Table 4.30 shows the frequencies of prospective elementary teachers' usage of these strategies to teach number patterns.

Table 4.30

The Frequencies of The Strategies for Planning a Lesson Regarding Functional Thinking

Strategies	Pre (n=9)	Post (n=9)
EXP	7	4
MIS	2	1
FT	0	4
Other	0	0

In the pre-interviews, as shown in Table 4.21, most participants thought of asking the students to find the next term of the given pattern, not going further than what was stated in the objective. For example, PET4 described a lesson for the objective as:

> "I would have students come the board. I would call two students first. Then I would add two more students, and it would be four. Later, when I added two more students, it would be 6. They would see that it increased and expanded as two students are added […]. All students would be on the board, and they would see it increased by two."

Moreover, two prospective teachers thought of using an activity that asked students to find the missing term in a number pattern. One of these participants, PET5, said that:

> "For example, it might go as 1, 3, 5. The differences always Increase by two. […]. It could be something like a puzzle where they would notice such patterns and then reach a conclusion. […] [Like a] fill-in-the-blank puzzle. [In that puzzle, we] would write 1, not 3, but write 5. [The students] will write 3 there."

On the other hand, different from the pre-interviews, in the post-interviews, 4 prospective elementary teachers were observed to go further than what was stated in the objective. They intended to create an activity that encourages functional thinking. Among these participants, some considered presenting a problem situation with quantities that vary together, rather than directly giving a number pattern to students. For instance, PET6 reported as:

PET6: I would show this through a problem. There was the question of the chair. If a chair has 4 legs, how many legs do 2 chairs have? We then convert this into a table. We let them see through the table. We can form a graph […].

R: What do you want students to do when you make a table or graph? What do you expect to hear?

PET6: Here, the students continue the pattern by counting at the beginning. This is what is desired in the objective. But other than that, I would make them see the amount of increase between them relationally. How many do you think increases when there are two chairs? How many will it increase compared to the first chair and two chairs, or how many will it increase for the other three chairs? They could see the difference between them and express themselves by establishing relationships […]. I would also ask the students questions and make them establish relationships, but I would make them establish covariational relationships.

As PET6 did, using problem situations or tables in a lesson for an objective that expects students to expand a number pattern was interpreted as an attempt to provide students with opportunities for functional thinking.

4.2.2 Knowledge of Content and Students (KCS)

During the early algebra lessons, besides the knowledge of instruction and representation, the prospective elementary teachers were expected to develop their knowledge related to students' possible conceptions and misconceptions as part of pedagogical content knowledge, namely knowledge of content and students. To understand the change in the participants' knowledge, they were asked to provide students' possible responses for some questions and interpret given students' responses. They responded to the questions for each big idea: equivalence and equation, generalized arithmetic, and functional thinking. The questions and the strategy codes used in the analysis will be presented in the order of big ideas.

For the first big idea, equivalence and equations, the missing value problem, 8 + 4 = [ ] + 5, was given to the prospective elementary teachers, and they were asked what kind of responses they expected from the students. The missing value problem and the common students' strategies are detailed in Table 4.31.

Table 4.31

Missing Value Item and Common Students' Strategies

A5. What correct and incorrect answers would you expect elementary students to the following question, and what would be their strategies for those answers?

8 + 4 = [   ] + 5

Codes	Definition	Example
Answer Comes Next – ACN	Student thinks that the answer is the result of the operation that comes right after the equal sign.	*8 + 4 = 12*
Add All the Numbers - ALL	Student adds all the numbers and does not consider where the equal sign appeared in the number sentences.	*8 + 4 + 5 = 17*
Compute – C	Student calculates the sum on the left side of the equation and finds a number to put in the box that when added to 5, it would give the same total.	*8 + 4 = 12; 12 – 5 = 7* *8 + 4 = 12 and 7 + 5 = 12*
Structure – S	Student considers the relation between the two addition expressions in the equation, not just the relation between the answers to the two calculations.	*5 is 1 more than 4, so the number in the blank must be 1 less than 8.*

Note: The problem and the codes were taken from Carpenter et al., 2003, pp. 9-13.

During the analysis of the item, all the student strategies expressed by the participants were coded. In this case, 9 participants put forward 18 possible student responses in the pre-interviews and 21 in the post-interviews. The distribution of the responses according to the strategies is presented in Table 4.32.

Table 4.32

The Distribution of the Answers According to the Strategies

Strategies	Pre (n=16)	Post (n=21)
ACN	5	7
ALL	0	4
C	7	8
S	3	2
Other	1	0

As shown in Table 4.32, “the answer comes next,” “compute,” and “structure” strategies were expected student responses for prospective teachers in both the pre- and post-interviews. While, in the pre-interviews, none of the participants expected students to add all the numbers, the number of participants who thought that the students could answer 17 using this strategy was recorded as 4 in the post-interviews. After asking participants’ expected student responses, they were presented with some student responses for the missing value problem, and they were asked to interpret them. In this way, the aim was to observe whether the prospective teachers noticed student’s way of thinking or approaches behind the student answers. Table 4.33 shows the students' sample responses and the participants' strategies.

Table 4.33

Participants Strategies of Interpretation of the Equality Item Responses

A6. What way of thinking might be behind the students' responses to the questions below? A6.1) 8 + 4 = [12] + 5 A6.2) 8 + 4 = [17] + 5 A6.3) 8 + 4 = [7] + 5 because if you take 1 away from the 8 and add it to the 4, you have 7 left

Table 4.33 (continued)

Codes	Definition	Example
Explaining Procedural Approach – P	Participant explains the mathematical procedures/steps correctly that the student follows to respond without mentioning the student's conception of the equal sign.	*The student found 12 by adding 4 + 8 and did not consider 5.*
Identifying Conceptions of Equal Sign – EQS	Participant explains the ways of thinking behind the given answers. Besides how students found 12, 17, and 7 as an answer, she explains how the student makes sense of the equal sign.	*They thought of writing the direct result by considering only one side, not the equality on both sides.* *Knows that both sides of the equal sign are equal, balancing on both sides.*
Misunderstanding Procedural/Thinking Approach – MISUN	Participant provides an explanation that is not correspondence of given students' thinking. They may misinterpret the student's way of thinking.	*I think the students who answered 17 rounded the numbers.*

The frequencies of prospective elementary teachers' usage of these strategies are presented in Table 4.34.

