FOSTERING PROSPECTIVE MATHEMATICS TEACHERS' CONCEPTIONS OF DEFINITE INTEGRAL IN THE CONTEXT OF ENGINEERING DESIGN ACTIVITIES: A DESIGN EXPERIMENT

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ABSTRACT

FOSTERING PROSPECTIVE MATHEMATICS TEACHERS' CONCEPTIONS OF DEFINITE INTEGRAL IN THE CONTEXT OF ENGINEERING DESIGN ACTIVITIES: A DESIGN EXPERIMENT

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The aim of the current study is to investigate the pre-service mathematics teachers' developments in the concept images about integral through a design experiments that was designed engineering design process activities. In this design experiment, a sequence of engineering design process activities was developed, tested, and evaluated in a real classroom environment in two cycles. Data were collected from three pre-service mathematics teachers enrolled and attending an elective course at a public university in Ankara during the spring semester of 2017-2018. In the data collection process multiple data collection sources were used. Qualitative methods were used in analyzing the data.

On the other hand during the analyzing the data a new method was used which is linkograph. The linkograph enabled researcher to see that what processes students went through while constructing a concept individually and within a group and how they constructed it. In this way, the stages of students' process of reaching mathematically accurate concept images were analyzed in more detail. moreover, thanks to the linkograph, students were visualized where and after which event and which link they made.

The data indicated that pre-service mathematics teachers various concept images such as partial-primitive, discrete and a hybrid concept image. In the progress of the design experiment sessions significant improvements were observed in pre-service mathematics teachers' concept images about integral. Namely, they developed mathematically accurate concept images about integral. Furthermore, they enriched their concept images about integral. Moreover, engineering design process activities provided them to experience the mathematics in the real life application. By this way their interpretation of the authentic problems and solving them become easier. The data also showed that engineering design process activities promoted pre-service mathematics teachers contextual understanding about integral and its sub-concepts.

Keywords: Engineering Design Process, Design- Based Research, Concept Image and Concept Definition, Integral, Mathematics Education

ÖZ

MATEMATİK ÖĞRETMENLİĞİ ÖĞRETMEN ADAYLARININ MÜHENDİSLİK TASARIMI AKTİVİTELERİ İLE BELİRLİ İNTEGRALİ ÖĞRENMELERİ BİR TASARIM DENEYİ

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Bu araştırmanın amacı, matematik öğretmen adaylarının integral ile ilgili kavram imgelerindeki değişimleri, mühendislik tasarım süreci ile tasarlanmış etkinlikleri bir tasarım deneyi aracılığıyla incelemektir. Bu tasarım deneyinde, bir dizi mühendislik tasarım süreci etkinliği gerçek bir sınıf ortamında iki kez uygulanarak geliştirildi, test edildi ve değerlendirildi. Veriler, 2017-2018 bahar döneminde Ankara'da bir devlet üniversitesinde seçmeli bir derse kayıt yaptıran ve devam eden üç tane matematik öğretmen adayından toplanmıştır. Veri toplama sürecinde birden çok veri toplama yöntemine başvurulmuştur. Verilerin analizinde nitel yöntemlerden faydalanılmıştır.

Verilerin analizi sırasında ise yeni bir yöntem olan linkograf yöntemi kullanılmıştır. Linkograf; araştırmacının, grubun gelişimini, öğretmen adaylarının bireysel gelişimlerini ve öğretmen ile grup ile etkileşim süreçlerindeki gelişimi analiz etmesini sağlamıştır. Ayrıca araştırmacıya öğretmen adaylarının kavrmaı inşa ederken hangi zihinsel süreçlerden geçtiğini ve kavramı nasıl yapılandırdıklarını görmesini sağlamıştır. Böylelikle öğrencilerin matematiksel

olarak doğru kavram imgelerine ulaşma süreçlerinin aşamalarının daha detaylı analiz edilmiştir. Dahası linkograf sayesinde araştırmacı öğretmen adaylarının hangi olaydan sonra hangi bağlantıyı kurdukları görselleştirilebilmiş böylece öğretmen adaylarının öğrenmelerini engelleyen ve kolaylaştıran faktörler belirleyebilmiştir. Böylelikle öğretilen kavramın inşaa süreci tamamen izlenebilmiştir.

