TIMELY COMMUNICATION FOR ENERGY-EFFICIENCY, DATA FRESHNESS AND TRACKING

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

BARAN TAN BACINOĞLU

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
ELECTRICAL AND ELECTRONICS ENGINEERING

SEPTEMBER 2021

Approval of the thesis:

TIMELY COMMUNICATION FOR ENERGY-EFFICIENCY, DATA FRESHNESS AND TRACKING

submitted by **BARAN TAN BACINOĞLU** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Electrical and Electronics Engineering Department, Middle East Technical University** by,

Prof.Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** __________

Prof.Dr. İlkay Ulusoy
Head of Department, **Electrical and Electronics Engineering** __________

Prof. Dr. Elif Uysal
Supervisor, **Electrical and Electronics Engineering, METU** __________

Examining Committee Members:

Prof. Dr. Erdal Arıkan
Electrical and Electronics Engineering, Bilkent University __________

Prof. Dr. Elif Uysal
Electrical and Electronics Engineering, METU __________

Prof. Dr. Kemal Leblebicioğlu
of Electrical and Electronics Engineering, METU __________

Prof. Dr. Ali Özgür Yılmaz
Electrical and Electronics Engineering, METU __________

Assoc. Prof. Dr. Tolga Girici
Electrical and Electronics Engineering, TOBB-ETU __________

Date: 01.09.2021

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname: Baran Tan Bacınoğlu

Signature :

ABSTRACT

TIMELY COMMUNICATION FOR ENERGY-EFFICIENCY, DATA FRESHNESS AND TRACKING

Bacınoğlu, Baran Tan
Ph.D., Department of Electrical and Electronics Engineering
Supervisor: Prof. Dr. Elif Uysal

September 2021, 135 pages

This thesis considers data transmission scenarios where timeliness of information transmission, or adapting to intermittently available resources is important. The first part of the thesis focuses on energy harvesting communication systems. For such systems, energy efficient scheduling algorithms that achieve certain throughput maximization and data freshness objectives are developed. The second part of the thesis considers data transmission for the purpose of tracking unstable sources through noisy channels. Conditions are developed on the rate of information transfer to satisfy order m moment trackability of these processes.

Keywords: timely communication, energy harvesting and energy-efficient communications, age of information, data freshness, trackability, tracking through noisy channels

ÖZ

ENERJİ VERİMLİLİĞİ, VERİ TAZELİĞİ VE TAKİP İÇİN ZAMANLI HABERLEŞME

Bacınoğlu, Baran Tan
Doktora, Elektrik ve Elektronik Mühendisliği Bölümü
Tez Yöneticisi: Prof. Dr. Elif Uysal

Eylül 2021 , 135 sayfa

Bu tez, bilgi iletiminde zamanlılığın veya kesikli olarak erişilebilir kaynaklara adaptasyonun önemli olduğu veri iletimi senaryolarını ele almaktadır. Tezin ilk kısmı, enerji hasatçı haberleşme sistemlerine odaklanmaktadır. Böyle sistemler için, belirli veri hacmi en yükseltilmesi ve veri tazeliği amaçlarına erişen enerji verimli çizelgeleme algoritmaları incelenmiştir. Tezin ikinci kısmı, stabil olmayan süreçleri gürültülü kanallar üzerinden göndermek amaçlı veri iletimini ele almaktadır. Bu süreçlerin m moment mertebesinde takip edilmesini sağlayan bilgi transferi hızı üzerine koşullar geliştirilmiştir.

Anahtar Kelimeler: zamanında haberleşme, enerji hasatçı ve enerji-verimli haberleşme, bilgi yaşı, veri tazeliği, takip edilebilirlik, gürültülü kanallar üzerinden takip

To my beloved family

ACKNOWLEDGMENTS

First and foremost, I would like to offer my deep gratitude to my supervisor, Prof. Dr. Elif Uysal. The completion of this thesis has been possible thanks to her valuable guidance and support. I would also like to express my thanks and gratitude to Assist. Prof. Dr. Yin Sun, Prof. Dr. Onur Kaya and Prof. Dr. Can Emre Köksal for their valuable guidance and their contributions to the research presented in this thesis. I would like to convey my gratitude to Prof. Dr. Kemal Leblebicioğlu and Assoc. Prof. Dr. Tolga Girici, who are the members of the thesis monitoring committee, for their valuable comments and discussions. I am also very grateful to Prof. Dr. Erdal Arıkan and Prof. Dr. Ali Özgür Yılmaz, who are the members of the thesis jury, for their valuable comments and suggestions. I would also like to thank Prof. Dr. Serdar Yüksel for his valuable comments and tutoring on the field of networked control.

I would like to extend my thanks to my friends and colleagues for their friendship and helpfulness. Finally, I would like to thank my family for their love and care.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

The modern understanding of digital communication heavily relies on Shannon's seminal work [1]. Remarkably, Shannon's original theory, i.e., *classical information theory*, captured the process of information transmission into a picture where the transmitted information can be characterized free of meaning and the transmission process can be characterized free of time. This picture showed that reliable transmission of information with a strictly positive rate is possible while there exists as fundamental limit on this rate. The evolution of digital communication systems followed these fundamental observations.

When digital communication systems evolve into more complex communication networks, studying the problem of communication on such a level requires more effort [2]. The main problem in communication networks is to balance demands and resources, both of which can be interfering and highly heterogeneously distributed in practice. One side of this problem is to establish coordination and adaptation which are required due to temporal changes in demands and resources. In accordance with these requirements, the timing of communication events plays a significant role in communication networks. Frequently, communication networks employ network schedulers for network traffic control. In packet switching networks, reducing congestion,latency and packet loss are the typical goals of network traffic control. Conventionally, such systems and mechanisms are studied as queuing systems [3].

The theory of queueing systems, i.e., *queueing theory*, is shown to be fruitful in analysis of steady-state characteristics of queueing lengths, delays, packet losses or throughput. On the other hand, the majority of theoretical work on queueing systems has concentrated on scheduling policies with limited adaptiveness. This is due to the following two reasons. First reason is that running highly adaptive scheduling algorithms/policies on queueing systems can be computationally intensive hence historically more affordable alternatives were preferred in practice. Second reason is also similar but from analytical perspective that less adaptive but simple scheduling policies are much more tractable in theory. In contrast, it can be argued that better adaptivity might be benefited given that corresponding solutions are practically feasible. Moreover, for some applications with time-sensitive goals, adaptivity might be a necessity rather than being an option for performance improvement. Consequently, studying communication systems with high adaptivity is important.

In this thesis, we will consider communication with high adaptivity under various settings and objectives. We will refer to such type of communication as *timely communication* as we consider adaptivity in temporal sense which translates into the *timeliness* of communication process.

In Part I of this thesis, we will mainly study timely communication for energy harvesting communication systems. We will show how energy harvesting poses a timeliness pressure on communication considering different settings and objectives. That is, the transmission performed for communication should adapt to time-varying energy income being neither too slow/delayed nor fast/rapid. While this is the main source of timeliness in the problems that will be investigated in Part I, we will address its interplay with other time-varying conditions. In Chapter 2, we will consider energy-efficient transmission for an energy harvesting point-to-point communication system with time-varying data arrival process and channel fading where its throughput is optimized in finite-horizon. In Chapter 3, we will apply the results from Chapter 2 to a wireless energy transfer communication (WET) system and compare transmitter-centric, receiver-centric and distributed scheduling scenarios in this system. In Chapter 4, we will survey update-based communication systems and a measure of freshness, namely *Age of Information (AoI)*, which sets an alternative objective for which it will be discussed why it is a more relevant objective for monitoring applications. In Chapter 5, we will study the optimization of transmission for an energy harvesting point-to-point communication system under the objective discussed in Chapter 4.

Chapter 4 and 5 emphasize both the usefulness of the transmitted information and the timeliness of the transmission process in the context of communication required for a monitoring application. Interestingly, these aspects are purposefully ignored in the fundamental information transmission problem of classical information theory. One reason is that these aspects are application dependent which makes them hard fit into an application independent theory. On the other hand, the need for monitoring applications suggests the studying these aspects of information transmission on the fundamental level.

In Part II of this thesis, we will investigate the fundamental role of timely communication for monitoring/tracking unstable processes. We will discuss why timeliness in communication is crucial for reliable tracking of particularly unstable processes. The reason is that losses and errors in transmission may have time dependent impacts on the reliability of tracking that should be stabilized as the transmitted/tracked process is unstable. While such an effect occurs only when the transmitted/tracked process is unstable, conventional communication methods and update-based approach fail to deal with this effect. In Chapter 6, we will survey the main challenges posed by this effect and solutions proposed in the literature in connection with the fundamental problem of networked control. In Chapter 7, we will consider information-theoretic requirements for tracking an unstable process based on causal information that derived from the process.

While Chapter 4 and 6 provide background information and review the literature on related subjects, Chapter 2,3,5 and 7 present original work. The contents of Chapter 2,3,5 and 7 have been covered or partly covered in [4], [5], [6] and [7], respectively.

Part I: Adaptive Scheduling Methods for Timely Communication under Energy Harvesting

CHAPTER 2

ENERGY-EFFICIENT TRANSMISSION OVER FADING CHANNELS UNDER RANDOM DATA ARRIVALS AND ENERGY HARVESTING

2.1 Introduction

In this chapter, we will consider a problem of energy-efficient data transmission with an *energy harvesting* transmitter where timeliness is crucial. The source of timeliness pressure in this problem is that the transmitter should adapt three time-varying exogenous processes that affect its transmission:

- Energy harvesting process which determines energy availability for realizing transmission at the transmitter.
- Data arrival process which determines the length of data backlogged for transmission.
- Channel fading process which determines the amount of data that can be transmitted at a particular time.

The goal of the problem is to maximize data throughput within a finite problem horizon, i.e., a finite duration of transmission. We will study two versions of this problem: The *offline* version where exogenous processes are completely known to the transmitter beforehand and the *online* version where exogenous processes are stochastic with known statistics and reveal to the transmitter causally as they occur.

Energy efficient packet scheduling with data arrival and deadline constraints has been the topic of numerous studies (e.g., [8–11]). Energy harvesting constraints have been incorporated within the offline and online formulations (e.g., [12–24].)

The offline problem of throughput maximization in energy harvesting communication systems with fading channels has been widely studied and structural properties of throughput maximizing solutions have been investigated. For the throughput maximization problem in [18] and [25], it was proved that the offline optimal solution can be expressed in terms of multiple distinct *water levels* (to be made precise later in this chapter) that are non-decreasing. In [19], this result is generalized to a continuous time system by introducing a *directional water-filling* interpretation of the offline solution. Similar results are also shown in [20], [21] and [22] for the throughput maximization problem over fading channels with energy harvesting transmitters. The proposed solutions for the online counterpart of the problem in [18–22] were either heuristic schemes unrelated to the offline optimal solution or direct applications of stochastic dynamic programming.

Asymptotically throughput optimal and delay optimal transmission policies were studied in [26] under stochastic data and energy arrivals. In [27], an online solution maximizing overall throughput was formulated using a Markov Decision Process (MDP) . The MDP approach was also used in [28] to obtain the performance limits of energy harvesting nodes with data and energy buffers. In [29], a learning theoretic approach was employed to maximize long term (infinite horizon) throughput. Another learning algorithm based on post-decision state-functions was introduced in [30] for optimal power control over fading channels with average delay and energy arrival constraints. The authors in [31] suggested an even simpler online power control, namely the Fixed Fraction policy, for an energy harvesting system with i.i.d. energy arrivals and finite battery, which was shown to maintain a constant-gap approximation to the optimal long term average throughput. In [32], the authors considered a MDP for throughput maximization with energy harvesting transmitters over time-correlated fading channels.

The online problem has been also considered under non-stochastic formulations. For example, competitive ratio analysis was used in [33] for a throughput maximization problem on an energy harvesting channel with arbitrary channel variation and a simple online policy with a competitive ratio equal to the number of remaining time slots (much below the average performance estimated by stochastic approaches) was shown.

The results that we will present in this chapter rely on per slot computation of optimal water levels. Together with the knowledge of channel fading levels, these optimal water levels determine optimal transmission power levels and corresponding transmission rates. These policies are throughput optimal in the sense that they maximize the throughput achieved in the problem horizon but also they are energy-efficient as they consume minimum energy to achieve maximum throughput. In offline problem, we will provide an explicit formulation of these policies which allows the computation of optimal water levels through fixed-point iteration. In online problem, we will formulate the solution using stochastic dynamic programming. Further, we will show an online approach that considers offline solution as a stochastic process to be tracked by an online solution. We will provide a method to derive performance guarantees comparing expected performances of suggested online solutions and optimal offline solutions.

2.2 System Model

In this section, we will describe the system model for an energy harvesting transmitter with time-varying data traffic and fading communication channel. Consider an energy harvesting transmitter S that sends data to a destination D through a fading point-to-point communication channel. (Fig. 2.1) Assume that S can adapt its transmission rate and power while observing three distinct exogenous processes, namely, energy harvesting, packet arrivals and channel fading. Consider the system in discrete time and over a finite horizon divided by equal time slots. Let $\{H_n\}$, $\{B_n\}$ and $\{\gamma_n\}$ be discrete time sequences over the finite horizon $n = 1, 2, \ldots N$, representing energy arrivals, packet arrivals and channel gain, respectively, over a transmission window of $N < \infty$ slots, where n is the time slot index. Particularly, H_n is the amount of energy that becomes available in slot n (harvested during slot $n - 1$), B_n is the amount of data that becomes available at the beginning of slot n and γ_n is the channel gain observed at slot n.

Let e_n and b_n be energy and data buffer levels at slot n, where transmit power ρ_n is used in slot n and the received power is $\rho_n \gamma_n$.

The transmit power and rate decisions ρ_n and r_n are assumed to obey a one-to-one relation $r_n = f(1 + \rho_n\gamma_n)$, [1] [2] where the function $f(\cdot)$ has the following properties:

- $f(x)$ is concave, increasing and differentiable.
- $f(1) = 0$, $f'(1 + x) < \infty$ and $\lim_{x\to\infty} f'(1 + x) = 0$.

The update equations for energy and data buffers[3] can be expressed as below:

Update Equation for the Energy Buffer:

$$e_{n+1} = e_n + H_n - \rho_n, \rho_n \leq e_n, \text{ for all } n. \tag{2.1}$$

Update Equation for the Data Buffer:

$$b_{n+1} = b_n + B_n - f(1 + \rho_n\gamma_n), f(1 + \rho_n\gamma_n) \leq b_n, \text{ for all } n. \tag{2.2}$$

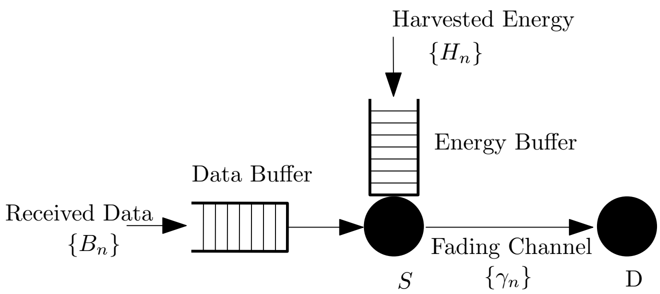


Figure 2.1: An illustration of the system model.

2.3 Offline Problem

We consider the following offline problem over a finite horizon of N slots:

$$\text{Maximize} \sum_{l=1}^{N} f(1 + \rho_l\gamma_l)$$

---

[1] The function $f(\cdot)$ is a general performance function as in [34].

[2] The choice of the function $f(\cdot)$, representing the relation $r_n = f(1 + \rho_n\gamma_n)$, has been made to simplify formulations and to signify the correspondence between the functions $f(\cdot)$ and $\log_2(\cdot)$.

[3] We do not assume any limit on the capacities of energy and data buffers. This assumption simplifies the problem and the characterization of the solution structure, and it is a reasonable assumption as we consider finite horizon scenarios. For example, in the design of the transmitter, the capacities of energy and data buffers could be chosen so large that overflow events do not occur within a typical range of transmission scenarios.

$$\text{subject to constraints in (2.1) and (2.2)}$$

As the problem is offline, we assume the sequence $\{H_n\}$, $\{B_n\}$ and $\{\gamma_n\}$ are known for $n \in [1, N]$. Accordingly, energy and data constraints can be completely determined as:

$$\sum_{l=n}^{n+u} \rho_l \leq e_n + \sum_{l=n+1}^{n+u} H_l, u = 1, 2, ..., (N-n), \tag{2.3}$$

$$\rho_n \leq e_n, \text{ for all } n.$$

$$\sum_{l=n}^{n+v} f(1 + \rho_l \gamma_l) \leq b_n + \sum_{l=n+1}^{n+v} B_l, v = 1, 2, ..., (N-n) \tag{2.4}$$

$$f(1 + \rho_n \gamma_n) \leq b_n, \text{ for all } n.$$

We make the following definitions to characterize offline policies and depict a clear distinction between the concepts of energy efficiency and throughput maximization.

Definition 1 *Any collection of power level decisions $\rho = (\rho_1, \rho_2, ..., \rho_N)$, satisfying energy and data constraints in (2.3) and (2.4), is a* feasible offline schedule.

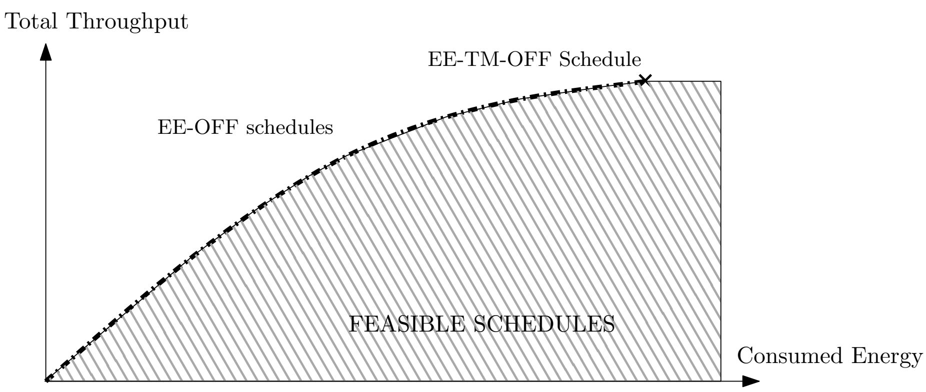


Figure 2.2: An illustration of feasible offline schedules in terms of achieved total throughput versus consumed energy.

Definition 2 *An* energy efficient offline (EE-OFF) transmission *schedule is a feasible offline schedule such that there is no other feasible offline schedule achieving higher throughput by consuming the same total amount of energy or achieving the same throughput by consuming less energy for a given realization of $\{H_n, B_n, \gamma_n, 1 \leq n \leq N.\}$.*

Definition 3 *Among all EE-OFF schedules, those that achieve the maximum throughput* [4] *are called* energy efficient thoughput maximizing offline transmission (EE-TM-OFF) *schedules.*

Note that EE-TM-OFF schedules are not only solutions to the offline problem but also energy-efficient solutions i.e., EE-OFF schedules. Hence, in case the energy harvested throughout the problem horizon is sufficient to transmit the received data completely, an EE-TM-OFF schedule leaves more energy at the end of the problem horizon than that of any other throughput maximizing schedule (Fig. 2.2).

We next define *water level* which will be useful in Theorem 1.

Definition 4 *Given a choice of power level ρ_n, a water level w_n is the unique solution of the following:*

$$\rho_n = \frac{1}{\gamma_n}\left[(f')^{-1}(\frac{1}{w_n\gamma_n}) - 1\right]^+ .$$

Proposition 1 *The water level w_n is non-decreasing in ρ_n and $f(1+\rho_l\gamma_l)$.*

Proof. As $f(\cdot)$ is increasing and concave, $(f')^{-1}(\frac{1}{w_n\gamma_n})$ is non-decreasing in w_n. ■

Remark 1 *For $\rho_n > 0$, the partial derivative of $f(1+\rho_n\gamma_n)$ with respect to ρ_n is equal to $\frac{1}{w_n}$.*

Clearly, any power level ρ_n can be obtained from a properly chosen water level w_n. Hence, any offline transmission schedule can be also defined by corresponding water levels $(w_1, w_2, ..., w_n)$.

For the solution of offline throughput maximization problem, it will be shown in Theorem 1 that the optimal water level for an EE-TM-OFF schedule is the maximum water level that barely empties data or energy buffer if it is applied continuously.

Theorem 1 *In an EE-OFF scheme, the water level w_n is bounded as:*

$$w_n \leq \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\},$$

[4] Note that not all feasible offline schedules that maximize the total throughput are EE-TM-OFF schedules. A schedule can be throughput optimal by delivering the data received during transmission but this can be done by consuming more energy than the corresponding EE-TM-OFF schedule.

where

$$w_n^{(e)}(w_n) = \min_{u=0,\ldots,(N-n)} \frac{e_n + \sum_{l=n+1}^{n+u} H_l + \sum_{l=n}^{n+u} K_l^{(e)}(w_n)}{u+1},$$

$$f\left(1 + \left[(f')^{-1}(\frac{1}{w_n^{(b)}(w_n)\gamma_n}) - 1\right]^+\right) = \min_{v=0,\ldots,(N-n)} \frac{b_n + \sum_{l=n+1}^{n+v} B_l + \sum_{l=n}^{n+v} K_l^{(b)}(w_n)}{v+1},$$

$$K_l^{(e)}(w_n) = w_n - \frac{1}{\gamma_l}\left[(f')^{-1}(\frac{1}{w_n\gamma_l}) - 1\right]^+,$$

$$K_l^{(b)}(w_n) = f\left(1 + \left[(f')^{-1}(\frac{1}{w_n\gamma_n}) - 1\right]^+\right) - f\left(1 + \left[(f')^{-1}(\frac{1}{w_n\gamma_l}) - 1\right]^+\right).$$

Particularly, water levels in an EE-TM-OFF schedule should satisfy the inequality above with equality, i.e. $w_n^* = \min\{w_n^{(e)}(w_n^*), w_n^{(b)}(w_n^*)\}$ *for all n in* $\{1, 2, \ldots.., N\}$.

Theorem 1 provides an explicit characterization of EE-TM-OFF schedules such that a particular water level w_n^* can be computed as the unique[5] fixed point of the function $\min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$. Accordingly, the optimal offline water level for each n can be obtained separately without making iterations over the entire schedule. The resulting offline schedule is similar to stair-case water-filling and directional water-filling with non-decreasing water levels(Fig. 2.3).

2.4 Offline Problem with Logarithmic Rate Function

In the offline problem, the throughput function $f(\cdot)$ could be chosen as $\frac{1}{2}\log_2(\cdot)$ that represents the AWGN capacity of the channel. The water level w_n in this case determines the power level ρ_n as $\rho_n = \frac{1}{\gamma_n}\left[\frac{\ln(2)}{2} w_n\gamma_n - 1\right]^+$. For this case, an EE-TM-OFF schedule can be obtained by setting the water level w_n[6] to $\min\{w_n^e, w_n^b\}$ for each time slot n where w_n^e and w_n^b are defined as follows:

$$w_n^e = \min_{u=0,\ldots,(N-n)} \frac{e_n + \sum_{l=n+1}^{n+u} H_l + \sum_{l=n}^{n+u} M_l^{(e)}(w_n)}{u+1}, \tag{2.5}$$

$$\log_2(w_n^b) = \min_{v=0,\ldots,(N-n)} \frac{b_n + \sum_{l=n+1}^{n+v} B_l + \frac{1}{2}\sum_{l=n}^{n+v} M_l^{(b)}(w_n)}{\frac{1}{2}(v+1)}, \tag{2.6}$$

[5] the existence and uniqueness of the fixed-point is due to the fact that $\min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$ is positive and monotone non-increasing in with decreasing w_n.

[6] Since $\frac{\ln(2)}{2}$ is a constant, in the rest, we will reset w_n to $\frac{\ln(2)}{2} w_n$ in order to simplify the notation.

where

$$M_l^{(e)}(w_n) = \min\left\{\frac{1}{\gamma_l}, w_n\right\}, M_l^{(b)}(w_n) = \log_2\left(\min\left\{\frac{1}{\gamma_l}, w_n\right\}\right).$$

The characterization of the offline optimal water level can be explicitly expressed as in the above. Due to the correction terms $M_l^{(e)}(w_n)$ and $M_l^{(b)}(w_n)$, the offline optimal water level w_n^* corresponding the unique fixed point of $\min\{w_n^e(w_n), w_n^b(w_n)\}$ should be computed iteratively with any fixed point iteration method. For example, w_n^* can be found by iteratively evaluating $\min\{w_n^e(w_n), w_n^b(w_n)\}$ as follows:

$$w_{n;k+1} = |_{w_n=w_{n;k}} \min\{w_n^e(w_n), w_n^b(w_n)\}, \tag{2.7}$$

where $w_{n;1} = w_n^{max}$ for some w_n^{max} guaranteed to be higher than w_n^*. (For example, $w_n^{max} = e_n + \frac{1}{\gamma_n}$ is always higher than w_n^*.) The proposition in the below states that the iteration in (2.7) converges.

Proposition 2 *The sequence of water level iterations, $w_{n;1}, w_{n;2}, ...$ converges to w_n^*.*

Proof. From (2.5),(2.6) and (2.7), $w_{n;k+1}$ is non-increasing with decreasing $w_{n;k}$. Accordingly, if $w_{n;k+1} < w_{n;k}$ for some k, then $w_{n;k+2} < w_{n;k+1}$ should be true [7] and setting $w_{n;1}$ to a value larger than w_n^* (e.g. $e_n + \frac{1}{\gamma_n}$) guarantees that $w_{n;2} < w_{n;1}$. As $w_{n;k}$'s are bounded below by zero, the iterations converge. Unless w_n^* is reached, the iterations have not stopped, hence the iterations will converge to w_n^* if $w_{n;1}$ is above w_n^*. ■

The offline optimal power level ρ_n^* that maximizes total throughput can be approached by computing the sequence, $w_{n;1}, w_{n;2}, ...$, which converges w_n^* by Proposition 2.

$$\rho_n^* = \lim_{k\to\infty}\left[w_{n;k} - \frac{1}{\gamma_n}\right]^+ \tag{2.8}$$

2.4.1 The complexity of finding w_n^*

Approximating the optimal offline water level w_n^* within an absolute error less than some $\varepsilon > 0$ has a linear complexity in N, i.e. $O(N)$. Given w_n, computing $\min\{w_n^e(w_n), w_n^b(w_n)\}$ can be done in $4(N-n)+2$ time steps since the computation of either $w_n^e(w_n)$ or $w_n^b(w_n)$ requires $2(N-n)+1$ steps. After the computation of $n = 0$ term in (2.5)(or (2.6)), the minimum until the next term can be evaluated by updating the current minimum

[7] unless $w_{n;k+2} = w_{n;k+1}$ which means $w_{n;k+2} = w_n^*$.

and computing $n = 1$ term based on $n = 0$ term. The procedure goes on in a similar iterative fashion computing the next term based on the previous term. The number of iterations (evaluations of $\min\{w_n^e(w_n), w_n^b(w_n)\}$) in (2.7) to approximate the fixed-point within some $\varepsilon > 0$, does not depend on N. One can see this by applying Banach's fixed-point theorem to the function $\min\{w_n^e(w_n), w_n^b(w_n)\}$. The function is non-decreasing in w_n, which means it maps a region $[0, a]$ into a region $[0, b]$ such that $b \leq a$. Also, it can be seen that the derivative $\frac{d}{dw_n} \min\{w_n^e(w_n), w_n^b(w_n)\}$ is bounded by some $q < 1$. This depends on the fraction of slots such that $w_n < \frac{1}{\gamma_l}$ for any $l \in [n, n+u]$ (or $l \in [n, n+v]$) when u (or v) is minimizing $w_n^e(w_n)$ (or $w_n^b(w_n)$). The largest of these fractions determines q which guarantees $| w_{n;k} - w_n^* | \leq \varepsilon$ when k is larger than $\frac{\log(\varepsilon(1-q)/|w_{n;2}-w_{n;1}|)}{\log(q)}$.

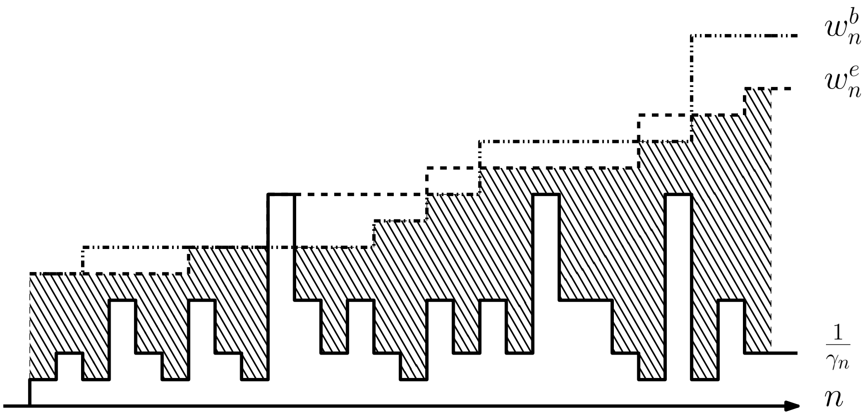


Figure 2.3: An illustration of an EE-TM-OFF policy.

2.5 Online Problem

The online problem formulation is an online counterpart of the offline problem with logarithmic [8] rate function. We formulate the problem as a dynamic program to maximize the expected total throughput. Let $x_n = (e_n, b_n, \gamma_n)$ be the state vector, $\theta_n = (H_1^n, B_1^n, \gamma_1^n)$ be the history and $X_n = (H_{n+1}, B_{n+1}, \gamma_{n+1} - \gamma_n)$ exogeneous processes at the slot n.

[8] For the sake of simplicity, the logarithmic rate function will be used in online formulations. However, similar formulations and results can be obtained also for the general concave function $f(\cdot)$.

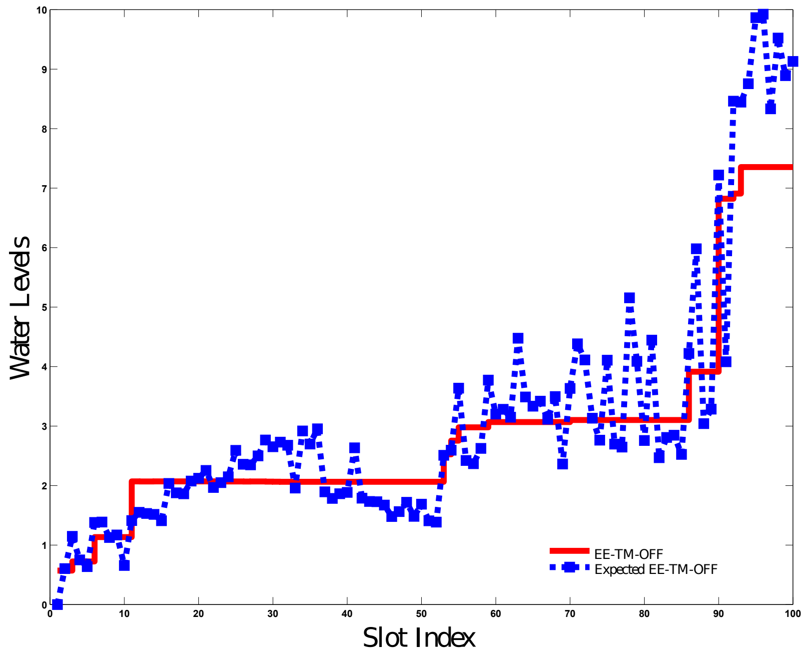


Figure 2.4: Sample water levels of an EE-TM-OFF schedule and the expected EE-TM-OFF given knowledge for a sample realization where energy harvesting, data arrival and channel fading processes are generated by 4 state DTMCs.

Define $A(x_n)$ as the set of admissible decisions such that $[w_n - \frac{1}{\gamma_n}]^+ \leq e_n$ and $[\log_2(w_n\gamma_n)]^+ \leq b_n$, $\forall w_n \in A(x_n)$. For $w_n \in A(x_n)$, the dynamic program for throughput maximization can be written as below:

$$\hat{V}^*_{n|\theta_n}(x_n) = \max_{w_n \in A(x_n)} \hat{V}_{n|\theta_n}(w_n, x_n) \tag{2.9}$$

$$\begin{aligned}\hat{V}_{n|\theta_n}(w_n, x_n) = [\log_2(w_n\gamma_n)]^+ + \\ E_{X_n}\,[\hat{V}^*_{n+1|\theta_{n+1}}(x_n + X_n - \phi(w_n;\gamma_n)) \mid x_n, \theta_n]\end{aligned} \tag{2.10}$$

where $\phi(w_n;\gamma_n) = ([w_n - \frac{1}{\gamma_n}]^+, [\log_2(w_n\gamma_n)]^+, 0)$ and $\psi_n = (H^N_{n+1}, B^N_{n+1}, \gamma^N_{n+1})$ represents the exogeneous processes for slots between n and N.

The solution of this dynamic programming formulation constitutes the online optimal policy maximizing expected total throughput to be achieved within the finite problem horizon. The drawback of this solution is that it suffers from the exponential time/memory computational complexity of the dynamic programming. On the other hand, when the vector $\psi_n = (H^N_{n+1}, B^N_{n+1}, \gamma^N_{n+1})$ is deterministic, the online problem is no different than the offline problem. The solution to the offline problem for the realization of ψ_n can be a reference for the online problem. It can be observed that a policy, which simply applies the statistical average of EE-TM-OFF water levels as its online water level at each and every time slot, typically closely follows the original EE-TM-OFF schedule (Fig. 2.4). Motivated by this observation, we consider EE-TM-OFF decisions as stochastic processes in the online problem domain. The next subsection will introduce an alternative dynamic programming formulation for minimizing the expected throughput loss of the online decisions with respect to the corresponding offline optimal decisions.

2.5.1 Online Schedule Based On the Offline Solution

Let $\tilde{w}^*_n = \tilde{w}^*_n(x_n)$ be the offline optimal water level which is a random variable generated over the realizations of ψ_n given the state vector x_n. Then, the total throughput achieved by applying offline optimal water levels until the end of transmission time window can be expressed as:

$$\tilde{V}^*_{n|\theta_n}(x_n) = [\log_2(\tilde{w}^*_n\gamma_n)]^+ + \tilde{V}^*_{n+1|\theta_{n+1}}(x_n + X_n - \phi(\tilde{w}^*_n;\gamma_n)) \tag{2.11}$$

The online throughput maximization problem can be reformulated by the following cost minimization problem:

$$J^*_{n|\theta_n}(x_n) = \min_{w_n \in A(x_n)} J_{n|\theta_n}(w_n, x_n) \tag{2.12}$$

where

$$J_{n|\theta_n}(w_n, x_n) = E_{\psi_n}[\tilde{V}^*_{n|\theta_n}(x_n) \mid x_n, \theta_n] - \hat{V}_{n|\theta_n}(w_n, x_n) \tag{2.13}$$

The cost function $J_{n|\theta_n}(w_n, x_n)$ can be separated into two parts:

- The expected throughput achieved by applying offline water levels for slots $[n, N]$ minus the expected throughput achieved by applying the decision w_n at the slot n, then applying offline optimal water levels for the rest, i.e. in $[n+1, N]$. Let $E_{\psi_n}[\tilde{F}_n(\tilde{w}^*_n, w_n) \mid x_n, \theta_n]$ represent this term.
- The expected throughput achieved by applying offline water levels for slots $[n+1, N]$ minus the expected total throughput achieved by online optimal decision for slots $[n+1, N]$ after w_n is applied at the slot n. Let $D^*_{n+1|\theta_{n+1}}(x_n+X_n-\phi(w_n;\gamma_n))$ represent this term. Then, we have:

$$J_{n|\theta_n}(w_n, x_n) = E_{\psi_n}[\tilde{F}_n(\tilde{w}^*_n, w_n) \mid x_n, \theta_n] + D^*_{n+1|\theta_{n+1}}(x_n + X_n - \phi(w_n;\gamma_n)). \tag{2.14}$$

Clearly, both of the terms are non-negative for any w_n since, by definition, EE-TM-OFF schedules are superior to online throughput maximizing schedules for any given realization. The first term $E_{\psi_n}[\tilde{F}_n(\tilde{w}^*_n(x_n), w_n) \mid x_n, \theta_n]$ is the conditional expectation of the variable $\tilde{F}_n(\tilde{w}^*_n, w_n)$ as follows:

$$\begin{aligned} \tilde{F}_n(\tilde{w}^*_n, w_n) = \; & (\log_2(\tilde{w}^*_n\gamma_n))^+ - (\log_2(w_n\gamma_n))^+ + \tilde{V}^*_{n+1|\theta_{n+1}}(x_n + X_n - \phi(\tilde{w}^*_n;\gamma_n)) \\ & - \tilde{V}^*_{n+1|\theta_{n+1}}(x_n + X_n - \phi(w_n;\gamma_n)) \end{aligned} \tag{2.15}$$

The equation in (2.14) can be rewritten as in below:

$$J_{n|\theta_n}(w_n, x_n) = E_{\psi_n}[F_n(\tilde{w}^*_n, w_n) \mid x_n, \theta_n] + D^*_{n+1|\theta_{n+1}}(x_n + X_n - \phi(w_n;\gamma_n)), \tag{2.16}$$

where $F_n(\tilde{w}^*_n, w_n) = E_{\psi_n}[\tilde{F}_n(\tilde{w}^*_n, w_n) \mid \tilde{w}^*_n, x_n, \theta_n]$.