Table 4.34

Frequencies of Participants' Interpretation Strategies for the Equality Item

	A6.1		A6.2		A6.3	
	Pre (n=9)	Post (n=9)	Pre (n=9)	Post (n=9)	Pre (n=9)	Post (n=9)
P	7	1	8	5	5	5
EQS	2	8	0	4	0	2
MISUN	0	0	1	0	2	2
NR					2	

For item A6.1, the answer of 12, which was due to the operational understanding of the equal sign, there were no participants who misunderstood the students' thinking or procedural approach in both the pre-and post-interviews. However, when they were asked how the student thinks while giving this answer, most of the participants, 7 out of 9, explained the mathematical procedures that the student followed without mentioning the student's conception of the equal sign. For example, PET3 interpreted this student's response as *"[the student] summed up 8 and 4. He may have seen the equality, but not the 5, or he may not have cared*." On the other hand, in the post-interviews, 8 out of 9 prospective teachers explained the ways of thinking behind the given answers and identified the student's conception of the equal sign. For instance, PET3 interpreted the student's answer this time as follows:

> The student looking at 8 + 4 thinks that equality does not help balance both sides, but it gives the result of the computation. In other words, s/he thinks that we will write the result directly without looking at the other side when we see an equality.

For item A6.2, the student adds all the numbers in the equation, 8 + 4 = __ + 5, and then gives the answer of 17, again due to the operational understanding of the equal sign. None of the prospective teachers mentioned how the student made sense of the equal sign in the pre-interviews. The number of participants who were able to do that was recorded as 4 in the post-interviews. For example, PET7 interpreted this students' thinking as:

> [The student] summed up all the numbers he saw. In other words, s/he says there is an addition operation, and whatever number I see in the addition operation, I must add before the equal sign. S/he saw 5 there. Maybe s/he thought we should add this too. He wrote 17 after the equal sign.

Lastly, for item A6.3, in which the student answers as 7 by recognizing the structure of the equation, participants were observed to have difficulty interpreting this student answer. Four participants in the pre-interviews and 2 participants in the post-interviews misinterpreted or did not understand the student's way of thinking. For example, in the pre-interviews, PET2 thought that the student rounded up the numbers. Nevertheless, while no participants identified that student's conception of the equal sign in the pre-interviews, 2 of them were able to do in the post-interviews. One of those participants, PET3, said that *"I think [this student] thought relationally. He also understands equality. He understands that both sides are equal. I think he knows that both sides of the equal sign are equal, keeping both sides balanced."*

The second big idea in which prospective elementary teachers' knowledge of content and students were examined was generalized arithmetic. Firstly, for this big idea, a set of operations were presented to the participants, and they were asked what they expected elementary students to notice when they completed these operations (item B6, see Figure 4.3). Regarding this question, all participants expressed that they expected students to realize that changing the order of numbers in addition does not change the result in both the pre-and post-interviews.

Figure 4.3

Set of Computation for Students' Thinking

Computation Task

Do the following computations.

12	27	45	23
+ 27	+ 12	+ 23	+ 45

Note: The task was taken from Blanton (2008, p. 13).

After that, the prospective elementary teachers were provided with some student responses to justify an arithmetic generalization that focused on the sum of three odd numbers. The aim was to examine whether the participants identified students' ways of justification by asking how they interpreted given students' responses. Similar to item A6, in the analysis of this item, item B7, whether the participants understood the responses and, if so, whether they noticed the justification approaches beyond the procedural steps while interpreting the ways of thinking were examined. However, as it was realized, some student responses presented were understood in different ways; only the participant responses that correctly addressed the student's justification approaches were coded during the analysis process. Table 4.35 shows the presented student responses and the justification approaches that expected the prospective teachers to address.

Table 4.35

Student Responses and the Justification Approaches

Student Answer	Justification Approach
“Derya says that the sum of any three odd numbers will be odd. Explain why this is true.” B7. Suppose the question of the sum of three odd numbers above was asked to elementary school students, and they gave the following responses. What kind of thinking might be behind these student responses?	
B7.1) *3 + 5 + 7 = 15 and 15 are odd, so Derya is right*	EMPIRICAL: The student thinks that the statement is correct because it works for one or more examples.
B7.2) *This is always true since an odd number is always one more than an even number. For example, if three 1s are put together, you get 3, which is an odd number because 3 added to an even is always an odd number.* B7.3) *This is true because two odd numbers equal an even plus another equals an odd because an even plus odd equals odd*	GENERAL ARGUMENT: The student uses general arguments to justify the argument. The student could use accepted arguments concerning the sums of even and odd numbers, the relation between odd and even numbers, or the definitions of even and odd numbers for justification.

Note: “The three odd numbers” problem was adapted from Isler et al. (2013, p. 141) and the justification approaches were adapted from Carpenter et al. (2003, p. 87).

For the response in which the student justifies the fact empirically, while in the pre-interviews, only 1 prospective teacher identified that the student was generalizing based on an example, in the post-interviews, 5 out of 9 participants identified the student thinking. For example, PET9 stated, “*It is interesting that s/he came to this conclusion directly from an example. I would probably ask the question: how do you know it will be the same for all numbers?”* Similarly, for the student responses that included general arguments, in the pre-interviews, while 1 prospective teacher correctly mentioned the student's way of justification for the response B7.2, none of them did that for the response B7.3. However, in the post-interviews, 6 participants for the response B7.2 and 2 participants for the response B7.3 correctly identified the students' justification ways. For instance, for B7.2, PET6 stated that "*This student saw the relationship between odd and even numbers. I understand that from his statement. He thought relationally and made sense of odd and even numbers.*"

Lastly, the prospective elementary teachers' knowledge of content and students was examined for the big idea of functional thinking. For this, the Saving for a Bicycle Problem was presented to the participants again, and it was asked what kind of responses they expected from the students to respond to the questions about this problem. The two questions prospective teachers were shown were: item C7.1. "Describe the patterns you see in the table" and item C7.2 "How do you describe the relationship between the number of weeks and the total amount of money?" While stating possible student responses, prospective teachers were anticipated to mention students' responses varying in the "Levels of sophistication describing generalization and representation of functional relationships" (Stephens et al., 2017, p. 153; see Table 4.19).