Analiz sonucunda elde edilen veriler tasarım deneyinin başlangıç aşamasında matematik öğretmen adaylarının kısmi-ilkel, ayrık ve karma kavram imgesi gibi çeşitli kavram imgelerine sahip olduklarını işaret etmektedir. Tasarım deneyi ilerleyen aşamalarında ise matematik öğretmen adaylarının integrale ilişkin kavram imgelerinde önemli gelişmeler gözlemlenmiştir. Yani, öğretmen adayları integral hakkında matematiksel olarak doğru kavram imgeleri geliştirmişlerdir. Dahası, integral ile ilgili kavram imgelerini zenginleştirmişlerdir. Ayrıca bu duruma ek olarak mühendislik tasarım süreci etkinlikleri, gerçek yaşam uygulamaları ile öğretmen adaylarının matematiği deneyimlemelerini sağlamıştır. Bu durum öğretmen adaylarının gerçek hayat problemlerini yorumlamalarını ve çözmeleri kolaylaştırmıştır. Veriler, ayrıca mühendislik tasarım süreci etkinliklerinin matematik öğretmen adaylarının integral ve alt kavramları hakkında kavramsal bağlantıları kurduklarını göstermiştir.

Anahtar Kelimeler: Mühendislik Tasarım Süreci, Tasarım- Tabanlı Araştırma,
Kavram İmge ve Kavram Tanımı, İntegral, Matematik Eğitimi

To my parents and my daughter

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CHAPTER 1

INTRODUCTION

Calculus, a field with wide usage in major sciences (Dickey, 1986), is a common area in which students have difficulties in two fundamental subjects which are differentiation and integration. Although earlier studies (Foley, 1994, Orton, 1983; Thomas, 1995) considered the topic "definite integral" as one entity to explain and find solutions to students' difficulties when learning definite integral. However, recent studies (Bezuindenhout & Oliver, 2000; Bressoud, 2011; Bricio-Barrios, Arceo-Díaz, Maravillas, 2020; Camacho, Depool & Santos-Trigo, 2004; Chhetri & Martin, 2013; Czarnocha, Dubinsky, Loch, Prabhu & Vidakovic, 2001; Jones, 2013; Sealey, 2008; Tall, 1993), or the Fundamental Theorem of Calculus (David, Hah-Roh &Sellers, 2000; Schnepp & Nemirovsky,2001; Thomas, 2003; Thompson, Byerley & Hatfield, 2013) revealed that the difficulties that students confronted might stem from other topics (e.g., accumulation, rate of change) which are necessary to have a robust understanding of the concept of definite integral. Studies about integral can be examined under three headings: integral, the Fundamental Theorem of Calculus, and the Riemann Sum. When studies on integral were concerned, it was seen that the studies focused on students' conceptual understanding, conception, relationship between integration and differentiation relation, accumulation, area under a curve, teaching integral using physics context. Integral is defined as "If is a continuous function defined for $a \leq x \leq b$ we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0\ (= a), x_1, x_2, \dots, x_n\ (= b)$ be the end points of these subintervals and we let $(x_1^*, x_2^*, \dots, x_n^*)$ e any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is $\int_a^b f(x)dx =$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$" (Stewart, 2009, p.381). The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is defined as the Riemann Sum. It can be seen that the concept of definite integral includes multiple ideas such as Riemann Sum, area, function, limit, derivative, multiplication, summation accumulation, and covariation In this context, instructors have problems in teaching integration (Bressoud, 2011; Pantozzi, 2009; Thomas, 2003; Thompson et al., 2013), and students have difficulties in learning the integral such as conceptualizing Riemann integral in terms of limit and infinitesimal structure in it (Sealey, 2008; Bressoud, 2011; Tall, 1993; Camacho, Depool & Santos-Trigo, 2004; Jones, 2013), or the Fundamental Theorem of Calculus (David, Hah-Roh &Sellers, 2000; Schnepp & Nemirovsky, 2001; Thomas, 2003; Thompson et al., 2013).