Accordingly, the function $F_n(\tilde{w}_n^*, w_n)$ can be seen as a loss term for the decision w_n since it corresponds to the throughput loss that cannot be recovered even if offline optimal decisions are applied in the rest of the time. The expectation of this loss term will be called as the *immediate loss* of the decision w_n as we define in below.

Definition 5 *Define* $E_{\psi_n}[F_n(\tilde{w}_n^*, w_n) \mid x_n, \theta_n]$ *as the* immediate loss *of the decision* w_n.

On the other hand, the second term $D^*_{n+1|\theta_{n+1}}(\cdot)$ can be expressed as :

$$D^*_{n+1|\theta_{n+1}}(x_{n+1}) = E_{X_n}[J^*_{n+1|\theta_{n+1}}(x_{n+1}) \mid x_n, \theta_n] \quad (2.17)$$

where $x_{n+1} = x_n + X_n - \phi(w_n; \gamma_n)$.

Therefore, the problem has the following dynamic programming formulation:

$$J^*_{n|\theta_n}(x_n) = \min_{w_n \in A(x_n)} E_{\psi_n}[F_n(\tilde{w}_n^*, w_n) \mid x_n, \theta_n] + E_{X_n}[J^*_{n+1|\theta_{n+1}}(x_n + X_n - \phi(w_n; \gamma_n)) | x_n, \theta_n]. \quad (2.18)$$

As this formulation is equivalent to the initial formulation in (2.9), its solution gives the online optimal policy. While the exact computation of this solution may also have exponential complexity, the formulation will lead us to define the *IF* metric which will be a vehicle toward the derivation of online solutions with performance guarantees.

2.5.2 Immediate Fill

The performance of any online policy w can be also evaluated by the ratio of its expected total throughput to the expected total throughput of the offline optimal policies.

Definition 6 *Define the* online-offline efficiency *of an online policy w as follows:*

$$\eta^w(x_n, \theta_n) = \frac{\hat{V}^w_{n|\theta_n}(x_n)}{E_{\psi_n}[\tilde{V}^*_{n|\theta_n}(x_n) \mid x_n, \theta_n]} \quad (2.19)$$

where $\hat{V}^w_{n|\theta_n}(x_n)$ *is the expected total throughput achieved by the online policy w given the present state* x_n *and the history* θ_n.

Any decision in the online schedule will incur an immediate throughput gain, let's call this immediate gain.

Definition 7 *Define the ratio of immediate gain to its sum with immediate loss as* Immediate Fill (IF). *For slot n, let $\mu_n^w(x_n, \theta_n)$ be the IF of policy w when the system is in state x_n with history θ_n.*

$$\mu_n^w(x_n, \theta_n) = \frac{(\log_2(w_n\gamma_n))^+}{(\log_2(w_n\gamma_n))^+ + E_{\psi_n}[F_n(\tilde{w}_n^*, w_n) \mid x_n, \theta_n]} \tag{2.20}$$

We will show that the minimal IF of the policy w lower bounds its online-offline efficiency.

Fig. 2.5 is an illustration of the IF approach in relation with the expectation of the achievable total throughput.

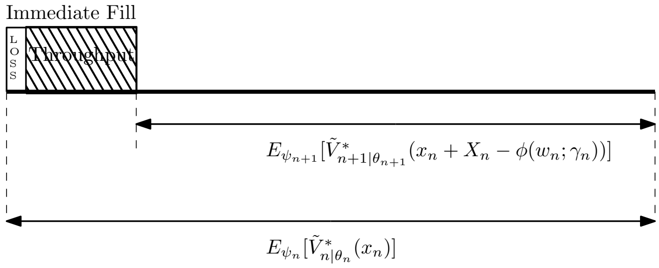


Figure 2.5: The expectation of the achievable total throughput by offline optimal decisions decreases as the state of the system changes due to an online decision. Each online decision incurs a gap from the expected throughput potential of the offline optimal policy and this gap is partially filled by the throughput gain achieved within the corresponding slot.

Theorem 2 *The efficiency of an online policy w with $w_N = \tilde{w}_N^*$ is lower bounded by the minimum IF observed by that policy:*

$$\eta^w(x_n, \theta_n) \geq \min_{m \geq n} \min_{(x_m, \theta_m)} \mu_m^w(x_m, \theta_m) \tag{2.21}$$

Theorem 2 provides an average performance lower bound for any policy considering a possible state and slot index at which the IF of the policy is worst. For the throughput value at any slot n, the state x_n can be considered to a "bad" state if e_n, b_n and γ_n have low values or a "good" state if they have high values. On the other hand, there is no obvious choice of a "bad" state for the IF (The states where any of e_n, b_n or γ_n is zero can be assumed to be perfectly "good" states for the IF as the IF is always 1 in those cases.) value. In many cases, rather the state it is the statistics of energy harvesting, packet arrival and channel fading processes are forcing the IF to be close to zero.

The approach could be useful in either of the following ways:

- To derive a lower bound for the online-offline performance gap of a given online policy. By Theorem 2, if an online policy guarantees a minimum value on the IF for all reachable states, then this minimum value bounds the online-offline performance gap of the policy.
- To design an online policy based on the optimization of the IF or a simpler metric which is guaranteed to be smaller. Such a policy could also use an approximation of the immedite fill based on Monte Carlo methods.

We will not display the latter use of the IF however, considering a special case, we will show that the performance gap estimated by this approach can be reasonably small.

The next section considers stochastic offline optimal decisions in a simpler case, namely the static channel case, in order to demonstrate how simple bounds on IF can be found and the distribution of offline optimal decisions can be characterized.

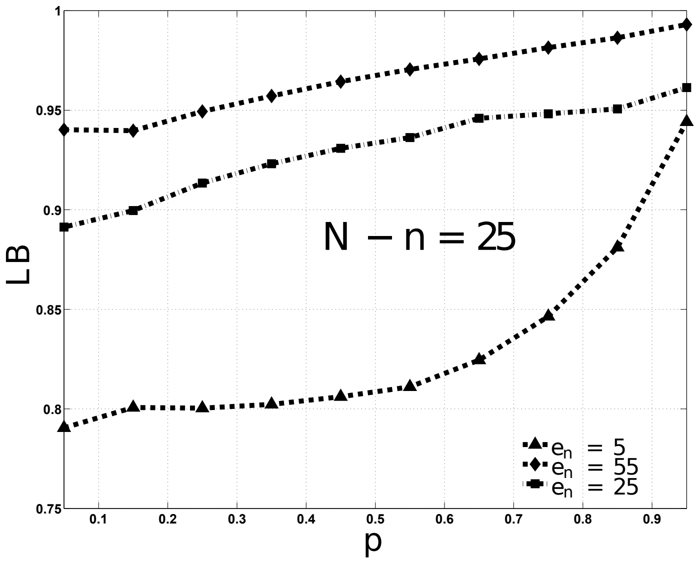
2.5.3 Results on the Static Channel Case

In this section, we focus on the case where the channel is static, i.e. $\gamma_n = 1$ for all n, and the data buffer is always full, i.e. $b_n = \infty$ for all n. Accordingly, the online power level and the offline optimal power level can be represented by $\rho_n = w_n - 1$ and $\tilde{\rho}_n^* = \tilde{w}_n^* - 1$. Then, the offline optimal power level at slot n can be expressed as:

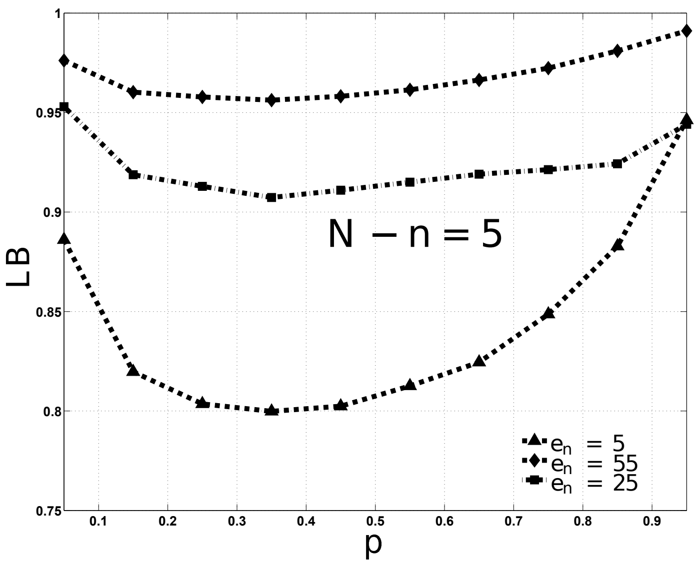
$$\tilde{\rho}_n^* = \min_{u=0,\dots,(N-n)} \frac{e_n + \sum_{l=n+1}^{n+u} H_l}{u+1} \tag{2.22}$$

Proposition 3 *Assuming that the channel is static i.e., $\gamma_n = 1$ for all n, and the data buffer is always full i.e., $b_n = \infty$ for all n, the IF is lower bounded as follows:*

$$\mu_n^w(x_n, \theta_n) \geq \frac{\ln(1+\rho_n)}{E\left[\ln(1+\tilde{\rho}_n^*)\right] + E\left[\frac{(\rho_n - \tilde{\rho}_n^*)^+}{1+\tilde{\rho}_{n+1}^{\triangleright}}\right]}$$



(a)



(b)

Figure 2.6: The lower bounds of the IF metric at (a) $N - n = 25$ (b) $N - n = 5$ for $\rho_n = E[\tilde{\rho}_n^*]$ policy where $\{H_n\}$ is a Bernoulli process with $\Pr(H_n = 0) = 1 - p$ and $\Pr(H_n = 24 \text{ units}) = p$.

where $\tilde{\rho}^{\triangleright}_{n+1}$ *is the offline optimal decision at slot* $n+1$ *after the decision* ρ_n *is made.*

Proposition 4 *Let* $\mu_n^{\check{w}}(x_n, \theta_n)$ *represent the maximum (achievable) IF at slot n, i.e,* $\mu_n^{\check{w}}(x_n, \theta_n) = \max_{w_n \in A(x_n)} \mu_n^{w}(x_n, \theta_n)$. *Then, the inequality below should hold:*

$$\mu_m^{\check{w}}(x_m, \theta_m) \geq \frac{1}{1 + \frac{E\left[\frac{(E[\tilde{\rho}_n^*] - \tilde{\rho}_n^*)^+}{1+\tilde{\rho}^{\triangleright}_{n+1}}\right]}{\ln(1+E[\tilde{\rho}_n^*])}}, (LB)$$

and it can be simplified as in the following:

$$\mu_m^{\check{w}}(x_m, \theta_m) \geq \frac{1}{1 + \frac{E\left[(E[\tilde{\rho}_n^*] - \tilde{\rho}_n^*)^+\right]}{\ln(1+E[\tilde{\rho}_n^*])}}$$

which implies $\mu_m^{\check{w}}(x_m, \theta_m) \geq \frac{1}{1+\frac{\sqrt{Var(\tilde{\rho}_n^*)}}{\ln(1+E[\tilde{\rho}_n^*])}}$.

In Fig. 2.6, the lower bound *LB* in Proposition 4 is plotted against varying arrival probabilities of a Bernoulli energy harvesting process [9] at different system states of energy level e_n and remaining number of slots $N-n$.

2.5.4 Online Heuristic

The online problem formulation in the previous sections assumes statistical information on exogeneous processes energy harvesting, packet arrival and channel fading. Then, the offline optimal decisions take these processes as their inputs in (2.5) and (2.6).

Alternatively, a heuristic policy could use (2.5) and (2.6) with estimated values of $\sum_{l=n+1}^{n+u} H_l$, $\sum_{l=n+1}^{n+u} B_l$, $\sum_{l=n}^{n+u} M_l^{(e)}(w_n)$ and $\sum_{l=n}^{n+u} M_l^{(b)}(w_n)$. We propose such a policy where $\sum_{l=n+1}^{n+u} H_l$, $\sum_{l=n+1}^{n+u} B_l$, $\sum_{l=n}^{n+u} M_l^{(e)}(w_n)$ and $\sum_{l=n}^{n+u} M_l^{(b)}(w_n)$ are estimated through observed time averages giving the estimated values of w_n^e and w_n^b as follows:

$$\hat{w}_n^e = \begin{cases} \frac{e_n - \bar{H}_n}{(N-n)} + \bar{H}_n + \bar{M}_n^{(e)}(w_n) & ; e_n \geq \bar{H}_n \\ e_n + \bar{M}_n^{(e)}(w_n) & ; \text{o.w.} \end{cases} \tag{2.23}$$

[9] Note that the lower bound *LB* is also valid for non-iid energy arrival processess however meaningful only with an assumption on arrival statistics. In case, the distribution of energy arrival is difficult to be known as in [24], one may take a distribution according to the principle of maximum entropy for example.

$$\log_2(\hat{w}_n^b) = \begin{cases} \frac{2(b_n-\bar{B}_n)}{(N-n)} + \bar{B}_n + \bar{M}_n^{(b)}(w_n) & ; b_n \geq \bar{B}_n \\ 2b_n + \bar{M}_n^{(b)}(w_n) & ; \text{o.w.} \end{cases} \quad (2.24)$$

where

$$\bar{H}_n = \frac{1}{n}\sum_{l=1}^{n} H_l, \bar{B}_n = \frac{1}{n}\sum_{l=1}^{n} B_l,$$

$$\bar{M}_n^{(e)}(w_n) = \frac{1}{n}\sum_{l=1}^{n} M_l^{(e)}(w_n), \bar{M}_n^{(b)}(w_n) = \frac{1}{n}\sum_{l=1}^{n} M_l^{(b)}(w_n).$$

The estimate of the throughput maximizing water level can be computed iteratively:

$$\hat{w}_n^{(k+1)} = |_{w_n=\hat{w}_n^{(k)}} \min\left\{\hat{w}_n^e, \hat{w}_n^b\right\}$$

where $\hat{w}_n^{(k)}$ is the kth iteration of the estimated value of throughput maximizing water level and $\hat{w}_n^{(1)} = \min\left\{e_n, 2^{2b_n}\right\}$.

2.6 Numerical Study of the Online vs Offline Policies

The purpose of the numerical study is to compare the online heuristic proposed in Section 3.4 with the EE-TM-OFF policy, under Markovian arrival processes. For the packet arrival process, a Markov model having two states as no packet arrival state and a packet arrival of constant size 10 KB per slot state with transition probabilities $q_{00} = 0.9$, $q_{01} = 0.1$, $q_{10} = 0.58$, $q_{11} = 0.42$ where slot duration is 1ms and the transmission window is $N = 100$ slots. Gilbert-Elliot channel is assumed where good ($\gamma^{good} = 30$) and bad ($\gamma^{bad} = 12$) states appear with equal probabilities i.e., $P(\gamma_n = \gamma^{good}) = P(\gamma_n = \gamma^{bad}) = 0.5$. Similarly, in energy harvesting process, energy harvests of 50nJs are assumed to occur with a probability of 0.5 at each slot.

For a typical sample realization of packet arrival, energy harvesting and channel fading processes, water level profiles of throughput maximizing offline optimal policy and online heuristic policy are shown in Fig. 2.7 (a) and (b). Fig. 2.7 (a) shows water level profiles when transmission window size N is set to 100 slots and Fig. 2.7 (b) shows water level profiles when transmission window size is extended to 200 slots. In the first 100 slot, water level profiles are similar to each other though, due to the relaxation of the deadline constraint, both optimal and heuristic water levels sligthly decrease when transmission window size is doubled.

To illustrate the effect of transmission window size, average throughput performances and energy consumption of throughput maximizing offline optimal policy and online heuristic are compared against varying transmission window size in Fig. 2.8 (a) and (b), respectively. The average performances of both offline optimal policy and online heuristic tend to saturate as transmission window size increases beyond 100 slots. The experiment is repeated in Fig. 2.9, for the case where energy harvesting process has a memory remaning in the same state with 0.9 probability and switching to other state with probability 0.1.

In Fig. 2.10 for (a) Gilbert-Elliot and (b) Rayleigh fading channels, the online heuristic is compared with the “Power-Halving" (PH) policy in [25]. The PH policy basically operates as follows: in each slot except the last one, it keeps half the stored energy in the battery, and uses the other half. It has been shown in [25], the average throughput performance of the PH policy can reach $80\% - 90\%$ of average throughput of offline optimal policy. On the other hand, the online heuristic uses casual information on energy-data arrivals and channels states to achieve average throughput rate much closer to offline optimal average throughput rates.

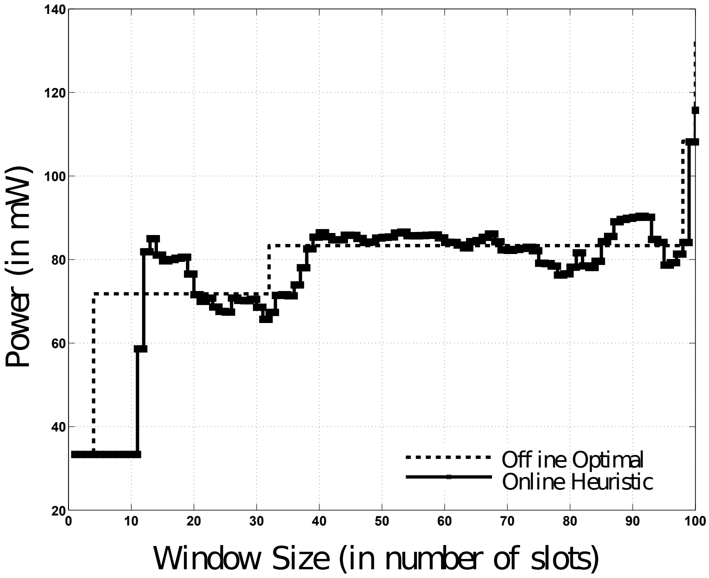
2.7 Appendix

2.7.1 The Proof of Theorem 1

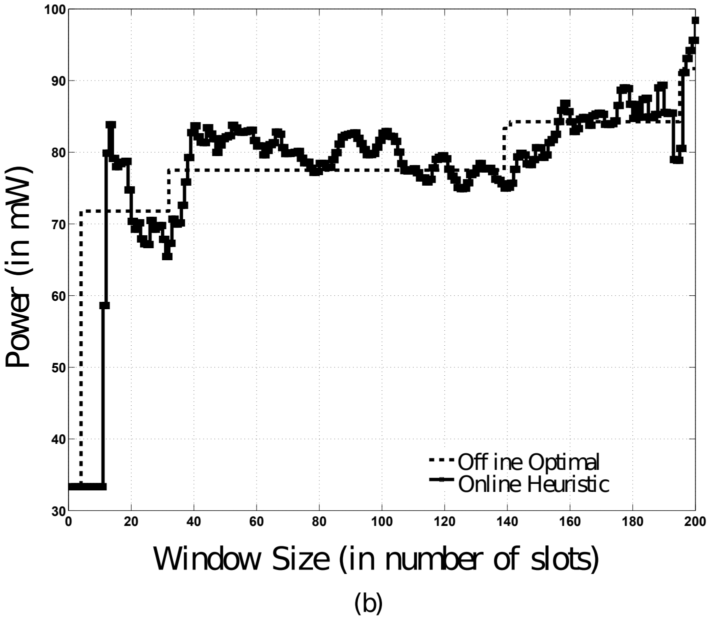
Proof. We divide the proof of Theorem 1 into two parts:

(i) We show that if the water level of any slot n is higher than the water level of the next slot $n + 1$ ($w_n > w_{n+1}$), then, there is an offline transmission schedule which achieves at least the same throughput or consumes at the most the same amount of energy with the initial schedule i.e., the initial schedule with $w_n > w_{n+1}$ for some slot n is not an EE-OFF schedule.

(ii) We show that in the offline optimal (EE-TM-OFF) policy, w_n is not lower than the maximum feasible level incurred by the inequalities resulting from the argument of part (i) i.e., $w_n = \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$ should be satisfied for any slot n in an EE-TM-OFF policy.

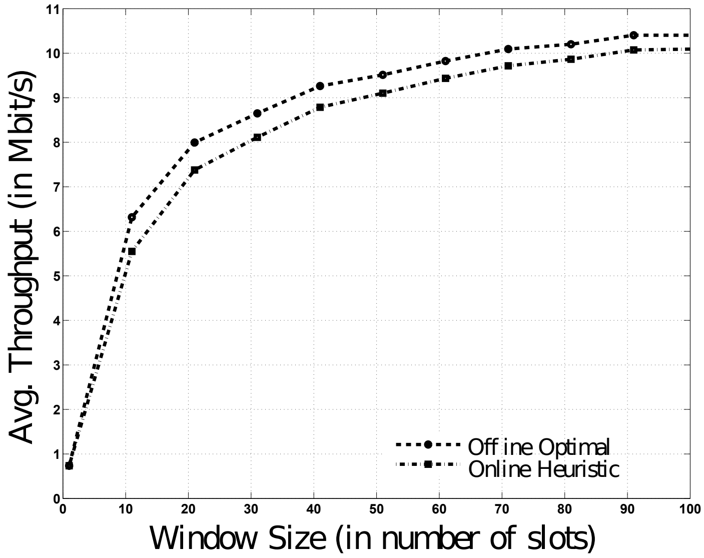


(a)

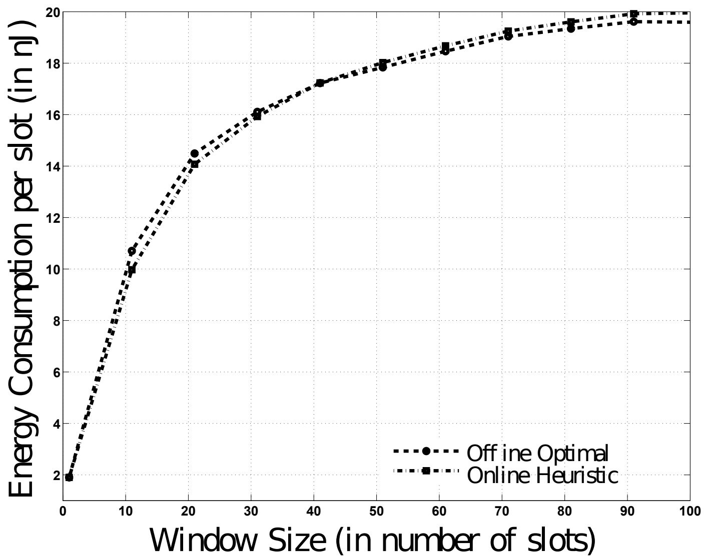


(b)

Figure 2.7: Water level profiles of throughput maximizing offline optimal policy and online heuristic policy for a sample realization of packet arrival, energy harvesting and channel fading processes when $N = 100$ (a) and $N = 200$ (b).



(a)



(b)

Figure 2.8: Average throughput (a) and energy consumption per slot (b) comparison of throughput maximizing offline optimal policy and online heuristic policy against varying transmission window size for stationary energy harvesting.

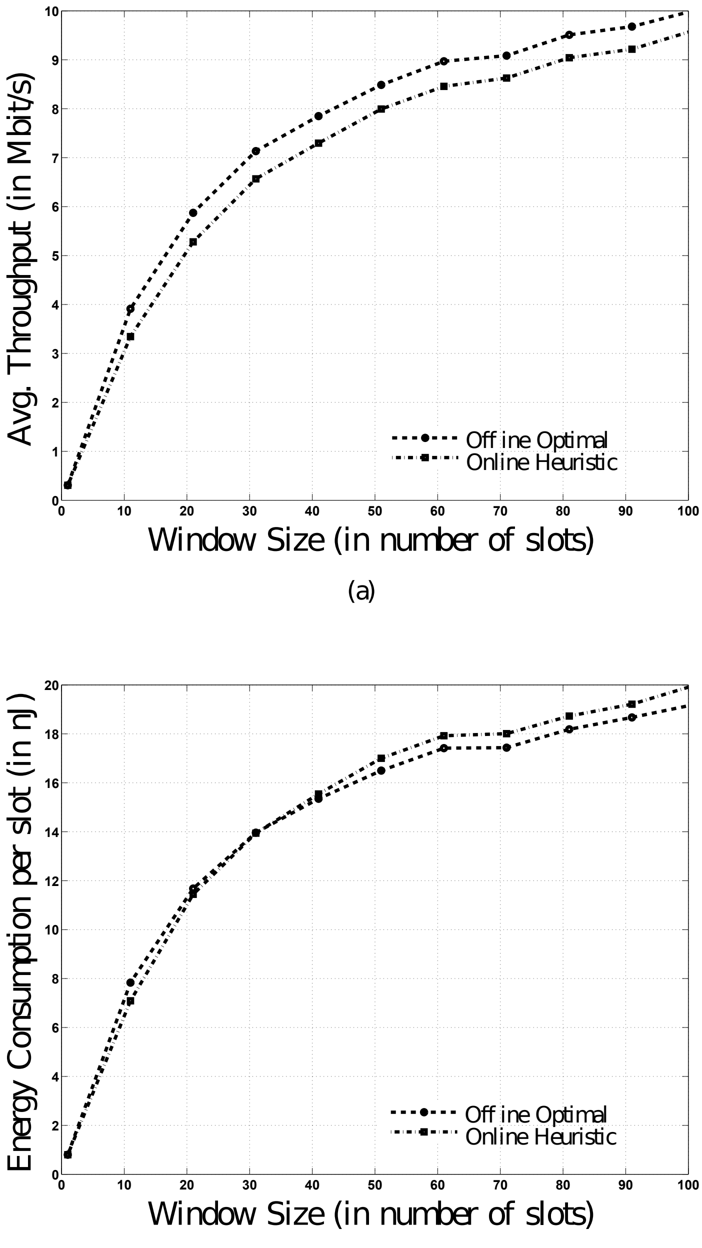
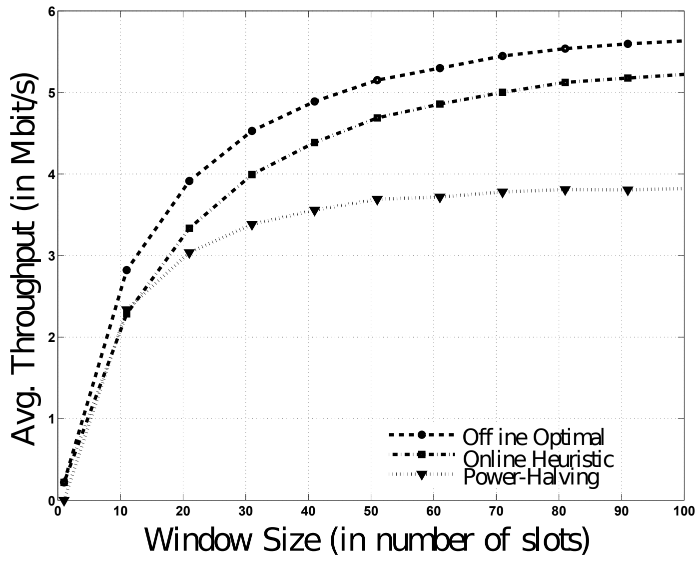
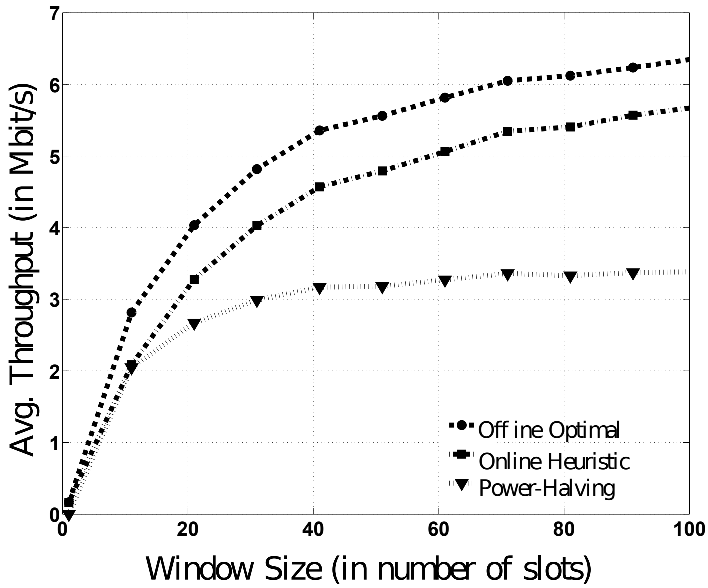


Figure 2.9: Average throughput (a) and energy consumption per slot (b) comparison of throughput maximizing offline optimal policy and online heuristic policy against varying transmission window size for energy harvesting with memory.



(a)



(b)

Figure 2.10: Average throughput comparison of throughput maximizing offline optimal policy and online heuristic policy and Power-Halving policy against varying transmission window size for stationary energy harvesting of 90nJs occuring with 0.1 probability in each time slot under (a) Gilbert-Elliot channel where good ($\gamma^{good} = 30$) and bad ($\gamma^{bad} = 12$) states appear with equal probabilities and (b) Rayleigh fading with average channel gain 20.

Part (i): Suppose that in a given transmission scheme π, $w_n > w_{n+1}$ for some n. π can be improved by reducing w_n and increasing w_{n+1} through one of the following: (Case a) move some data form slot n to slot $n+1$ while keeping the total throughput achieved during $(n, n+1)$ fixed, (Case b) move some energy from slot n to slot $n+1$ while keeping the total energy consumed during $(n, n+1)$ fixed. Let ρ_n^π and ρ_{n+1}^π be the transmission power levels for slots $(n, n+1)$ belonging to the scheme π.

(Case a): Consider the following convex optimization problem for slots $(n, n+1)$:

$$\begin{aligned} \min_{\rho_n, \rho_{n+1}} \ & \rho_n + \rho_{n+1} \\ \text{s.t.} \quad & f(1+\rho_n\gamma_n) + f(1+\rho_{n+1}\gamma_{n+1}) = D_{n,n+1}, \\ & \rho_n \geq 0, \rho_{n+1} \geq 0. \end{aligned} \tag{2.25}$$

where $D_{n,n+1}$ corresponds to the total throughput obtained by the scheme π during $(n, n+1)$ i.e., $f(1+\rho_n^\pi\gamma_n) + f(1+\rho_{n+1}^\pi\gamma_{n+1})$. The Lagrangian of the problem in (2.25) can be written as:

$$\mathcal{L}(\rho_n,\ \rho_{n+1}, \lambda, \mu_n, \mu_{n+1}) = -(\rho_n + \rho_{n+1}) + \lambda(f(1+\rho_n\gamma_n) + f(1+\rho_{n+1}\gamma_{n+1}) - D_{n,n+1})$$

$$-\mu_n\rho_n - \mu_{n+1}\rho_{n+1},$$

which yields $\gamma_n f'(1+\rho_n\gamma_n) = \frac{\mu_n+1}{\lambda}$ by setting $\dfrac{\partial \mathcal{L}}{\partial \rho_n} = 0$. Also, considering the complementary slackness for μ_n, μ_n should be set to zero whenever $\rho_n \geq 0$. Therefore, the optimal solution ρ_n^* can be expressed as $\rho_n^* = \frac{1}{\gamma_n}\left[(f')^{-1}(\frac{1}{\lambda\gamma_n}) - 1\right]^+$. Similarly, the optimal ρ_{n+1}^* satisfies $\rho_{n+1}^* = \frac{1}{\gamma_{n+1}}\left[(f')^{-1}(\frac{1}{\lambda\gamma_{n+1}}) - 1\right]^+$. Accordingly, $(\rho_n + \rho_{n+1})$ is minimized when both water levels w_n and w_{n+1} are set to λ that satisfies the total throughput constraint.

When $w_n > w_{n+1}$, the optimal water level should be inside (w_n, w_{n+1}) as the total throughput strictly decreasing with decreasing w_n as long as $\rho_n > 0$. Therefore, the water levels w_n and w_{n+1} can always be equalized by transferring some data from slot n to $n+1$. This does not violate data causality as the throughput at slot n is reduced while the total throughput achieved during $(n, n+1)$ is preserved by increasing the throughput at slot $n+1$ to compensate.

(Case b): Similarly, we consider the following optimization problem:

$$\max_{\rho_n, \rho_{n+1}} f(1+\rho_n\gamma_n) + f(1+\rho_{n+1}\gamma_{n+1})$$
$$\text{s.t.} \quad \rho_n + \rho_{n+1} = E_{n,n+1},$$
$$\rho_n \geq 0, \rho_{n+1} \geq 0. \tag{2.26}$$

where $E_{n,n+1}$ corresponds to the total energy consumption by the scheme π during $(n, n+1)$ i.e., $E_{n,n+1} = \rho_n^\pi + \rho_{n+1}^\pi$.

The Lagrangian of the problem in (2.26) can be written as follows:

$$\mathcal{L}(\rho_n, \rho_{n+1}, \lambda, \mu_n, \mu_{n+1}) = f(1+\rho_n\gamma_n) + f(1+\rho_{n+1}\gamma_{n+1}) + \lambda((\rho_n + \rho_{n+1}) - E_{n,n+1})$$

$$-\mu_n\rho_n - \mu_{n+1}\rho_{n+1},$$

which yields $\gamma_n f'(1+\rho_n\gamma_n) = \mu_n - \lambda$ by setting $\dfrac{\partial\mathcal{L}}{\partial\rho_n} = 0$. After setting the KKT multiplier μ_n to zero where $\rho_n \geq 0$, we get $\rho_n^* = \frac{1}{\gamma_n}\left[(f')^{-1}(\frac{\lambda}{\gamma_n}) - 1\right]^+$ for ρ_n^*. Similarly, for optimizing ρ_{n+1} setting $\dfrac{\partial\mathcal{L}}{\partial\rho_{n+1}} = 0$ gives $\rho_{n+1}^* = \frac{1}{\gamma_{n+1}}\left[(f')^{-1}(\frac{\lambda}{\gamma_{n+1}}) - 1\right]^+$ for ρ_{n+1}^*.

When both water levels w_n and w_{n+1} are equalized to $\frac{1}{\lambda}$ that satisfies the total energy constraint, the total throughput achieved during the slots $(n, n+1)$ is maximized and this can be done whenever $w_n > w_{n+1}$ by transferring energy from n and $n+1$ without violating energy causality or total energy constraints. Therefore in an EE-OFF schedule, w_ns are non-decreasing with increasing n.

Part (ii): By the energy causality, total energy consumption is bounded as follows:

$$\sum_{l=n}^{n+u} \rho_l \leq e_n + \sum_{l=n+1}^{n+u} H_l, u = 1, 2, ..., (N-n).$$

Expressing ρ_l using water levels, we get the following from energy causality constraints:

$$\sum_{l=n}^{n+u} \frac{1}{\gamma_l}\left[(f')^{-1}(\frac{1}{w_l\gamma_l}) - 1\right]^+ \leq e_n + \sum_{l=n+1}^{n+u} H_l. \tag{2.27}$$

In an EE-OFF schedule, $w_n \leq w_m$ for any slot $m > n$ as it is proven in Part (i), thus we have:

$$\sum_{l=n}^{n+u} \frac{1}{\gamma_l}\left[(f')^{-1}(\frac{1}{w_n\gamma_l}) - 1\right]^+ \leq \sum_{l=n}^{n+u} \frac{1}{\gamma_l}\left[(f')^{-1}(\frac{1}{w_l\gamma_l}) - 1\right]^+. \tag{2.28}$$

Combining (2.27) and (2.28) gives the following:

$$\sum_{l=n}^{n+u} \frac{1}{\gamma_l} \left[(f')^{-1}(\frac{1}{w_n \gamma_l}) - 1 \right]^+ \leq e_n + \sum_{l=n+1}^{n+u} H_l \tag{2.29}$$

The above inequality should be satisfied for any $u = 1, 2, ..., (N-n)$ and it can be seen that w_n is bounded by its lowest value for which the inequality holds with equality for some $u = 1, 2, ..., (N-n)$. To find the energy bound value for w_n, the inequality can be transformed into the following form:

$$w_n \leq \frac{e_n + \sum_{l=n+1}^{n+u} H_l + \sum_{l=n}^{n+u} K_l^{(e)}(w_n)}{u+1}.$$

The maximum value of w_n that satisfies the energy causality is given by the following:

$$w_n^{(e)}(w_n) = \min_{u=0,...,(N-n)} \frac{e_n + \sum_{l=n+1}^{n+u} H_l + \sum_{l=n}^{n+u} K_l^{(e)}(w_n)}{u+1}.$$

Similarly, the data causality bounds the water level w_n as follows:

$$f\left(1 + \left[(f')^{-1}(\frac{1}{w_n^{(b)}(w_n)\gamma_n}) - 1 \right]^+ \right) = \min_{v=0,...,(N-n)} \frac{b_n + \sum_{l=n+1}^{n+v} B_l + \sum_{l=n}^{n+v} K_l^{(b)}(w_n)}{v+1}.$$

Any EE-TM-OFF schedule is EE-OFF by definition, hence $w_n \leq \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$ for any EE-TM-OFF schedule. We will show that, in EE-TM-OFF schedule, w_n should not be smaller than $\min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$ i.e., $w_n \geq \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$.