Similar to item A5, all expected student responses that were indicated by participants were coded during the analysis of the item. In this circumstance, for C7.1, 9 participants put forward a total of 16 possible student responses in the pre-interviews and 20 in the post-interviews. As for C7.2, 9 possible student responses in the pre-interviews and 11 in the post-interviews were provided. The distribution of the responses according to the functional thinking levels is presented in Table 4.36.

Table 4.36

The Distribution of the Responses According to the Functional Thinking Levels

		C7.1		C7.2	
Levels		**Pre (n=16)**	**Post (n=20)**	**Pre (n=9)**	**Post (n=11)**
Variational	RP-P	3	2	0	0
	RP-G	6	6	0	0
Covariational	CR	0	3	1	2
Correspondence	FR-P	0	0	0	0
	FR-B	2	0	0	3
	FR-E	1	1	3	2
	FR-C	2	3	2	4
Other		2	5	3	0

As shown in Table 4.36, while working on the Saving for a Bicycle problem, for the question which asked the pattern students could see in the table, the prospective teachers expected student responses in the levels of variational and correspondence thinking in both the pre-and post-interviews. However, in the pre-interviews, none of the participants expected students to use covariational thinking. In the post-interviews, 3 student responses put forward by the participants were coded as covariational; for instance, "*as the number of weeks increases by one, the total amount of money increases by three*" (PET8).

For the question which asked how students describe the relationship between the number of weeks and the total amount of money, the student responses in variational level were not provided as possible student response for the prospective teachers neither in the pre- nor in the post-interviews. On the other hand, compared to the pre-interviews, more participants expected to hear a student response categorized as FR-C.

After asking participants expected students' responses for the Saving for a Bicycle problem, they were shown student responses to the problem and were asked to interpret them. In this way, participants were aimed to observe whether they could decide the functional thinking levels of the students. Table 4.37 shows students' responses and the participants' strategies of interpretation.

Table 4.37

Participants Strategies of Interpretation of the Functional Thinking Item

C8. What way of thinking might be behind the students' following answers for the question of how they describe the relationship between the number of weeks and the total amount of money?

C8.1) The total amount of money goes by three
C8.2) Each week, the total money increases by three
C8.3) The total amount of money is equal to three times the number of weeks

Codes	Definition	Example
Explaining Procedural Approach – P	Participant explains the mathematical procedures/steps correctly that the student follows to respond without mentioning the student's approach related to functional thinking.	*He added three each week. He may have used rhythmic counting again by adding three to it.*
Identifying Functional Thinking Approach – FT	Participant explains the students' ways to establish a relationship. She realizes correctly which variable(s) the student is considering. She may also explain the mathematical procedures/steps that the student follows to give that response.	*This student only thought about the amount of money and did not establish a relationship with the other.* *The student established a relationship between both the week and the amount of money.*

Table 4.37 (continued)

Misunderstanding Procedural/Thinking Approach – MISUN	Participant provides an explanation that is not a correspondence of given students’ thinking. They may think that the student’s answer is wrong, or they may misinterpret the student's way of thinking.	*The student who said that as the number of weeks increases by one, the total amount of money increases by three, thought that the total amount of money increased by three by multiplying the number of weeks by three.*

Table 4.38 shows the frequencies of prospective elementary teachers’ usage of these strategies.

Table 4.38

Frequencies of Participants’ Interpretation Strategies for the Functional Thinking Item

	C8.1		C8.2		C8.3	
	Pre (n=9)	Post (n=9)	Pre (n=9)	Post (n=9)	Pre (n=9)	Post (n=9)
P	9	4	6	2	4	1
FT	0	5	1	7	1	6
MISUN	0	0	1	0	3	2
Other	0	0	1	0	1	0

For item C8.1, the student answer that describes the relationship as “the total amount of money goes by three” using the variational thinking, in both the pre- and post-interviews, there were no participants who misunderstood the student thinking or procedural approach. However, in the pre-interviews, all participants addressed the student’s procedural approach without identifying the functional thinking approach when asked to interpret that student’s answer. For example, PET5 stated, “*Since the total amount of money goes up 3, 6, 9, [the student] might have given such an answer because [s/he] saw that the difference between them was constantly increasing by three*.” On the other hand, 5 out of 9 prospective teachers explained the students' ways to establish a relationship in the post-interviews. For instance, PET8 reported that *"This student thinks recursively. So, he/she only goes through a single variable. S/he only considers the total amount of money. She does not coordinate the number of weeks with the total amount of money."*

For item C8.2, in which the student used covariational thinking, "each week the total money increases by three," in the pre-interviews, most participants, 6 out of 9, did not identify that in the response the two variables (number of weeks and amount of total money) were coordinated rather than mentioned separately. For example, PET4 stated, "S/*he was adding three each week. [The student] may have used rhythmic counting again by adding three to it."* Different from the pre-interviews, after the early algebra lessons, in the post-interviews, 7 prospective teachers pointed out that the number of weeks and the total amount of money were *coordinated* with this student response. As an example, PET3 said, "*[This student] also coordinated [the total amount of money] with the week. In other words, she/he tried to show that the total money is also related to the week. He looked at both sides and coordinated them."*

Lastly, for item C8.3, the student answer stated that "the total amount of money is equal to three times the number of weeks," participants were expected to address that the student coordinated the two variables. While reflecting on this student's answer, the participants could elaborate on the coordination of variables and/or say that a direct relationship/rule was established between the number of weeks and the total amount of money. In this case, similar to previous items, more prospective teachers were recorded to identify the student's functional thinking in the post-interviews than the pre-interviews. While in the post-interviews, 6 out of 9 prospective teachers asserted that this student considered two variables together, the number of participants who responded in this way in the pre-interviews was 1. For example, after the early algebra lessons, PET7 explained the student's thinking as "*[This student] completely coordinated the total amount of money with the number of weeks. S/he said it as a rule.*" In addition to these, in the pre-interviews, 5 out of 9 prospective teachers thought that the responses of the students who said that "the total amount of money is increasing by three” and “the total amount of money is increasing by three each week” were the same. In other words, they did not identify the differences between variational and covariational thinking in the pre-interviews. However, the number of participants who thought in this way was recorded as 2 in the post-interviews.