As a result of not overcoming these problems, students often lack a conceptual understanding of the definite integral concept. Although students knew what to do in integral related tasks, "when questioned about their method, they did not know why they were doing it" (Orton, 1983, p.8). Hence, they confuse integration when algebra is involved in summing rectangular areas (Orton,1983). Students' conceptual learning and argued that the difficulties faced by the students were due to a lack of knowledge (Foley,1994). There have also been some other studies focused on the role of integral and derivative relation in students' integral learning (Abdul-Rahman, 2005; Contreas, 2002), students' errors, misconceptions, and conceptions while learning integrals (Rasslan & Tall, 2002; Fisher, Samuels, & Wangberg, 2016; Li, Julaihi, & Eng, 2017), and the conceptual understanding regarding the calculations related to area, accumulation, and function in definite integral (Oberg, 2000). In this respect, it is risky to define the integral as an area, as this can give rise to different initial concepts evoke (Calvo,1997). That means that students, both low and high achievers, need assistance and compensation for their insufficient knowledge to properly construct the notion of integral. From this perspective, the Riemann Sum should be taught first when teaching integral as a prerequisite concept in conceptualizing integral (Sealey, 2008). Teaching integral using under a curve representation for integral is not enough, and it does not include all of the sub-concepts of integral. Sealey (2008) argues that to compensate those gaps, Riemann sum should be taught first since (i) not all functions have antiderivatives, so the Fundamental Theorem cannot be used in every problem (ii) Riemann sum is the basis for understanding the numerical methods (iii) understanding the Riemann sum will provide students with an intuition when they are setting up appropriate definite integral (Sealey, 2008). It is also necessary to emphasize that while teaching the Riemann sum, its relations with the sub-concepts (eg., accumulation, rate of change, accumulation rate of change, summation, limit) that make up the integral should also be introduced.

My review of relevant literature revealed that the integral has been taught purely without context, preventing students from learning it conceptually and causes students to see it as an operation (e.g., Foley, 1994, Orton, 1983; Thomas, 1995). Therefore, later studies focused on teaching integral in the physics context (Bajracharya, 2012; Cui, 2006; Christensen & Thompson, 2010; Ferrini-Mundy & Graham,1994; Jones, 2013; Marrongelle, 2001; Nguyen & Rebello, 2011; Pantozzi, 2009, Wagner, 2018). These studies showed that students could not transfer their integral knowledge into physics. Moreover, students have difficulty due to the lack of integral conceptualization, even if their physics knowledge is enough (Marrongelle, 2001; Nguyen & Rebello, 2011). The common problem among the students was not to decide on the appropriate mathematical method but not to interpret the mathematical variables in different contexts, especially in physics (Jones, 2013; Marrongelle, 2001; Nguyen & Rebello, 2011). Thus, a student who has difficulty understanding integral cannot correctly understand the physics subjects (such as electricity, center of gravity) that use or are constructed through the notion of integral. Also, they could not apply the concepts of the area under the curve, integral, and accumulation to physics contexts in the given problems (Marrongelle, 2001; Nguyen & Rebello, 2011). The reason for not using these concepts in a different context is that students cannot learn the integral subject conceptually and do not make sense of the given notations, which was one of the reasons that hinder transfer of integral knowledge since students were not able to interpret the meaning of the integral notation and had confusion about their relations (Jones, 2013). The common suggestion of the conducted studies is to solve integral problems in different contexts and do more scaffolding while teaching integral (Bajracharya, 2012; Cui, 2006; Christensen & Thompson, 2010; Ferrini-Mundy & Graham,1994; Jones 2013; Marrongelle 2001; Nguyen and Rebello, 2011; Pantozzi, 2009, Wagner, 2018). As a result, the common consensus is that students' understanding of integral is not deep, and their knowledge is predominantly procedural.