Consider an EE-OFF schedule where $w_n = \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$ for slot n and $w_m \leq \min\{w_m^{(e)}(w_m), w_m^{(b)}(w_m)\}$ for slots $m > n$ since the schedule is EE-OFF. The selection of w_n only affects the throughput achieved during the slots n to N, hence if the reselection of w_n as $w_n < \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$ could improve the throughput achieved by the schedule within $[n, N]$ while keeping EE-OFF property, then the modified schedule could be EE-TM-OFF. This is not possible due to the observation in Remark 1. When $w_n = \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$, to improve the total throughput achieved in later slots $n+1, n+2, ..., N$, some energy/data can be moved from n to later slots, however the throughput decrease in slot n would be larger than the possible increase in some later slot $m > n$ as the derivative of the throughput with respect to power level (Remark 1) decreases with increasing water level and $w_m \geq w_n$ in an EE-OFF policy. Hence, selecting the water level as $w_n = \min\{w_n^{(e)}(w_n), w_n^{(b)}(w_n)\}$ always maximizes the total throughput as long as $w_m \geq w_n$ for $m > n$ which means all of w_ms after n should be also selected as $w_m = \min\{w_m^{(e)}(w_m), w_m^{(b)}(w_m)\}$. ■

2.7.2 The Proof of Theorem 2

Proof. Consider the inequality for $n = N$:

$$\eta^w(x_N, \theta_N) \geq \min_{m \geq N} \min_{(x_m, \theta_m)} \mu_m^w(x_m, \theta_m) \tag{2.30}$$

which means, $\eta^w(x_N, \theta_N) \geq \min_{(x_N, \theta_N)} \mu_N^w(x_N, \theta_N)$ The inequality (2.30) always holds as the offline optimal water level of the last slot $\tilde{w}_N^*$ is deterministic given x_N implying that $\eta^w(x_N, \theta_N)$ and $\mu_N^w(x_N, \theta_N)$ are both equal to 1 if $w_N = \tilde{w}_N^*$ for any x_N and θ_N.

Now, consider the following inequality:

$$\eta^w(x_{n+1}, \theta_{n+1}) \geq \min_{m \geq n+1} \min_{(x_m, \theta_m)} \mu_m^w(x_m, \theta_m) \tag{2.31}$$

We will show that the inequality (2.31) implies the inequality (2.21). The efficiency of the online policy w can be expressed as follows:

$$\begin{aligned}
\eta^w(x_n, \theta_n) &= \frac{(\log_2(w_n\gamma_n))^+ + E_{X_n}[\hat{V}^*_{n+1|\theta_{n+1}}(x_{n+1}) \mid x_n, \theta_n]}{(\log_2(w_n\gamma_n))^+ + E_{\psi_n}[F_n(\tilde{w}_n^*, w_n) + \tilde{V}^*_{n+1|\theta_{n+1}}(x_{n+1}) | x_n, \theta_n]} \\
&\geq \min\left\{\mu_n^w(x_n, \theta_n), \frac{E_{X_n}[\hat{V}^*_{n+1|\theta_{n+1}}(x_{n+1}) \mid x_n, \theta_n]}{E_{\psi_n}[\tilde{V}^*_{n+1|\theta_{n+1}}(x_{n+1}) \mid x_n, \theta_n]}\right\} \\
&\geq \min\left\{\mu_n^w(x_n, \theta_n), \min_{(x_{n+1}, \theta_{n+1})} \eta^w(x_{n+1}, \theta_{n+1})\right\} \\
&= \min_{m \geq n} \min_{(x_m, \theta_m)} \mu_m^w(x_m, \theta_m)
\end{aligned} \tag{2.32}$$

Similarly, by the backward induction, the inequality (2.30) implies the inequality (2.21). ■

2.7.3 The Proof of Proposition 3

Proof. To obtain the lower bound in Proposotion 3 for the IF of the decision $\rho_n = w_n - 1$, we first consider the immediate loss term $E_{\psi_m}[F_m(\tilde{w}_m^*, w_m) \mid x_m, \theta_m]$ which is basically the expected throughput difference between the schedules $(\rho_n, \tilde{\rho}^{\triangleright}_{n+1}, ..., \tilde{\rho}^{\triangleright}_N)$ and $(\tilde{\rho}^*_n, \tilde{\rho}^*_{n+1}, ..., \tilde{\rho}^*_N)$ where $\tilde{\rho}^{\triangleright}_{n+1}, ..., \tilde{\rho}^{\triangleright}_N$ are offline optimal power levels following ρ_n. The immediate loss is the expectation of the throughput difference in below:

$$\log_2(1 + \tilde{\rho}^*_n) - \log_2(1 + \rho_n) + \xi(\rho_n), \tag{2.33}$$

where $\xi(\rho_n) = \sum_{k=n+1}^{N} \log_2(1 + \tilde{\rho}^*_k) - \sum_{k=n+1}^{N} \log_2(1 + \tilde{\rho}^{\triangleright}_k)$.

Then, we can upper bound the term $\xi(\rho_n)$ as follows:

$$\begin{aligned} \xi(\rho_n) &= \sum_{k=n+1}^{N} \log_2\left(1 + \frac{\tilde{\rho}^*_k - \tilde{\rho}^{\triangleright}_k}{1 + \tilde{\rho}^*_k - (\tilde{\rho}^*_k - \tilde{\rho}^{\triangleright}_k)}\right) \\ &\le \max_{\Delta \in \mathbf{S}(\rho_n)} \sum_{k=n+1}^{N} \log_2\left(1 + \frac{\Delta_k}{1 + \tilde{\rho}^*_k - \Delta_k}\right) \end{aligned} \tag{2.34}$$

where Δ is the vector $[\Delta_{n+1}, \Delta_{n+2}, ..., \Delta_N]$ and $\mathbf{S}(\rho_n)$ is the set of all Δ vectors for which Δ is a possible instance of the vector $[\tilde{\rho}^*_{n+1} - \tilde{\rho}^{\triangleright}_{n+1}, \tilde{\rho}^*_{n+2} - \tilde{\rho}^{\triangleright}_{n+2}, ..., \tilde{\rho}^*_N - \tilde{\rho}^{\triangleright}_N]$. We know the following facts for any Δ vector in the set $\mathbf{S}(\rho_n)$: If $\Delta \in \mathbf{S}(\rho_n)$, $0 \le \Delta \le [\tilde{\rho}^*_{n+1}, \tilde{\rho}^*_{n+1}, ..., \tilde{\rho}^*_N]$ and $||\Delta||_1 = (\rho_n - \tilde{\rho}^*_n)$ since the energy consumption of both $(\rho_n, \tilde{\rho}^{\triangleright}_{n+2}, ..., \tilde{\rho}^{\triangleright}_N)$ and $(\tilde{\rho}^*_n, \tilde{\rho}^*_{n+1}, ..., \tilde{\rho}^*_N)$ schedules should be equal. Now, consider the case $\rho_n < \tilde{\rho}^*_n$. Clearly, $\xi(\rho_n) < 0$ for this case since the offline optimal decisions $\tilde{\rho}^{\triangleright}_k$s have more energy to spend than the offline optimal decisions $\tilde{\rho}^*_k$s. Therefore, we can upper bound $\xi(\rho_n)$ considering the instances of $\tilde{\rho}^*_n$ where $\rho_n \ge \tilde{\rho}^*_n$:

$$\begin{aligned} \xi(\rho_n) &\le \max_{\substack{\Delta \in \mathbf{S}(\rho_n) \\ \tilde{\rho}^*_n \le \rho_n}} \sum_{k=n+1}^{N} \log_2\left(1 + \frac{\Delta_k}{1 + \tilde{\rho}^{\triangleright}_k}\right) \\ &\le \max_{\substack{\Delta \in \mathbf{S}(\rho_n) \\ \tilde{\rho}^*_n \le \rho_n}} \sum_{k=n+1}^{N} \log_2\left(1 + \frac{\Delta_k}{1 + \tilde{\rho}^{\triangleright}_{n+1}}\right) \\ &\le \max_{\substack{||\Delta||_1 = (\rho_n - \tilde{\rho}^*_n) \\ \tilde{\rho}^*_n \le \rho_n}} \sum_{k=n+1}^{N} \log_2\left(1 + \frac{\Delta_k}{1 + \tilde{\rho}^{\triangleright}_{n+1}}\right) \\ &= (N - n) \log_2\left(1 + \frac{\frac{(\rho_n - \tilde{\rho}^*_n)}{N-n}}{1 + \tilde{\rho}^{\triangleright}_{n+1}}\right) \\ &\le \sup_{\substack{N \in \mathbb{N}^+ \\ \tilde{\rho}^*_n \le \rho_n}} (N - n) \log_2\left(1 + \frac{\frac{(\rho_n - \tilde{\rho}^*_n)}{N-n}}{1 + \tilde{\rho}^{\triangleright}_{n+1}}\right) \\ &= \lim_{\substack{N \to +\infty \\ \tilde{\rho}^*_n \le \rho_n}} (N - n) \log_2\left(1 + \frac{\frac{(\rho_n - \tilde{\rho}^*_n)}{N-n}}{1 + \tilde{\rho}^{\triangleright}_{n+1}}\right) \end{aligned} \tag{2.35}$$

Therefore, $\xi(\rho_n)$ can be upper bounded as $\xi(\rho_n) \leq \ln(2)\frac{(\rho_n - \tilde{\rho}_n^*)^+}{1+\tilde{\rho}_{n+1}^{\triangleright}}$ since $\xi(\rho_n) < 0$ for $\rho_n < \tilde{\rho}_n^*$. Accordingly,

$$E_{\psi_m}[F_m(\tilde{w}_m^*, w_m) \mid x_m, \theta_m] \leq E[\log_2(1+\tilde{\rho}_n^*)] - \log_2(1+\rho_n) + \ln(2)E\left[\frac{(\rho_n - \tilde{\rho}_n^*)^+}{1+\tilde{\rho}_{n+1}^{\triangleright}}\right],$$

which implies:

$$\begin{aligned}\mu_n^w(x_n, \theta_n) &= \frac{\log_2(1+\rho_n)}{\log_2(1+\rho_n) + E_{\psi_m}[F_m(\tilde{w}_m^*, w_m) \mid x_m, \theta_m]} \\ &\geq \frac{\ln(1+\rho_n)}{E\left[\ln(1+\tilde{\rho}_n^*)\right] + E\left[\frac{(\rho_n - \tilde{\rho}_n^*)^+}{1+\tilde{\rho}_{n+1}^{\triangleright}}\right]}.\end{aligned} \tag{2.36}$$

■

2.7.4 The Proof of Proposition 4

Proof. The bound in Proposition 3 can be simplified as follows,

$$\mu_n^w(x_n, \theta_n) \geq \frac{\ln(1+\rho_n)}{E\left[\ln(1+\tilde{\rho}_n^*)\right] + E\left[(\rho_n - \tilde{\rho}_n^*)^+\right]}$$

When $\rho_n = E[\tilde{\rho}_n^*]$, $E\left[\ln(1+\tilde{\rho}_n^*)\right] \leq \ln(1+E[\tilde{\rho}_n^*])$ due to Jensen's inequality. Therefore,

$$\mu_m^{\check{w}}(x_m, \theta_m) \geq \frac{1}{1 + \frac{E\left[(E[\tilde{\rho}_n^*] - \tilde{\rho}_n^*)^+\right]}{\ln(1+E[\tilde{\rho}_n^*])}} \geq \frac{1}{1 + \frac{E\left[\sqrt{(E[\tilde{\rho}_n^*] - \tilde{\rho}_n^*)^2}\right]}{\ln(1+E[\tilde{\rho}_n^*])}} \geq \frac{1}{1 + \frac{\sqrt{Var(\tilde{\rho}_n^*)}}{\ln(1+E[\tilde{\rho}_n^*])}},$$

where the last step used Jensen's inequality on $\sqrt{\cdot}$ function. ■

CHAPTER 3

ENERGY-EFFICIENT TRANSMISSION WITH WIRELESS ENERGY TRANSFER

3.1 Introduction

In this chapter, we will show an application of EH transmission optimization methods in Chapter 2 to communication systems with *wireless energy transfer* (WET). Mechanisms of WET, which involve the transmission of electrical energy without wires using time-varying electric, magnetic, or electromagnetic fields, have been demonstrated as viable options for various communication systems [35–37]. Such technologies are of particular interest in the case of sensing and data collection applications with a simple sensor that does not have a large battery or energy source, which can simply sample, encode and transmit data on demand to the receiver of the data using the RF energy sent to it by the receiver, as illustrated in the system scenario in Figure 3.1.

The transmission optimization in the wireless energy transfer scenario, where the receiver supplies energy to the transmitter, is distinguished from the general literature in the EH transmission optimization problem in the following aspect: when the energy transfer and data transfer are made in subsequent time intervals in the same frequency band as in time division duplexing, the energy sent from the RX to the TX, and the data transmission sent from the TX to the RX, may experience a correlated, and in many cases nearly equal channel gain. This phenomenon introduces an inherent correlation in the energy and data transfer decisions. Another aspect that distinguishes the WET transmission optimization problem from earlier formulations is the interesting question of where the optimization is carried out: if the transmitter is doing the

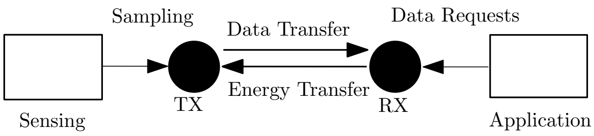


Figure 3.1: System Model.

optimization, it can determine how much data to transmit at a given time, by considering the energy it has received so far (e.g., energy stored in its battery). On the other hand, if the optimization is carried out on the receiver side, the receiver can also *optimize* the amount of energy it will inject to the transmitter. This means, equipped with the channel state information, and the knowledge of the amount of data to be pulled from the transmitter, the receiver may be able to solve the optimization problem in a larger solution space.

The motivation in this chapter is to address the differences in transmitter-centric, receiver-centric and distributed scheduling approaches regarding possible gains in terms of throughput and energy efficiency. First, we will solve transmitter-centric and receiver centric WET and transmission optimization problems in an offline setting. Secondly, we will propose online algorithms for both formulations that are based on computing estimates of the offline policy. Next, we will formulate a more general distributed problem where energy transfers are scheduled at the receiver, and data transmissions are scheduled at the transmitter. We will show that under static channels, distributed and receiver centric solutions are equivalent, and are potentially superior to transmitter centric solutions.

3.2 System Model

We consider a point to point channel consisting of a transmitter receiver pair. Variable rate external data requests are assumed to arrive intermittently at the receiver during the transmission. The receiver in turn forwards these requests to the transmitter, e.g., a sensor, to pull the data at the desired rate. The transmitter is powered solely by RF energy harvesting, and the energy required for its operation is assumed to be wirelessly transmitted by its designated receiver, also during the transmission. Let γ_n^{rt} denote the channel gain on the energy transfer channel from the receiver to the transmitter and γ_n^{tr} denote the channel gain on the data link channel from the transmitter to the receiver. To keep the model general, we do not assume a particular relation between γ_n^{rt} and γ_n^{tr}. However, if one further assumes that both the uplink data transmit channel and the downlink energy transfer channel suffer from frequency flat slow fading, it is possible to make use of channel reciprocity, i.e., set $\gamma_n^{rt} = \alpha\gamma_n^{tr}$, where $\alpha < 1$ captures the energy transfer/harvesting inefficiency. Such an assumption can be justified whether we assume that wireless energy transfer from the receiver takes place on a frequency band which is orthogonal, but close to the band used by the transmitter for data transmission (hence the loss due to slightly different frequency of operation is negligible), or we assume that it takes place on orthogonal consecutive time intervals. A model with channel reciprocity has the additional advantage that the energy transfer channel can be used to estimate the instantaneous channel state for the transmitter-receiver link.

The model is simplified in the sense that energy transfer by RF harvesting is assumed to be linear and transmission power solely reflects the energy consumption in the transmitter. The first assumption is practical when the RF harvester operates in a power regime where its harvesting efficiency does not depend on the received power. The second assumption can be justified as we consider the sensor is always ON during the problem interval and the power consumption for sensing is negligible compared to the power required for wireless transmission.

As in Chapter 2, we will use a slotted transmission model: the scheduling strategy remains constant for each T second long slot, and potentially changes at the beginning of the next slot. The coherence time of the channel is chosen to be large enough so that the channel state remains constant during the slot. The total transmission takes place for N slots. In each slot n, the receiver gets a data request of B_n bits. For technical reasons, particularly to facilitate comparison of various problem formulations, we will assume that the receiver (energy supplier) has a total of E_0 units of energy available to it in the beginning of the transmission window.

We consider two offline models, with varying scheduling capabilities at the transmitter and the receiver. The objective is to find energy efficient throughput maximizing scheduling policies. Energy efficiency is defined as in Chapter 2 (see Definition 2):

Definition 8 *A throughput optimal offline policy is* energy efficient *if there is no other*

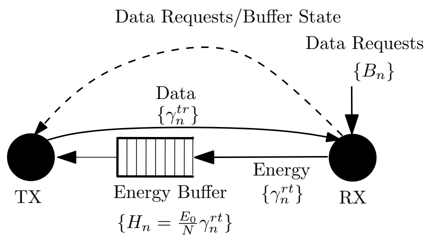


Figure 3.2: Transmitter-Centric System Model.

feasible offline schedule that achieves the same maximum throughput by consuming less energy for a given realization of energy arrival, fading and data request patterns.

Hence, the optimal policies that we are looking for are energy efficient, throughput maximizing offline policies, i.e. EE-TM-OFF policies as defined in Chapter 2. In the rest of this section, after stating the offline problem formulations and discussing their solutions, we build on these solutions to obtain good policies for their online counterparts.

3.2.1 Transmitter-Centric Scheduling

In the first model, called transmitter-centric scheduling, the receiver wirelessly transfers energy to the transmitter at a fixed rate, $Q_n = Q \triangleq E_0/N$. The received power at the transmitter side is attenuated by a factor proportional to the channel gain γ_n^{rt}, as shown in Figure 3.2. Accordingly, we assume that the transmitter harvests $H_n = Q\gamma_n^{rt}$ units of energy during slot $n \in \{1, \ldots, N\}$. The receiver also sends the size B_n of the data requests to the transmitter over a low rate feedback link, so that the transmitter has knowledge of B_ns. The transmitter samples the required amount of data and selects its optimal transmit powers, P_n, that provide an energy efficient solution (see

Definition 8) to the following offline problem[1]:

$$
\begin{aligned}
(\mathcal{P}_1): \quad \max_{P_n} \quad & \sum_{n=1}^{N} \frac{1}{2} \log_2 (1 + \gamma_n^{tr} P_n) \\
\text{s.t.} \quad & \sum_{\ell=1}^{n} P_\ell \leq \sum_{\ell=1}^{n} \frac{E_0}{N} \gamma_\ell^{rt}, \quad \forall n && (3.1) \\
& \sum_{\ell=1}^{n} \frac{1}{2} \log_2 (1 + \gamma_\ell^{tr} P_\ell) \leq \sum_{\ell=1}^{n} B_\ell, \quad \forall n && (3.2) \\
& 0 \leq P_n, \quad \forall n. && (3.3)
\end{aligned}
$$

3.2.2 Receiver-Centric Scheduling

In the second model, called receiver-centric scheduling and illustrated in Figure 3.3, the receiver determines the amount of energy transfers to its designated transmitter, jointly taking into account the backlogged data requests, its available energy, and the channel state. The transmitter does not make any decisions, it simply uses the energy it has received without storing it, to transmit the data that it samples on demand, at a rate dictated by the receiver. The receiver seeks an energy efficient solution which solves the following problem:

$$
\begin{aligned}
(\mathcal{P}_2): \max_{Q_n} \quad & \sum_{n=1}^{N} \frac{1}{2} \log_2 (1 + \gamma_n^{rt} \gamma_n^{tr} Q_n) \\
\text{s.t.} \quad & \sum_{n=1}^{N} Q_n \leq E_0, && (3.4) \\
& \sum_{\ell=1}^{n} \frac{1}{2} \log_2 (1 + \gamma_\ell^{rt} \gamma_\ell^{tr} Q_\ell) \leq \sum_{\ell=1}^{n} B_\ell, \ \forall n && (3.5) \\
& 0 \leq Q_n, \quad \forall n. && (3.6)
\end{aligned}
$$

As previously stated, maximization of the objective (total throughput) in the problems stated above is not sufficient for finding a solution, as among all throughput optimal schedules, only the ones that use minimum energy per bit are also energy efficient. For $\mathcal{P}_1$, energy efficiency is defined at the transmitter while for $\mathcal{P}_2$ (and in $\mathcal{P}_3$, to be proposed in Section 3.5) it is defined in the end-to-end sense. In the next section, we

[1] In general, there will be a solution set containing policies $\{P_l\}$ that achieve the same objective function value (total throughput). Among these there is at least one that minimizes the LHS of (3.15), i.e., that uses least energy, and that is the solution according to Definition 8.

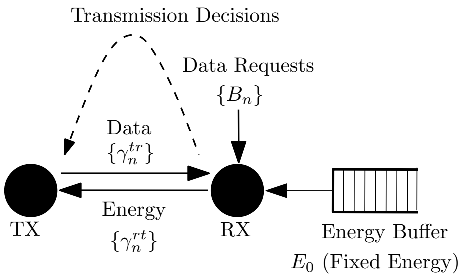


Figure 3.3: Receiver-Centric System Model.

characterize the EE-TM-OFF solutions for these problems. In Section 3.4, we will present online algorithms developed so as to mimic the offline solutions.

3.3 Optimal Offline Schedules

The main goal of this section is to provide offline solutions to problems $\mathcal{P}_1$ and $\mathcal{P}_2$, in a format which is amenable for online implementation. Therefore, it is desirable to express the offline solutions in terms of current energy, data and fading states of the system model of choice. To this end, we define the current energy state, e_n of the system for all $n \in \{1, \ldots, N\}$ through the following update equations for $n \geq 0$, for each system model $\mathcal{P} \in \{\mathcal{P}_1, \mathcal{P}_2\}$:

$$e_{n+1} = \begin{cases} e_n + \gamma_n^{rt} E_0/N - P_n, & e_0 = 0, \quad \mathcal{P} = \mathcal{P}_1; \\ e_n - Q_n, & e_0 = E_0, \quad \mathcal{P} = \mathcal{P}_2; \end{cases} \tag{3.7}$$

Note that, for $\mathcal{P}_1$, e_n denotes the energy state at the transmitter, which is initialized as $e_0 = 0$, whereas for $\mathcal{P}_2$, e_n is the energy state at the receiver, thus initialized at $e_0 = E_0$.

Similarly, the backlog on data requests at the beginning of slot n is denoted by b_n, $n \geq 0$, with $b_0 = 0$, and

$$b_{n+1} = \begin{cases} b_n + B_n - \frac{1}{2}\log_2\left(1 + \gamma_n^{tr} P_n\right), & \mathcal{P} = \mathcal{P}_1; \\ b_n + B_n - \frac{1}{2}\log_2\left(1 + \gamma_n^{rt}\gamma_n^{tr} Q_n\right), & \mathcal{P} = \mathcal{P}_2. \end{cases} \tag{3.8}$$

3.3.1 Optimal Offline Transmitter-Centric Schedule

In Chapter 2, the EE-TM-OFF schedule was shown to be solutions for the following problem, will refer to it as $\mathcal{P}_0$ here for convenience:

$$
\begin{aligned}
(\mathcal{P}_0): \quad & \max_{P_n} \quad \sum_{n=1}^{N} \frac{1}{2} \log_2 (1 + \gamma_n P_n) \\
& \text{s.t.} \quad \sum_{\ell=1}^{n} P_\ell \leq \sum_{\ell=1}^{n} H_l, \quad \forall n && (3.9) \\
& \qquad \sum_{\ell=1}^{n} \frac{1}{2} \log_2 (1 + \gamma_\ell P_\ell) \leq \sum_{\ell=1}^{n} B_\ell, \quad \forall n && (3.10) \\
& \qquad 0 \leq P_n, \quad \forall n. && (3.11)
\end{aligned}
$$

The solution of $\mathcal{P}_1$ rests on recognizing that with an appropriate choice of parameter values, the solution of $\mathcal{P}_0$ provides the solution of $\mathcal{P}_1$. To map $\mathcal{P}_1$ to $\mathcal{P}_0$, γ_ℓs are replaced with γ_l^{rt}s and H_ls are replaced with $\frac{E_0}{N}\gamma_l^{rt}$ values.

3.3.2 Optimal Offline Receiver-Centric Schedule

The solution of $\mathcal{P}_2$ similarly proceeds by mapping it to $\mathcal{P}_0$ by replacing γ_ℓs with $\gamma_l^{rt}\gamma_l^{tr}$s, setting $e_0 = E_0$ and all H_ls to zero. Therefore, for a particular slot n, given the energy level e_n and the backlog on data requests b_n ($b_0 = 0$) at that slot, the optimal offline value for Q_n can be computed as $Q_n = [w_n - \frac{1}{\gamma_n^{rt}\gamma_n^{tr}}]^+$ where w_n satisfies the following:

$$
\begin{aligned}
w_n &= \min\{w_n^e(w_n), w_n^b(w_n)\} \\
w_n^e &= \min_{u=0,\ldots,(N-n)} \frac{e_n + \sum_{l=n}^{n+u} M_l^{(e,r)}(w_n)}{u+1} && (3.12) \\
\log_2(w_n^b) &= \min_{v=0,\ldots,(N-n)} \frac{b_n + \sum_{l=n+1}^{n+v} B_l + \frac{1}{2}\sum_{l=n}^{n+v} M_l^{(b,r)}(w_n)}{\frac{1}{2}(v+1)} && (3.13) \\
M_l^{(e,r)}(w_n) &= \min\left\{\frac{1}{\gamma_l^{rt}\gamma_l^{tr}}, w_n\right\} \\
M_l^{(b,r)}(w_n) &= \log_2\left(\min\left\{\frac{1}{\gamma_l^{rt}\gamma_l^{tr}}, w_n\right\}\right)
\end{aligned}
$$

3.3.3 Minimum Energy Demand Profile in WET

Consider an EE-TM-OFF schedule defined via successive water levels. Now suppose that for a given data arrival profile $\{B_n\}$, water levels are never to be limited by energy arrivals $\{H_n\}$ so that $w_n^e \geq w_n^b$ for all n:

$$\min_{u=0,\ldots,(N-n)} \frac{e_n + \sum_{l=n+1}^{n+u} H_l + \sum_{l=n}^{n+u} M_l^{(e)}(w_n)}{u+1} \geq \min_{v=0,\ldots,(N-n)} 2^{\frac{b_n + \sum_{l=n+1}^{n+v} B_l + \frac{1}{2}\sum_{l=n}^{n+v} M_l^{(b)}(w_n)}{\frac{1}{2}(v+1)}} ; \forall n \tag{3.14}$$

Definition 9 *When (3.14) is satisfied with equality for all n, the energy arrival profile corresponds the* minimum demand *of the given data arrival profile.*

A minimum demand energy profile always exists and is unique: the corresponding H_ns can be found by backward induction starting from $n = N$ as long as H_ns are allowed to take arbitrary non-negative values. Let $\{H_n^{(demand)}\}_1^N$ denote the sequence of energy arrivals satisfying (3.14) with equality for every slot n

Accordingly, any energy arrival profile $\{H_n\}$ that does not satisfy the following conditions,

$$\sum_{n=1}^{m} H_n \geq \sum_{n=1}^{m} H_n^{(demand)}; m \in [1, N-1]$$

$$\sum_{n=1}^{N} H_n = \sum_{n=1}^{N} H_n^{(demand)}$$

is either insufficient to transmit all received data or provides more energy than needed. In other words, for any energy arrival profile $\{H_n\}$ that does not satisfy above conditions, energy or data buffers become non-empty at the end of the problem horizon.

This observation leads to the following result, which says that, when channel states are constant, receiver-centric optimization, which is able to adjust the energy transfer minimally according to the data demand, operates in a solution space that subsumes transmitter-centric optimization.

Theorem 3 *Under time invariant channel conditions, i.e.,* $\gamma_n^{tr} = c_1$, $\gamma_n^{rt} = c_2$, *where* $c_1 > 0$ *and* $c_2 > 0$ *are arbitrary constants, the objective value attained by solving* $\mathcal{P}_2$ *is greater than or equal to that obtained by solving* $\mathcal{P}_1$.

Proof. Let $\gamma_n^{tr} = c_1$ and $\gamma_n^{rt} = c_2$, and let $P_n = c_2 Q'_n$ for some $Q'_n \geq 0$. Then $\mathcal{P}_1$ becomes,

$$\begin{aligned}
(\mathcal{P}_{1,c}): \quad \max_{Q'_n} \quad & \sum_{n=1}^{N} \frac{1}{2} \log_2 (1 + c_1 c_2 Q'_n) \\
\text{s.t.} \quad & \sum_{\ell=1}^{n} Q'_\ell c_2 \leq \sum_{\ell=1}^{n} \frac{E_0}{N} c_2, \quad \forall n && (3.15) \\
& \sum_{\ell=1}^{n} \frac{1}{2} \log_2 (1 + c_1 c_2 Q'_\ell) \leq \sum_{\ell=1}^{n} B_\ell, \quad \forall n && (3.16) \\
& 0 \leq c_2 Q'_n, \quad \forall n. && (3.17)
\end{aligned}$$

As $c_2 > 0$, the constraints in the above definition of $\mathcal{P}_{1,c}$ can be rearranged and $\mathcal{P}_{1,c}$ can be rewritten as follows:

$$\begin{aligned}
(\mathcal{P}_{1,c}): \quad \max_{Q'_n} \quad & \sum_{n=1}^{N} \frac{1}{2} \log_2 (1 + c_1 c_2 Q'_n) \\
\text{s.t.} \quad & \sum_{\ell=1}^{n} Q'_\ell \leq \sum_{\ell=1}^{n} \frac{E_0}{N}, \quad \forall n < N && (3.18) \\
& \sum_{\ell=1}^{N} Q'_\ell \leq E_0, && (3.19) \\
& \sum_{\ell=1}^{n} \frac{1}{2} \log_2 (1 + c_1 c_2 Q'_\ell) \leq \sum_{\ell=1}^{n} B_\ell, \quad \forall n && (3.20) \\
& 0 \leq Q'_n, \quad \forall n. && (3.21)
\end{aligned}$$

Now, it can be seen that $\mathcal{P}_{1,c}$ is $\mathcal{P}_2$ evaluated at $\gamma_n^{tr} = c_1$, $\gamma_n^{rt} = c_2$, with additional constraints in (3.18), and is therefore stricter, and the result follows. ■

3.4 Online Policies

As $\mathcal{P}_1$ and $\mathcal{P}_2$ can both be mapped to P_0 by an appropriate choice of parameter values, we will consider online algorithms based on the online algorithm developed in Chapter 2 for P_0.

3.4.1 Transmitter-Centric Online Policy

For the transmitter-centric case, we extend the online algorithm in Chapter 2 to the transmitter-centric problem formulation. This policy uses (2.5) and (2.6) with estimated values of $\sum_{l=n+1}^{n+u} \gamma_l^{rt}$, $\sum_{l=n+1}^{n+v} B_l$, $\sum_{l=n}^{n+u} M_l^{(e)}(w_n)$ and $\sum_{l=n}^{n+v} M_l^{(b)}(w_n)$. The estimated average on energy arrivals is $\bar{H}_n = \frac{E_0}{Nn} \sum_{l=1}^{n} \gamma_l^{rt}$. The values of $\sum_{l=n+1}^{n+u} H_l$, $\sum_{l=n+1}^{n+v} B_l$, $\sum_{l=n}^{n+u} M_l^{(e,t)}(w_n)$ and $\sum_{l=n}^{n+v} M_l^{(b,t)}(w_n)$ are estimated through observed time averages giving the estimated values of w_n^e and w_n^b as follows:

$$\hat{w}_n^e = \begin{cases} \frac{e_n - \bar{H}_n}{N-n} + \bar{H}_n + \bar{M}_n^{(e,t)}(w_n) & ; e_n \geq \bar{H}_n \\ e_n + \bar{M}_n^{(e,t)}(w_n) & ; \text{o.w.} \end{cases}$$

$$\log_2(\hat{w}_n^b) = \begin{cases} \frac{2(b_n - \bar{B}_n)}{N-n} + \bar{B}_n + \bar{M}_n^{(b,t)}(w_n) & ; b_n \geq \bar{B}_n \\ 2b_n + \bar{M}_n^{(b,t)}(w_n) & ; \text{o.w.} \end{cases}$$

Here,

$$\bar{H}_n = \frac{E_0}{Nn} \sum_{l=1}^{n} \gamma_l^{rt}, \bar{B}_n = \frac{1}{n} \sum_{l=1}^{n} B_l$$

$$\bar{M}_n^{(e,t)}(w_n) = \frac{1}{n} \sum_{l=1}^{n} M_l^{(e,t)}(w_n)$$

$$\bar{M}_n^{(b,t)}(w_n) = \frac{1}{n} \sum_{l=1}^{n} M_l^{(b,t)}(w_n)$$

The estimate of the throughput maximizing water level can be computed iteratively:

$$\hat{w}_n^{(k+1)} = |_{w_n = \hat{w}_n^{(k)}} \min \left\{ \hat{w}_n^e, \hat{w}_n^b \right\}$$

where $\hat{w}_n^{(k)}$ is the kth iteration of the estimate of the throughput maximizing water level and $\hat{w}_n^{(1)} = \min \left\{ e_n, 2^{2b_n} \right\}$.

3.4.2 Receiver-Centric Online Policy

For the receiver-centric problem setting, the corresponding estimation-based online policy is the following:

$$\hat{w}_n^e = \frac{e_n}{N-n} + \bar{M}_n^{(e,r)}(w_n).$$

$$\log_2(\hat{w}_n^b) = \begin{cases} \frac{2(b_n - \bar{B}_n)}{N-n} + \bar{B}_n + \bar{M}_n^{(b,r)}(w_n) & ; b_n \geq \bar{B}_n \\ 2b_n + \bar{M}_n^{(b,r)}(w_n) & ; \text{o.w.} \end{cases}$$

$$\bar{B}_n = \frac{1}{n}\sum_{l=1}^{n} B_l$$

$$\bar{M}_n^{(e,r)}(w_n) = \frac{1}{n}\sum_{l=1}^{n} M_l^{(e,r)}(w_n)$$

$$\bar{M}_n^{(b,r)}(w_n) = \frac{1}{n}\sum_{l=1}^{n} M_l^{(b,r)}(w_n)$$

3.5 Distributed Scheduling

Before evaluating the performance of the offline and online algorithms, we propose a third scheduling problem formulation, namely, distributed scheduling.In this formulation, the scheduling decision is formed partly at the TX and partly at the RX. The amounts of energy transfers are determined by the receiver. On the other hand, the transmitter chooses its transmit powers based on its energy level (i.e. battery state), channel state, and the data requests coming from the receiver. The resulting optimization problem is,

$$(\mathcal{P}_3): \max_{\{P_n, Q_n\}} \quad \sum_{n=1}^{N} \frac{1}{2}\log_2\left(1+\gamma_n^{tr} P_n\right)$$

$$\text{s.t.} \quad \sum_{n=1}^{N} Q_n \le E_0, \tag{3.22}$$

$$\sum_{\ell=1}^{n} P_\ell \le \sum_{\ell=1}^{n} Q_\ell \gamma_\ell^{rt}, \quad \forall n \tag{3.23}$$

$$\sum_{\ell=1}^{n} \frac{1}{2}\log_2\left(1+\gamma_\ell^{tr} P_\ell\right) \le \sum_{\ell=1}^{n} B_\ell,\ \forall n \tag{3.24}$$

$$0 \le Q_n, \quad 0 \le P_n, \quad \forall n. \tag{3.25}$$

Distributed scheduling requires less feedback between the transmitter and the receiver. Moreover, it allows the RX to send energy at times that γ_ℓ^{rt} takes a large value, whereas the TX can optimize its own transmissions according to γ_n^{tr}. This flexibility can result in a higher overall throughput than that achieved by TX-centric and RX-centric solutions when γ_ℓ^{rt} and γ_n^{tr} differ significantly (e.g., have low correlation.) On the other hand, under a fixed channel state, this problem reduces to the receiver-centric formulation, as summarized in Theorem 4.