CHAPTER 5

DISCUSSION AND IMPLICATIONS

The current study investigated how prospective elementary teachers' knowledge of teaching algebra in early grades might have developed through case discussions. In this regard, individual interviews examined participants' subject matter knowledge and pedagogical content knowledge in the big ideas of equivalence and equations, generalized arithmetic, and functional thinking. Following this purpose, in this chapter, the findings will be discussed under the big ideas. Specifically, the first section will discuss the development of prospective elementary teachers' (PETs) knowledge to teach equivalence and equations as a core algebraic concept. The second section will discuss the development of PETs' knowledge to teach generalized arithmetic. Then, development in PETs' knowledge to teach functional relationships will be discussed in the third section. Lastly, the implications of the findings will be presented.

5.1 The Developments in PETs' Knowledge to Teach Equivalence and Equations

In their study with the middle school students that focused on the equal sign's meaning and its relation to their ability to solve algebraic equations, Knuth et al. (2005) asked, "why might middle school students hold an operational view of the equal sign?" (p. 309). They thought that the answer to this question might be that traditionally the equal sign was only presented in the first years of elementary school, and there is no direct instruction on the meaning of the equal sign in later grades. Besides this, there might be another answer to this question: teachers, who are expected to present to students the relational meaning of the equal sign, might see the equal sign as a "do something" signal. The findings of this study showed that prospective teachers might hold an operational view of the equal sign. Before the early algebra lessons, 5 out of 9 prospective teachers displayed an operational understanding of the equal sign when they were asked what the equal sign symbol meant. Other findings supported this result. While they were thinking about the correctness of the statement "If 16+15=31, the expression of 16+15-9=31-9 is also true" or the statement "The equations $3x – 12 = 51$ and $3x – 12 +3 = 51 + 3$ have the same solution", the fact that they need to do calculations to make a decision can be interpreted as their lack of understanding of the equal sign as relational.

After the early algebra lessons in which prospective elementary teachers were presented with some cases, including students' different conceptions of the equal sign and discussion on students' thinking and possible instructional approaches, the participants' views of the equal sign changed. In the post-interviews, 8 out of 9 prospective teachers explained the meaning of the equal sign with a relational understanding. Different from the pre-interviews, the majority of the participants found the missing values in the equations 8 + 4 = [ ] + 5 and 67 + 83 = [ ] + 82 by recognizing the structure in the equations and responded them without the need for a calculation. This development was also observed in the True / False items. Moreover, in pre-interviews, three participants stated that the expression 17 = 17 was not mathematically meaningful by thinking the same with Ana, who was a second-grade student, and said that "Well, yes, eight equals eight, but you just shouldn't write it that" (Falkner et al., 1999, p. 235). However, they changed their opinions in the post-interviews and mostly thought that it was an expression of equality. Based on these findings, it can be concluded that the early algebra lessons might have helped pre-service teachers develop a relational understanding of the equal sign.

A similar conclusion can be made for the findings related to the pedagogical content knowledge regarding equivalence. After reading and discussing classroom cases, including different students' conceptions of the equal sign, the prospective teachers' knowledge of content and students, specifically, their knowledge related to students' conception and misconceptions of the equal sign, seemed to have been developed. Since the problem 8 + 4 = [ ] + 5 was trivial for many teachers (Falkner et al., 1999), they may not have expected students to answer 12 or 17. For example, in the study of Asquith et al. (2007), when teachers were asked to predict how middle school students define the equal sign, they predicted that students would show the relational understanding of the equal sign at all grade levels. Likewise, before the early algebra lessons, no participants thought that 17 might be a student answer for that problem. Different from the pre-interviews, more participants thought that students could have an operational view of the equal sign and give the answer 12 or 17 for the missing value in item 8 + 4 = [   ] + 5.

Furthermore, when they were presented with some student responses and asked to explain the students' ways of thinking, in the pre-interviews, just a few of them could deduce students' understanding of the equal sign from their answers; they instead explained the mathematical procedures or steps. Similar findings were found by Vermeulen and Meyer (2017). In their study, most prospective teachers could not identify students' errors and misconceptions related to the equal sign due to a lack of knowledge of content and students. However, in this study, after the early algebra lessons, participants' knowledge related to students' thinking has been observed to be improved. In the post-interviews, more participants could recognize the students' conceptions and misconceptions of the equal sign based on their answers.

Besides students' thinking, the prospective teachers' knowledge of content and teaching regarding the big idea of equivalence and equations was also enhanced after the early algebra lessons. In the pre-interviews, 4 out of 9 prospective teachers asserted an activity that was appropriate to provide students the relational meaning of the equal sign. While, in the post-interviews, 8 out of 9 prospective teachers aimed to guide students to recognize that the equal sign refers to the balance of quantities in their activities regarding the objective related to the meaning of the equal sign. Moreover, as Van de Walle et al. (2013) recommended, 6 prospective teachers, 5 more than the pre-interviews, considered using a pan balance to provide students the relational meaning of the equal sign. Consequently, showing prospective teachers different student understandings related to the meaning of the equal sign through classroom cases and discussion on students' thinking and appropriate instruction might have helped them develop pedagogical content knowledge around teaching equivalence.

5.2 The Developments in PETs' Knowledge to Teach Generalized Arithmetic

Helping children identify, describe, and justify patterns and regularities in operations and properties of numbers is the basis of generalized arithmetic (Blanton, 2008). It is believed that in elementary grades, which are dominated by arithmetic, creating a learning environment focusing on these practices enables students to think algebraically and prepare them for later algebra learning (Russell et al., 2011). Studies have shown that with appropriate instruction, students' ability to generalize over operations and numbers can be improved, and students' algebraic thinking can be supported by teachers who offer opportunities for generalization in the elementary school curriculum (e.g., Hunter, 2010; Russell et al., 2011). Thus, teachers should be aware of these generalization processes and need the required knowledge to identify, generalize, represent, and justify the arithmetic relationships. This study showed that prospective elementary teachers might not be ready to create such a learning environment because of their lack of knowledge and experiences in the conjecturing and generalizing processes. Before the early algebra lessons, 4 out of 9 prospective teachers could not make a mathematically correct conjecture from a set of computations. Afterward, only 1 of these 4 participants answered whether the arithmetic relationship was true for all numbers by considering the structure of the relationships or using general arguments. When they were asked to describe the relationship in variables, none of the prospective elementary teachers stated a complete expression corresponding to their conjectures. Similarly, while justifying the conjectures, most participants did not use general arguments or generic examples identified as mathematically appropriate ways of justification (Carpenter et al., 2003). These findings support previous studies which documented that prospective teachers struggle to generalize and describe arithmetic relationships (e.g., Ding et al., 2013; Monandi, 2018).