Theorem 4 *Under time invariant channel conditions, i.e.,* $\gamma_n^{tr} = c_1$, $\gamma_n^{tr} = c_2$, *where* $c_1 > 0$ *and* $c_2 > 0$ *are arbitrary constants,* $\mathcal{P}_2$ *and* $\mathcal{P}_3$ *are equivalent.*

Proof. Let $\gamma_n^{tr} = c_1$, $\gamma_n^{rt} = c_2$, and let $P_n = c_2 Q'_n$. Then, $\mathcal{P}_3$ becomes,

$$
\begin{aligned}
(\mathcal{P}_{3,c}) : \max_{\{Q'_n, Q_n\}} \quad & \sum_{n=1}^{N} \frac{1}{2} \log_2 (1 + c_1 c_2 Q'_n) \\
\text{s.t.} \quad & \sum_{n=1}^{N} Q_n \leq E_0, && (3.26) \\
& \sum_{\ell=1}^{n} Q'_\ell c_2 \leq \sum_{\ell=1}^{n} Q_\ell c_2, \quad \forall n && (3.27) \\
& \sum_{\ell=1}^{n} \frac{1}{2} \log_2 (1 + c_1 c_2 Q'_\ell) \leq \sum_{\ell=1}^{n} B_\ell, \ \forall n && (3.28) \\
& 0 \leq Q_n, \quad 0 \leq c_2 Q'_n, \quad \forall n. && (3.29)
\end{aligned}
$$

Let the pair of sequences $\{Q'_n, Q_n\}$ constitute an optimal solution for $\mathcal{P}_{3,c}$, for which (3.27) is not tight. We can set $Q_n = Q'_n$ for all n, without changing the value of the objective function, and without violating (3.26). This reduces the problem to $\mathcal{P}_2$. Therefore, among the optimal solutions, Q'_n, of $\mathcal{P}_{3,c}$, there exists one which also solves $\mathcal{P}_2$ when $\gamma_n^{tr} = c_1$ and $\gamma_n^{rt} = c_2$. ■

While we postpone further discussion of $\mathcal{P}_3$ and its solution to future work, we will plot numerical solutions of it along with the other algorithms in the next section.

3.6 Numerical and Simulation Results

In this section, we compare the average throughput performances of offline optimal schedules for $\mathcal{P}_1$,$\mathcal{P}_2$ and $\mathcal{P}_3$, and the proposed online policies.

First, we consider a reciprocal Rician channel ($\kappa = 0.5$) on both uplink and downlink, with a path loss of -70dB in each direction. We assume a random data request in each slot, which is distributed uniformly with mean 5bits/Hz. For this simulation, we assume that the energy buffer of the transmitter is not limited. In Figure 3.4, we compare the throughput of all policies in question. The distributed optimal offline policy serves as a benchmark for other policies. We observe that the transmitter centric offline policy performs better than the receiver centric offline policy, due to its ability

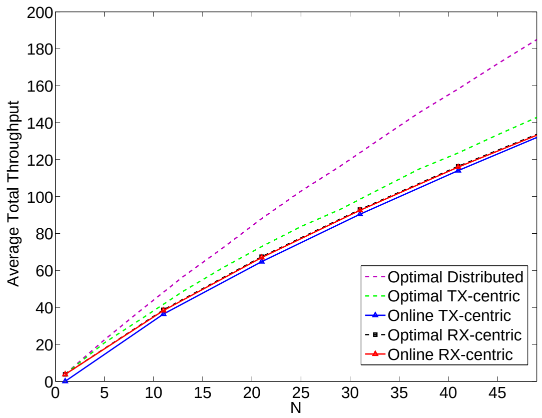


Figure 3.4: Average total throughput (in Mbits) versus horizon length N (in seconds) under each policy.

to store power at the transmitter, for future better channel states. However, the online receiver centric schedule outperforms its transmitter centric counterpart, and achieves nearly identical performance to optimal receiver centric schedule, with its ability to adaptively control the energy packets. In any case, the online policies can be said to closely follow their offline counterparts.

CHAPTER 4

AGE OF INFORMATION: A MEASURE OF FRESHNESS FOR QUALITY OF SERVICE IN TIMELY COMMUNICATION

4.1 Introduction

In Chapter 2 and 3, we merely considered communication problems with constraints and objectives but not the actual use of communication. The use of communication may vary from application to application however, mainly, one can consider two types of uses. The first is the use where shared information can be treated as a record being persistently useful or having a long-term value. The second is the use where an end-user is in need of following a process, i.e., a source of interest, for the sake of curiosity or in order to achieve a goal. In this chapter, we will focus on this second use and refer to it as *monitoring* in the widest sense.

4.2 Update-based Systems

When communication is about monitoring, a basic solution is to simply deliver status updates from the source of interest. Particularly, such a communication takes place in existing applications of sensor networks, industrial manufacturing, telerobotics, Internet of Things (IoT) and social networks. In these applications, communication can be designed and viewed as an exchange of status updates. We will refer to communication systems designed in this way as *update-based systems*.

One aspect of update-based systems is that their performance can be characterized by generation and successful delivery times of updates from the source to the end-user [1].

[1] Here, we assume that both sides see their clocks ticking at the same rate while being synchronized.

Assuming that the updates capture the status of the source of interest precisely at their generation instant, it will be sufficient to measure the quality of service for monitoring on a temporal basis. Ideally, we require both the duration between generation and delivery times, i.e., delay, to be small and the frequency of updates, i.e., throughput, to be large. On the other hand, neither of these describes timeliness which actually determines the success of monitoring. From the perspective of the end-user, timeliness can be understood in terms of the freshness of available information.

4.3 Data Freshness

4.3.1 Age of Information

In monitoring, one typically requires fresh updates as usefulness of old updates degrades over time. Moreover, when there is no need to store old updates, they get obsolete upon the arrival of an update with a more recent time-stamp. In that case, a measure of data freshness at a particular instant is the time elapsed since the most recent update available to the end-user was generated at the source. This measure the *Age of Information* (AoI), or simply the age, which was proposed in [38, 39]. Precisely, the age is defined as:

$$\Delta(t) = t - U(t), \tag{4.1}$$

where $U(t)$ is the time-stamp of the most recent update.

Let S_i and D_i denote generation and delivery times of an update with index i, respectively. Then, $U(t)$ can be expressed as follows:

$$U(t) = \max\{S_i : D_i \leq t\}. \tag{4.2}$$

From its definition, it can be seen that the age is a process that grows linearly with unit slope and if an update providing a smaller age, i.e., a fresher update, arrives, the age drops to that level (Fig. 4.1). Considering generation and delivery times that are random, the age is a stochastic process. In that case, a reasonable objective is to minimize the *average age* which is defined as follows:

$$\bar{\Delta} = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Delta(t)dt\right]. \tag{4.3}$$

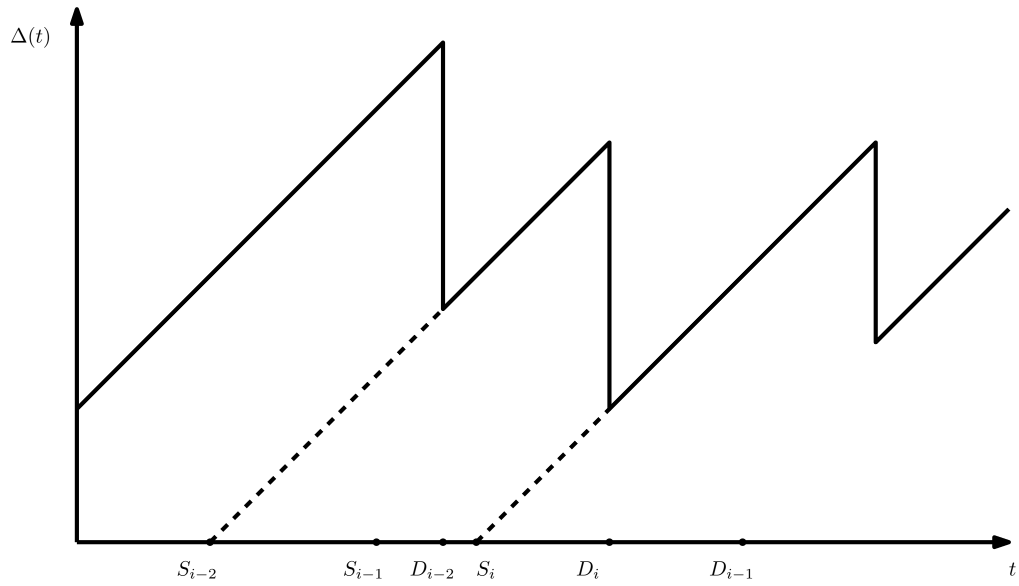


Figure 4.1: The Age of Information versus time.

Minimization of the average age $\bar{\Delta}$ is a particularly relevant goal for an update-based system where the usefulness of status updates typically degrades linearly proportional to the elapsed time. The average age can be connected most commonly used measures in remote estimation such as the time-average mean-square error (MSE). An example of this is the result in [40] where it was shown that remote estimation of a Wiener process minimizing MSE reduces to minimizing average age when the sampling times at the transmitting side are independent from the process. On the other hand, in many scenarios, the success of monitoring may be related to *non-linear* (see [41–48]) functions of the age.

4.3.2 Non-linear Functions of the Age

In general, the change in the usefulness of updates can be less/more significant as the age grows. If there is a fixed pattern which characterizes this change for all updates, then one can model the usefulness of available information as a non-linear function $p(\Delta(t))$ of the age $\Delta(t)$, i.e., the *age-penalty* where the function $p : [0, \infty) \to [0, \infty)$ is non-deceasing such that available information becomes less useful as the age grows. The relevant goal for such cases can be the minimization of the *average age-penalty* which is defined as follows:

$$\bar{p} = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T p(\Delta(t))dt\right]. \tag{4.4}$$

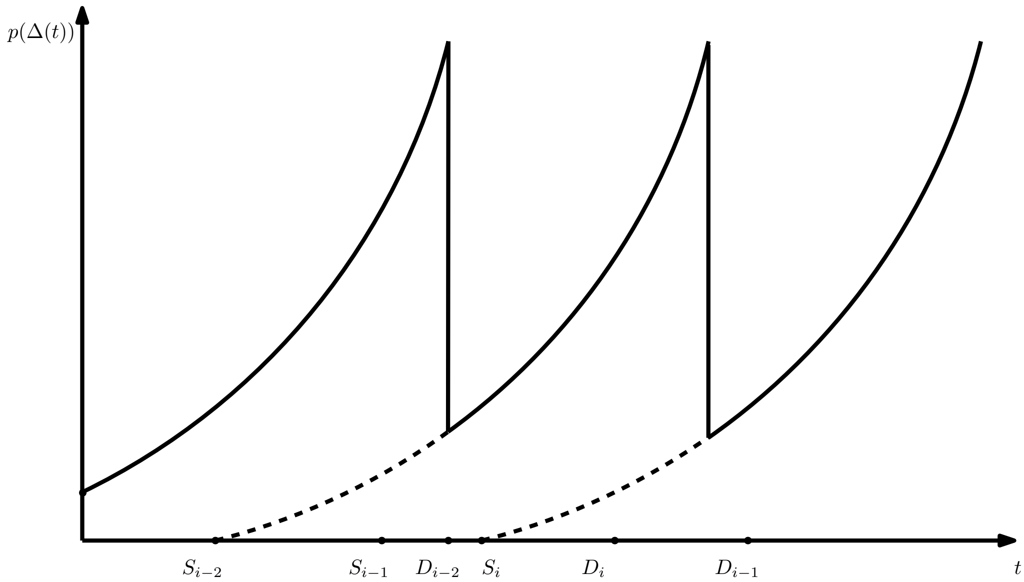


Figure 4.2: The Age-penalty versus time.

Similar to age-penalty functions, one can characterize the usefulness of updates using utility functions of age, i.e., *age-utility* functions, which decrease with increasing age. Some examples of age-penalty and age-utility functions include: The age-penalty for online learning [43], the age-penalty based on autocorrelation [44], the age-penalty based on the probability of mission failure [47], the age-utility based on mutual information [45], the age-penalty based on quadratic error in the estimation of a multi-dimensional linear system [48].

4.3.3 The Age of Information and Queueing

Several studies on AoI considered this performance metric under various queueing system models comparing service disciplines and queue management policies (e.g., [49–57]). A common observation in these studies was that many queueing/service policies that are throughput and delay optimal but are often suboptimal with respect to AoI, while AoI-optimal policies can be throughput and delay optimal, at the same time. This showed that AoI optimization is quite different than optimization with respect to classical performance metrics. This required many queueing models to be re-addressed under respect to age related objectives. Moreover, some studies considered the adjustment of physical layer parameters for source and channel coding [58–69] . However, these formulations typically assume no precise control on the transmission or generation times of status updates. Indeed, such control is important for age optimization [42, 43].

4.3.4 Timely Updating

A direct control on the generation times of status updates is possible through a control algorithm that runs at the source. This is the *“generate-at-will"* assumption formulated in [70, 71] and studied in [40, 42, 43]. In [70], the problem of AoI optimization for a source, which is constrained by an arbitrary sequence of energy arrivals was studied. In [71], AoI optimization was considered for a source that harvests energy at a constant rate under stochastic delays experienced by the status update packets. The results in these studies showed suboptimality of work-conserving transmission schemes. Often, introducing a waiting time before sending the next update is optimal. That is, for maximum freshness, one may sometimes send updates at a rate lower than one is allowed to which may be counter-intuitive at first sight.

CHAPTER 5

TIMELY COMMUNICATION FOR OPTIMIZING AGE OF INFORMATION UNDER ENERGY HARVESTING

5.1 Introduction

Sensor networks for monitoring applications are arguably the most important use cases of update-based systems. As we discussed in Chapter 4, the optimization for data freshness appears to be a more relevant goal for such applications, than throughput and delay optimization. This suggests the significance of studying sensor networks under data freshness measures. Another aspect of sensor networks is that they are composed of “nodes" which are envisioned to be ideally simple and self-sustainable (or more specifically energy harvesting) devices.

Motivated by these practical concerns, several studies [70–79] were dedicated to AoI optimization for energy harvesting communication.The common assumption in these studies is that energy harvesting process is considered as an arrival process where each energy arrival carries the energy required for an update. The goal of AoI optimization in such formulations is to find an optimal timing of update instants (under “generate-at-will" assumption) in order to minimize average AoI while transmission opportunities are subject to the availability of energy. Energy arrivals occur irregularly or randomly, which models an energy harvesting scenario. The main challenge in optimizing time average expected age under random energy arrivals is that in the case of an energy outages (empty battery), the transmitter must idle for an unknown duration of time. If it is the case that such random durations are inevitable, they introduce a tension for the regulation of inter-update durations. Another challenge is due to the finiteness of battery sizes. Theoretically, it is possible to achieve asymptotically optimal average AoI by employing simple schemes assuming infinite [72] or sufficiently large battery [73] sizes. However, when the battery size is comparable to the energy required per update, such simple schemes do not allow performance guarantees. Consequently, it is important to explore optimal policies under such regimes where performance depends heavily on the statistics of energy arrivals and the battery size.

In this chapter, we will consider an AoI optimization capturing both the randomness of energy arrivals and finite energy storage capability. In addition capturing both challenges we go further, by optimizing not only average age itself, but an average age-penalty (see Chapter 4.3.2). Under the assumption of Poisson energy arrivals, we will show the structure of solutions for the age-penalty optimization problem. The structure of the optimal solution reflects a basic intuition about the optimal strategy: Updates should be sent when the update is valuable (when the age is high) and the energy is cheap (the battery level is high). We show that the optimal solution is given by a stopping rule according to which an update is sent when its immediate cost is surpassed by the expected future cost. For Poisson energy arrivals, this stopping rule can be found in the set of policies that we refer as *monotone threshold policies*. Monotone threshold policies have the property that each update is sent only when the age is higher than a certain threshold which is a non-increasing function of the instantaneous battery level. One of the key results is that the value of the age-penalty function at the optimal threshold corresponding to the full battery level is exactly equal to the optimal value of the average age-penalty.

5.2 System Model

In this section, we will describe the system model for an energy harvesting update-based system and the problem of average age-penalty minimization. Consider an energy harvesting transmitter that sends update packets to a receiver, as illustrated in Fig 5.1. Suppose that the transmitter has a finite battery which is capable of storing up to B units of energy. Similar to [72], we assume that the transmission of an update packet consumes one unit of energy. The energy that can be harvested arrive in units according to a Poisson process with rate μ_H. Let $E(t)$ denote the amount of energy stored in the battery at time t such that $0 \leq E(t) \leq B$. The timing of status updates are controlled by a sampler which monitors the battery level $E(t)$ for all t. We assume that the initial age and the initial battery level are zero, i.e., $\Delta(0) = 0$ and $E(0) = 0$.

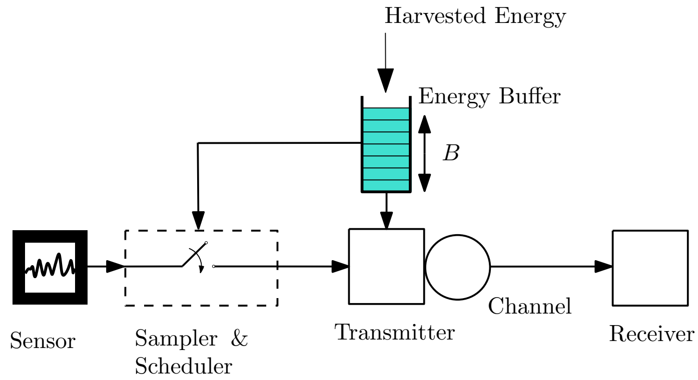


Figure 5.1: System Model.

Let $H(t)$ and $A(t)$ denote the number of energy units that have arrived during $[0, t]$ and the number of updates sent out during $[0, t]$, respectively. Hence, $\{H(t), t \geq 0\}$ and $\{A(t), t \geq 0\}$ are two counting processes. If an energy unit arrives when the battery is full, it is lost because there is no capacity to store it.

The system starts to operate at time $t = 0$. Let Z_k denote the generation time of the k-th update packet such that $0 = Z_0 \leq Z_1 \leq Z_2 \leq \ldots$. An update policy is represented by a sequence of update instants $\pi = (Z_0, Z_1, Z_2, ...)$. Let X_k represent the inter-update duration between updates $k-1$ and k, i.e., $X_k = Z_k - Z_{k-1}$. In many status-update systems (e.g., a sensor reporting temperature [80]), update packets are small in size and are only sent out sporadically. Typically, the duration for transmitting a packet is much smaller than the difference between two subsequent update times, i.e., X_ks are typically large compared to the duration of a packet transmission. With such systems in mind, in our model, we will approximate the packet transmission durations as zero. In other words, once the k-th update is generated and sent out at time $t = Z_k$, it is immediately delivered to the receiver. Hence, the age of information $\Delta(t)$ at any time $t \geq 0$ is

$$\Delta(t) = t - \max\{Z_k : Z_k \leq t\}, \tag{5.1}$$

which satisfies $\Delta(t) = 0$ at each update time $t = Z_k$. Because an update costs one unit of energy, the battery level reduces by one upon each update, i.e.,

$$E(Z_k) = E(Z_k^-) - 1, \tag{5.2}$$

where Z_k^- is the time immediately before the k-th update. Further, because the battery size is B, the battery level evolves according to

$$E(t) = \min\{E(Z_k) + H(t) - H(Z_k), B\}, \tag{5.3}$$

when $t \in [Z_k, Z_{k+1})$ is between two subsequent updates.

In terms of energy available to the scheduler, we can define update policies, that do not violate causality, as in the following:

Definition 10 A policy π is said to be *energy-causal* if updates only occur when the battery is non-empty, that is, $E(Z_k^-) \geq 1$ for each packet k.

Another restriction on update instants is due to the information available to the scheduler which we define as follows,

Definition 11 Information on the energy arrivals and updates by time t is represented by the filtration [1] $\mathcal{F}_t = \sigma(\{(H(t'), A(t')), 0 \leq t' < t\})$ which is the σ-field generated by the sequence of energy arrivals and updates, i.e., $\{(H(t'), A(t')), 0 \leq t' < t\}$.

Similar to the definition of energy-causal policies, in the policy space that we will consider we merely assume the causality of available information besides energy causality. To formulate this assumption, we use the definition of $\mathcal{F}_t$. In terms of information available to the scheduler, any random time instant θ does not violate causality if and only if $\{\theta \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. We will refer such random instants as *Markov times* [81] and consider update times as Markov times based on the filtration $\mathcal{F}_t$ in general. Notice that such update times do not have to be finite, however, we will refer Markov times that are also finite with probability 1 (w.p.1.) as *stopping times* [81]. For a policy trying to regulate age, it is legitimate to assume that update instants are always finite w.p.1. as otherwise the age may grow unbounded with a

[1] Note that the filtration is right continuous as both $H(t)$ and $A(t)$ are right continuous.

positive probability. With this in mind, we will consider only the update instants that are stopping times.

Accordingly, we can define the *online* update policies combining the causality assumptions on available energy and information as follows:

Definition 12 A policy is said to be *online* if (i) it is energy causal, (ii) no update instant is determined based on future information, i.e., all update times are stopping (finite Markov) times based on $\mathcal{F}_t$, i.e., Z_k is finite w.p.1. while $\{Z_k \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$ and $k \geq 1$.

Let Π^{online} denote the set of online update policies. To evaluate the performance of online policies, we consider an *age-penalty function* that relates the age at a particular time to a cost which increases by the age. This function is defined as in below:

We consider an *age-penalty function* $p(\cdot)$ that maps the age $\Delta(t)$ at time t to a penalty $p(\Delta(t))$:

Definition 13 A function $p : [0, \infty) \to [0, \infty)$ of the age is said to be an *age-penalty function* if

- $\lim_{\Delta \to \infty} p(\Delta) = \infty$.
- $p(\cdot)$ is a non-decreasing function.
- $\int_0^\infty p(t)e^{-\alpha t}dt < \infty$ for all $\alpha > 0$.

Observe that the definition of age-penalty functions covers any non-decreasing function of age that is of sub-exponential order[2] and grows to infinity.

The time-average expected value of the age-penalty or simply the *average age-penalty* can be expressed as

$$\bar{p} = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T p(\Delta(t))dt\right]. \tag{5.4}$$

[2] This is due to the third property in the definition, which is a technical requirement for the proofs.

Let $\bar{p}_\pi$ denote the average age-penalty achieved by a particular policy π. We will try to find the optimal update policy for minimizing the average age-penalty, which is formulated as

$$\min_{\pi \in \Pi^{\text{online}}} \bar{p}_\pi. \tag{5.5}$$

5.3 Main Results

We begin with a result guaranteeing that the space of threshold-type policies (see Definition 14) contains optimal update policies hence we can focus our attention to these policies for finding solutions to (5.5).

Note that at time $t = Z_k$, the age $\Delta(t)$ is equal to 0. In the meanwhile, the battery level $E(t)$ will grow as more energy is harvested. In threshold policies, the threshold $\tau_{E(t)}$ changes according to the battery level $E(t)$ and a new sample is taken at the earliest time that the age $\Delta(t)$ exceeds the threshold $\tau_{E(t)}$. We define such policies as follows:

Definition 14 When $E(t) \in \{\ell = 1, ..., B\}$ represents the battery level at time t, an online policy is said to be a *threshold policy* if there exists τ_ℓ for $\ell = 1, ..., B$ s.t.

$$Z_{k+1} = \inf\left\{t \geq Z_k : \Delta(t) \geq \tau_{E(t)}\right\}, \tag{5.6}$$

Note that a policy is said to be *stationary* if its actions depend only on a current state while being independent of time. An immediate observation is that given $\Delta(t)$ and $E(t)$ threshold policies do not depend on time, hence:

Proposition 1 *All threshold policies are stationary.*

Proof. By definition, the update instants of a threshold policy only depend on the time elapsed since the last update, i.e., $\Delta(t)$, and the current battery level. ■

We expect that such stationary policies can minimize $\bar{\Delta}$ among all online policies as energy arrivals follow a Poisson process which is memoryless. Due to the memorylessness of energy arrivals, the evolution of the system can be understood through a renewal type behaviour which suggests that an optimal policy should be stationary.

Indeed, we note the following as the first key result,

Theorem 5 *There exists a threshold policy that is optimal for solving (5.5).*

Proof. See Appendix 5.5.1. ■

One significant challenge in the proof of Theorem 5 is that (5.5) is an infinite time-horizon time-averaged MDP which has an uncountable state space. When the state space is countable, one can analyze infinite time-horizon time-averaged MDP by making a unichain assumption. However, this method cannot be directly applied when state space is uncountable. To resolve this, we use a modified version of the “vanishing discount factor" approach [82] to prove Theorem 5 in two steps:

1. Show that for every $\alpha > 0$, there exists a threshold policy that is optimal for solving

$$\min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_0^\infty e^{-\alpha(t-a)} p(\Delta(t)) dt\right].$$

2. Prove that this property also holds when the discount factor α vanishes to zero.

In our search for an optimal policy, we can further reduce the space of policies:

Definition 15 A threshold policy is said to be a *monotone threshold* policy if $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_B$.

Note that the definition of monotone threshold policies refers only to the case of thresholds that non-increasing in battery levels as opposed to the non-decreasing case.

Let Π^{MT} be the set of monotone threshold policies, then, the following is true:

Theorem 6 *There exists a monotone threshold policy $\pi \in \Pi^{\text{MT}}$ that is optimal for solving (5.5).*

Proof. See Appendix 5.5.2. ■

Theorem 6 implies that in the optimal update policy, update packets are sent out more frequently when the battery level is high and less frequently when the battery level is low. This result is quite intuitive: If the battery is full, arrival energy cannot be harvested; if the battery is empty, update packets cannot be transmitted when needed and the age increases. Hence, both battery overflow and outage are harmful. Monotone threshold policies can address this issue. When the battery level l is high, the threshold τ_l is small to reduce the chance of battery overflow; when the battery level l is low, the threshold τ_l is high to avoid battery outage.

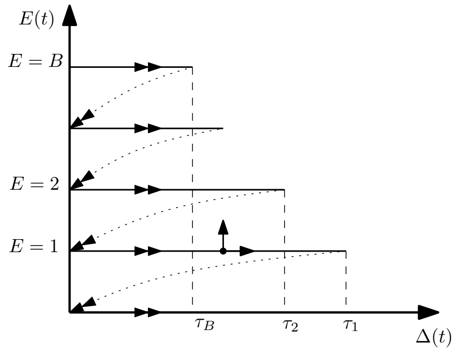


Figure 5.2: An illustration of a monotone threshold policy.

For a policy in Π^{MT}, the state $(\Delta(t), E(t))$ does not spend a measurable amount of time anywhere $\Delta(t) \geq \tau_{E(t)}$ in which an update is sent out instantly reducing the battery level. Otherwise, the battery level is incremented upon energy harvests while the age is increasing linearly in time. The illustration in Fig. 5.2 shows the time evolution of the state $(\Delta(t), E(t))$ for policies in Π^{MT}. If the energy level is $E(Z_k) = j$ upon the previous update, then the inter-update time $X_{k+1} \in [\tau_m, \tau_{m-1}]$ holds if and only if $m - j$ packets arrive during the inter-update time. In other words, reaching the battery state m or higher is necessary and sufficient for the next inter-update duration being shorter than some x when $x \in [\tau_m, \tau_{m-1})$. Let Y_i denote the duration required for $i \geq 1$ successive energy arrivals, which obeys the Erlang distribution at rate μ_H with parameter i,

$$P(Y_i \leq x) = 1 - \sum_{v=0}^{i-1} \frac{1}{v!} e^{-\mu_H x} (\mu_H x)^v, \tag{5.7}$$

and let $Y_i = 0$ for $i \leq 0$.

Accordingly, for policies in Π^{MT}, the cumulative distribution function (CDF) of inter-update durations, can be expressed as

$$\Pr(X_{k+1} \leq x \mid E(Z_k) = j) = \begin{cases} 0, & \text{if } x < \tau_B \\ \Pr(Y_{m-j} \leq x), & \text{if } \tau_m \leq x < \tau_{m-1}, \forall m \in \{2, ..., B\}, \\ \Pr(Y_{1-j} \leq x), & \text{if } \tau_1 \leq x, \end{cases} \tag{5.8}$$

From (5.8), an expression for the transition probability $\Pr(E(Z_{k+1}) = i \mid E(Z_k) = j)$ for $i = 0, 1, ..., B-1$ can be derived[3]

$$\Pr(E(Z_{k+1}) = i \mid E(Z_k) = j) = \begin{cases} \Pr(Y_{B-j} \leq \tau_{B-1}), & \text{if } i = B-1, \\ \Pr(Y_{1+i-j} \leq \tau_i) - \Pr(Y_{2+i-j} \leq \tau_{i+1}), & \text{if } i < B-1, \end{cases} \tag{5.9}$$

Hence, energy states sampled at update instants can be described as a Discrete Time Markov Chain (DTMC) with the transition probabilities in (5.9) (See Fig. 5.3). When thresholds are finite, this DTMC is ergodic as any energy state is reachable from any other energy state in $B-1$ steps with positive probability.

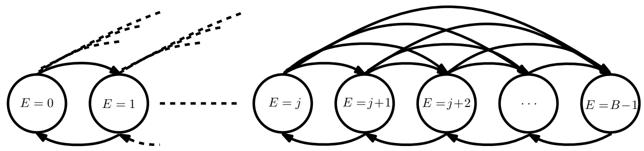


Figure 5.3: The DTMC for energy states sampled at update times.

Any optimal policy in Π^{MT} has the following property:

Theorem 7 *An optimal policy for solving (5.5) is a monotone threshold policy that satisfies the following*

$$p(\tau_B^*) = \bar{p}_{\pi^*} = \min_{\pi \in \Pi^{\text{online}}} \bar{p}_\pi. \tag{5.10}$$

where π^* is a monotone threshold policy solving (5.5) and τ_B^* is its age threshold for the full battery case.

[3] Note that the event $E(Z_{k+1}) = i$ happens if and only if $X_{k+1} \in [\tau_{i+1}, \tau_i)$, accordingly $\Pr(E(Z_{k+1}) = i \mid E(Z_k) = j) = \Pr(X_{k+1} \leq \tau_i \mid E(Z_k) = j) - \Pr(X_{k+1} \leq \tau_{i+1} \mid E(Z_k) = j)$.

Proof. See Appendix 5.5.5. ■

The result in Theorem 7 exhibits a structural property of optimal policies which also appears in the sampling problem that was studied in [45]. The sampling problem in [45] considered sources without energy harvesting, where the packet transmission times were *i.i.d.* and non-zero. On the one hand, the optimal sampling policy in Theorem 1 of [45] is a threshold policy on an expected age penalty term, and the threshold is exactly equal to the optimal objective value. On the other hand, we consider a sampling problem for an energy harvesting source with zero packet transmission time. The optimal sampling policy in Theorem 7 can be rewritten as

$$Z_{k+1} = \inf\left\{t \geq Z_k : p(\Delta(t)) \geq p(\tau^*_{E(t)})\right\}$$

which is a multi-threshold policy on the age penalty function, each threshold $p(\tau^*_\ell)$ corresponding to a battery level ℓ. Further, the threshold $p(\tau^*_B)$ associated with a full battery size $E(t) = B$ is equal to the optimal objective value. The results in these two studies are similar to each other. Together, they provide a unified view on optimal sampler design for sources both with and without energy harvesting capability. The proof techniques in these two studies are of fundamental difference.

5.3.1 Average Age Case

If we take the age-penalty function as an identity function, i.e., $p(\Delta) = \Delta$, then (5.5) becomes the problem of minimizing the time-average expected age. In this case, the result in Theorem 7 implies that in optimal monotone threshold policies, inter-update durations can be small as much as the minimum average AoI only when the battery is full. From results in [72] and [73], we know that the minimum average AoI for the infinite battery case is $\frac{1}{2\mu_H}$ and this can be achieved asymptotically using the best-effort scheme in [73] or with a threshold policy [72] where all thresholds are nearly equal to $\frac{1}{\mu_H}$. On the other hand, according to Theorem 7, the optimal threshold for the full battery level tends to $\frac{1}{2\mu_H}$ as the battery capacity increases. This shows that the optimal monotone threshold policies remain structurally dissimilar to asymptotically optimal policies when the battery capacity is approaching to infinity. The result is more useful when the battery capacity is finite as it may lead to the optimal threshold values of the other battery levels. We will use this in an algorithm for finding near optimal policies for any given integer sized battery capacity. In addition, the special case of Theorem 7 for average age [83] can be derived from a more general result which we provide in Lemma 1. This result shows a relation between the partial derivatives of a non-negative random variable with respect to the thresholds determining the random variable in a similar way to the inter-update duration case.

Lemma 1 *Suppose X is a r.v. that satisfies the following:*

$$\Pr(X \le x) = \begin{cases} 0 & \text{if } x < \tau_B, \\ F_i(x) & \text{if } \tau_i \le x < \tau_{i-1}, \forall i \in \{2, ..., B\}, \\ F_1(x) & \text{if } \tau_1 \le x, \end{cases}$$

where $0 < \tau_B \le ... \le \tau_2 \le \tau_1$ *and for each* $i \in \{1, ..., B\}$ $F_i(x)$ *is the CDF of a non-negative random variable. Then:*

$$\frac{\partial}{\partial \tau_i}\mathbb{E}\left[X^2\right] = 2\tau_i \frac{\partial}{\partial \tau_i}\mathbb{E}\left[X\right].$$

Proof. See Appendix 5.5.3. ■

Corollary 1 *The inter-update intervals, X, for any* $\pi \in \Pi^{\mathrm{MT}}$ *satisfy the following:*

$$\frac{\partial}{\partial \tau_i}\mathbb{E}\left[X^2 \mid E = j\right] = 2\tau_i \frac{\partial}{\partial \tau_i}\mathbb{E}\left[X \mid E = j\right], \forall (i, j) \in \{1, 2, ..., B\}^2, \tag{5.11}$$

where $\mathbb{E}\left[X \mid E = j\right] \triangleq \mathbb{E}\left[X_k \mid E(Z_k) = j\right]$ *and* $\mathbb{E}\left[X^2 \mid E = j\right] \triangleq \mathbb{E}\left[X_k^2 \mid E(Z_k) = j\right]$.

Note that the transition probabilities (5.9) do not depend on τ_B hence the steady-state probabilities obtained from (5.9) also do not depend on τ_B. This leads to a property of τ_B the average age case of Theorem 7 as shown in [83]. The unit-battery case , i.e., $B = 1$ case was solved in [72] and [73]. For completeness, this result is summarized in Theorem 8.

Theorem 8 *When* $B = 1$, *the average age* $\bar{\Delta}$ *can be expressed as*

$$\bar{\Delta} = \frac{\frac{1}{2}(\mu_H \tau_1)^2 + e^{-\mu_H \tau_1}(\mu_H \tau_1 + 1)}{\mu_H(\mu_H \tau_1 + e^{-\mu_H \tau_1})}, \tag{5.12}$$

and $\tau_1^* = \bar{\Delta}_{\pi^*} = \frac{1}{\mu_H} 2W(\frac{1}{\sqrt{2}})$ *where* $W(\cdot)$ *is the Lambert-W function.*

Proof. See Appendix 5.5.6. ■

Theorem 9 *When $B = 2$, the average age $\bar{\Delta}$ can be expressed as:*

$$\bar{\Delta} = \frac{\frac{\alpha_2^2}{2} + e^{-\alpha_2}[\alpha_2 + 1 + \rho_1(\alpha_2^2 + 2\alpha_2 + 2)] - e^{-\alpha_1}[\alpha_1 + 1 + \rho_1(\alpha_1^2 + \alpha_1 + 1)]}{\mu_H\big(\alpha_2 + e^{-\alpha_2}[1 + \rho_1(\alpha_2 + 1)] - e^{-\alpha_1}[1 + \rho_1\alpha_1]\big)}, \tag{5.13}$$

where

$$\rho_1 = \frac{e^{-\alpha_1}}{1 - e^{-\alpha_1}\alpha_1},$$

and

$$\alpha_1 = \mu_H\tau_1, \alpha_2 = \mu_H\tau_2.$$

Proof. See Appendix 5.5.7. ■

5.3.2 An Algorithm for Finding Near Optimal Policies

We propose an algorithm to find a near optimal policy $\pi \in \Pi^{\mathrm{MT}}$ such that $\bar{\Delta}_\pi - \bar{\Delta}_{\pi^*} \leq \frac{1}{2^{q+1}\mu_H}$ for any given B and $q \in \mathbb{Z}^+$. Let $m_1(\tau_1, \tau_2, ..., \tau_B)$ and $m_2(\tau_1, \tau_2, ..., \tau_B)$ denote the functions such that:

$$m_1(\tau_1, \tau_2, ..., \tau_B) = \sum_{j=0}^{B-1} \mathbb{E}[X \mid E = j] \Pr(E = j), \tag{5.14}$$

$$m_2(\tau_1, \tau_2, ..., \tau_B) = \sum_{j=0}^{B-1} \mathbb{E}\left[X^2 \mid E = j\right] \Pr(E = j), \tag{5.15}$$

where $\Pr(E = j)$ is the steady-state probability for energy state j, $\mathbb{E}[X \mid E = j] \triangleq \mathbb{E}[X_k \mid E(Z_k) = j]$ and $\mathbb{E}\left[X^2 \mid E = j\right] \triangleq \mathbb{E}\left[X_k^2 \mid E(Z_k) = j\right]$.