On the other hand, it might be concluded that the early algebra lessons might have helped prospective elementary teachers to develop their knowledge of generalizing, representing, and justifying arithmetic relationships to some extent. In the post-interviews, 2 more participants came up with mathematically correct conjectures, a total of 7, and 2 more participants considered the mathematical structure with general arguments to explain why that arithmetic relationship was true for all numbers. In other words, after the early algebra lessons, relatively more participants described an arithmetical relationship (e.g., if we subtract a number from another number and add it, the result will be the first number) and showed that this holds true for all numbers, not empirically but with general arguments (e.g., adding a number and then subtracting the same number means adding zero and adding zero always gives the starting number). Additionally, unlike the pre-interviews, 2 participants could describe their complete conjectures using variables. Lastly, 1 more participant, a total of 3 in the post-interviews, used generic examples or general arguments to justify her conjecture in the post-interviews. However, from the pre-interviews to the post-interviews, the increase in the number of prospective elementary teachers was relatively low, and that most of the participants still did not reveal the required knowledge and skills specified in the early algebra lessons' learning goals. This may indicate that the teaching offered may not have been effective enough in the field of generalized arithmetic.

Building, expressing, and justifying conjectures about mathematical structure and relationships are the requirement of algebraic thinking and are seen as a "habit of mind" (Blanton & Kaput, 2004, p. 142). Therefore, it can be said that the two weeks (4 lessons) that focused on generalized arithmetic and including case discussions related to generalizing and justifying arithmetic relationships might not have been enough for developing such a habit of mind since most elementary teachers had little experience related to algebra after high school algebra which focuses on symbol manipulations, solving equations, simplifying expressions, and so on (Blanton & Kaput, 2003).

As for the prospective elementary teachers' knowledge of content and students, it can be said that more data is needed to come up with a clear conclusion. The aim was to observe the change in the participants' expectations related to students' generalization from a set of computations, but in both interviews, all participants expected to hear from the students the same conjecture, changing the order of addends does not change the sum. Therefore, it seemed not possible to talk about any change. Changing the interview questions focusing on all processes of conjecturing, describing, and justifying an arithmetical relationship, instead of only generalizing, may be beneficial to gain insights about participants' knowledge of content and students related to arithmetic generalizations. On the other hand, prospective elementary teachers made progress in identifying students' thinking for justification. In the post-interviews, more participants could identify whether the given students' justifications were based on empirical or general arguments. Whereas, in the pre-interviews, almost all participants evaluated whether the students' thinking was true or false without providing any insight related to students' ways of justification. Studies showed that although elementary students mostly tend to rely on the examples to verify the truth of a statement (Knuth et al., 2002), they were found capable of moving beyond the empirical justifications with a supportive instruction (e.g., Bastable & Schifter, 2008; Isler et al., 2013). Thus, to create a learning environment that students can learn to justify their thinking, teachers should be able to identify students' ways of thinking. In this direction, "engaging teachers in discussions focused on the details of students' competencies in justifying and proving may provide a basis for enhancing both teachers' own understandings of proof and their perspectives regarding proof in school mathematics" (Knuth et al., 2002, p. 1698). As a result, including such discussions in teacher education may be beneficial to prospective teachers.

Besides the prospective elementary teachers' pedagogical content knowledge regarding students' thinking, participants' knowledge of content and teaching related to generalized arithmetic was also the focus of this study. After the early algebra lessons, the findings showed that prospective elementary teachers changed their instructional approaches to address the arithmetic relationships. According to Blanton, "instruction that supports children's algebraic thinking is marked by rich conversation in which children make and explore mathematical conjectures, build arguments to establish or refute these conjectures and treat established conjectures (generalization) as important pieces of shared classroom knowledge" (2008, p. 93). Parallel with this idea, when the participants were asked to design a lesson for teaching an arithmetic relationship, 6 out of 9 prospective teachers proposed creating an environment to lead students to make conjectures in the post-interviews. In contrast, all participants focused on using one or more examples to show that the arithmetic relationship was true in the pre-interviews. Hence, it can be concluded that the early algebra lessons might have helped prospective teachers develop opportunities for their prospective students to conjecture, generalize, and justify.

5.3 The Developments in PETs' Knowledge to Teach Functional Relationships

Functional thinking is a significant strand of algebraic thinking (Kaput, 2008). It is seen as a critical entry point into algebra for early graders (Carraher & Schliemann, 2007). Contrary to this belief and the studies showing that elementary school students, even at kindergarten (Blanton & Kaput, 2011), can generalize and represent functional relationships (e.g., Blanton, 2008; Cooper & Warren, 2011), elementary school curriculum is not rich in terms of content to support functional thinking (Stephens et al., 2017), except for one topic: patterns. It is also true for the mathematics curriculum in Turkey (see MoNE, 2018). However, Smith (2003) stated that "elementary school teachers may create rich classroom experiences around patterns, yet not have a sense of how this topic ties into the ongoing mathematical development of their students, much less into the topic of functions" (p. 136). The findings of the pre-interviews conducted within the scope of this study showed that this idea might be correct; moreover, teachers may not have sufficient knowledge to generalize and represent functional relationships themselves. When all of the prospective teachers were asked to describe the relationships in the table created for the Saving for a Bicycle problem, 5 of them described a relation as a recursive pattern without the coordination of two variables. Similarly, in studies conducted with students, it was seen that students focused on recursive patterns created with a single variable, not on the relationships between variables (e.g., Carraher et al., 2008; Lannin et al., 2006), and this was also the case for the studies with prospective teachers (e.g., Alajmi, 2016; Yesildere & Akkoc, 2010). For example, Polo-Blanco et al. (2019) figured out that teacher candidates tend to focus on recursive patterns while generalizing functional relationships. Their study investigated Spanish and Portuguese prospective elementary school teachers' ways of identifying and expressing generalization from a geometric pattern; recursive strategy for obtaining distant terms was quite common in both countries. Likewise, Yesildere and Akkoc (2010) found that prospective elementary mathematics teachers tend to find the general rules for linear and quadratic growing patterns by considering constant differences between the patterns' terms.

In the current study, when the direct relationship between the number of weeks and the total amount of money was asked, 2 out of 9 participants stated a relation in the most sophisticated level, FR-C, in the pre-interviews. Furthermore, only 1 prospective teacher described the relationship between these two variables using covariational thinking, which some researchers thought to define the concept of function more appropriately (Confrey & Smith, 1994). However, in the post-interviews, all prospective elementary teachers defined the relationship they saw in the table regarding the problem situation with correspondence thinking, 5 of which were at the FR-C level. In addition, there was also an increase in the number of participants expressing the relationship as "when the number of weeks goes up by 1, the amount of money goes up by 3," thus using covariational thinking. According to these findings, it can be concluded that prospective teachers might have made progress in coordinating quantities that change together after the early algebra lessons.

Although according to the mathematics curriculum (MoNE, 2018), students are expected to find the rule of the number patterns and express them with variables for the first time in the 6th grade, studies have shown that elementary school students can use variables to express quantities that change together (e.g., Brizuela et al., 2015). Expressing function rules with variables was also noted to be easier for students than using words (Blanton, Stephens, et al., 2015). Therefore, to gradually prepare students for the use of algebraic notation, teachers are expected to have sufficient content knowledge in expressing relations and generalized pattern rules using variables. However, the current study's findings showed that prospective elementary teachers might not have the necessary background to guide elementary school students to express relationships using variables. In the pre-interviews, when the participants were asked to describe the relationship between the number of weeks and the total amount of money in variables, only 2 participants were able to do that, and they were FR-E, which involved description of the transformation on one variable but not explicitly relating it to the other. This result supports studies showing that prospective teachers have difficulty expressing pattern generalizations algebraically (e.g., Ozyildirim Gumus, 2021; Zazkis & Liljedahl, 2002). On the other hand, although only 2 out of 9 participants wrote a complete function rule in variables in the post-interviews, the increase from 2 to 4 in the number of stating the rule as an expression rather than an equation, using FR-E, can be interpreted as progress. Based on these findings, it can be said that after the early algebra lessons, the prospective teachers showed some improvement in expressing relationships with variables.

Regarding the justification of the correctness of the relationships stated by the participants, 2 more participants were recorded to use the function rule instead of table or problem context in the post-interviews. Related to the generalized arithmetic big idea, it can be said that prospective elementary teachers were not familiar with justifications of functional relationships. A similar result emerged in Tanışlı et al.'s

(2017) study, examining pre-service elementary mathematics teachers' knowledge of generalizations and justifications about patterns. One of the conclusions of their study was that "the pre-service teachers' justifications were limited to using empirical evidence and taking support from external authority" (p. 195). Furthermore, in this study, when PETs were asked to reason with the relationship and think about the possible price of the bicycle, 2 more prospective teachers could recognize that regardless of the week Mert bought the bicycle, the price must be a multiple of three after the early algebra lessons. Whereas in the pre-interviews, those 2 participants had thought that the information given was not sufficient to decide on the price of the bicycle because the number of weeks that Mert bought the bicycle should be known. In the light of these findings, it may be concluded that the early algebra lessons, including case discussions, might have helped prospective elementary teachers enhance their knowledge related to reasoning with the relationships between the quantities that change together.

Besides the prospective elementary teachers' common content knowledge related to functional thinking, their knowledge of content and students and knowledge of content and teaching were also investigated. In order to support elementary students' functional thinking processes, teachers need to know how the students coordinate the variables. It is well documented that elementary students can describe the relationship between the variables as variational, covariational, and correspondence relations (e.g., Martinez & Brizuela, 2006; Warren et al., 2006). Moreover, Confrey and Smith asserted that the covariational approach was "easier and more intuitive" for students (1994, p. 33). Contrary to this idea, none of the prospective elementary teachers expected students to describe a pattern or relation that they see in the table created for the Saving for a Bicycle problem as covariational before the early algebra lessons. Nonetheless, it can be inferred that the early algebra lessons supported the participants' knowledge of students' functional thinking approaches, as 3 out of 9 participants in the post-interviews expected the students to define a relationship covariationally. Another finding supporting this inference is that the prospective elementary teachers noticed the students' functional thinking approaches after the early algebra lessons. When the PETs were asked to interpret the relationships formed by the students using variational, covariational, and correspondence approaches in the pre-interviews, while almost none of the participants have shown any awareness regarding them, most participants noticed students' different ways of approaching these relationships in the post-interviews. The fact that the participants could not identify the functional thinking approaches of the students in the preliminary interview supports the findings of the study examining the teachers' noticing skills. For example, the study by Dogan Coskun (2021), which investigated how prospective elementary teachers noticed students' ways of algebraic thinking in their written solutions, found that while the PETs attended to students' responses, they struggled to provide solid evidence from the students' works. Nonetheless, the current study revealed that such skills could be developed with supportive interventions, which provided a context for collaborative learning.

As mentioned earlier, patterns can be used to create a learning environment to support students' functional thinking. However, as a single variable data set (Blanton & Kaput, 2004), commonly "patterns are used to find generalizations within the elements themselves: What comes next? Which part is repeating? Which part is missing?" (Warren & Cooper, 2006, p. 9). Similarly, in this study, when prospective elementary teachers were asked to design a lesson about the number patterns provided a learning objective, before the early algebra lessons, the participants mentioned activities in which students were expected to either extend a given pattern or find the terms that were not given in the pattern. This finding parallels Ozyildirim Gumus's (2021) study, which investigated pre-service elementary mathematics teachers' use of pattern and pattern problems in lesson plans. She found that pre-service mathematics teachers could not create pattern problems different from the routine ones found in textbooks. However, in the post-interviews of this study, even though the participants did not explicitly state that they aimed to support functional thinking, 4 prospective teachers attempted to guide their students to develop functional thinking using number patterns regarding the given objective. Although these lessons might be thought insufficient to create opportunities for functional thinking, they were categorized as having the potential to support functional thinking. For instance, in proposed lessons including creating and expanding number patterns, using problem situations with quantities that change together (e.g., number of chairs and number of chair legs), or considering using tables with multiple variables and expanding data might be interpreted as the attempts to provide students with opportunities for functional thinking (Blanton, 2008).

5.4 Implications, Recommendations and Further Research

The current study was not conducted to "simply document teacher weaknesses but to inform the design of teacher education in particular aspects of early algebraic reasoning" (Stephens, 2008, p. 275). The findings provide researchers and teacher educators with information on the development of prospective elementary teachers' knowledge of teaching algebra in the early grades. This section will discuss some implications.