Note that it is straight forward to derive $m_1(\tau_1, \tau_2, ..., \tau_B)$ and $m_2(\tau_1, \tau_2, ..., \tau_B)$ using (5.8) and (5.9), hence we assume these functions are available for any B.

In the below theorem , we state the main result that we will use in an algorithm for finding near optimal policies:

Theorem 10 *For $B > 1$, the equation*

$$2\tau_B m_1(\tau_1, \tau_2, ..., \tau_B) - m_2(\tau_1, \tau_2, ..., \tau_B) = 0, \tag{5.16}$$

has a solution with monotone non-increasing thresholds, i.e., $\tau_B \leq ... \leq \tau_2 \leq \tau_1$ if and only if $\tau_B \geq \bar{\Delta}_{\pi^*}$.

Proof. See Appendix 5.5.8. ■

Algorithm 1 uses this result to find a near optimal policy $\pi \in \Pi^{\text{MT}}$ such that $\bar{\Delta}_\pi - \bar{\Delta}_{\pi^*} \leq \frac{1}{2^{q+1}\mu_H}$. Each iteration in Algorithm 1 halves the interval where the minimum average AoI can be found based on the existence of solution to (10) with the current estimate of the smallest threshold $\hat{\tau}_B$. Accordingly, it is guaranteed that Algorithm 1 finds a solution within a gap to the optimal value that is $\frac{1}{2^{q+1}\mu_H}$.

Algorithm 1 assumes a numerical solver that can solve the transcendental equation in (5.16), however, the exact solution is required only once at the final step while iterations only require verifying the existence of a solution to (10).

Algorithm 1 Find $\pi \in \Pi^{\text{MT}}$ such that $\bar{\Delta}_\pi - \bar{\Delta}_{\pi^*} \leq \frac{1}{2^{q+1}\mu_H}$

Require: $B \geq 1 \wedge q \geq 1$

Ensure: $\bar{\Delta}_\pi - \bar{\Delta}_{\pi^*} \leq \frac{1}{2^{q+1}\mu_H}$

$\tau_B^- \leftarrow \frac{1}{2\mu_H}, \tau_B^+ \leftarrow \frac{1}{\mu_H}$

for $i = 1, 2, ..., q$ **do**

$\hat{\tau}_B \leftarrow \frac{\tau_B^- + \tau_B^+}{2}$

if $\exists \tau_{B-1} \leq ... \leq \tau_2 \leq \tau_1$ s.t. $\tau_{B-1} \geq \hat{\tau}_B$ and $2\hat{\tau}_B m_1(\tau_1, \tau_2, ..., \hat{\tau}_B) - m_2(\tau_1, \tau_2, ..., \hat{\tau}_B) = 0$ **then**

$\tau_B^+ \leftarrow \hat{\tau}_B$

else

$\tau_B^- \leftarrow \hat{\tau}_B$

end if

end for

Solve $2\hat{\tau}_B m_1(\tau_1, \tau_2, ..., \hat{\tau}_B) - m_2(\tau_1, \tau_2, ..., \hat{\tau}_B) = 0$

Return: $\pi = (\tau_1, \tau_2, ..., \hat{\tau}_B)$

5.4 Numerical Results

For battery sizes $B = 1, 2, 3, 4$, the policies in Π^{MT} are numerically optimized giving AoI versus energy arrival rate (Poisson) curves in Fig 5.4. We give the corresponding threshold values in Table 5.1. These results were obtained through exhaustive search for possible threshold values, and Monte Carlo analysis for approximating AoI values in the simulation of the considered system and policies without relying on analytical results. It can be seen that these optimal thresholds and corresponding AoI values (in Table 5.1) agree with Theorem 7. Fig. 5.5 and 5.6 show the dependency of AoI on threshold values τ_1 and τ_2 which is consistent with the result in Theorem 9 for the special case of $B = 2$.

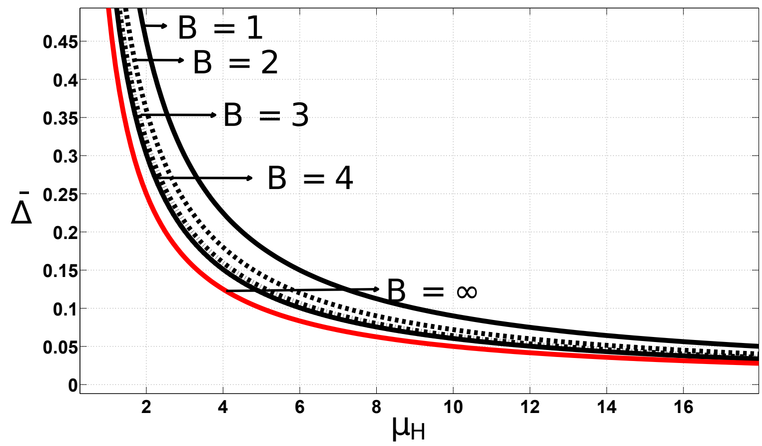


Figure 5.4: AoI versus energy arrival rate (Poisson) for different battery sizes $B = 1, 2, 3, 4$.

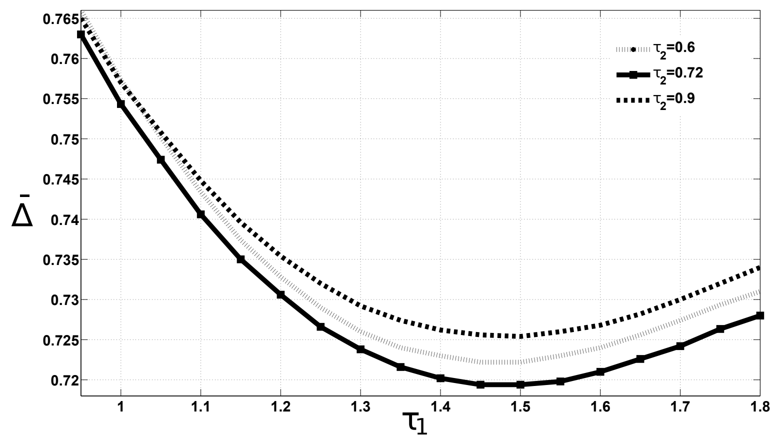


Figure 5.5: AoI versus τ_1 against various τ_2 values for $B = 2$ and $\mu_H = 1$.

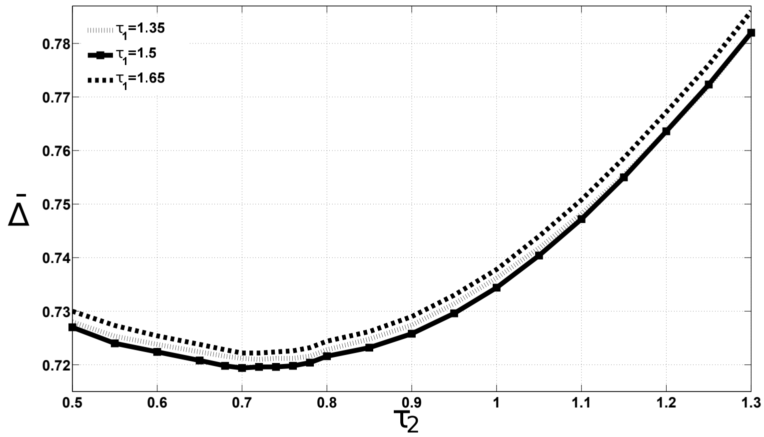


Figure 5.6: AoI versus τ_2 against various τ_1 values for $B = 2$ and $\mu_H = 1$.

Table 5.1: Optimal thresholds for different battery sizes for $\mu_H = 1$

	τ_1	τ_2	τ_3	τ_4	$\bar{\Delta}_{\pi^*}$
$B = 1$	0.90	-	-	-	0.90
$B = 2$	1.5	0.72	-	-	0.72
$B = 3$	1.5	1.2	0.64	-	0.64
$B = 4$	1.5	1.2	0.86	0.604	0.604

5.5 Appendix

5.5.1 The Proof of Theorem 5

In order to prove Theorem 5, we use a modified version of the “vanishing discount factor" approach [82] which consists of 2 steps:

Step 1. Show that for every $\alpha > 0$, there exists a threshold policy that is optimal for solving

$$\min_{\pi\in\Pi^{\text{online}}} \mathbb{E}\left[\int_0^\infty e^{-\alpha t} p(\Delta(t))dt\right].$$

Step 2. Prove that this property still holds when the discount factor α vanishes to zero.

We first discuss *Step 1.* Recall that $\mathcal{F}_t$ represents the information about the energy arrivals and the update policy during $[0, t]$. Given $\mathcal{F}_a$, we are interested in finding the optimal online policy during $[a, \infty)$, which is formulated as

$$\min_{\pi\in\Pi^{\text{online}}} \mathbb{E}\left[\int_a^\infty e^{-\alpha(t-a)} p(\Delta(t))dt\middle|\mathcal{F}_a\right]. \tag{5.17}$$

Observe that, in (5.17), the term $e^{-\alpha(t-a)}$ ensures that the exponential decay always starts from unity so that the problem is independent of a given $\mathcal{F}_a$. In addition, this problem has the following nice property:

Lemma 2 *There exists an optimal solution to (5.17) that depends on $\mathcal{F}_a$ only through* $(\Delta(a), E(a))$*. That is,* $(\Delta(a), E(a))$ *is a sufficient statistic for solving (5.17).*

Proof. In Problem (5.17), the age evolution $\{\Delta(t), t \geq a\}$ is determined by the initial age $\Delta(a)$ at time a and the update policy during $[a, \infty)$. Further, the update policy during $[a, \infty)$ is determined by the initial age $\Delta(a)$, the initial battery level $E(a)$, and the energy counting process $\{H(t) - H(a), t \geq a\}$. Hence, $\{\Delta(t), t \geq a\}$ is determined by $\Delta(a)$, $E(a)$, and $\{H(t) - H(a), t \geq a\}$.

Recall that $\Delta(0)$ and $E(0)$ are fixed. Hence, for any online update policy, the online update decisions during $[0, a]$ depends only on $\{H(t), t \leq a\}$. Hence, $\mathcal{F}_a$ is determined by $\{H(t), t \leq a\}$. Because $\{H(t), t \geq 0\}$ is a compound Poisson process, $\{H(t) - H(a), t \geq a\}$ is independent of $\{H(t), t \leq a\}$. Hence, $\{\Delta(t), t \geq a\}$ depends on $\mathcal{F}_a$ only through $\Delta(a)$ and $E(a)$. By this, $(\Delta(a), E(a))$ is a sufficient statistic for solving (5.17). ■

By using Lemma 2, we can simplify (5.17) as (7.29) and define a cost function $J_\alpha(\Delta(a), E(a))$ which is the optimal objective value of (7.29):

$$\begin{aligned} J_\alpha(\Delta(a), E(a)) :=& \min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^\infty e^{-\alpha(t-a)} p(\Delta(t))dt \middle| \mathcal{F}_a\right] \\ =& \min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^\infty e^{-\alpha(t-a)} p(\Delta(t))dt \middle| \Delta(a), E(a)\right]. \end{aligned} \tag{5.18}$$

Furthermore, one important question is: Given that the previous update occurs at $Z_k = a$, how to choose the next update time Z_{k+1}. This can be formulated as

$$\min_{(Z_1,\ldots,Z_k=a,Z_{k+1},\ldots) \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^\infty e^{-\alpha(t-a)} p(\Delta(t))dt \middle| Z_k = a, \Delta(a) = 0, E(a)\right] , \tag{5.19}$$

where we have used the fact that if $Z_k = a$, then $\Delta(a) = \Delta(Z_k) = 0$.

According to the definition of Π^{online}, Z_{k+1} is a finite Markov time, i.e., stopping time, hence the problem of finding Z_{k+1} for a solution to (5.19) can be formulated as an infinite horizon optimal stopping problem in the interval $[a, \infty)$. We will consider a *gain* [81] process $G = (G_t)_{t \geq a}$ adapted to the filtration $\mathcal{F}_t$ where a stopping time Z_{k+1} for a solution to (5.19) maximizes $\mathbb{E}[G_{Z_{k+1}} \mid \mathcal{F}_a]$ when we choose Z_{k+1} from a family of stopping times based on $\mathcal{F}_t$. Let $\mathfrak{M}_a$ denote this family of Z_{k+1}s which can be expressed as:

$$\mathfrak{M}_a = \{Z_{k+1} \geq a : \Pr(Z_{k+1} < \infty) = 1, \{Z_{k+1} \leq t\} \in \mathcal{F}_t, \forall t \geq a\}.$$

Note that a stopping time in $\mathfrak{M}_a$ may violate energy causality however our definition of the gain process will guarantee that those stopping times cannot be optimal.

We will define the gain process $(G_t)_{t \geq a}$ based on the value of the discounted cost when an update is sent at a particular time t. The gain process $(G_t)_{t \geq a}$ for $E(t) > 0$

corresponds to the additive inverse of this cost and can be written as follows:

$$G_t = -\min_{\pi\in\Pi^{\text{online}}} \mathbb{E}\left[\int_a^\infty e^{-\alpha(w-a)} p(\Delta(w))dw \middle| Z_k = a, Z_{k+1} = t, E(t)\right],$$
$$E(t) > 0. \tag{5.20}$$

Note that the stopping time cannot be at time t when $E(t) = 0$ as there is no energy to send another update in that case. To cover this case, we set G_t to $-\infty$ so that a stopping time Z_{k+1} maximizing $\mathbb{E}[G_{Z_{k+1}} \mid \mathcal{F}_a]$ should satisfy energy causality hence belongs to an online policy. In other words, the stopping time Z_{k+1} in a solution to (5.19) maximizes $\mathbb{E}[G_{Z_{k+1}} \mid \mathcal{F}_a]$ among all the stopping times in $\mathfrak{M}_a$.

Alternatively, the gain process $(G_t)_{t\geq a}$ can be expressed in terms of the cost defined in (7.29) as follows

$$\begin{aligned} G_t &= -\int_a^t e^{-\alpha(w-a)} p(w-a)dw - \mathbb{E}\left[\int_t^\infty e^{-\alpha(w-a)} p(\Delta(w))dw \middle| Z_k = a, Z_{k+1} = t, E(t)\right] \\ &= -\int_a^t e^{-\alpha(w-a)} p(w-a)dw - e^{-\alpha(t-a)} J_\alpha(0, E(t)-1), \end{aligned} \tag{5.21}$$

for $t \geq a$ and $E(t) > 0$.

Let's define $J(0, -1) := \infty$ so that (5.21) holds for the $E(t) = 0$ as well. Notice that the process G_t is driven by the random process $E(t)$ which is not conditioned on any particular value of $E(a)$ while being adapted to the filtration $\mathcal{F}_t$. However, for a policy solving (5.19), the stopping time Z_{k+1} depends on $E(a)$ as it maximizes $\mathbb{E}[G_{Z_{k+1}} \mid \mathcal{F}_a]$ which depends on $E(a)$ through the filtration $\mathcal{F}_a$.

Accordingly, we define the stopping problem of maximizing the expected gain in the given interval $[a, \infty)$ as in the following:

$$\max_{t\in\mathfrak{M}_a} \mathbb{E}[G_t \mid \mathcal{F}_a]. \tag{5.22}$$

Based on this formulation, we will show that the optimal stopping time exists and is given by the following stopping rule for Z_{k+1}:

$$Z_{k+1} = \inf\{t \geq Z_k = a : G_t = S_t\}, \tag{5.23}$$

where S is the Snell envelope [81] for G:

$$S_t = \operatorname*{ess\,sup}_{t'\in\mathfrak{M}_t} \mathbb{E}[G_{t'} \mid \mathcal{F}_t]. \tag{5.24}$$

Showing that Z_{k+1} in (5.23) is finite w.p.1 is sufficient to prove the existence of the optimal stopping time and the optimality of the stopping rule in (5.23)(see [81, Theorem 2.2.]). Consider the lemma below and its proof in order to see the finiteness of Z_{k+1} in (5.23):

Lemma 3 *For the stopping rule in (5.23) Z_{k+1} is finite w.p.1, i.e.,* $\Pr(Z_{k+1} < \infty) = 1$.

Proof. Consider the Markov time Q_{k+1} which is defined as follows:

$$Q_{k+1} := \inf\{t \geq Z_k = a : E(t) = B, G_t = S_t\}. \tag{5.25}$$

Clearly, the stopping time Z_{k+1} chosen in (5.23) is earlier than Q_{k+1} as Q_{k+1} has an additional stopping condition $E(t) = B$. This means that if $\Pr(Q_{k+1} < \infty) = 1$, then $\Pr(Z_{k+1} < \infty) = 1$.

Accordingly, for the proof of this lemma, it is sufficient to show that Q_{k+1} is finite w.p.1. We will show this by showing the finiteness of (i) the first time $t \geq Z_k = a$ such that $E(t) = B$, and (ii) the duration between this time and the Markov time Q_{k+1}. Note that $E(t) = B$ condition is always satisfied after it reached for the first time. Let R_{k+1} be the Markov time representing the first time when $E(t) = B$ is satisfied:

$$R_{k+1} := \inf\{t \geq Z_k = a : E(t) = B\}. \tag{5.26}$$

(i) Observe that the Markov time R_{k+1} is finite w.p.1 as it is stochastically dominated by $a + Y_B$ where Y_B is an Erlang distributed random variable with parameter B which obeys (5.7) and $\Pr(Y_B < \infty) = 1$.

(ii) In order to see that $Q_{k+1} - R_{k+1}$ is also finite, consider the time period after R_{k+1}, i.e., $[R_{k+1}, \infty)$. As $E(t) = B$ for any $t \geq R_{k+1}$, the evolution of G_t becomes deterministic after $t \geq R_{k+1}$:

$$G_t = -\int_a^t e^{-\alpha(w-a)} p(w-a)dw - e^{-\alpha(t-a)} J_\alpha(0, B-1) \quad , \tag{5.27}$$

for $t \geq R_{k+1}$.

On the other hand, for $t \geq R_{k+1}$, the Snell envelope is $S_t = \operatorname{ess\,sup}_{t' \in \mathfrak{M}_t} G_{t'} = \sup_{t' \geq t} G_{t'}$. We will show that G_t is always non-increasing after some finite time so that $S_t = G_t$ is always satisfied after that time.

In order to see this, consider the change in G_t for $t \geq R_{k+1}$. As

$$-\frac{\partial}{\partial t}\left[\int_a^t e^{-\alpha(w-a)}p(w-a)dw + e^{-\alpha(t-a)}J_\alpha(0,B-1)\right] = \\ e^{-\alpha(t-a)}\left(\alpha J_\alpha(0,B-1) - p(t-a)\right), \tag{5.28}$$

and $p(t-a)$ is non-decreasing, for $t \geq R_{k+1}$, G_t is non-increasing if $t \geq t_c$ for some t_c such that

$$t_c := \inf\{t \geq a : p(t-a) = \alpha J_\alpha(0,B-1)\}. \tag{5.29}$$

This implies that, for $t \geq \max\{R_{k+1}, t_c\}$, $G_t = \sup_{t' \geq t} G_{t'}$ and hence $S_t = G_t$. Accordingly, the stopping conditions of Q_{k+1} are satisfied for the first time when $t = \max\{R_{k+1}, t_c\}$ which means $Q_{k+1} = \max\{R_{k+1}, t_c\}$.

As $\alpha J_\alpha(0,B-1)$ is finite, t_c is finite which implies Q_{k+1} is finite w.p.1 as R_{k+1} is finite w.p.1. This completes the proof. ■

We just showed that the Markov time in (5.23) is finite w.p.1 and this means that it is the optimal stopping time by [81, Theorem 2.2.]. Next, we show that the optimal stopping rule in (5.23) is a threshold policy by using the properties of the cost function in (7.29). To relate the optimal stopping time and the cost function in (7.29), we will express the Snell envelope in an alternative way.

Notice that the Snell envelope can be written by substituting (5.20) in (5.24) as follows:

$$S_t = \operatorname*{ess\,sup}_{t' \in \mathfrak{M}_t} - \min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^\infty e^{-\alpha(w-a)}p(\Delta(w))dw \middle| Z_k = a, Z_{k+1} = t', \mathcal{F}_t\right]. \tag{5.30}$$

Hence,

$$S_t = -\min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^\infty e^{-\alpha(w-a)}p(\Delta(w))dw \middle| Z_k = a, Z_{k+1} \geq t, \mathcal{F}_t\right]. \tag{5.31}$$

Accordingly, using the definition of $J_\alpha(\Delta(a), E(a))$, we can write

$$S_t = -\int_a^t e^{-\alpha(w-a)}p(w-a)dw + e^{-\alpha(t-a)}J_\alpha(\Delta(t), E(t)). \tag{5.32}$$

Therefore, as the first terms in (5.27) and (5.32) are identical, the optimal stopping rule in (5.23) is equivalent to

$$Z_{k+1} = \inf\{t \geq Z_k = a : J_\alpha(0, E(t)-1) = J_\alpha(\Delta(t), E(t))\}. \tag{5.33}$$

Next, we show that the stopping rule in (5.33) is a threshold rule in age. In order to show this, let us define the function $\rho_\alpha(\cdot) : \{0, 1, ..., B\} \to [0, \infty)$ such that:

$$\rho_\alpha(\ell) := \inf\{w \geq 0 : J_\alpha(0, \ell - 1) = J_\alpha(w, \ell)\}. \tag{5.34}$$

We can show that for any $\Delta \geq \rho_\alpha(\ell)$, it is guaranteed that $J_\alpha(0, \ell - 1) = J_\alpha(\Delta, \ell)$ due to the following reasons:

- For any Δ and $\ell \in \{0, 1, 2, .., B\}$, $J_\alpha(\Delta, \ell)$ is smaller than or equal to $J_\alpha(0, \ell - 1)$ as :

$$\begin{aligned} J_\alpha(\Delta, \ell) &= \min_{\pi \in \Pi^{\text{online}}} e^a \mathbb{E}\left[\int_a^\infty e^{-\alpha w} p(\Delta(w)) dw \middle| Z_k = t_a - \Delta, Z_{k+1} \geq t_a, E(t_a) = \ell\right] \\ &\leq \min_{\pi \in \Pi^{\text{online}}} e^a \mathbb{E}\left[\int_a^\infty e^{-\alpha w} p(\Delta(w)) dw \middle| Z_k = t_a - \Delta, Z_{k+1} = a, E(t) = \ell\right] \\ &= J_\alpha(0, \ell - 1), \end{aligned}$$

  where the inequality is true as the expectation is conditioned on policies with $Z_{k+1} = t_a$.

- For any $\ell \in \{0, 1, 2, .., B\}$, $J_\alpha(\Delta, \ell)$ is non-decreasing in Δ as :

$$\begin{aligned} J_\alpha(\Delta, \ell) = &\min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^{Z_{k+1}} e^{-\alpha(w-a)} p(w + \Delta - t_a) dw \middle| \theta(\Delta)\right] \\ &+ \mathbb{E}\left[\int_{Z_{k+1}}^\infty e^{-\alpha(w-a)} p(\Delta(w)) dw \middle| \theta(\Delta)\right] \\ \leq &\min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^{Z_{k+1}} e^{-\alpha(w-a)} p(w + \Delta' - t_a) dw \middle| \theta(\Delta)\right] \\ &+ \mathbb{E}\left[\int_{Z_{k+1}}^\infty e^{-\alpha(w-a)} p(\Delta(w)) dw \middle| \theta(\Delta)\right] \\ = &\min_{\pi \in \Pi^{\text{online}}} \mathbb{E}\left[\int_a^{Z_{k+1}} e^{-\alpha(w-a)} p(\Delta(w)) dw \middle| \theta(\Delta')\right] \\ &+ \mathbb{E}\left[\int_{Z_{k+1}}^\infty e^{-\alpha(w-a)} p(\Delta(w)) dw \middle| \theta(\Delta')\right] \\ = &J_\alpha(\Delta', \ell), \end{aligned}$$

  for any $\Delta' \geq \Delta$ and $\theta(\Delta) := (Z_k = t_a - \Delta, Z_{k+1} \geq t_a, E(t_a) = \ell)$ where the inequality follows from the fact that $p(\cdot)$ is non-decreasing and the second equality is due to that, given Z_{k+1}, the integrated values are conditionally independent from Z_k.

Accordingly, $J_\alpha(\Delta, \ell) = J_\alpha(0, \ell-1)$ for any $\ell \in \{0, 1, 2, .., B\}$ and $\Delta \geq \rho_\alpha(\ell)$. Therefore, the stopping rule in (5.33) is equivalent to:

$$Z_{k+1} = \inf\{t \geq Z_k = a : \Delta(t) \geq \rho_\alpha(E(t))\}, \tag{5.35}$$

for $\ell \in \{0, 1, 2, .., B\}$.

We showed that the stopping rule in (5.35) gives the optimal stopping time Z_{k+1} for a policy solving (5.19). Now, we can start discussing *Step 2* in order to show that the optimal stopping rule with the same structure also gives a solution to (5.5).

In this part (*Step 2*) of the proof, we will consider the optimal stopping rules in (5.35) while the discount factor α is vanishing to zero. Notice that the policy solving (5.19) is identified by $\rho_\alpha(\ell)$ due to (5.35). Let π_α and $\Delta_{\pi_\alpha}(t)$ be a policy obeying (5.35) and solving (5.19) for discount factor α and the age at time t for that policy, respectively. We will show the following

$$\lim_{\beta\downarrow 0} \lim_{t_f\to\infty} \frac{\int_0^{t_f} \mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right]dt}{t_f} = \inf_{\pi\in\Pi^{\text{online}}} \limsup_{t_f\to\infty} \frac{\int_0^{t_f} \mathbb{E}\left[p(\Delta_\pi(t))\right]dt}{t_f}, \tag{5.36}$$

which implies that for any $\{\beta_n\}_{n\geq 1} \downarrow 0$ sequence, π_{β_n} converges to the policy solving (5.5).

To prove the equivalence in (5.36), we will use Feller's Tauberian theorem [84] (also see the Tauberian theorem in [85]) which can be stated as follows:

Theorem 11 ***(Feller 1971)*** *Let $f(t)$ be a Lebesgue-measurable, bounded, real function. Then,*

$$\begin{aligned}
\liminf_{t_f\to\infty} \frac{\int_0^{t_f} f(t)dt}{t_f} &\leq \liminf_{\alpha\downarrow 0} \alpha \int_0^{t_f} e^{-\alpha(t-a)} f(t)dt \\
\leq \limsup_{\alpha\downarrow 0} \alpha \int_0^{t_f} e^{-\alpha(t-a)} f(t)dt &\leq \limsup_{t_f\to\infty} \frac{\int_0^{t_f} f(t)dt}{t_f}.
\end{aligned} \tag{5.37}$$

Moreover, if the central inequality is an equality, then all inequalities are equalities.

This theorem can be applied for the function $f(t) = \mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right]$ where $\beta > 0$ [4]. To simplify the inequalities for this case, let's define a function $J_{\alpha;\beta}(\Delta(a), E(a))$ for $\beta > 0$

[4] Note that the function $\mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right]$ is Lebesgue-measurable (as $p(\cdot)$ is non-decreasing) and bounded (as X_ks are bounded w.p.1 for a policy obeying (5.35)).

such that:

$$J_{\alpha;\beta}(\Delta(a), E(a)) := \int_a^{\infty} e^{-\alpha(t-a)} \mathbb{E}\left[p(\Delta_{\pi_\beta}(t)) \middle| \Delta(a), E(a)\right] dt. \tag{5.38}$$

Note that for $a = 0$:

$$J_{\alpha;\beta}(0, 0) := \int_a^{\infty} e^{-\alpha(t-a)} \mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right] dt.$$

Accordingly, we can apply Feller's Tauberian theorem for $f(t) = \mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right]$ when $a = 0$ giving:

$$\begin{aligned} &\liminf_{t_f\to\infty} \frac{\int_0^{t_f} \mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right] dt}{t_f} \leq \liminf_{\alpha\downarrow 0} \alpha J_{\alpha;\beta}(0, 0) \leq \\ &\limsup_{\alpha\downarrow 0} \alpha J_{\alpha;\beta}(0, 0) \leq \limsup_{t_f\to\infty} \frac{\int_0^{t_f} \mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right] dt}{t_f}. \end{aligned} \tag{5.39}$$

We can show that the inequalities in (5.39) are satisfied with equality for any π_β with $\beta > 0$ as $\lim_{t_f\to\infty} \frac{\int_0^{t_f} \mathbb{E}\left[\Delta_{\pi_\beta}(t)\right] dt}{t_f}$ exists for any π_β with $\beta > 0$. To see this, consider the following lemma:

Lemma 4 *For $\alpha > 0$ and $\{Z_{k+1}, k \geq 0\}$ with Z_{k+1} as in (5.35), the following holds:*

$$\lim_{t_f\to\infty} \frac{\int_0^{t_f} \mathbb{E}\left[p(\Delta_{\pi_\alpha}(t))\right] dt}{t_f} = \frac{\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{E}[p(X_k)]}{\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} X_k} \quad \text{w.p.1.} \tag{5.40}$$

Proof. The proof of Lemma 3 showed that for $Z_k = a$ and optimal stopping time solving (5.22) it is true that $\Pr(X_{k+1} \geq x) \leq \Pr(t_c - t_a + Y_B \geq x)$ where t_c is the deterministic time defined in (5.29) and Y_B is an Erlang distributed with parameter B which obeys (5.7). Accordingly, $\mathbb{E}[p(X_{k+1})]$ is finite as $\mathbb{E}[p(\alpha J_\alpha(0, B) + Y_B)]$ is finite for $\alpha > 0$. On the other hand, $\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} X_k < \infty$ w.p.1 and $\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} X_k > \frac{1}{\mu_H}$ w.p.1 due to the energy causality constraint. Therefore, we can apply the derivation steps in [86, Theorem 5.4.5] and obtain (5.40). This completes the proof. ■

Lemma 4 and (5.39) imply the following for for $a = 0$ and $\beta > 0$:

$$\lim_{\alpha\downarrow 0} \alpha J_{\alpha;\beta}(0, 0) = \lim_{t_f\to\infty} \frac{\int_0^{t_f} \mathbb{E}\left[p(\Delta_{\pi_\beta}(t))\right] dt}{t_f}. \tag{5.41}$$

Now, consider an arbitrary online policy π for which $\mathbb{E}\left[p(\Delta_\pi(t))\right]$ is Lebesgue-measurable and bounded, then apply Feller's Tauberian theorem for $f(t) = \mathbb{E}\left[p(\Delta_\pi(t))\right]$ giving the following inequality when $t_a = 0$:

$$\limsup_{\alpha \downarrow 0} \alpha \int_0^\infty e^{-\alpha(t-a)} \mathbb{E}\left[p(\Delta_\pi(t))\right] dt \leq \limsup_{t_f \to \infty} \frac{\int_0^{t_f} \mathbb{E}\left[p(\Delta_\pi(t))\right] dt}{t_f}. \tag{5.42}$$

Note that for $\alpha > 0$, $J_{\alpha;\beta}(0,0)$ is minimized for $\alpha = \beta$, hence:

$$\lim_{\beta \downarrow 0} \lim_{\alpha \downarrow 0} \alpha J_{\alpha;\beta}(0,0) = \inf_{\beta > 0} \lim_{\alpha \downarrow 0} \alpha J_{\alpha;\beta}(0,0) \leq$$
$$\limsup_{\alpha \downarrow 0} \alpha \int_0^\infty e^{-\alpha(t-a)} \mathbb{E}\left[p(\Delta_\pi(t))\right] dt. \tag{5.43}$$

Combining (5.41), (5.42) and (5.43), we get (5.36). This completes the proof.

5.5.2 The Proof of Theorem 6

Theorem 6 follows from the proof of Theorem 5. To prove the theorem it is sufficient to show that for any $\alpha > 0$, $\rho_\alpha(\ell)$ (see (5.34)) is non-increasing in ℓ as this guarantees that the monotonicity of optimal thresholds holds for any sequence of α values that vanishes to zero. To see this, consider the following lemma and the argument provided below its proof:

Lemma 5 *For $J(\cdot,\cdot)$ is the function defined in (7.29), $J_\alpha(0,\ell) - J_\alpha(0,\ell+1)$ is non-increasing in $\ell \in \{0, 1, ..., B-1\}$ for any $\alpha \geq 0$.*

Proof.

First, consider the alternative formulation of $J_\alpha(r, \ell+1)$ in below:

$$J_\alpha(r, \ell+1) = \min_{\pi \in \Pi^{\text{online}}} e^a \mathbb{E}[\mathbb{E}\left[\int_a^\infty e^{-\alpha t} p(\Delta(t)) dt \middle| Z_{k+1}, \Delta(t_a) = r, E(t_a) = \ell + 1\right]],$$

where the outer expectation is taken over Z_{k+1}.

Let

$$K_{r,\ell+1}(z,\sigma) := \Pr\left(Z_{k+1} = z, H(z) - H(a) = \sigma | \Delta(t_a) = r, E(t_a) = \ell + 1\right)$$

be the joint distribution of $Z_{k+1} \in \mathfrak{M}_a$ and the energy harvested during $[a, z]$. Then, we can write $J_\alpha(r, \ell + 1)$ as follows:

$$J_\alpha(r, \ell+1) = \min_{Z_{k+1}\in\mathfrak{M}_a} \sum_{\sigma=0}^{\infty} \int_{t_a}^{\infty} K_{r,\ell+1}(z,\sigma) e^a \left[\int_a^z e^{-\alpha t} p(\Delta(t))dt + e^{-\alpha z} J_\alpha(0, \min\{\ell+\sigma, B-1\}) \right] dz. \tag{5.44}$$

Similarly,

$$J_\alpha(r, \ell+2) = \min_{Z_{k+1}\in\mathfrak{M}_a} \sum_{\sigma=0}^{\infty} \int_{t_a}^{\infty} K_{r,\ell+2}(z,\sigma) e^a \left[\int_a^z e^{-\alpha t} p(\Delta(t))dt + e^{-\alpha z} J_\alpha(0, \min\{\ell+1+\sigma, B-1\}) \right] dz. \tag{5.45}$$

Now, let $K^*_{r,\ell+2}(z, \sigma)$ be the distribution corresponding to the update time $Z_{k+1} \in \mathfrak{M}_a$ that is optimal in (5.45), which means:

$$J_\alpha(r, \ell+2) = \sum_{\sigma=0}^{\infty} \int_{t_a}^{\infty} K^*_{r,\ell+2}(z,\sigma) e^a \left[\int_a^z e^{-\alpha t} p(\Delta(t))dt + e^{-\alpha z} J_\alpha(0, \min\{\ell+1+\sigma, B-1\}) \right] dz. \tag{5.46}$$

Clearly, $K^*_{r,\ell+2}(z, \sigma)$ is not necessarily the joint distribution corresponding the update time $Z_{k+1} \in \mathfrak{M}_a$ that is optimal for (5.44), hence:

$$J_\alpha(r, \ell+1) \leq \sum_{\sigma=0}^{\infty} \int_{t_a}^{\infty} K^*_{r,\ell+2}(z,\sigma) [\int_a^z e^{-\alpha(t-a)} p(\Delta(t))dt + e^{-\alpha(z-a)} J_\alpha(0, \min\{\ell+\sigma, B-1\})] dz. \tag{5.47}$$

Combining (5.46) and (5.47) gives:

$$J_\alpha(r, \ell+1) - J_\alpha(r, \ell+2) \leq \sum_{\sigma=0}^{\infty} \int_a^{\infty} K^*_{r,\ell+2}(z,\sigma) e^{-\alpha(z-a)} \times$$
$$[J_\alpha(0, \min\{\ell+\sigma, B-1\}) - J_\alpha(0, \min\{\ell+1+\sigma, B-1\})] dz. \tag{5.48}$$

which implies :

$$J_\alpha(r, \ell+1) - J_\alpha(r, \ell+2) \leq \max_{\sigma\in\{0,1,\ldots,B-\ell\}} J_\alpha(0, \min\{\ell+\sigma, B-1\}) - J_\alpha(0, \min\{\ell+1+\sigma, B-1\})$$

Now, consider the case when $r = 0$ and $\ell = B - 2$ for (5.48):

$$J_\alpha(0, B-1) - J_\alpha(0, B) \leq$$
$$\sum_{\sigma=0}^{\infty} \int_a^{\infty} K^*_{r,\ell+2}(z,\sigma) e^{-\alpha(z-a)} [J_\alpha(0, \min\{B-2+\sigma, B-1\}) -$$
$$J_\alpha(0, \min\{B-1+\sigma, B-1\})] dz, \tag{5.49}$$

which implies:

$$J_\alpha(0, B-1) - J_\alpha(0, B) \leq J_\alpha(0, B-2) - J_\alpha(0, B-1). \tag{5.50}$$

Suppose that the inequality below is true for $j \geq \ell + 1$:

$$J_\alpha(0, j+1) - J_\alpha(0, j+2) \leq J_\alpha(0, j) - J_\alpha(0, j+1). \tag{5.51}$$

Then, we have:

$$\begin{aligned}
&J_\alpha(0, \ell+1) - J_\alpha(0, \ell+2) \leq \\
&\leq \sum_{\sigma=0}^{\infty} \int_a^\infty K^*_{r,\ell+2}(z, \sigma) e^{-\alpha(z-a)} \times \\
&[J_\alpha(0, \min\{\ell+\sigma, B-1\}) - J_\alpha(0, \min\{\ell+1+\sigma, B-1\})]dz \\
&\leq \int_a^\infty K^*(z, 0) e^{-\alpha(z-a)} [J_\alpha(0, \ell) - J_\alpha(0, \ell+1)]dz + \sum_{\sigma=1}^{\infty} \\
&\int_a^\infty K^*_{r,\ell+2}(z, \sigma) e^{-\alpha(z-a)} [J_\alpha(0, \ell+1) - J_\alpha(0, \ell+2)]dz \\
&\leq J_\alpha(0, \ell) - J_\alpha(0, \ell+1).
\end{aligned} \tag{5.52}$$

This means that the inequality (5.51) is also true for $j = \ell$ so is for any $j = 0, 1, ..., B-2$ by induction. Combining this and (5.49):

$$J_\alpha(r, \ell+1) - J_\alpha(r, \ell+2) \leq J_\alpha(0, \ell) - J_\alpha(0, \ell+1), \tag{5.53}$$

for $\alpha \geq 0$, $r \geq 0$ and ■

Lemma 5 shows that $\rho_\alpha(\ell)$ is non-increasing in ℓ for $\alpha > 0$. It is sufficient to consider (5.53) when $r = \rho_\alpha(\ell)$:

$$0 = J_\alpha(0, \ell-1) - J_\alpha(\rho_\alpha(\ell), \ell) \leq J_\alpha(0, \ell-2) - J_\alpha(\rho_\alpha(\ell), \ell-1), \tag{5.54}$$

which implies $\rho_\alpha(\ell-1) \geq \rho_\alpha(\ell)$ combining

$$J_\alpha(0, \ell-2) - J_\alpha(\rho_\alpha(\ell-1), \ell-1)$$

and that $J_\alpha(r, \ell-1)$ is non-decreasing [5] in r. Accordingly, the optimal policies solving (5.17) are monotone threshold policies, i.e., $\pi_\alpha \in \Pi^{MT}$ for any $\alpha > 0$.