According to the findings of this study, although elementary school students can think algebraically when they are provided with the appropriate instruction (e.g., Bastable & Schifter, 2008; Blanton et al., 2011; Carpenter et al., 2003), teachers may not be as ready as they are. This conclusion is parallel with the studies that revealed that prospective teachers do not have sufficient knowledge related to students’ ways of mathematical thinking and their conceptions/misconceptions (e.g., Philipp, 2008; Ubuz & Yayan, 2010). Early algebra studies have well revealed the necessity and importance of algebraic thinking at the elementary school level. However, we do not have enough information about the difficulties that prospective teachers may face. We expect them to create an entirely different algebraic thinking environment from the algebra learning they were exposed to in their student life and even during their teaching education program. Therefore, including teaching early algebra in teacher education and professional development programs is significant in enhancing elementary school teachers' learning and teaching. Regarding this necessity of searching the ways in which prospective elementary teachers have the opportunities to develop their knowledge, the findings of the current study could inform the implications of preparing elementary teachers to teach algebra in early grades. The subjects of equivalence, equations, arithmetical relationships, and patterns are already provided in the national mathematics curriculum, and prospective teachers are prepared to teach them. However, the missing part is seeing these subjects as a big idea for teaching algebra and developing ways to create algebraic thinking opportunities using these contents. Hence, based on the findings, it is suggested that while designing the mathematics method courses in elementary school teacher education programs, the algebraic thinking opportunities in the existing contents, especially the ones that are determined as the big ideas for early algebra learning, should be considered and emphasized.

Findings also indicated that engaging prospective elementary teachers in case discussions and enabling them to "think like a teacher" (Kleinfeld, 1992, p. 33) could develop their knowledge. This finding supports the previous studies, which revealed that cases could be used to provide opportunities for prospective teachers to develop their mathematical and pedagogical knowledge (Henningsen, 2008; Pang, 2011; Steele, 2008). Similar to the current study, Henningsen (2008) reported that reading and discussing narrative cases related to hexagon pattern tasks in mathematics method courses helped pre-service teachers enhance their mathematical knowledge. As before the case discussion, while 38% of participants could describe how the pattern was growing, 67% could do that after the discussion. Moreover, the excerpts presented by Steele (2008) were an example of how case discussions could encourage prospective teachers' pedagogical content knowledge. Additionally, the findings of this study support the idea that including case discussions in teacher education programs might be fruitful to engage prospective teachers in thinking about student thinking and instructional approaches. The reason for this conclusion is that classroom cases can be used for presenting the complex and dynamic nature of the learning and teaching environments and putting knowledge of teaching into practice (Butler et al., 2006). In other words, cases help prospective teachers to connect the theory and practices (J. Shulman, 2002). Besides being an alternative to overly theoretical teacher education, the case discussions also provide prospective teachers a collaborative learning environment in which they share and compare knowledge with peers, discuss different perspectives, and ultimately develop a shared understanding (Cobb, 1994). Hence, teacher educators are suggested to use classroom cases and case discussions in their teacher education programs.

There are also some recommendations for the teacher educators and the researchers who would like to implement the early algebra lessons developed in this study. Having the prospective elementary teachers engage in more case discussions related to each big idea, especially for the big idea of generalized arithmetic, might result in more salient development of knowledge. As documented in findings in this study, the development of the participants' knowledge in some points related to common content knowledge or pedagogical content knowledge was less than hoped. It is believed that the reason for such the result was that the course time was short for the content presented. Thus, spending more time on discussions related to early algebra content might be more beneficial for prospective elementary teachers. Besides, as mentioned earlier, this study mainly focused on the prospective teachers' pedagogical content knowledge, but their common content knowledge was also examined as a prerequisite for this (Agathangelou & Charalambous, 2020). However, the participants' mathematics backgrounds and so the readiness of studying on teaching algebra, namely common content knowledge, needed to be supported more than planned. This situation caused more time to be devoted to content knowledge and less time to pedagogical content knowledge than scheduled during the intervention. Therefore, before designing a lesson that includes a discussion on how to teach algebra in early grades, getting explicit information related to prospective teachers' general mathematics knowledge and algebra might be useful.

Lastly, there are also some recommendations for future research. This study investigated the development of prospective elementary teachers' knowledge to teach early algebra through case discussion. Three of the five big ideas identified as the content of early algebra, equivalence, and equations, generalized arithmetic, and functional thinking were examined within the scope of this study. Future research might consider investigating the prospective teachers' knowledge of all five big ideas to see a more comprehensive picture. On the other hand, focusing on only one big idea and examining the knowledge development of prospective teachers in this field can also provide us with more profound and detailed information.

The participants of the current study constituted a mixed group in terms of achievement, and this situation may affect the development of their SMK and PCK. Further research could prefer to choose the participants at different levels of achievement.

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APPENDICES

A. APPENDIX A: INTERVIEW PROTOCOL

PART A – EQUAL SIGN &VARIABLE

A1. Aşağıdaki ikinci sınıf kazanımı için nasıl bir ders planı hazırlamanız gerektiğini düşünün. Dersi planlarken neleri göz önüne alırdınız? Dersin giriş, orta ve sonuç kısımlarında neler yapardınız? Kullanacağız örnekler, materyaller neler olurdu?

M.2.1.3.5. Eşit işaretinin matematiksel ifadeler arasındaki "eşitlik" anlamını fark eder

A2. 3 + 4 = 7
↑

Ok ile gösterilen sembolün adı nedir?
Bu işaret ne anlama gelmektedir? Açıklayınız.

A3. Aşağıdaki eşitliklerin doğru olması için [ ] ile gösterilen değerleri bulunuz. Cevaplarınızı açıklayınız.

8 + 4 = [ ] + 5

A4. Aşağıdaki ifadeler için Doğru veya Yanlış' olarak değerlendiriniz. Cevaplarınızın nedenini açıklayınız.

- 16 + 15 = 31 ise 16 + 15-9 = 31-9 ifadesi de doğrudur
- 3x - 12 = 51 ve 3x - 12 +3 = 51 + 3 denklemleri aynı çözüme sahiptir
- 17 = 17 ifadesi matematiksel olarak anlamlıdır

A5. Aşağıda verilen sorular için ilkokul öğrencilerinin verebileceği doğru veya yanlış cevaplar neler olabilir? Bu cevapları verirken kullandığı stratejiler neler olabilir?