[5] This fact is provided in the proof of Theorem 5.

5.5.3 The Proof of Lemma 1

Let $\tau_{B+1} = 0$. Then, consider:

$$
\begin{aligned}
\frac{\partial}{\partial \tau_i}\mathbb{E}\left[X^2\right] &= \frac{\partial}{\partial \tau_i}\int_0^{\infty} 2x\Pr(X \geq x)dx \\
&= \frac{\partial}{\partial \tau_i}\sum_{i=0}^{B}\int_{\tau_{i+1}}^{\tau_i} 2x\Pr(X \geq x)dx \\
&= 2\frac{\partial}{\partial \tau_i}\left[\int_{\tau_{i+1}}^{\tau_i} x\Pr(X \geq x)dx + \int_{\tau_i}^{\tau_{i-1}} x\Pr(X \geq x)dx\right] \\
&= 2\tau_i\frac{\partial}{\partial \tau_i}\int_{\tau_{i+1}}^{\tau_{i-1}} \Pr(X \geq x)dx \\
&= 2\tau_i\frac{\partial}{\partial \tau_i}\sum_{i=0}^{B}\int_{\tau_{i+1}}^{\tau_i} \Pr(X \geq x)dx = 2\tau_i\frac{\partial}{\partial \tau_i}\mathbb{E}\left[X\right],
\end{aligned}
$$

for $i = 0, 1, ..., B$.

5.5.4 Useful Results for Asymptotic Properties

Lemma 6, 7 and 8 provide some useful results that combine ergodicity properties and renewal-reward theorem for a DTMC with transition probabilities in (5.9).

Lemma 6 *The DTMC with the transition probabilities in (5.9) is ergodic for a monotone threshold policy where τ_1 is finite.*

Proof. Consider an energy state j in $[0, B-1]$. We will show that any other energy state i is reachable from j in at most $B-1$ steps with a positive probability. For $i \geq j$, the higher energy state i is reachable from j in one step with a positive probability as for $i = B-1$, $\Pr(Y_{B-j} \leq \tau_{B-1})$ is strictly positive and for $j \leq i < B-1$:

$$
\begin{aligned}
&\Pr(Y_{1+i-j} \leq \tau_i) - \Pr(Y_{2+i-j} \leq \tau_{i+1}) \geq \\
&\Pr(Y_{1+i-j} \leq \tau_{i+1}) - \Pr(Y_{2+i-j} \leq \tau_{i+1}) > 0,
\end{aligned}
$$

as $\tau_{i+1} \leq \tau_i$ and $i - j \geq 0$.

Similarly, the energy state $i = j-1$ for $j = 1, ..., B-1$ can be reached from j with a probability $1 - \Pr(Y_1 \leq \tau_j)$ which is strictly positive as τ_j is finite. This means that any state $i < j$ can be reached from j in at most $B-1$ steps with a positive probability.
■

Lemma 7 *For monotone threshold policies with finite τ_1, the following is true:*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} X_k = \sum_{j=0}^{B-1} \mathbb{E}\left[X \mid E = j\right] \Pr(E = j) \quad \textit{w.p.1.} \tag{5.55}$$

$$\lim_{n\to+\infty} \frac{1}{2n} \sum_{k=0}^{n} \mathbb{E}[X_k^2] = \frac{1}{2} \sum_{j=0}^{B-1} \mathbb{E}\left[X^2 \mid E = j\right] \Pr(E = j), \tag{5.56}$$

where $\Pr(E = j)$ *is the steady-state probability for energy state j,* $\mathbb{E}\left[X \mid E = j\right] \triangleq \mathbb{E}\left[X_k \mid E(Z_k) = j\right]$ *and* $\mathbb{E}\left[X^2 \mid E = j\right] \triangleq \mathbb{E}\left[X_k^2 \mid E(Z_k) = j\right]$.

Proof. Consider:

$$\frac{1}{n} \sum_{k=0}^{n} X_k = \frac{1}{n} \sum_{j=0}^{B-1} \sum_{\substack{k\in[0,n] \\ E(Z_k)=j}} X_k = \frac{1}{n} \sum_{j=0}^{B-1} \sum_{\ell=0}^{L_j} X_{\ell;j},$$

where L_j is the number of ks in $[0, n]$ such that $E(Z_k) = j$ and $X_{\ell;j}$ is a r.v. with the CDF $\Pr(X_{\ell;j} \leq x) = \Pr(X_k \leq x \mid E(Z_k) = j)$ for some k.

Note that the sequence $X_{0;j}, X_{1;j}, ..., X_{L_j;j}$ is i.i.d. for any j and their mean is bounded as all thresholds are finite, hence:

$$\lim_{L_j\to\infty} \frac{1}{L_j} \sum_{\ell=0}^{L_j} X_{\ell;j} = \mathbb{E}\left[X \mid E = j\right], w.p.1.$$

Due to the ergodicity of $E(Z_k)$s (Lemma 6):

$$\lim_{n\to\infty} \frac{L_j}{n} = \Pr(E = j), w.p.1.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} X_k &= \lim_{n\to\infty} \sum_{j=0}^{B-1} \frac{L_j}{n} \left(\frac{1}{L_j} \sum_{\ell=0}^{L_j} X_{\ell;j}\right), \\ &= \sum_{j=0}^{B-1} \mathbb{E}\left[X \mid E = j\right] \Pr(E = j), w.p.1.\end{aligned}$$

Similarly,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\mathbb{E}[X_k^2] = \lim_{n\to\infty}\sum_{j=0}^{B-1}\frac{L_j}{n}(\frac{1}{L_j}\sum_{\ell=0}^{L_j}X_{\ell;j}^2)$$
$$= \sum_{j=0}^{B-1}\mathbb{E}\left[X^2 \mid E=j\right]\Pr(E=j), w.p.1.$$

■

Lemma 8 *For a threshold policy where τ_1 is finite, the average age $\bar{\Delta}$ is finite (w.p.1) and given by the following expression.*

$$\bar{\Delta} = \frac{\lim_{n\to+\infty}\frac{1}{2n}\sum_{k=0}^{n}\mathbb{E}[X_k^2]}{\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n}X_k} \quad w.p.1. \tag{5.57}$$

Proof. The proof is a generalization of Theorem 5.4.5 in [86] for the case where X_ks are non-i.i.d. but the limits still exist (w.p.1). When X_ks are i.i.d. with $\mathbb{E}[X_k] < \infty$ and $\mathbb{E}[X_k^2] < \infty$, the convergence (w.p.1) of the limits is guaranteed. ■

5.5.5 The Proof of Theorem 7

Theorem 7 follows from the proof of Theorem 5. The proof of Lemma 3 shows that given that $Z_k = a$ is the last update time and $E(t') = B$ for some $t' > a$, the condition $S_t = G_t$ is satisfied for the first time when $t \geq \{t', t_c\}$ (see (5.29)). This means that $\rho_\alpha(B) = \alpha J_\alpha(0, B-1)$ for $\rho_\alpha(E(t))$ in (5.35). Accordingly,

$$p(\tau_B^*) = \lim_{\alpha\downarrow 0}\rho_\alpha(B) = \lim_{\alpha\downarrow 0}\alpha J_\alpha(0, B-1)$$
$$= \min_{\pi\in\Pi^{\text{online}}}\limsup_{t_f\to\infty}\frac{\int_0^{t_f}\mathbb{E}\left[p(\Delta_\pi(t)) \mid E(0)=B\right]dt}{t_f} = \bar{p}_{\pi^*},$$

which follows from the application of Feller's Tauberian theorem (applying Theorem 11 for $f(t) = \mathbb{E}\left[p(\Delta_\pi(t)) \mid E(0)=B\right]$). This completes the proof.

5.5.6 The Proof of Theorem 8

By Lemma 8 and Lemma 7, $\bar{\Delta}$ for $B = 1$ can be computed as follows

$$\bar{\Delta} = \frac{1}{2}\frac{\mathbb{E}\left[X^2 \mid E=0\right]\Pr(E=0)}{\mathbb{E}\left[X \mid E=0\right]\Pr(E=0)}, \tag{5.58}$$

where $\Pr(E=0)=1$, $\mathbb{E}\left[X^2 \mid E=0\right] = \tau_1^2+(\frac{2}{\mu_H^2}+\frac{2}{\mu_H}\tau_1)e^{-\mu_H\tau_1}$ and $\mathbb{E}\left[X \mid E=0\right] = \tau_1 + \frac{1}{\mu_H}e^{-\mu_H\tau_1}$. Accordingly, $\bar{\Delta}$ is given by (5.12). By Theorem 7, $\tau_1^* = \bar{\Delta}_{\pi^*}$ and combining this with (5.12) results in

$$\mu_H\tau_1^* = \frac{\frac{1}{2}(\mu_H\tau_1^*)^2 + e^{-\mu_H\tau_1^*}(\mu_H\tau_1 + 1)}{\mu_H\tau_1^* + e^{-\mu_H\tau_1^*}}. \tag{5.59}$$

Solving (5.59) gives that $(\tau_1^*)^2 = \frac{2}{\mu_H}e^{-\mu_H\tau_1^*}$ which means $\tau_1^* = \frac{1}{\mu_H}2W(\frac{1}{\sqrt{2}})$.

5.5.7 The Proof of Theorem 9

By Lemma 8 and Lemma 7, $\bar{\Delta}$ for $B=2$ is the following:

$$\bar{\Delta} = \frac{1}{2}\frac{\mathbb{E}\left[X^2 \mid E=0\right]\Pr(E=0) + \mathbb{E}\left[X^2 \mid E=1\right]\Pr(E=1)}{\mathbb{E}\left[X \mid E=0\right]\Pr(E=0) + \mathbb{E}\left[X \mid E=1\right]\Pr(E=1)}. \tag{5.60}$$

The probability of being in $E=1$, i.e. $\Pr(E=1)$ can be solved using:

$$\Pr(E=1) = \sum_{j=0}^{1}\Pr(E(Z_{k+1})=1 \mid E(Z_k)=j)\Pr(E=j). \tag{5.61}$$

Combining (5.61) and (5.8),

$$\Pr(E=1) = \frac{e^{-\mu_H\tau_1}}{1-e^{-\mu_H\tau_1}\mu_H\tau_1}. \tag{5.62}$$

Now, we can obtain $\mathbb{E}\left[X^2 \mid E=j\right]$, $\mathbb{E}\left[X \mid E=j\right]$ using (5.8). Combining these with (5.62) and substituting in (5.60) gives (9).

5.5.8 The Proof of Theorem 10

First, we show that $\tau_B \geq \bar{\Delta}_{\pi^*}$ is *necessary* to find a solution to (5.16) with monotonic non-increasing thresholds. Then, we show that this condition is also *sufficient*.

The *necessity* part of the proof follows from the fact that $\tau_B = \bar{\Delta}_\pi$ for any solution of (5.16), as $\bar{\Delta}_\pi = m_1(\tau_1,\tau_2,...,\tau_B)/2m_2(\tau_1,\tau_2,...,\tau_B)$ by Lemma 8 and Lemma 7. Therefore, by the optimality of $\bar{\Delta}_{\pi^*}$, $\tau_B \geq \bar{\Delta}_{\pi^*}$ must hold for any solution of (5.16).

Now, we consider the *sufficiency* part of the proof where it is useful to define a function $\phi : [0,\infty)^B \to \mathbb{R}$ as follows:

$$\phi(\tau_B, \tau_{B-1}-\tau_B, ..., \tau_1-\tau_2) \triangleq 2\tau_B m_1(\tau_1,\tau_2,...,\tau_B) - m_2(\tau_1,\tau_2,...,\tau_B).$$

Using this definition, (5.16) can be written as,

$$\phi(\tau_B, \tau_{B-1} - \tau_B, ..., \tau_1 - \tau_2) = 0.$$

We need to show that given $\tau_B \geq \bar{\Delta}_{\pi^*}$, one can find a set of non-negative real numbers $d_1, ..., d_{B-1}$ such that $\phi(\tau_B, d_{B-1}, ..., d_1) = 0$. Accordingly, τ_B and $d_1, ..., d_{B-1}$ constitute a solution to (5.16) with monotonic non-decreasing thresholds where $\tau_i = \tau_{i+1} + d_i$, for $i = 1, ..., B-1$. In order to prove this, let us start with the optimal policy $\pi^* = (\tau_1^*, \tau_2^* ..., \tau_B^*)$ where we know that $\tau_B^* = \bar{\Delta}_{\pi^*}$ by Theorem 7. Starting from the optimal policy π^*, the policy will be modified following the procedure below:

- *Phase 1:* Modify the policy $\pi^{(+)} = (\tau_1^{(+)}, \tau_2^{(+)} ..., \tau_B^{(+)})$ from the previous phase to the policy $\pi^{(-)} = (\tau_1^{(-)}, \tau_2^{(-)} ..., \tau_B^{(-)})$ so that $\tau_B^{(-)} = \min\{\tau_{B-1}^{(+)}, \tau_B\}$ while $\tau_i^{(-)} = \tau_i^{(+)}$, for $i = 1, ..., B-1$. Then, go to *Phase 2* with policy $\pi^{(-)}$.
- *Phase 2:* Modify the policy $\pi^{(-)} = (\tau_1^{(-)}, \tau_2^{(-)} ..., \tau_B^{(-)})$ from the previous phase to the policy $\pi^{(+)} = (\tau_1^{(+)}, \tau_2^{(+)} ..., \tau_B^{(+)})$ so that $\tau_B^{(+)} = \tau_B^{(-)}$ while $\tau_i^{(+)} = \tau_i^{(-)} + x$ for $i = 1, ..., B-1$ where $x > 0$ is the solution of the following:

$$\phi(\tau_B^{(-)}, \tau_{B-1}^{(-)} - \tau_B^{(-)} + x, ..., \tau_1^{(-)} - \tau_2^{(-)} + x) = 0. \tag{5.63}$$

  If $\tau_B^{(-)} = \tau_B$, the procedure stops and (5.63) gives the solution that $\phi(\tau_B, d_{B-1}, ..., d_1) = 0$, otherwise go to *Phase 1* with policy $\pi^{(+)}$.

It can be shown that the procedure always stops with a solution that $\phi(\tau_B, d_{B-1}, ..., d_1) = 0$. To see this, first observe that (5.63) always has a solution as long as:

$$\phi(\tau_B^{(-)}, \tau_{B-1}^{(-)} - \tau_B^{(-)}, ..., \tau_1^{(-)} - \tau_2^{(-)}) > 0. \tag{5.64}$$

This is due to the following facts about the function $\phi(\tau_B^{(-)}, \tau_{B-1}^{(-)} - \tau_B^{(-)} + x, ..., \tau_1^{(-)} - \tau_2^{(-)} + x)$: (i) it is a continuous function of x, (ii) it goes to $-\infty$ as x grows.

Next, observe that (5.64) always holds, i.e.,

$$\phi(\tau_B^{(-)}, \tau_{B-1}^{(-)} - \tau_B^{(-)}, ..., \tau_1^{(-)} - \tau_2^{(-)}) = \underbrace{\phi(\tau_B^{(+)}, \tau_{B-1}^{(+)} - \tau_B^{(+)}, ..., \tau_1^{(+)} - \tau_2^{(+)})}_{=0 \text{ due to the } Phase\ 2 \text{ or the initial/optimal policy}} + \int_{\pi^{(+)}}^{\pi^{(-)}} d\phi,$$

is positive. This can be seen by considering:

$$\frac{\partial \phi}{\partial \tau_B} = 2m_1(\tau_1, \tau_2, ..., \tau_B) + \sum_{j=0}^{B-1} \left[2\tau_B \frac{\partial}{\partial \tau_B} \mathbb{E}\left[X \mid E = j\right] - \frac{\partial}{\partial \tau_B} \mathbb{E}\left[X^2 \mid E = j\right] \right] \Pr(E = j),$$

which follows from the fact that $\Pr(E = j)$ does not depend on τ_B (see (5.9)) and can be further simplified by Lemma 1, hence:

$$\frac{\partial \phi}{\partial \tau_B} = 2m_1(\tau_1, \tau_2, ..., \tau_B).$$

Accordingly, we have:

$$\phi(\tau_B^{(-)}, \tau_{B-1}^{(-)} - \tau_B^{(-)}, ..., \tau_1^{(-)} - \tau_2^{(-)}) = \int_{\pi^{(+)}}^{\pi^{(-)}} d\phi = 2\int_{\tau_B^{(+)}}^{\tau_B^{(-)}} m_1(\tau_1^{(+)}, \tau_2^{(+)}, ..., \tau) d\tau > 0,$$

where the inequality follows from the fact that $m_1(\tau_1^{(+)}, \tau_2^{(+)}, ..., \tau)$ being the average inter-update time is always positive.

Therefore, (5.63) can be always satisfied in *Phase 2*. Also, as the second smallest threshold is strictly increased in *Phase 2*, the smallest threshold can be moved toward τ_B in *Phase 1*. Also, it can be shown that the procedure does not converge any policy other than the policy that $\phi(\tau_B, d_{B-1}, ..., d_1) = 0$. This can be seen considering the following:

$$\begin{aligned}
&\frac{d}{dx} m_2(\tau_1 + x, \tau_2 + x, ..., \hat{\tau}_B) \mid_{x=0} \\
&< \lim_{x\to 0} \lim_{n\to+\infty} \frac{1}{nx} \sum_{k=0}^{n} \left(\mathbb{E}[(X_k + x)^2] - \mathbb{E}[X_k^2] \right) \\
&= \lim_{n\to+\infty} \frac{2}{n} \sum_{k=0}^{n} \mathbb{E}[X_k] = 2m_1(\tau_1, \tau_2, ..., \hat{\tau}_B),
\end{aligned}$$

hence,

$$\begin{aligned}
&\frac{d}{dx}\phi(\hat{\tau}_B, \tau_{B-1} - \hat{\tau}_B + x, ..., \tau_1 - \tau_2 + x) \mid_{x=0} + \frac{d}{dx}\phi(\hat{\tau}_B + x, \tau_{B-1} - \hat{\tau}_B - x, ..., \tau_1 - \tau_2) \mid_{x=0} \\
&> 2\hat{\tau}_B \frac{d}{dx} m_1(\tau_1 + x, \tau_2 + x, ..., \hat{\tau}_B) \mid_{x=0}, \qquad (5.65)
\end{aligned}$$

which implies that the procedure cannot converge to a policy with $\tau_B^{(+)} < \tau_B$ as the RHS of (5.65) is positive [6] and does not vanish for a finite set of thresholds. Therefore, as the smallest threshold of the policies modified by the procedure is increased up to τ_B, a solution that $\phi(\tau_B, d_{B-1}, ..., d_1) = 0$ is eventually reached. This completes the proof.

[6] This follows from the fact that any increase in thresholds causes an increase in the battery overflow probability which means an increase in the average inter-update duration, i.e, $m_1(\tau_1, \tau_2, ..., \hat{\tau}_B)$.

Part II: The Role of Timely Communication for Tracking Stochastic Processes through Noisy Channels

CHAPTER 6

THE NECESSITY OF TIMELY COMMUNICATION FOR TRACKING AND NETWORKED CONTROL

6.1 Introduction

The setting and the objective considered in Chapter 5 rely on a common assumption in the related literature of update-based systems: While updates have identical sampling/communication costs, they capture the status of the source equally well, at the moment when they are generated. This assumption is reasonable when it is practically feasible that updates either carry the exact status, i.e., all bits required to describe the status, or an approximation (quantization) of the status such that updates can be treated as equally precise samples. On the other hand, the assumption is problematic when the status is an unstable (or non-stationary) process.

For example, an unstable Markov process can drift arbitrarily far from any reference point given sufficient time. Accordingly, capturing its exact state or an approximation with guaranteed precision may require growing [1] amount of resources. Hence, sampling the actual state can be impractical for such a source. One solution is *differential encoding* which captures the change in the process rather than its actual state. However, in that case, the number of bits to describe the change in the unstable process is likely to grow with time [2]. This means that status updates can be more costly as we wait.

[1] In fact, for a practical scenario, one can set a finite horizon and apply a fixed quantizer for the range of states which can be possibly reached in that horizon, such that each quantization requires a fixed resource, i.e., number of bits. However, considering the time evolution of the source, one can see that this solution can be highly inefficient and more efficient solutions should exist.

[2] This is due to that the difference between the unstable process and its sample at a particular time is also an unstable process, hence it produces entropy with a positive rate.

The need for differential encoding suggests that the information transmitted for the status of an unstable Markov process should benefit from earlier transmissions. This arises another problem when the transmission is prone to errors. This problem is also due to the process being unstable. That is, any transmission error occurred at a certain time contributes to an estimate of the process, and because of the positive feedback in unstable modes of the process, the impact of this contribution may grow to be significant in long term. Accordingly, past errors should be corrected by new transmissions to prevent the estimate of the process from diverging away from the actual state of the process.

The aforementioned observations exhibit the main challenges of designing communication schemes for monitoring or tracking an unstable stochastic process. While models of unstable stochastic processes are reasonable under a certain[3] regimes, they appear ubiquitously in many theories. For example, the Wiener process is used to model the physical diffusion process known as Brownian motion [87], option pricing in financial analysis [88], phase noise in communication channels [89] and forms a basis for analysis tools such as Feynman-Kac formula [90]. One important application of unstable processes is that they are used to model uncontrolled systems in control scenarios where the goal of the controller is to stabilize the system. The design of control and communication schemes for stabilizing systems where observers and controllers are separated with communication links is the main problem of the area known as *networked control* [91].

6.2 Networked Control

6.2.1 Networked Control Problem with a Communication Link

A basic problem of networked control is as follows (see Fig. 6.1). Consider a sensor which takes measurements (S_t) from a dynamical system and encodes its measurements to symbols X_t for communicating with a controller. Then, the controller applies control inputs (U_t) to stabilize the dynamical system using the information obtained

[3] The statistics of an unstable process can grow to infinity. Any model with underlying infinity assumption is practically questionable yet its model can be understood as a representation under a certain regime.

from the outputs (Y_t) of the communication channel. The sensor, the encoder and the controller should agree with the causality structure of the setting.

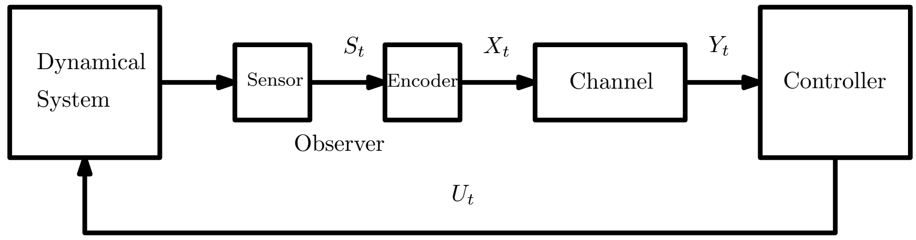


Figure 6.1: An illustration of the basic networked control setting.

This problem differs from conventional control problems due to the communication channel between the observer and the controller, and the limitedness of this communication channel poses the main challenge. In other words, together with control objectives, the problem is characterized by the limitations on the communication channel. This problem of stabilizing a dynamical system with limited communication has been extensively studied from stochastic control [92–100], rate-distortion theory [101–105] and joint source-channel coding [106] perspectives for linear systems and from the perspectives of metric and topological entropy for non-linear dynamical systems [107]. Some of these studies actually considers tracking problems relying on the fact that the *separation principle* applies to the considered scenario.

When separation principle, which suggests that optimal estimation and optimal control can be decoupled, applies, then the networked control problem can be studied through a dual problem which is about the tracking of the dynamical system (see Fig. 6.2). This tracking problem can be understood from the perspective of information theory as it considers the flow of information from measurements (S_t) to estimates ($\hat{S}_t$).

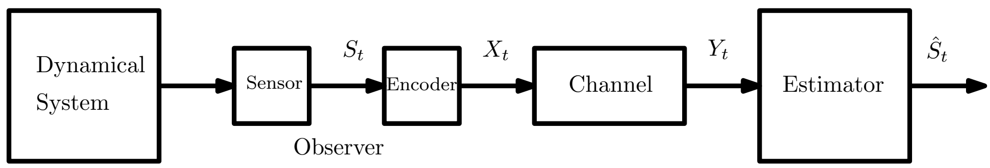


Figure 6.2: An illustration of the tracking problem setting.

6.2.2 Causal Source Coding of Unstable Processes

A significant amount of research has focused on studying minimum communication rate requirements for stabilizing dynamical systems through noiseless channels. Finding such requirements is essentially related to the causal source coding of an unstable process that represents the dynamical system.

Considering the Wiener process in particular, a source coding theorem was provided in [108]. The difficulty of source coding for general non-stationary autoregressive processes was observed in [109] where only a finite-horizon result was shown. The rate-distortion function for non-stationary Gaussian autoregressive processes was studied in [110]. These studies and other related studies considered the problem from the perspective of information theory usually for scalar sources.

From the perspective of control theory, a practically important case is the stabilization of multidimensional linear systems. In this context, [93] suggested and studied adaptive quantizers for the stabilization of a multidimensional linear system. It was shown that adaptive quantizers which “zoom-in" and “zoom-out" following the system evolution can stabilize the system asymptotically. However, the considered scheme targeted the asymptotic stability only rather than using the minimum communication rate needed. In [94] and [95], a lower bound on the minimum rate needed for stabilizing a multidimensional linear system was given and shown to be tight when only an average rate constraint is considered [95]. The result showed that the minimum rate is the sum of logarithms of unstable eigenvalues of the system evolution matrix which is independent from the statistics of the process and observer noises. In [96], this rate bound was generalized for different stability criteria. Similar rate bounds were studied for channels with packet losses in [97, 102, 111] and for links with random communication delays in [104]. In [102] and [105], non-asymptotic bounds on the minimum rate requirement were provided. A rate-cost function, considering both system distortion and control cost, was formulated in [103].

6.2.3 Anytime Reliability and Capacity

Channel requirements for tracking or stabilizing unstable processes can be more detailed than rate constraints. This side of the problem was considered in [112], which introduced the notions of *anytime reliability* and *anytime capacity*. In particular, the anytime reliability is defined as follows:

Definition 16 *Suppose* $\{B_t\}_{t=1,2,...}$ *is a fixed-rate binary source of rate R. Then, an encoder-decoder pair is said to be* (α, R)*-anytime reliable if we have the following for some* $K < \infty$

$$\Pr\left(B_\tau \neq \hat{B}_{\tau|t}\right) \leq K 2^{-\alpha(t-\tau)}, \forall t, \tag{6.1}$$

where $\hat{B}_{\tau|t}$ *is an estimate of* B_τ *based on information available up to t.*

Then, we have the following definition for the α-anytime capacity of a channel:

Definition 17 *The* α*-anytime capacity of a channel is the maximum R such that there exists an* (α, R)*-anytime reliable encoder-decoder pair over the channel.*

It was shown that anytime capacity provides the necessary and sufficient condition on the rate of an unstable scalar Markov source that can be tracked in the finite mean-squared error sense. Accordingly, it was claimed that anytime capacity, which is upper bounded by Shannon capacity, is the correct figure of merit to measure the quality of a channel on the purpose of tracking an unstable source and also controlling through an unreliable channel [113]. In that study, anytime capacity was shown to be strictly positive for memoryless channels, however a closed-form expression for anytime capacity was not shown in similar way that Shannon capacity can be expressed as an optimization of mutual information. Yet, anytime capacity of particular channels such as erasure channels with feedback [114] and Markov channels [113] were derived.

6.2.4 Anytime Reliable Codes

In [112], the existence of anytime reliable codes on memoryless channels was shown [4] using a random coding argument considering random ensembles of infinite tree codes. Anytime reliable codes on erasure channels was shown to exist with high probability among random ensembles of linear causal codes with time-invariance property in [116, 117]. In [106], an error-exponent was provided for the same ensemble of linear codes and particularly under sequential decoding to guarantee feasible expected complexity. Convolutional extensions of low density parity check (LDPC) codes [118, 119], spatially coupled codes [120] and causal codes based on chaotic maps [121] were also shown to have anytime reliability.

[4] In fact, this result was shown in [115, Theorem 7] where “forced" decoding of input symbols at arbitrary times was considered.

CHAPTER 7

ON THE TRACKABILITY OF STOCHASTIC PROCESSES

In this chapter, we will consider the requirements for tracking an unstable process through a noisy communication channel. As we discussed in Chapter 6, such requirements are particularly relevant in network control setting with a communication link. In networked control, an important stability condition is the *m-th moment stability* of the system. In accordance with this condition, the study will be centered on a definition of reliable estimation which we refer as *order m moment trackability*. Based on this definition, we study the estimation of integer-valued [1] stochastic processes which may represent linear or non-linear discrete-time systems.

We will show two moment-entropy inequalities for integer-valued random variables inspired from the inequality for the moments of guessing random variables in [122]. One of these bounds is for bounded integer-valued random variables (see Lemma 9) while the other (see Lemma 10) is valid for integer-valued random variables that do not necessarily have finite support. Based on these moment-entropy inequalities, we will provide necessary conditions (see Theorem 12 and Theorem 13) for tracking integer-valued sources using causal information. Corollaries of Theorem 12 are upper bounds on anytime capacity based on Gallager's reliability function and the Gartner-Ellis limit of the information density between channel inputs and outputs. Then, we will provide sufficient conditions for tracking integer-valued sources using causal information in Theorem 14 and Theorem 15, and also an achievable bound on the estimation error, which can be used to obtain sufficient conditions for tracking, in Theorem 16.

[1] Our results are for integer-valued sources, however, note that this is not restrictive for digital systems where data is represented using integers.

7.1 System Model

In this section, we will describe the general setting of trackability. Consider the problem of tracking a scalar discrete-time and discrete-valued stochastic process $\{X_t\}_{t=1,2,\ldots}$ based on causal knowledge of another stochastic process $\{Y_t\}_{t=1,2,\ldots}$. At any time t, the estimator generates a guess $\hat{X}_t = f_t(Y_{1:t})$ of the current value X_t, where $f_t(\cdot)$ is a function and $Y_{1:t} = (Y_1, Y_2, \ldots, Y_t)$ is the information that is available at time t.

Definition 18 *For any $m > 0$, $\{X_t\}_{t=1,2,\ldots}$ is said to be order m moment trackable based on $\{Y_t\}_{t=1,2,\ldots}$ if there exists a family of functions $\{f_t(\cdot)\}_{t=1,2,\ldots}$ such that $\hat{X}_t = f_t(Y_{1:t})$ and*

$$\sup_{t>0} \mathbb{E}\left[|X_t - \hat{X}_t|^m\right] < \infty. \tag{7.1}$$

The first goal is to find necessary conditions and sufficient conditions for the m-th moment trackability of process $\{X_t\}_{t=1,2,\ldots}$ based on the process $\{Y_t\}_{t=1,2,\ldots}$. In [112], the *anytime capacity* of a noisy channel was shown to be a necessary and sufficient quality measure of a channel to allow order m moment trackability of a Markov source $\{S_t\}_{t=1,2,\ldots}$ based on the channel output $\{Y_t\}_{t=1,2,\ldots}$, The second goal is to find new bounds of the anytime capacity, based on the trackability results.

7.2 Main Results

7.2.1 Necessary Conditions for Trackability

We provide two necessary conditions for order m moment trackability, which are expressed in terms of Rényi entropy and information density. The Rényi entropy of order α, where $\alpha \geq 0$ and $\alpha \neq 1$, is defined as [123]

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \mathbb{E}\left[P_X(X)^{\alpha-1}\right] \tag{7.2}$$

$$= \frac{1}{1-\alpha} \log \left[\sum_{x\in\mathcal{X}} P_X(x)^\alpha\right]. \tag{7.3}$$

Given joint distribution P_{XY}, the information density function is defined as [124]

$$i(x; y) = \log\left[\frac{P_{XY}(x, y)}{P_X(x)P_Y(y)}\right]. \tag{7.4}$$

The first necessary condition that we present is as follows:

Theorem 12 *If $\{X_t\}_{t=1,2,...}$ is an integer-valued stochastic process that satisfies*

$$|X_t| \leq c_t, \tag{7.5}$$

$$\lim_{t\to\infty} \frac{1}{t} \log(\log(c_t)) = 0, \tag{7.6}$$

then $\{X_t\}_{t=1,2,...}$ is order m moment trackable based on $\{Y_t\}_{t=1,2,...}$, where $Y_t \in \mathcal{Y}$ and $|\mathcal{Y}| < \infty$, only if the following inequality holds, for all $\rho \in (0, m]$ and $q > \rho + 1$,

$$\liminf_{t\to\infty} -\frac{1}{\rho t} \log \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{\rho}{q} i(X_t;Y_{1:t})} \middle| Y_{1:t}\right]^q\right] \geq \limsup_{t\to\infty} \frac{1}{t} H_{\frac{q-1}{q-\rho-1}}(X_t). \tag{7.7}$$

Proof. See Appendix 7.3.1. ■

The proof of Theorem 12 uses the following moment-entropy inequality for the Rényi entropy, which is inspired by Theorem 1 in [122].

Lemma 9 *If X is an integer-valued random variable that takes values from the set $\mathcal{X} = \{-M_-, \ldots, -1, 0, 1, \ldots, M_+\}$ where M_- and M_+ are positive integers, then for all $\rho \geq 0$*

$$\mathbb{E}[|X|^\rho] + 1 \geq [3 + \log(M_- M_+)]^{-\rho} e^{\rho H_{\frac{1}{1+\rho}}(X)}. \tag{7.8}$$

Proof. See Appendix 7.3.2. ■

Lemma 9 requires that M_- and M_+ are finite. As a result, Theorem 12 only applies to stochastic processes that satisfy (7.5) and (7.6). Next, we will provide a necessary condition for the trackability of unbounded stochastic processes in Theorem 13, which is based on the following moment-entropy inequality.

Lemma 10 *If X is an integer-valued random variable, then for all $\rho \in (0, m)$*

$$\mathbb{E}[|X|^m] + 1 \geq \left[1 + 2\zeta\left(\frac{m}{\rho}\right)\right]^{-\rho} e^{\rho H_{\frac{1}{1+\rho}}(X)}, \tag{7.9}$$

where $\zeta(\cdot)$ is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{7.10}$$

Proof. See Appendix 7.3.3. ■

Theorem 13 *An integer-valued stochastic process* $\{X_t\}_{t=1,2,...}$ *is order m moment trackable based on* $\{Y_t\}_{t=1,2,...}$ *, where* $Y_t \in \mathcal{Y}$ *and* $|\mathcal{Y}| < \infty$*, only if* (7.7) *holds for all* $\rho \in (0, m)$ *and* $q > \rho + 1$.

Proof. The proof is identical to the proof of Theorem 12, except that it uses Lemma 10 instead of Lemma 9. Note that $\zeta(\frac{m}{\rho})$ is finite for all $\rho \in (0, m)$. ■

Theorem 13 requires a weaker condition than Theorem 12. Accordingly, the result of Theorem 13 is weaker than that of Theorem 12. For this, notice that $\rho = m$ is not allowed in Theorem 13.

7.2.2 Upper Bounds of Anytime Capacity

Now, we show that (7.7) implies two inequalities that provide upper bounds on anytime capacity. First one can be expressed in terms of Gallager's reliability function which is defined as [125]

$$E_0(\rho, P_{Y|X}, P_X) = -\log \sum_{y\in\mathcal{Y}} \left(\sum_{x\in\mathcal{X}} P_X(x)[P_{Y|X}(y|x)]^{\frac{1}{1+\rho}} \right)^{1+\rho}. \tag{7.11}$$

In [126], an alternative expression for Gallager's reliability function was used as follows

$$E_0(\rho, P_{Y|X}, P_X) = -\log \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{1}{1+\rho} i(\bar{X};Y)} \middle| Y \right]^{1+\rho} \right], \tag{7.12}$$

where $P_{XY\bar{X}}(x, y, \bar{x}) = P_X(x)P_{Y|X}(y|x)P_X(\bar{x})$ is the joint density for X, Y and $\bar{X}$.

We find the following expression of Gallager's reliability function convenient, due to its connection with the LHS of (7.7).

$$E_0(\rho, P_{Y|X}, P_X) = -\log \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{\rho}{1+\rho} i(X;Y)} \middle| Y \right]^{1+\rho} \right]. \tag{7.13}$$

Using (7.13), one can observe that the LHS of (7.7) becomes the Gallager's reliability function as q reduces to $\rho + 1$. Based on this observation, we derive the following corollary of Theorem 12:

Corollary 2 *Suppose that $S_t \to X_{1:t} \to Y_{1:t}$ is a Markov chain for each t. If $\{S_t\}_{t=1,2,...}$ is an integer-valued stochastic process that satisfies*

$$|S_t| \leq c_t, \tag{7.14}$$

$$\lim_{t\to\infty} \frac{1}{t} \log(\log(c_t)) = 0, \tag{7.15}$$

then $\{S_t\}_{t=1,2,...}$ is order m moment trackable based on $\{Y_t\}_{t=1,2,...}$, where $Y_t \in \mathcal{Y}$ and $|\mathcal{Y}| < \infty$, only if

$$\liminf_{t\to\infty} \frac{1}{mt} E_0(m, P_{Y_{1:t}|X_{1:t}}, P_{X_{1:t}}) \geq \limsup_{t\to\infty} \frac{1}{t} H_\infty(S_t). \tag{7.16}$$

Proof. Apply Theorem 12 for S_t considering $\rho = m$ and the limit that q reduces to $\rho + 1$ yields

$$\liminf_{t\to\infty} \frac{1}{mt} E_0(m, P_{Y_{1:t}|S_t}, P_{S_t}) \geq \limsup_{t\to\infty} \frac{1}{t} H_\infty(S_t). \tag{7.17}$$

Observe that (7.17) implies (7.16) as $E_0(m, P_{Y_{1:t}|S_t}, P_{S_t})$ is upper bounded by $E_0(m, P_{Y_{1:t}|X_{1:t}}, P_{X_{1:t}})$ due to data-processing inequality for Rényi divergence (see [126, Theorem 5]). ■

Corollary 2 can be related to the α-anytime capacity of a channel (see [127, Definition 3.2]) when we consider the following communication system. Let $Y_{1:t}$ be the outputs of a channel given by $P_{Y_{1:t}|X_{1:t}}(y_{1:t}|x_{1:t})$ with $X_{1:t}$ being inputs that encode a source $\{S_t\}$ [2]. As the outputs of the channel depend on the source process only through the channel inputs, the system follows $S_t \to X_{1:t} \to Y_{1:t}$. For ease of analysis, we will consider the type of source representing a stream of bits with fixed rate as follows:

Definition 19 *For R being a positive integer, a discrete-time process $\{S_t\}$ is said to be a rate-R source if it obeys:*

$$S_{t+1} = 2^R S_t + W_t, \tag{7.18}$$

where $\{W_t\}$ is an i.i.d. process such that W_t is uniformly chosen from the set $\left\{0, 1, ..., 2^R - 1\right\}$, and $S_0 = 0$.

Note that a rate-R source satisfies $|S_t| \leq 2^{Rt}$ almost surely and $H_\infty(S_t) = Rt\log(2)$ as it has a uniform distribution for all t. Accordingly, we can apply Corollary 2 to a rate-R source and show the following

[2] A causal and general communication system as such is given by $P_{Y_{1:t}|X_{1:t}}(y_{1:t}|x_{1:t}) = \prod_{t'=1}^{t} P_{Y_{t'}|X_{1:t'},Y_{1:t'-1}}(y_{t'}|x_{1:t'}, y_{1:t'-1})$ and $P_{X_{1:t}|S_{1:t}}(x_{1:t}|s_{1:t}) = \prod_{t'=1}^{t} P_{X_{t'}|S_{1:t'}}(x_{t'}|s_{1:t'})$ if we describe the encoding in terms of conditional probabilities for ease of description.

Corollary 3 *If $C_{any}(\alpha)$ is the α-anytime capacity of a discrete memoryless channel (DMC) without feedback, R is an positive integer, $m > 0$ is an arbitrary positive number, and*

$$R\log(2) \leq C_{any}(mR), \tag{7.19}$$

then

$$R\log(2) \leq \frac{E_0(m)}{m}, \tag{7.20}$$

where $E_0(m) = \sup_{P_X} E_0(m, P_{Y|X}, P_X)$ for given transition probabilities $P_{Y|X}$ of the channel.

Proof. First suppose that (7.19) holds which means a rate-R source is order m moment trackable though a DMC with anytime capacity $C_{any}(\alpha)$ (see [127, Theorem 3.3]). On the other hand, if a rate-R source is order m moment trackable through a DMC, the following should also hold:

$$\liminf_{t\to\infty} \frac{1}{mt} E_0(m, P_{Y_{1:t}|X_{1:t}}, P_{X_{1:t}}) \geq R\log(2), \tag{7.21}$$

which follows from Corollary 2. Moreover, this implies (7.20) as $E_0(m, P_{Y_{1:t}|X_{1:t}}, P_{X_{1:t}}) \leq tE_0(m)$ (see [125, Theorem 5]) for DMCs without feedback. ■

A result that is similar to Corollary 3 was shown (see [114, Theorem 3.3.2]) for symmetric DMCs with feedback based on sphere packing exponent. On the other hand, Corollary 3 holds both for asymmetric and symmetric DMCs without feedback.

The second inequality that we provide can be obtained [3] from (7.16) while considering a rate-R source for S_t. Accordingly, when $Y_{1:t}$ are the outputs of a channel with inputs $X_{1:t}$ that encode a rate-R source, the source is order m trackable based on $Y_{1:t}$ only if:

$$\liminf_{t\to\infty} \frac{1}{\rho t} \log \mathbb{E}\left[e^{\rho i(X_{1:t};Y_{1:t})}\right] \geq R\log(2). \tag{7.22}$$

In fact, the LHS of (7.22) is the Gartner-Ellis limit of $i(X_{1:t};Y_{1:t})$ which provides another upper bound for anytime capacity if we use (7.22) instead of (7.21) in the proof of Corollary 3. Also, observe that both (7.21) and (7.22) can be applied for channels other than DMCs without feedback.

[3] See Appendix 7.3.6 for the proof.

7.2.3 Sufficient Conditions for Trackability

Next, we provide two sufficient conditions for order m moment trackability. The first one is based on MAP estimators.

Definition 20 *An estimator $\hat{X}_t^{(MAP)}$ is said to be a maximum a posteriori (MAP) estimator if*

$$\hat{X}_t^{(MAP)} = \arg\max_{x \in \mathcal{X}} P_{X_t|Y_{1:t}}(x|Y_{1:t}), \tag{7.23}$$

with ties in the maximization broken arbitrarily.

We will use the following lemma to derive a sufficient condition for order m moment trackability based on MAP estimators:

Lemma 11 *For an integer-valued stochastic process $\{X_t\}$, a discrete-valued stochastic process $\{Y_t\}$ and $d(\cdot,\cdot)$ being a distance metric such that $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^{\geq 0}$ we have the following for arbitrary real numbers $\rho > 0$ and $s > 1$:*

$$\mathbb{E}\left[d(X_t, \hat{X}_t^{(MAP)})\right] \leq \zeta(s) \sum_{y_{1:t}} P_{Y_{1:t}}(y_{1:t}) \sum_{x} [P_{X_t|Y_{1:t}}(x|y_{1:t})]^{\frac{1}{\rho+1}} \left[\sum_{x'} [P_{X_t|Y_{1:t}}(x'|y_{1:t})]^{\frac{1}{\rho+1}} d(x,x')^{\frac{s}{\rho}}\right]^{\rho}, \tag{7.24}$$

Proof. See Appendix 7.3.4 ■

A sufficient condition for order m moment trackability using Lemma 11 is as follows:

Theorem 14 *Let*

$$\tau(x, y_{1:t}) = \mathbb{E}\left[P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{m}{m+1}} |X_t - x|^s \mid Y_{1:t} = y_{1:t}\right], \tag{7.25}$$

where $s > 1$ is an arbitrary real number and m is an integer. Then, the integer-valued stochastic process $\{X_t\}_{t=1,2,\ldots}$ is order m moment trackable using $\{Y_t\}_{t=1,2,\ldots}$ if

$$\sup_{t>0} \mathbb{E}\left[\tau(X_t, Y_{1:t})^m P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{m}{m+1}}\right] < \infty. \tag{7.26}$$

Proof. Apply Lemma 11 for $d(x,x') = |x - x'|^m$ and $\rho = m$, then observe that

$$\mathbb{E}\left[\left|X_t - \hat{X}_t^{(\mathrm{MAP})}\right|^m\right] \leq \zeta(s)\mathbb{E}\left[\tau(X_t, Y_{1:t})^m P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{m}{m+1}}\right]. \tag{7.27}$$

■

In addition to MAP estimators, we consider another type of estimators which are defined below:

Definition 21 *For $\rho > 0$ being an arbitrary real number, let $\{\hat{X}_t^{(\rho)}(Y_{1:t})\}$ be a family of estimators such that $\hat{X}_t^{(\rho)}(y_{1:t})$ is uniformly chosen from the set $\mathcal{A}_t(\rho, y_{1:t}, J_t(\rho, y_{1:t}))$ where*

$$\mathcal{A}_t(\rho, y_{1:t}, c) = \left\{x : \frac{P_{X_t|Y_{1:t}}(x|y_{1:t})}{P_{X_t|Y_{1:t}}(x'|y_{1:t})} \geq c|x - x'|^\rho, \forall x'\right\} \tag{7.28}$$

and

$$J_t(\rho, y_{1:t}) = \sup\{c \geq 0 : \mathcal{A}_t(\rho, y_{1:t}, c) \neq \emptyset\}. \tag{7.29}$$

Observe that, as opposed to MAP estimators, the estimator $\hat{X}_t^{(\rho)}$ has a notion of distance and it requires that a possible value to be less likely proportional with its distance to the estimate. This requirement is natural as more likely values cluster around the estimate value. Accordingly, considering the family of estimators $\{\hat{X}_t^{(\rho)}\}$ yields

Theorem 15 *If $p > 1$ and $s > 1$ are arbitrary real numbers, and m is a positive integer, then, the integer-valued stochastic process $\{X_t\}_{t=1,2,...}$ is order m moment trackable based on $\{Y_t\}_{t=1,2,...}$ if*

$$\sup_{t>0} \mathbb{E}\left[\mathbb{E}\left[\left[P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{m}{m+1}} \mid Y_{1:t}\right]^{m+1} - 1\right]^p\right]^{\frac{1}{p}} \mathbb{E}\left[J_t(sm(m+1), Y_{1:t})^{\frac{mp}{(m+1)(1-p)}}\right]^{\frac{p-1}{p}} < \infty, \tag{7.30}$$

where $J_t(\rho, y_{1:t})$ is as defined in (7.29) .

Proof. See Appendix 7.3.5. ■

Note that the first term in (7.30) can be expressed in terms of conditional Rényi [4] entropy when $p = 1$ while J_t function in the second term can be considered as a measure for the shape of the conditional distribution $P_{X_t|Y_{1:t}}(x|Y_{1:t})$.

Next, we consider an achievable bound on the estimation error together with a formulation of causal encoding and channel models. Let $\{S_t\}_{t=1,2,...}$, where $S_t \in \mathbb{Z}$, be

[4] We consider the definition of conditional Rényi entropy that fits our case.

a discrete-time and discrete-valued process, i.e., the source, to be tracked based on causal information obtained through a communication channel . Let $X_t \in \mathcal{X}$ and $Y_t \in \mathcal{Y}$ be the input and the output of the channel at time t where $\mathcal{X}$ and $\mathcal{Y}$ represent the input and the output alphabet of the channel, respectively. Then, $\{X_t\}_{t=1,2,...}$ and $\{Y_t\}_{t=1,2,...}$ represent the stochastic processes as the inputs and the outputs of the channel, respectively. We will consider a causal channel such that the following holds for all $x_{1:t} \in \mathcal{X}^t$, $y_{1:t} \in \mathcal{Y}^t$ and $t > 1$:

$$P_{Y_{1:t}|X_{1:t}}(y_{1:t}|x_{1:t}) = \prod_{t'=1}^{t} P_{Y_{t'}|X_{1:t'},Y_{1:t'-1}}(y_{t'}|x_{1:t'}, y_{1:t'-1}). \tag{7.31}$$

Accordingly, the conditional probabilities $P_{Y_1|X_1}(y_1|x_1), P_{Y_2|X_{1:2},Y_1}(y_2|x_{1:2}, y_1)\cdots$ characterize the channel.

An encoder observes $\{S_t\}_{t=1,2,...}$ and applies channel inputs causally such that X_t is determined based on $S_{1:t}$ and a codebook c as the output of the function $\varepsilon_{t;c} : \mathbb{Z}^t \to \mathcal{X}$, i.e., $X_t = \varepsilon_{t;c}(S_{1:t})$ [5]. The family of functions $\{\varepsilon_{t;c}(\cdot)\}_{t=1,2,\cdots}$ defines the codebook c. Let $\varepsilon_{1:t;c}(S_{1:t})$ represent the sequence of coded inputs corresponding to $S_{1:t}$, i.e., $\varepsilon_{1:t;c}(S_{1:t}) = X_{1:t}$ where $X_t = \varepsilon_{t;c}(S_{1:t})$.

On the other end of the channel, an estimator observes $\{Y_t\}_{t=1,2,...}$ and produce the estimate $\hat{S}_t$ for S_t based on $Y_{1:t}$ as the output of the function $\theta_t : \mathcal{Y}^t \to \mathbb{Z}$, i.e., $\hat{S}_t = \theta_t(Y_{1:t})$. The family of functions $\{\theta_t(\cdot)\}_{t=1,2,\cdots}$ defines the estimator.

We will consider MAP estimators for $S_{1:t}$ such that:

$$\hat{S}_{1:t}^{(\mathrm{MAP})} = \arg\max_{s_{1:t}\in\mathbb{Z}^t} P_{S_{1:t}|Y_{1:t}}(s_{1:t}|Y_{1:t}). \tag{7.32}$$

For such estimators, we have the following bound on the estimation error:

Theorem 16 *For a random causal code encoding the source $S_{1:t}$, $\hat{S}_t$ is the estimate at time t for the MAP estimator of $S_{1:t}$, i.e., $\hat{S}_{1:t}^{(MAP)}$ in (7.32), $d(\cdot,\cdot)$ being a distance metric such that $d : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}^{\geq 0}$ and $r \geq 0$, we have*

$$\begin{aligned} &\Pr\left(d(S_t, \hat{S}_t) > r\right) \\ \leq\ &\mathbb{E}\left[\min\{1, M_t(r,\sigma,q,S_{1:t})\kappa_t(p,\sigma,Y_{1:t},S_{1:t})\}\right], \end{aligned} \tag{7.33}$$

[5] Technically, t

where

$$M_t(r, \sigma, q, S_{1:t}) = \mathbb{E}\left[\mathbb{I}_{\{d(S_t, \bar{S}_t) > r\}} \frac{\left(P_{S_{1:t}}(\bar{S}_{1:t})\right)^{q(\sigma-1)}}{\left(P_{S_{1:t}}(S_{1:t})\right)^{q\sigma}} \Big| S_{1:t}\right]^{\frac{1}{q}}, \tag{7.34}$$

$$\kappa_t(p, \sigma, Y_{1:t}, S_{1:t}) = \mathbb{E}\left[\prod_{t'=1}^{t}\left(\frac{P_{Y_{t'}|X_{1:t'}, Y_{1:t'-1}}(Y_{t'}|\varepsilon_{1:t';C}(\bar{S}_{1:t'}), Y_{1:t'})}{P_{Y_{t'}|X_{1:t'}, Y_{1:t'-1}}(Y_{t'}|\varepsilon_{1:t';C}(S_{1:t'}), Y_{1:t'})}\right)^{p\sigma} \Big| Y_{1:t}, S_{1:t}\right]^{\frac{1}{p}}, \tag{7.35}$$

where $\{\varepsilon_{t;C}(\cdot)\}_{t=1,2,\cdots}$ *is a family of encoding functions,* $\bar{S}_t$*s are chosen independently from* $Y_{1:t}$ *having the same distribution as* S_t*,* (p, q) *are Hölder conjugates, and* $\sigma > 0$*.*

Proof. See Appendix 7.3.7 ■

A sufficient condition for order m moment trackability of the source $\{S_t\}_{t=1,2,...}$ or other achievability results based on long-term estimation objectives can be obtained from Theorem 16. Besides, we can gain insights from this result through a comparison with other random coding bounds.

Theorem 16 is partly inspired by the random coding union bound introduced in [128, Theorem 16] and hence has a similar form [6]. Due to this similarity, one can compare [7] an achievable estimation performance in the causal communication setting to an achievable decoding error performance in the communication setting of finite block-length channel coding where the codes are not necessarily causal and the information source, i.e., the message, is fully known to the transmitter beforehand.

Note that the term $M_t(r, \sigma, q, S_{1:t})$ quantifies a property of the source combining the evolution of its uncertainty with the estimation goal, while the term $\kappa_t(p, \sigma, Y_{1:t}, S_{1:t})$ quantifies the uncertainty reduction due to the causal information obtained through the channel. Accordingly, one can observe a small estimation error if the reduction in $\kappa_t(p, \sigma, Y_{1:t}, S_{1:t})$ can compensate the growth in $M_t(r, \sigma, q, S_{1:t})$.

[6] Theorem 16 is also closely related to the bound in [129, Theorem 1] which considers joint source-channel coding with MAP criterion as in our case.

[7] The term $M_t(r, \sigma, q, S_{1:t})$ is comparable to the term representing the message set size and the term $\kappa_t(p, \sigma, Y_{1:t}, S_{1:t})$ is comparable to the conditional probability term in the upper bound of [128, Theorem 16].

7.3 Appendix

7.3.1 The Proof of Theorem 12

Consider arbitrary estimators $\{\hat{X}_t\}$ such that $|\hat{X}_t| \leq c_t$ for $t > 0$ [8] . Let us define estimators $\{\hat{X}_t^{(c)}\}$ such that $\hat{X}_t^{(c)} = \lceil \hat{X}_t \rceil$ where $\lceil \cdot \rceil$ is the ceiling function. If $m \in (1, \infty)$,

$$\mathbb{E}\left[|X_t - \hat{X}_t^{(c)}|^m\right]^{\frac{1}{m}} \leq \mathbb{E}\left[|X_t - \hat{X}_t|^m\right]^{\frac{1}{m}} + 1, \tag{7.36}$$

where the inequality follows from Minkowski's inequality and that $\mathbb{E}\left[|\hat{X}_t - \hat{X}_t^{(c)}|^m\right] \leq 1$. If $m \in (0, 1]$,

$$\begin{aligned}
&\mathbb{E}\left[|X_t - \hat{X}_t^{(c)}|^m\right] \\
\leq &\mathbb{E}\left[\left(|X_t - \hat{X}_t| + |\hat{X}_t - \hat{X}_t^{(c)}|\right)^m\right] \\
\leq &\mathbb{E}\left[|X_t - \hat{X}_t|^m + |\hat{X}_t - \hat{X}_t^{(c)}|^m\right] \\
\leq &\mathbb{E}\left[|X_t - \hat{X}_t|^m\right] + 1,
\end{aligned} \tag{7.37}$$

where the first inequality is due to triangle inequality, the second inequality follows from the inequality that $(a + b)^m \leq a^m + b^m$ for $a, b \geq 0$ when $m \in (0, 1]$, and the third inequality is due to $\mathbb{E}\left[|\hat{X}_t - \hat{X}_t^{(c)}|^m\right] \leq 1$. Hence, combining (7.36) and (7.37), we conclude that:

$$\sup_{t>0} \mathbb{E}\left[|X_t - \hat{X}_t|^m\right] < \infty \tag{7.38}$$

holds only if

$$\sup_{t>0} \mathbb{E}\left[|X_t - \hat{X}_t^{(c)}|^m\right] < \infty. \tag{7.39}$$

Accordingly, (7.39) is a necessary condition to satisfy (7.38).

Now, we find a necessary condition for (7.39). Let $E_t := X_t - \hat{X}_t^{(c)}$ be estimation error for estimators $\{\hat{X}_t^{(c)}\}$. As $|X_t| \leq c_t$ for $t > 0$ and $\hat{X}_t^{(c)}$ is integer-valued, E_t is an integer valued random variable taking values in $[-2c_t, 2c_t]$.

Using Lemma 9 for E_t being conditioned on $Y_{1:t}$, we have:

$$\begin{aligned}
&\mathbb{E}\left[|E_t|^m \mid Y_{1:t} = y_{1:t}\right] + 1 \geq \\
&\mathbb{E}\left[|E_t|^\rho \mid Y_{1:t} = y_{1:t}\right] + 1 \geq (3 + 2\log(2c_t))^{-\rho}\, \mathbb{E}\left[P_{E_t|Y_{1:t}}(E_t|Y_{1:t})^{-\frac{\rho}{\rho+1}} \mid Y_{1:t} = y_{1:t}\right]^{\rho+1}
\end{aligned} \tag{7.40}$$

[8] Clearly, any estimator $\hat{X}_t$ which can take values that are outside of $[-c_t, c_t]$ is suboptimal for minimizing $\mathbb{E}\left[|X_t - \hat{X}_t|^m\right]$.

where$P_{E_t|Y_{1:t}}$ is the conditional distribution for E_t conditioned on $Y_{1:t}$ and the first inequality is due to that E_t is integer-valued and the second inequality is due to Lemma 9.

As $(E_t, Y_{1:t}) \to (X_t, Y_{1:t})$ is a bijective transformation when both X_t and $\hat{X}_t^{(c)}$ are integer-valued, (7.40) becomes:

$$\mathbb{E}\left[|E_t|^m \mid Y_{1:t} = y_{1:t}\right] + 1 \geq (3 + 2\log(2c_t))^{-\rho}\, \mathbb{E}\left[P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{\rho}{\rho+1}} \mid Y_{1:t} = y_{1:t}\right]^{\rho+1}. \tag{7.41}$$

Taking expectations over $Y_{1:t}$ on both sides in (7.41) gives:

$$\mathbb{E}\left[|E_t|^m\right] + 1 \geq (3 + 2\log(2c_t))^{-\rho}\, \mathbb{E}\left[\mathbb{E}\left[P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{\rho}{\rho+1}} \mid Y_{1:t}\right]^{\rho+1}\right]. \tag{7.42}$$

Now, consider

$$\begin{aligned} & \mathbb{E}\left[P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{\rho}{\rho+1}} \mid Y_{1:t}\right]^{\rho+1} \\ \geq\ & \mathbb{E}\left[e^{-\frac{\rho}{p(\rho+1)} i(X_t;Y_{1:t})} \mid Y_{1:t}\right]^{p(\rho+1)} \mathbb{E}\left[P_{X_t}(X_t)^{\frac{\rho}{(p-1)(\rho+1)}} \mid Y_{1:t}\right]^{(1-p)(\rho+1)}, \end{aligned} \tag{7.43}$$

where the inequality follows from the reverse Hölder inequality for $p \in (1, \infty)$ and $i(X_t; Y_{1:t})$ is the information density for $P_{X_t Y_{1:t}}$. Then, we can get:

$$\begin{aligned} & \frac{1}{\rho}\log \mathbb{E}\left[\mathbb{E}\left[P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{\rho}{\rho+1}} \mid Y_{1:t}\right]^{\rho+1}\right] \\ \geq\ & \frac{1}{\rho}\log \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{\rho}{p(\rho+1)} i(X_t;Y_{1:t})} \mid Y_{1:t}\right]^{p(\rho+1)}\right] + H_\alpha(X_t), \end{aligned} \tag{7.44}$$

where $\alpha = (p(\rho + 1) - 1)/((p - 1)(\rho + 1))$.

Combining (7.42) and (7.44):

$$\begin{aligned} & \frac{1}{\rho}\log\left(\mathbb{E}\left[|E_t|^m\right] + 1\right) \\ \geq\ & \frac{1}{\rho}\log \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{\rho}{p(\rho+1)} i(X_t;Y_{1:t})} \mid Y_{1:t}\right]^{p(\rho+1)}\right] \\ & + H_\alpha(X_t) - \log\left(3 + 2\log(c_t)\right). \end{aligned} \tag{7.45}$$

As $\lim_{t\to\infty} \log(\log(c_t))/t = 0$,

$$\limsup_{t\to\infty} \frac{-1}{t}\log\left(3 + 2\log(2c_t)\right) = 0. \tag{7.46}$$

Therefore, combining (7.45) and (7.46) implies that:

$$\limsup_{t\to\infty} \frac{1}{\rho t} \log \left(\mathbb{E}\left[|E_t|^m\right] + 1\right) < \infty \tag{7.47}$$

holds only [9] if

$$\liminf_{t\to\infty} -\frac{1}{\rho t} \log \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{\rho}{p(\rho+1)} i(X_t;Y_{1:t})} \mid Y_{1:t}\right]^{p(\rho+1)}\right] \geq \limsup_{t\to\infty} \frac{1}{t} H_\alpha(X_t). \tag{7.48}$$

In addition, if (7.39) holds then (7.47) holds. Hence (7.48) is a necessary condition for (7.39). Therefore, (7.48) is a necessary condition for (7.38), i.e., $\{X_t\}$ being order m moment trackable though process $\{Y_t\}$. As $p > 1$ is arbitrary, $p(\rho + 1)$ can be replaced with an arbitrary q such that $q > \rho + 1$.

7.3.2 The Proof of Lemma 9

We have two methods to prove Lemma 9. The first method follows the proof of Theorem 1 in [122], with the guessing function replaced by $A(x)$ defined in (7.49) below and some other necessary changes. In the sequel, we provide a second proof method, which is based on the reverse Hölder inequality approach used in [122, Lemma 1] and in [130, Theorem 2.1].

Let us define the following function:

$$A(x) = \begin{cases} |x| & \text{if } x \neq 0, \\ \epsilon & \text{if } x = 0, \end{cases} \tag{7.49}$$

where ϵ is an arbitrary positive real number. Accordingly, observe that

$$\begin{aligned} &\mathbb{E}[|X|^\rho] + \epsilon^\rho P_X(0) \\ &= \sum_{x\in\mathcal{X}} P_X(x) A(x)^\rho \\ &\geq \left[\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{p}}\right]^p \left[\sum_{x\in\mathcal{X}} A(x)^{\frac{-\rho}{p-1}}\right]^{-(p-1)}, \end{aligned} \tag{7.50}$$

[9] Here, it is possible that (7.47) holds when both limits in (7.48) diverge. However, observe that the LHS of (7.48) converges as $i(X_t; Y_{1:t})$ is uniformly bounded by $\log(|\mathcal{Y}|)$ and $|\mathcal{Y}|$ is finite.

where the inequality is due to the reverse Hölder inequality for $p \in (1, \infty)$. Considering $p = 1 + \rho$ in (7.50), we get

$$
\begin{aligned}
&\mathbb{E}[|X|^{\rho}] + \epsilon^{\rho} P_X(0) \\
\geq &\left[\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right]^{1+\rho} \left[\sum_{x \in \mathcal{X}} A(x)^{-1}\right]^{-\rho} \\
\geq &\left[\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right]^{1+\rho} \left[2 + \frac{1}{\epsilon} + \log(M_- M_+)\right]^{-\rho}, \qquad (7.51)
\end{aligned}
$$

where the second inequality is due to

$$
\begin{aligned}
\sum_{x=-M_-}^{M_+} A(x)^{-1} &= \epsilon^{-1} + \sum_{i=1}^{M_-} i^{-1} + \sum_{j=1}^{M_+} j^{-1} \\
&\leq 2 + \epsilon^{-1} + \log(M_- M_+).
\end{aligned}
$$

Letting $\epsilon = 1$, combining (7.51) with $P_X(0) \leq 1$ and

$$
e^{\rho H_{\frac{1}{1+\rho}}(X)} = \left[\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right]^{1+\rho}, \qquad (7.52)
$$

we obtain (7.8).

7.3.3 The Proof of Lemma 10

The proof is similar to the proof of Lemma 9 where $A(x)$ is defined as in (7.49). We have

$$
\begin{aligned}
&\mathbb{E}[|X|^{m}] + 1 \\
\geq &\left[\sum_{x=-\infty}^{\infty} P_X(x)^{\frac{1}{1+\rho}}\right]^{1+\rho} \left[\sum_{x=-\infty}^{\infty} A(x)^{-\frac{m}{\rho}}\right]^{-\rho} \\
\geq &\left[\sum_{x=-\infty}^{\infty} P_X(x)^{\frac{1}{1+\rho}}\right]^{1+\rho} \left[1 + 2\sum_{n=1}^{\infty} \frac{1}{n^{\frac{m}{\rho}}}\right]^{-\rho}, \qquad (7.53)
\end{aligned}
$$

where the first inequality follows from reverse Hölder inequality and the second inequality follows from the choice of $\epsilon = 1$.

7.3.4 The Proof of Lemma 11

We first consider the following:

$$\mathbb{E}\left[d(X_t, \hat{X}_t^{(\text{MAP})})\right] = \sum_{r=1}^{\infty} \Pr\left(d(X_t, \hat{X}_t^{(\text{MAP})}) \geq r\right). \tag{7.54}$$

Then, observe that:

$$\begin{aligned}
& \Pr\left(d(X_t, \hat{X}_t^{(\text{MAP})}) \geq r \mid X_t = x, Y_{1:t} = y_{1:t}\right) \\
\leq\ & \min\left\{1, \sum_{x' : d(x,x') \geq r} \mathbb{I}_{\left\{P_{X_t|Y_{1:t}}(x|y_{1:t}) \leq P_{X_t|Y_{1:t}}(x'|y_{1:t})\right\}}\right\} \\
\leq\ & \left(\sum_{x'} \mathbb{I}_{\{d(x,x') \geq r\}} \mathbb{I}_{\left\{P_{X_t|Y_{1:t}}(x|y_{1:t}) \leq P_{X_t|Y_{1:t}}(x'|y_{1:t})\right\}}\right)^{\rho} \\
\leq\ & \left(\sum_{x'} \left(\frac{d(x,x')}{r}\right)^{\frac{s}{\rho}} \left(\frac{P_{X_t|Y_{1:t}}(x'|y_{1:t})}{P_{X_t|Y_{1:t}}(x|y_{1:t})}\right)^{\frac{1}{\rho+1}}\right)^{\rho}
\end{aligned} \tag{7.55}$$

where $\mathbb{I}_{\{\cdot\}}$ is the indicator function, the first inequality is due to the definition of MAP estimators, the second inequality follows from the inequality $\min\{1, x\} \leq x^{\rho}$ for any $\rho \geq 0$ when x is either 0 or larger than equal to 1, and the third inequality follows from the inequality $\mathbb{I}_{\{x \geq r\}} \leq (x/r)^s$ for any $x \geq 0, r > 1, s \geq 0$. Combining (7.55) and (7.54) gives (7.24).

7.3.5 The Proof of Theorem 15

Consider $\hat{X}_t^{(\rho)}$ for $\rho = sm(m+1)$, and the following:

$$\mathbb{E}\left[|X_t - \hat{X}_t^{(\rho)}|^m\right] = \sum_{r=1}^{\infty} \Pr\left(|X_t - \hat{X}_t^{(\rho)}|^m \geq r\right). \tag{7.56}$$

Then observe that:

$$
\begin{aligned}
& \Pr\left(|X_t - \hat{X}_t^{(\rho)}|^m \geq r \mid X_t = x, Y_{1:t} = y_{1:t}\right) \\
\leq\ & \min\left\{1, \sum_{x':|x-x|^m \geq r} \mathbb{I}_{\left\{Q(x,x') \leq P_{X_t|Y_{1:t}}(x'|y_{1:t})\right\}}\right\} \\
\leq\ & \left(\sum_{x'} \mathbb{I}_{\{|x-x|^m \geq r\}} \mathbb{I}_{\left\{Q(x,x') \leq P_{X_t|Y_{1:t}}(x'|y_{1:t})\right\}}\right)^{\rho} \\
\leq\ & \left[\sum_{x' \neq x} \left(\frac{|x-x|^m}{r}\right)^s \left(\frac{P_{X_t|Y_{1:t}}(x'|y_{1:t})}{Q(x,x')}\right)^{\frac{1}{m+1}}\right]^m, \\
=\ & \frac{J_t(\rho, y_{1:t})^{\frac{-m}{m+1}}}{r^s} \left[\sum_{x'} \left(\frac{P_{X_t|Y_{1:t}}(x'|y_{1:t})}{P_{X_t|Y_{1:t}}(x|y_{1:t})}\right)^{\frac{1}{m+1}} - 1\right]^m, \\
\leq\ & \frac{J_t(\rho, y_{1:t})^{\frac{-m}{m+1}}}{r^s} \left[\left[\sum_{x'} \left(\frac{P_{X_t|Y_{1:t}}(x'|y_{1:t})}{P_{X_t|Y_{1:t}}(x|y_{1:t})}\right)^{\frac{1}{m+1}}\right]^m -1\right],
\end{aligned}
\tag{7.57}
$$

where $Q(x, x') = J_t(\rho, y_{1:t})|x - x'|^{\rho} P_{X_t|Y_{1:t}}(x|y_{1:t})$, $\mathbb{I}_{\{\cdot\}}$ is the indicator function, the first ,second and third inequalities follow similarly as (7.55), and the last inequality is due to Jensen's inequality. Combining (7.57) and (7.56) gives:

$$
\mathbb{E}\left[|X_t - \hat{X}_t^{(\rho)}|^m\right] \leq \zeta(s)\mathbb{E}[J_t(\rho, Y_{1:t})^{\frac{-m}{m+1}} \left[\mathbb{E}\left[P_{X_t|Y_{1:t}}(X_t|Y_{1:t})^{-\frac{m}{m+1}} \mid Y_{1:t}\right]^{m+1} - 1\right]].
\tag{7.58}
$$

Applying Hölder inequality to the RHS of (7.58) gives the sufficient condition.

7.3.6 The Proof of (7.22)

Consider (7.16) for a rate-R source for S_t:

$$
\liminf_{t\to\infty} \frac{1}{\rho t} - \log \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{\rho}{1+\rho} i(X_{1:t};Y_{1:t})}\middle| Y\right]^{1+\rho}\right] \geq R \log(2).
\tag{7.59}
$$

Then, we have:

$$
\begin{aligned}
& \mathbb{E}\left[\mathbb{E}\left[e^{-\frac{\rho}{1+\rho} i(X_{1:t};Y_{1:t})}\middle| Y\right]^{1+\rho}\right]^{-1} \\
\leq & \mathbb{E}\left[\mathbb{E}\left[e^{\rho i(X_{1:t};Y_{1:t})}\middle| Y\right]^{-1}\right]^{-1} \\
\leq & \mathbb{E}\left[\mathbb{E}\left[e^{\rho i(X_{1:t};Y_{1:t})}\middle| Y\right]\right],
\end{aligned}
\tag{7.60}
$$

where the inequalities follow from Jensen's inequality. Combining (7.60) with (7.59) gives (7.22).