- 8 + 4 = [ ] + 5

A6. Aşağıdaki sorulara verilen öğrenci cevaplarının arkasındaki düşünme şekilleri için neler söyleyebilirsiniz?

- 8 + 4 = [*12*] + 5
- 8 + 4 = [17] + 5
- 8 + 4 = [*7*] + 5, çünkü 8'den bir alıp 4'e eklerseniz 7 kalır

A7. 3n veya n + 6 ifadelerinden hangisinin daha büyük olduğunu söyleyebilir misiniz? Cevabınızı açıklayınız.

PART B – GENERALIZED ARITHMETIC

B1. Aşağıdaki dördüncü sınıf kazanımı için nasıl bir ders planı hazırlamanız gerektiğini düşünün. Dersi planlarken neleri göz önüne alırdınız? Dersin giriş, orta ve sonuç kısımlarında neler yapardınız? Kullanacağız örnekler, materyaller neler olurdu?

M.4.1.4.2. *Üç doğal sayı ile yapılan çarpma işleminde sayıların birbirleriyle çarpılma sırasının değişmesinin, sonucu değiştirmediğini gösterir.*

İşlem Etkinliği

Aşağıdaki işlemleri yapınız.

17 – 8 + 8 =	98 – 29 + 29 =
12 – 12 + 71 =	13 – 13 + 72 =

B2. Hesaplamalarda ne fark ettiniz? Hesaplamalar ile ilgili varsayımınızı kelimelerle açıklayınız.
B3. Varsayımınız hangi sayılar için doğrudur? Tüm sayılar için doğru mudur?
B4. Varsayımınızı değişkenler kullanarak nasıl yazarsınız?
B5. Varsayımınızın neden doğru olduğunu açıklayınız.
B6. Aşağıdaki işlem etkinliğinin ilkokul öğrencilerine sunulduğunu ve aşağıdaki soruları cevaplamalarının istendiğini varsayalım. Öğrencilerin cevapları ve bu cevaplar için düşünme biçimleri ne olabilir?

İşlem Etkinliği

Aşağıdaki işlemleri yapınız.

12	27	45	23
+ 27	+ 12	+ 23	+ 45

- Hesaplamalarda ne fark ettiniz? Hesaplamalar ile ilgili varsayımınızı kelimelerle açıklayınız
- Varsayımınız hangi sayılar için doğrudur? Tüm sayılar için doğru mudur?

B7. Üç tek sayı toplamı sorusunun ilkokul öğrencilerine sorulduğunu ve aşağıdaki cevapları verdiklerini varsayalım. Bu öğrenci cevaplarının arkasında nasıl bir düşüme şekli olabilir?

- *3 + 5 + 7 = 15 ve 15 tek, bu yüzden Derya haklı*
- *Bu her zaman doğrudur, çünkü bir tek sayı her zaman bir çift sayıdan bir büyüktür, bu yüzden üç tane 1 toplanırsa, 3 elde edersiniz, bu da bir tek sayı oluşturur Çift sayılar toplamı her zaman çifttir ve çifte 3 eklendiğinde her zaman tek sayıdır.*
- *Bu doğrudur çünkü iki tek sayı toplamı çift olur. Bir tek sayı daha eklendiğinde tek sayı olur çünkü çift sayı ile tek sayının toplamı tektir*

PART C – FUNCTIONAL THINKING

C1. Aşağıdaki üçüncü sınıf kazanımı için nasıl bir ders planı hazırlamanız gerektiğini düşünün. Dersi planlarken neleri göz önüne alırdınız? Dersin giriş, orta ve sonuç kısımlarında neler yapardınız? Kullanacağız örnekler, materyaller neler olurdu?

M.3.1.1.7. *Aralarındaki fark sabit olan sayı örüntüsünü genişletir ve oluşturur.*

Bisiklet Problemi

Mert'in babası her hafta ev işlerine yardım ettiği için ona 3 lira veriyor. Mert, bisiklet almak için parasını biriktiriyor. İki hafta sonra ne kadar para biriktirir? Üç hafta sonra? Aşağıdaki tabloyu doldurun.

Hafta	Toplam Para
1	
2	
3	
4	
5	

C2. Tabloda gördüğünüz örüntüyü açıklayınız.
C3. Hafta sayısı ile toplam para miktarı arasındaki ilişkiyi (kuralı) nasıl tanımlarsınız?
C4. Değişkenler kullanarak hafta sayısı ile toplam para miktarı arasındaki ilişkiyi (kuralı) nasıl tanımlarsınız?
C5. Tanımladığınız ilişkinizin doğru olduğunu nasıl anlarsınız?
C6. Mert'in herhangi bir haftanın sonunda tüm parasını bir bisiklet almak için harcadığı biliniyorsa, bu bisikletin fiyatı aşağıdakilerden hangisi olabilir? Bisikletini hangi haftada aldı? Cevabınızı açıklayınız.

a) 110 TL b) 120 TL c) 130 TL d) 140 TL

C7. Bisiklet probleminin ilkokul öğrencilerine sunulduğunu ve aşağıdaki soruları cevaplamalarının istendiğini varsayalım. Öğrencilerin cevapları ve bu cevaplar için düşünme biçimleri ne olabilir?

- Tabloda gördüğünüz örüntüyü açıklayınız.
- Hafta sayısı ile toplam para miktarı arasındaki ilişkiyi nasıl tanımlarsınız?

C8. Hafta sayısı ile toplam para miktarı arasındaki ilişkiyi nasıl tanımlarsınız sorusuna verilen aşağıdaki öğrenci cevaplarının arkasında nasıl bir düşüme şekli olabilir?

- Toplam para miktarı üçer üçer artıyor
- Her hafta toplam para miktarı üçer artıyor
- Toplam para miktarı hafta sayısının üç katıdır

B. APPENDIX B: APPROVAL OF THE UNIVERSITY HUMAN SUBJECTS ETHICS COMMITTEE

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

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26 TEMMUZ 2021

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın *Işıl İşler BAYKAL*

Danışmanlığını yaptığınız Nejla ÖZTÜRK'ün "Sınıf Öğretmeni Adaylarının Durum Tartışmaları Yoluyla Erken Cebir Öğretimine Yönelik Bilgilerinin Gelişiminin İncelenmesi " başlıklı araştırmanız İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve 292-ODTU-2020 protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

Prof.Dr. Mine MISIRLISOY
İAEK Başkanı