7.3.7 The Proof of Theorem 16

Consider the MAP estimator for $S_{1:t}$, i.e., $\hat{S}_{1:t}^{(\mathrm{MAP})}$, and that $\hat{S}_t$ is the estimate at time t for this estimator, which gives the following:

$$\begin{aligned}
&\Pr\left(d(S_t,\hat{S}_t) > r\right) = \mathbb{E}\left[\Pr\left(d(S_t,\hat{S}_t) > r \mid C\right)\right] \\
\leq\ & \mathbb{E}\left[\mathbb{E}\left[\min\left\{1, \sum_{s_{1:t}:d(S_t,s_t)>r} \mathbb{I}_{\{P_{S_{1:t}|Y_{1:t}}(s_{1:t}|Y_{1:t})\geq P_{S_{1:t}|Y_{1:t}}(S_{1:t}|Y_{1:t})\}}\right\}\middle|Y_{1:t}, S_{1:t}, C\right]\right] \\
=\ & \mathbb{E}\left[\mathbb{E}\left[\min\left\{1, \sum_{s_{1:t}:d(S_t,s_t)>r} \mathbb{I}_{\{P_{S_{1:t}|Y_{1:t}}(s_{1:t}|Y_{1:t})\geq P_{S_{1:t}|Y_{1:t}}(S_{1:t}|Y_{1:t})\}}\right\}\middle|Y_{1:t}, S_{1:t}\right]\right] \\
\leq\ & \mathbb{E}\left[\min\left\{1, \mathbb{E}\left[\sum_{s_{1:t}:d(S_t,s_t)>r} \mathbb{I}_{\{P_{S_{1:t}|Y_{1:t}}(s_{1:t}|Y_{1:t})\geq P_{S_{1:t}|Y_{1:t}}(S_{1:t}|Y_{1:t})\}}\middle|Y_{1:t}, S_{1:t}\right]\right\}\right] \\
\leq\ & \mathbb{E}\left[\min\left\{1, \mathbb{E}\left[\sum_{s_{1:t}:d(S_t,s_t)>r} \left(\frac{P_{S_{1:t}|Y_{1:t}}(s_{1:t}|Y_{1:t})}{P_{S_{1:t}|Y_{1:t}}(S_{1:t}|Y_{1:t})}\right)^{\sigma}\middle|Y_{1:t}, S_{1:t}\right]\right\}\right] \\
=\ & \mathbb{E}\left[\min\left\{1, \mathbb{E}\left[\sum_{s_{1:t}:d(S_t,s_t)>r} \left(\frac{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|s_{1:t})P_{S_{1:t}}(s_{1:t})}{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|S_{1:t})P_{S_{1:t}}(S_{1:t})}\right)^{\sigma}\middle|Y_{1:t}, S_{1:t}\right]\right\}\right] \\
=\ & \mathbb{E}\left[\min\left\{1, \mathbb{E}\left[\left(\frac{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|\bar{S}_{1:t})}{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|S_{1:t})}\right)^{\sigma} \mathbb{I}_{\{d(S_t,\bar{S}_t)>r\}} \frac{\left(P_{S_{1:t}}(\bar{S}_{1:t})\right)^{\sigma-1}}{(P_{S_{1:t}}(S_{1:t}))^{\sigma}}\middle|Y_{1:t}, S_{1:t}\right]\right\}\right],
\end{aligned} \tag{7.61}$$

where $\mathbb{I}_{\{\cdot\}}$ is the indicator function, the first inequality is due to the MAP estimation of $S_{1:t}$ (as in (7.32)), the second inequality follows from Jensen's inequality for $\min\{1, x\}$, the third inequality follows from the inequality $\mathbb{I}_{\{x\geq r\}} \leq (x/r)^{\sigma}$ for any $x \geq 0, r > 1, s \geq 0$, and $\bar{S}_{1:t}$ are chosen independently from $Y_{1:t}$ having the same distribution with $S_{1:t}$.

Now, consider:

$$
\begin{aligned}
& \mathbb{E}\left[\left(\frac{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|\bar{S}_{1:t})}{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|S_{1:t})}\right)^{\sigma} \mathbb{I}_{\{d(S_t,\bar{S}_t)>r\}} \frac{\left(P_{S_{1:t}}(\bar{S}_{1:t})\right)^{\sigma-1}}{(P_{S_{1:t}}(S_{1:t}))^{\sigma}} \Big| Y_{1:t}, S_{1:t}\right] \\
\leq\ & \mathbb{E}\left[\left(\frac{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|\bar{S}_{1:t})}{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|S_{1:t})}\right)^{p\sigma} \Big| Y_{1:t}, S_{1:t}\right]^{\frac{1}{p}} \mathbb{E}\left[\mathbb{I}_{\{d(S_t,\bar{S}_t)>r\}} \frac{\left(P_{S_{1:t}}(\bar{S}_{1:t})\right)^{q(\sigma-1)}}{(P_{S_{1:t}}(S_{1:t}))^{q\sigma}} \Big| Y_{1:t}, S_{1:t}\right]^{\frac{1}{q}} \\
=\ & \mathbb{E}\left[\left(\frac{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|\bar{S}_{1:t})}{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|S_{1:t})}\right)^{p\sigma} \Big| Y_{1:t}, S_{1:t}\right]^{\frac{1}{p}} \mathbb{E}\left[\mathbb{I}_{\{d(S_t,\bar{S}_t)>r\}} \frac{\left(P_{S_{1:t}}(\bar{S}_{1:t})\right)^{q(\sigma-1)}}{(P_{S_{1:t}}(S_{1:t}))^{q\sigma}} \Big| S_{1:t}\right]^{\frac{1}{q}} \\
=\ & \mathbb{E}\left[\left(\frac{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|\bar{S}_{1:t})}{P_{Y_{1:t}|S_{1:t}}(Y_{1:t}|S_{1:t})}\right)^{p\sigma} \Big| Y_{1:t}, S_{1:t}\right]^{\frac{1}{p}} M_t(r,\sigma,q,S_{1:t}) \\
=\ & \mathbb{E}\left[\left(\frac{P_{Y_{1:t}|X_{1:t}}(Y_{1:t}|\varepsilon_{1:t;C}(\bar{S}_{1:t}))}{P_{Y_{1:t}|X_{1:t}}(Y_{1:t}|\varepsilon_{1:t;C}(S_{1:t}))}\right)^{p\sigma} \Big| Y_{1:t}, S_{1:t}\right]^{\frac{1}{p}} M_t(r,\sigma,q,S_{1:t}) \\
=\ & \mathbb{E}\left[\prod_{t'=1}^{t}\left(\frac{P_{Y_{t'}|X_{1:t'},Y_{1:t'-1}}(Y_{t'}|\varepsilon_{1:t';C}(\bar{S}_{1:t'}), Y_{1:t'})}{P_{Y_{t'}|X_{1:t'},Y_{1:t'-1}}(Y_{t'}|\varepsilon_{1:t';C}(S_{1:t'}), Y_{1:t'})}\right)^{p\sigma} \Big| Y_{1:t}, S_{1:t}\right]^{\frac{1}{p}} M_t(r,\sigma,q,S_{1:t}) \\
=\ & \kappa_t(p,\sigma,Y_{1:t},S_{1:t}) M_t(r,\sigma,q,S_{1:t}) \qquad (7.62)
\end{aligned}
$$

where the inequality follows from Hölder's inequality.

Combining (7.61) and (7.62) gives (7.33).

Conclusion: Summary and Research Directions

CHAPTER 8

CONCLUSION

8.1 Summary

Ubiquitous connectivity for reliable sensing and control is one of the ultimate goals of communication networks. Engineering communication systems and schemes to be more adaptive or responsive is a natural direction towards this goal. Communicating agents may adapt their data transmission in accordance with observable dynamic conditions affecting the efficiency of their transmission. The intermittent availability of resources, the usefulness of the transmitted data, or a dynamic information source from which transmitted data are obtained are such conditions. In this thesis, we studied communication problems where communication schemes are not oblivious to either of these conditions.

In Chapter 2, we considered a problem of energy efficient data transmission for an energy harvesting transmitter. The transmitter was modelled as a device which consumes its accumulated energy to forward its accumulated data with an efficiency determined by its transmission power and the fading level of the channel. Throughput maximizing and energy efficient offline transmission schedules for finite horizons were characterized through a per-slot based water-filling solution where water levels are dynamic and adapting to energy and data arrival processes. The structure of these schedules exhibits the intuition that while a late consumption of energy and data is undesirable for throughput maximization over a finite horizon, an early consumption is also undesirable for energy efficiency which in turn limits the achievable throughput. This suggests that the transmission should be timely in order to adapt to these dynamics. On the other hand, such a timely transmission is highly sensitive to future arrivals of energy, data and channel fading which are not known in advance. For this reason, we studied online schedules and proposed an approach where online schedules are derived from the estimates based on the possible outcomes of their offline counterparts. The content of Chapter 2 has been covered in [4].

In Chapter 3, we studied energy-efficient transmission schedules in communication systems where the energy income of the transmitter is controlled by the receiver through wireless energy transfer and affected by the channel gain of the energy transfer channel. We compared energy-efficient transmission under transmitter-centric, receiver-centric and distributed scheduling scenarios that differ depending on which side of the transmitter-receiver pair controls the transmission and whether they adapt their energy transmission/consumption. This comparison showed the effectiveness of energy aware and channel aware transmission power control adaptations. The content of Chapter 3 has been covered in [5].

In Chapter 4, we described and discussed the AoI and its variants as performance measures for data freshness or the usefulness of the transmitted data. These measures are particularly suitable for evaluating and optimizing update-based systems where the freshness of the updates available to the end-user is essential. Neither of throughput maximization and delay minimization objectives, which are conventionally used in the optimization of communication networks, may match the end-user's actual demand. On the other hand, the non-linear functions of the age provide a high degree of freedom for constructing an optimization objective close to the actual demand. For machine to machine communication, this is rather doable as communicating agents may have well-defined and quantifiable performance goals[1]. Although being an application centric approach, the optimization of these measures can be done solely through the timing of update transmissions which makes resulting communication schemes extendable and manageable.

In Chapter 5, we considered a problem of timely update transmission for an energy harvesting transmitter. We assumed that the transmission of update packets is error-free and takes zero delay which is reasonable for scenarios where the update packets are transmitted over a direct link with high packet success rate and small delay. The

[1] For example, consider the communication between two devices that takes place as a part of an automated process.

optimal policy was shown to be an age-based and energy-dependent threshold policy for a age-penalty objectives based on a general class of non-decreasing and non-linear functions of the age. As in the problem of energy efficient data transmission, an early consumption of update transmission opportunities can be undesirable as well as their late consumption. This explains why the optimal threshold is positive even for the highest energy level. Due to this property, optimal policies cannot consume energy harvests that occur at the highest energy level hence attain a lower throughput than what is possible. Yet, this can be viewed as an advantage of optimal policies a they produce a lower packet traffic under utilizing network resources. The content of Chapter 5 has been covered in [6].

In Chapter 6, we discussed what happens when the information source is an unstable (or non-stationary) dynamic process and reviewed the related literature. In the case of an unstable source, past transmission errors may become more significant over time as opposed to update-based communication where only the error incurred by the freshest update matters. This is the main challenge in the transmission of unstable sources through a noisy system. Particularly, this challenge emerges in networked control systems where communicating agents are in need of stabilizing a dynamical system. The advancement of networked control systems and applications motivates overcoming this challenge with practical communication mechanisms.

In Chapter 7, we studied when the tracking of an unstable source can be maintained with the help of causal and noisy information encoding the source. We formulated the communication setting considering only the basic operations and making minimal assumptions besides causality in order to focus on what is possible statistically. The channel can be the model of a communication system itself. For example, the encoding scheme with the goal of tracking can be considered as an outer code to short-term error control mechanisms. An application layer error control scheme relying on error control and transmission capabilities provided by the lower layer protocols of the network stack, can be designed for that purpose. In that case, the latency and noisiness due to the network infrastructure characterize the channel model in the communication setting which has the goal of tracking. The limits of such settings can be studied through the provided results. The content of Chapter 7 has been partly covered in [7].

The need for timeliness in communication was the common theme in all settings considered in this thesis. Though these settings, we showed why the transmission of information may need to be studied and engineered in a timely manner in the existence of dynamic conditions.

8.2 Research Directions

We suggest the following research directions for further advances in the study of communication systems considered in this thesis:

- A general formulation of the condition that is both necessary and sufficient for reliable tracking through noisy communication, and its study is an important research direction. This formulation may provide a framework for practical schemes.
- The complexity of timely communication schemes, that achieve reliable tracking, may be considered as a practically important research direction. A formulation that incorporates the goal of tracking with complexity based measures may reveal the achievable tracking performance for communicating agents with limited complexities.
- The statistics of relevant processes may not be a priorly known to communicating agents. This makes the study of universal timely communication schemes, that attain performance guarantees without relying on a priorly known statistics, another practically important research direction.

REFERENCES

[1] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379–423, Jul 1948.

[2] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, Dec 2011.

[3] R. Srikant, *Communication networks : an optimization, control, and stochastic networks perspective*. Cambridge, United Kingdom New York: Cambridge University Press, 2014.

[4] B. T. Bacinoglu, E. Uysal-Biyikoglu, and C. E. Koksal, "Finite-horizon energy-efficient scheduling with energy harvesting transmitters over fading channels," *IEEE Trans. on Wireless Communications*, vol. 16, pp. 6105–6118, Sept 2017.

[5] B. T. Bacinoglu, O. Kaya, and E. Uysal-Biyikoglu, "Energy efficient transmission scheduling for channel-adaptive wireless energy transfer," in *2018 IEEE Wireless Communications and Networking Conference (WCNC)*, pp. 1–6, 2018.

[6] B. T. Bacinoglu, Y. Sun, E. Uysal, and V. Mutlu, "Optimal status updating with a finite-battery energy harvesting source," *Journal of Communications and Networks*, vol. 21, no. 3, pp. 280–294, 2019.

[7] B. T. Bacinoglu, Y. Sun, and E. Uysal, "On the trackability of stochastic processes based on causal information," in *2020 IEEE International Symposium on Information Theory (ISIT)*, pp. 2228–2233, 2020.

[8] E. Uysal-Biyikoglu, B. Prabhakar, and A. E. Gamal, "Energy-efficient packet transmission over a wireless link," *IEEE Trans. on Networking*, vol. 10, pp. 487–499, Aug. 2002.

[9] R. A. Berry and R. G. Gallager, "Communication over fading channels with

delay constraints," *IEEE Trans. on Information Theory*, vol. 48, pp. 1135–1149, May 2002.

[10] P. Nuggehalli, V. Srinivashan, and R. R. Rao, "Delay constrained energy efficient transmission strategies for wireless devices," in *Proc. IEEE INFOCOM*, vol. 3, pp. 1765–1772, June 2002.

[11] M. A. Zafer and E. Modiano, "A calculus approach to energy-efficient data transmission with quality of service constraints," *IEEE Trans. on Networking*, vol. 17, pp. 898–911, June 2009.

[12] J. Yang and S. Ulukus, "Optimal packet scheduling in an energy harvesting communication system," *IEEE Trans. on Communications*, vol. 60, pp. 220–230, January 2012.

[13] K. Tutuncuoglu and A. Yener, "Optimum transmission policies for battery limited energy harvesting nodes," *IEEE Trans. on Wireless Communications*, vol. 11, pp. 1180–1189, March 2012.

[14] M. A. Antepli, E. Uysal-Biyikoglu, and H. Erkal, "Optimal packet scheduling on an energy harvesting broadcast link," *IEEE Journal on Selected Areas in Communications*, vol. 29, pp. 1721–1731, September 2011.

[15] S. Chen, P. Sinha, N. B. Shroff, and C. Joo, "Finite-horizon energy allocation and routing scheme in rechargeable sensor networks," in *Proc. IEEE INFOCOM*, pp. 2273–2281, April 2011.

[16] O. Ozel, K. Tutuncuoglu, J. Yang, S. Ulukus, and A. Yener, "Adaptive transmission policies for energy harvesting wireless nodes in fading channels," in *Information Sciences and Systems (CISS), 2011 45th Annual Conference on*, pp. 1–6, March 2011.

[17] M. Gatzianas, L. Georgiadis, and L. Tassiulas, "Control of wireless networks with rechargeable batteries," *IEEE Trans. on Communications*, vol. 9, pp. 581–593, Feb. 2010.

[18] C. K. Ho and R. Zhang, "Optimal energy allocation for wireless communications powered by energy harvesters," in *Proc. IEEE Intl. Symposium on Information Theory*, pp. 2368–2372, June 2010.

[19] O. Ozel, K. Tutuncuoglu, J. Yang, S. Ulukus, and A. Yener, “Transmission with energy harvesting nodes in fading wireless channels: Optimal policies,” *Selected Areas in Communications, IEEE Journal on*, vol. 29, pp. 1732 –1743, september 2011.

[20] O. Ozel, J. Yang, and S. Ulukus, “Optimal scheduling over fading broadcast channels with an energy harvesting transmitter,” in *Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP), 2011 4th IEEE International Workshop on*, pp. 193–196, Dec 2011.

[21] O. Orhan, D. Gunduz, and E. Erkip, “Throughput maximization for an energy harvesting communication system with processing cost,” in *Information Theory Workshop (ITW), 2012 IEEE*, pp. 84–88, Sept 2012.

[22] N. Roseveare and B. Natarajan, “A structured approach to optimization of energy harvesting wireless sensor networks,” in *Consumer Communications and Networking Conference (CCNC), 2013 IEEE*, pp. 420–425, Jan 2013.

[23] D. Gunduz, K. Stamatiou, N. Michelusi, and M. Zorzi, “Designing intelligent energy harvesting communication systems,” *Communications Magazine, IEEE*, vol. 52, pp. 210–216, January 2014.

[24] H. Li, J. Xu, R. Zhang, and S. Cui, “A general utility optimization framework for energy-harvesting-based wireless communications,” *IEEE Communications Magazine*, vol. 53, pp. 79–85, April 2015.

[25] C. K. Ho and R. Zhang, “Optimal energy allocation for wireless communications with energy harvesting constraints,” *Signal Processing, IEEE Trans. on*, vol. 60, pp. 4808–4818, Sept 2012.

[26] V. Sharma, U. Mukherji, V. Joseph, and S. Gupta, “Optimal energy management policies for energy harvesting sensor nodes,” *Wireless Communications, IEEE Trans. on*, vol. 9, no. 4, pp. 1326–1336, 2010.

[27] A. Sinha and P. Chaporkar, “Optimal power allocation for a renewable energy source,” in *Communications (NCC), 2012 National Conference on*, pp. 1–5, Feb 2012.

[28] R. Srivastava and C. Koksal, “Basic performance limits and tradeoffs in energy-harvesting sensor nodes with finite data and energy storage,” *Networking, IEEE/ACM Trans. on*, vol. 21, pp. 1049–1062, Aug 2013.

[29] P. Blasco, D. Gunduz, and M. Dohler, “A learning theoretic approach to energy harvesting communication system optimization,” *Wireless Communications, IEEE Trans. on*, vol. 12, pp. 1872–1882, April 2013.

[30] I. Ahmed, K. T. Phan, and T. Le-Ngoc, “Optimal stochastic power control for energy harvesting systems with delay constraints,” *IEEE Journal on Selected Areas in Communications*, vol. 34, pp. 3512–3527, Dec 2016.

[31] D. Shaviv and A. Ozgur, “Universally near-optimal online power control for energy harvesting nodes,” in *2016 IEEE International Conference on Communications (ICC)*, pp. 1–6, May 2016.

[32] M. Salehi Heydar Abad, O. Ercetin, and D. Gündüz, “Channel Sensing and Communication over a Time-Correlated Channel with an Energy Harvesting Transmitter,” *ArXiv e-prints*, Mar. 2017.

[33] R. Vaze, R. Garg, and N. Pathak, “Dynamic power allocation for maximizing throughput in energy-harvesting communication system,” *IEEE/ACM Trans. on Networking*, vol. 22, pp. 1621–1630, Oct 2014.

[34] V. Rodriguez and R. Mathar, “Generalised water-filling: costly power optimally allocated to sub-carriers under a general concave performance function,” in *Information Sciences and Systems (CISS), 2010 44th Annual Conference on*, pp. 1–3, March 2010.

[35] N. Shinohara, “Development of rectenna with wireless communication system,” in *Proc. European Conf. Ant. Prop.*, pp. 3970–3973, April 2011.

[36] Z. Popovic, “E-wehp: A batteryless embedded sensor platform wirelessly powered from ambient digital-tv signal,” *IEEE Microw. Mag.*, vol. 14, pp. 55–62, 2013.

[37] Y. K. R. J. Vyas, B. Cook and M. M.Tentzeris, “Cut the cord: Low-power far-field wireless powering,” *IEEE Trans. Microwave Th. Tech.*, vol. 61, pp. 2491–2505, 2013.

[38] S. Kaul, M. Gruteser, V. Rai, and J. Kenney, "Minimizing age of information in vehicular networks," in *Sensor, Mesh and Ad Hoc Communications and Networks (SECON), 2011 8th Annual IEEE Communications Society Conference on*, pp. 350–358, June 2011.

[39] S. Kaul, R. Yates, and M. Gruteser, "Real-time status: How often should one update?," in *INFOCOM 2012*, pp. 2731–2735.

[40] Y. Sun, Y. Polyanskiy, and E. Uysal-Biyikoglu, "Remote estimation of the wiener process over a channel with random delay," in *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 321–325, June 2017.

[41] J. Cho and H. Garcia-Molina, "Effective page refresh policies for web crawlers," *ACM Trans. Database Syst.*, vol. 28, pp. 390–426, Dec. 2003.

[42] Y. Sun, E. Uysal-Biyikoglu, R. Yates, C. E. Koksal, and N. B. Shroff, "Update or wait: How to keep your data fresh," in *IEEE INFOCOM 2016*, pp. 1–9, April 2016.

[43] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, "Update or wait: How to keep your data fresh," *IEEE Transactions on Information Theory*, vol. 63, pp. 7492–7508, Nov 2017.

[44] A. Kosta, N. Pappas, A. Ephremides, and V. Angelakis, "Age and value of information: Non-linear age case," in *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 326–330, June 2017.

[45] Y. Sun and B. Cyr, "Sampling for data freshness optimization: Non-linear age functions," *Journal of Communications and Networks - Special Issue on Age of Information, in press*, 2019.

[46] S. Razniewski, "Optimizing update frequencies for decaying information," in *Proceedings of the 25th ACM International on Conference on Information and Knowledge Management*, CIKM '16, (New York, NY, USA), pp. 1191–1200, ACM, 2016.

[47] A. Garnaev, W. Zhang, J. Zhong, and R. D. Yates, "Maintaining information freshness under jamming," in *IEEE INFOCOM 2019 - IEEE Conference*

on Computer Communications Workshops (INFOCOM WKSHPS), IEEE, Apr 2019.

[48] O. Ayan, M. Vilgelm, M. Klügel, S. Hirche, and W. Kellerer, "Age-of-information vs. value-of-information scheduling for cellular networked control systems," in *Proceedings of the 10th ACM/IEEE International Conference on Cyber-Physical Systems*, ACM, Apr 2019.

[49] C. Kam, S. Kompella, and A. Ephremides, "Age of information under random updates," in *IEEE ISIT*, pp. 66–70, July 2013.

[50] M. Costa, M. Codreanu, and A. Ephremides, "Age of information with packet management," in *IEEE ISIT*, pp. 1583–1587, June 2014.

[51] L. Huang and E. Modiano, "Optimizing age-of-information in a multi-class queueing system," in *IEEE ISIT*, pp. 1681–1685, June 2015.

[52] N. Pappas, J. Gunnarsson, L. Kratz, M. Kountouris, and V. Angelakis, "Age of information of multiple sources with queue management," in *2015 ICC*, pp. 5935–5940, June 2015.

[53] C. Kam, S. Kompella, G. D. Nguyen, and A. Ephremides, "Effect of message transmission path diversity on status age," *IEEE Transactions on Information Theory*, vol. 62, pp. 1360–1374, March 2016.

[54] E. Najm and R. Nasser, "Age of information: The gamma awakening," in *IEEE ISIT*, pp. 2574–2578, July 2016.

[55] R. D. Yates and S. K. Kaul, "The age of information: Real-time status updating by multiple sources," *IEEE Transactions on Information Theory*, vol. 65, pp. 1807–1827, March 2019.

[56] E. Najm, R. Yates, and E. Soljanin, "Status updates through m/g/1/1 queues with harq," in *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 131–135, June 2017.

[57] S. Farazi, A. G. Klein, and D. R. Brown, "Average age of information for status update systems with an energy harvesting server," in *IEEE INFOCOM 2018 -*

IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS), pp. 112–117, April 2018.

[58] J. Zhong and R. D. Yates, "Timeliness in lossless block coding," in *2016 Data Compression Conference (DCC)*, IEEE, Mar 2016.

[59] R. D. Yates, E. Najm, E. Soljanin, and J. Zhong, "Timely updates over an erasure channel," in *2017 IEEE International Symposium on Information Theory (ISIT)*, IEEE, Jun 2017.

[60] P. Parag, A. Taghavi, and J.-F. Chamberland, "On real-time status updates over symbol erasure channels," in *2017 IEEE Wireless Communications and Networking Conference (WCNC)*, IEEE, Mar 2017.

[61] P. Mayekar, P. Parag, and H. Tyagi, "Optimal lossless source codes for timely updates," in *2018 IEEE International Symposium on Information Theory (ISIT)*, IEEE, Jun 2018.

[62] J. Zhong, R. D. Yates, and E. Soljanin, "Timely lossless source coding for randomly arriving symbols," in *2018 IEEE Information Theory Workshop (ITW)*, IEEE, Nov 2018.

[63] H. Sac, T. Bacinoglu, E. Uysal-Biyikoglu, and G. Durisi, "Age-optimal channel coding blocklength for an m/g/1 queue with harq," in *2018 IEEE 19th International Workshop on Signal Processing Advances in Wireless Communications (SPAWC)*, IEEE, Jun 2018.

[64] R. Devassy, G. Durisi, G. C. Ferrante, O. Simeone, and E. Uysal-Biyikoglu, "Delay and peak-age violation probability in short-packet transmissions," in *2018 IEEE International Symposium on Information Theory (ISIT)*, IEEE, Jun 2018.

[65] S. Bhambay, S. Poojary, and P. Parag, "Fixed length differential encoding for real-time status updates," *IEEE Transactions on Communications*, vol. 67, pp. 2381–2392, Mar 2019.

[66] S. Feng and J. Yang, "Age-optimal transmission of rateless codes in an erasure channel," in *ICC 2019 - 2019 IEEE International Conference on Communications (ICC)*, IEEE, May 2019.

[67] E. T. Ceran, D. Gunduz, and A. Gyorgy, “Average age of information with hybrid arq under a resource constraint,” *IEEE Transactions on Wireless Communications*, vol. 18, pp. 1900–1913, Mar 2019.

[68] J. Ostman, R. Devassy, G. Durisi, and E. Uysal, “Peak-age violation guarantees for the transmission of short packets over fading channels,” in *IEEE INFOCOM 2019 - IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)*, IEEE, Apr 2019.

[69] A. Arafa, K. Banawan, K. G. Seddik, and H. V. Poor, “On timely channel coding with hybrid arq,” in *2019 IEEE Global Communications Conference (GLOBECOM)*, IEEE, Dec 2019.

[70] T. Bacinoglu, E. T. Ceran, and E. Uysal-Biyikoglu, “Age of information under energy replenishment constraints,” in *Proc. Info. Theory and Appl. Workshop*, Feb. 2015.

[71] R. D. Yates, “Lazy is timely: Status updates by an energy harvesting source,” in *2015 IEEE International Symposium on Information Theory (ISIT)*, Jun. 2015.

[72] T. Bacinoglu and E. Uysal-Biyikoglu, “Scheduling status updates to minimize age of information with an energy harvesting sensor,” in *2017 IEEE International Symposium on Information Theory (ISIT)*, Jun. 2017.

[73] X. Wu, J. Yang, and J. Wu, “Optimal status update for age of information minimization with an energy harvesting source,” *IEEE Transactions on Green Communications and Networking*, vol. 2, pp. 193–204, March 2018.

[74] A. Arafa, J. Yang, S. Ulukus, and H. V. Poor, “Age-Minimal Online Policies for Energy Harvesting Sensors with Incremental Battery Recharges,” *ArXiv e-prints*, Feb. 2018.

[75] A. Arafa, J. Yang, S. Ulukus, and H. V. Poor, “Age-minimal transmission for energy harvesting sensors with finite batteries: Online policies,” *CoRR*, vol. abs/1806.07271, 2018.

[76] S. Feng and J. Yang, “Minimizing age of information for an energy harvesting source with updating failures,” in *2018 IEEE International Symposium on Information Theory (ISIT)*, pp. 2431–2435, June 2018.

[77] A. Arafa, J. Yang, S. Ulukus, and H. V. Poor, "Online timely status updates with erasures for energy harvesting sensors," in *2018 56th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pp. 966–972, Oct 2018.

[78] O. Ozel, "Timely status updating through intermittent sensing and transmission," *CoRR*, vol. abs/2001.01122, 2020.

[79] E. Gindullina, L. Badia, and D. Gündüz, "Age-of-information with information source diversity in an energy harvesting system," *CoRR*, vol. abs/2004.11135, 2020.

[80] A. R. Al-Ali, I. Zualkernan, and F. Aloul, "A mobile gprs-sensors array for air pollution monitoring," *IEEE Sensors Journal*, vol. 10, pp. 1666–1671, Oct 2010.

[81] G. Peskir and A. Shiryaev, *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics. ETH Zürich, Birkhäuser Basel, 2006.

[82] X. Guo and O. Hernandez-Lerma, *Continuous-Time Markov Decision Processes*. Berlin: Springer, Science Pres, 2009.

[83] B. T. Bacinoglu, Y. Sun, E. Uysal-Biyikoglu, and V. Mutlu, "Achieving the age-energy tradeoff with a finite-battery energy harvesting source," in *2018 IEEE International Symposium on Information Theory (ISIT)*, pp. 876–880, June 2018.

[84] W. Feller, *An Introduction to Probability Theory and Its Applications, Volume 2, 2nd Edition*. John Wiley and Sons, New York, 1971.

[85] D. V. Widder, *The Laplace transform*. Princeton: Princeton University Press, 1946.

[86] R. Gallager, *Stochastic Processes: Theory for Applications*. Cambridge University Press, 2013.

[87] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus (Graduate Texts in Mathematics)*. New York, NY, USA: Springer, 1991.

[88] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, vol. 39. Springer, 1998.

[89] G. J. Foschini and G. Vannucci, "Characterizing filtered light waves corrupted by phase noise," *IEEE Transactions on Information Theory*, vol. 34, pp. 1437–1448, Nov 1988.

[90] K. B. Kac, Mark and M. D. Donsker, *Mark Kac: probability, number theory, and statistical physics : selected papers*. Cambridge, Mass, MIT Press, 1979.

[91] S. Yuksel and T. Basar, *Stochastic Networked Control Systems*. Springer New York, 2013.

[92] R. Bansal and M. Basar, "Simultaneous design of measurement and control strategies for stochastic systems with feedback," *Automatica*, 9 1989.

[93] R. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, pp. 1279–1289, Jul 2000.

[94] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1056–1068, Jul 2004.

[95] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, pp. 413–436, Jan 2004.

[96] S. Yuksel and T. Basar, "Minimum rate coding for lti systems over noiseless channels," *IEEE Transactions on Automatic Control*, vol. 51, pp. 1878–1887, Dec 2006.

[97] O. Imer, S. Yüksel, and M. Basar, "Optimal control of lti systems over unreliable communication links," *Automatica*, 9 2006.

[98] A. Mahajan and D. Teneketzis, "Optimal design of sequential real-time communication systems," *IEEE Transactions on Information Theory*, vol. 55, pp. 5317–5338, Nov 2009.

[99] S. Yuksel, "Stochastic stabilization of noisy linear systems with fixed rate limited feedback," *IEEE Transactions on Automatic Control*, 2010.

[100] S. Yuksel, "Characterization of information channels for asymptotic mean stationarity and stochastic stability of nonstationary/unstable linear systems," *IEEE Transactions on Information Theory*, vol. 58, pp. 6332–6354, Oct 2012.

[101] C. D. Charalambous, C. K. Kourtellaris, and P. Stavrou, "Stochastic control over finite capacity channels: Causality, feedback and uncertainty," in *Proceedings of the 48th IEEE Conference on Decision and Control, CDC*, pp. 5889–5894, 2009.

[102] A. Khina, V. Kostina, A. Khisti, and B. Hassibi, "Tracking and control of gauss-markov processes over packet-drop channels with acknowledgments," *IEEE Transactions on Control of Network Systems*, pp. 1–1, 2018.

[103] V. Kostina and B. Hassibi, "Rate-cost tradeoffs in control," *IEEE Transactions on Automatic Control*, vol. 64, pp. 4525–4540, Nov 2019.

[104] J. Zhang and C. Wang, "On the rate-cost of gaussian linear control systems with random communication delays," in *2018 IEEE International Symposium on Information Theory (ISIT)*, pp. 2441–2445, June 2018.

[105] P. A. Stavrou, M. Skoglund, and T. Tanaka, "Sequential source coding for stochastic systems subject to finite rate constraints," *CoRR*, vol. abs/1906.04217, 2019.

[106] A. Khina, E. R. Garding, G. M. Pettersson, V. Kostina, and B. Hassibi, "Control over gaussian channels with and without source-channel separation," *IEEE Transactions on Automatic Control*, vol. 64, pp. 3690–3705, Sep. 2019.

[107] C. Kawan and S. Yuksel, "Metric and topological entropy bounds for optimal coding of stochastic dynamical systems," *IEEE Transactions on Automatic Control*, pp. 1–1, 2019.

[108] T. Berger, "Information rates of wiener processes," *IEEE Transactions on Information Theory*, vol. 16, pp. 134–139, Mar 1970.

[109] R. Gray, "Information rates of autoregressive processes," *IEEE Transactions on Information Theory*, vol. 16, pp. 412–421, Jul 1970.

[110] T. Hashimoto and S. Arimoto, "On the rate-distortion function for the nonstationary gaussian autoregressive process," *IEEE Transactions on Information Theory*, vol. 26, pp. 478–480, Jul 1980.

[111] K. You and L. Xie, "Minimum data rate for mean square stabilizability of linear systems with markovian packet losses," *IEEE Transactions on Automatic Control*, vol. 56, pp. 772–785, Apr 2011.

[112] A. Sahai, "Anytime information theory," in *Ph.D. Dissertation, MIT*, 2001.

[113] P. Minero and M. Franceschetti, "Anytime capacity of a class of markov channels," *IEEE Transactions on Automatic Control*, vol. 62, pp. 1356–1367, March 2017.

[114] H.T.Simsek, "Anytime channel coding with feedback," in *Ph.D. Dissertation, University of California, Berkeley*, 2004.

[115] J. David Forney, G., "Convolutional codes ii. maximum-likelihood decoding," *Information and Control*, vol. 25, pp. 222–266, Jul 1974.

[116] R. T. Sukhavasi and B. Hassibi, "Linear error correcting codes with anytime reliability," in *2011 IEEE International Symposium on Information Theory Proceedings*, IEEE, Jul 2011.

[117] R. T. Sukhavasi and B. Hassibi, "Linear time-invariant anytime codes for control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 61, pp. 3826–3841, Dec 2016.

[118] L. Dossel, L. K. Rasmussen, R. Thobaben, and M. Skoglund, "Anytime reliability of systematic ldpc convolutional codes," in *2012 IEEE International Conference on Communications (ICC)*, IEEE, Jun 2012.

[119] L. Grosjean, L. K. Rasmussen, R. Thobaben, and M. Skoglund, "Systematic ldpc convolutional codes: Asymptotic and finite-length anytime properties," *IEEE Transactions on Communications*, vol. 62, pp. 4165–4183, Dec 2014.

[120] M. Noor-A-Rahim, K. D. Nguyen, and G. Lechner, "Anytime characteristics of spatially coupled code," in *2013 51st Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, IEEE, Oct 2013.

[121] A. Tarable and F. J. Escribano, "Chaos-based anytime-reliable coded communications," *IEEE Systems Journal*, vol. 14, pp. 2214–2224, Jun 2020.

[122] E. Arikan, "An inequality on guessing and its application to sequential decoding," *IEEE Transactions on Information Theory*, vol. 42, pp. 99–105, Jan 1996.

[123] A. Renyi, "On measures of entropy and information," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pp. 547–561, University of California Press, 1961.

[124] M. Pinsker, *Information and information stability of random variables and processes*. Holden-Day series in time series analysis, Holden-Day, 1964.

[125] R. Gallager, "A simple derivation of the coding theorem and some applications," *IEEE Transactions on Information Theory*, vol. 11, pp. 3–18, January 1965.

[126] Y. Polyanskiy and S. Verdu, "Arimoto channel coding converse and renyi divergence," in *2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pp. 1327–1333, Sep. 2010.

[127] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link-part i: Scalar systems," *IEEE Transactions on Information Theory*, vol. 52, pp. 3369–3395, Aug 2006.

[128] Y. Polyanskiy, H. V. Poor, and S. Verdu, "Channel coding rate in the finite blocklength regime," *IEEE Transactions on Information Theory*, vol. 56, no. 5, pp. 2307–2359, 2010.

[129] A. Tauste Campo, G. Vazquez-Vilar, A. Guillén i Fàbregas, and A. Martinez, "Random-coding joint source-channel bounds," in *2011 IEEE International Symposium on Information Theory Proceedings*, pp. 899–902, 2011.

[130] J. Li, "Large deviations for conditional guesswork," *CoRR*, vol. abs/1809.10921, 2018.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Bacınoğlu, Baran Tan
Nationality: Turkish (TC)

EDUCATION

Degree	Institution	Year of Graduation
M.S.	METU Electrical and Electronics Engineering	2013
B.S.	METU Electrical and Electronics Engineering	2009

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2014-2019	METU	Research Assistant