Riemann sum is considered as an effective way to develop a conceptual understanding of the integral (Bezuindenhout & Oliver, 2000; Bricio-Barrios, Arceo-Díaz, Maravillas, 2020; Chhetri & Martin, 2013; Czarnocha, Dubinsky, Loch, Prabhu, Vidakovic,2001; Hamdan, 2019; Hanifah, 2021; Jones, 2015a; Jones, 2015b; Maravillas, Arceo, Salazar, Andrade, Montes, 2020; Nisa, Waluya, and Mariani, 2020; Sealey, 2008; 2014; Sealey & Oehrtman, 2005; Thomas, 1996). However, studies indicated that students had problems with the Riemann sum caused by a lack of understanding of the limit concept (Cui, 2006; Fisher, Samuel & Wangberg, 2016; Sauerheber and Muñoz, 2020). Students' misconceptions in Riemann sum will affect the understanding of the Fundamental Theorem of Calculus (Bezuindenhout & Oliver, 2000; Thompson & Silverman, 2008). It can be argued that without trying to construct, represent and understand Riemann Sums, students cannot understand accumulation and its importance in the Fundamental Theorem of Calculus. Moreover, students had partial learning about Riemann Sum and area and function schema, and when the students went through graphical representations, they could interpret an advanced function concept, albeit indirectly (Brijlall & Bansilal, 2010). Riemann Sum is advantageous to calculate the pressure on the surface area in physics problems or useful in calculating volume. However, when asked, students explain the integral primarily based on area and inverse derivative, while they do not explain it over Riemann sum (Jones, 2015a; 2015b). Because teachers do not spend enough time explaining the Riemann sum and quickly go over to the Fundamental Theorem of Calculus and their explanation of the Riemann sum focus only on calculations, their students perceive the Riemann Sum as a technique requiring only complicated calculations (Jones, 2015a; 2015b). To overcome this kind of problem, it is suggested that when describing the integral, it should start with the Riemann Sum, and the benefit of the Riemann sum in integral interpretations should be prioritized (Jones, 2015a; 2015b; Jones, Lim, & Chandler, 2017).

Another critical issue is the inverse relationship between integral, and differentiation (Abdul-Rahman, 2005; Contreas, 2002; Delice & Sevimli, 2010; Ely, 2017; Hall, 2010; Kaplan, 2005; Salleh & Zakaria, 2012; Serhan, 2015; Yasin & Enver, 2007). Studies revealed that students have two types of difficulties building relations between derivative and integral. They had fewer problems when going from derivative to integral than integral to derivative because of students' perceptions that the integral is an inverse process (Abdul-Rahman, 2005; Contreas, 2002). Thus, students cannot correctly define the definite integral and have difficulties interpreting an area in calculus problems such as definite integrals in different contexts. Also, they found that they cannot find a relationship between the theory of integral and its application (Rasslan & Tall, 2002). Hence, if students cannot understand the area under a curve and the inverse relationship, they have difficulties transiting this knowledge to other integral concepts such as the Fundamental Theorem of Calculus.

Understanding the Fundamental Theorem of Calculus is essential as it plays a bridge role between theory and application and involves the main components of integral (Brosseoud, 2011). As a result of this, it is essential for those who study integral to conceptualize the proper form of the Fundamental Theorem of Calculus since it consists of Riemann sum, which bases on the area under a curve and anti-differentiation in its nature and, all science areas are somehow related with integration (Bressoud, 2011; Carlson et al., 2003; Davidson, 1990; Dickey, 1986; Pantozzi, 2009; Prenowitz,1953; Thomas, 2003; Thompson, 2008; Thompson, Byerley, & Hatfield 2013; Upshaw, 1993). Some detected learning difficulties arise from lack of knowledge about rate of change, accumulation, problems on construction of function, and multiplication of quantities which are $f(c)$ and Δx (Carlson et al., 2003; Thompson,1994; Thompson & Silverman, 2008; Thompson, 2008, Smith, 2008).

Another difficulty that students experience is that while learning the Fundamental Theorem of Calculus, they usually lose focus when they take the limit of the function (Ely, 2012). Thompson (1994) argues that students' understanding of the Riemann Sum affects their understanding of the Fundamental Theorem of Calculus (FTC). Hence students have difficulty in interpreting FTC and they could not explain their reasoning by relating the Fundamental Theorem of Calculus with the Riemann Sum. Besides, students also have problems interpreting the notations without reasoning their interpretations and they just explain how the given expression was written (Jones, 2015a). In this regard, to help students to construct definite integral, it is suggested that accumulation, rate of change, covariation (Thompson, 1994), the infinitesimal concept and limit (Ely, 2012), relationship between multi representations such as graphical, algebraic, visual (Mundy, 1984; Pantozzi, 2009; Thompson & Silverman, 2007), connections between the changes of dy and dx relative to each other and their rate of change (Thompson & Dreyfus, 2016) and derivative (Ely, 2012) need to be taught to students.

When all this literature is examined, it can be concluded that the main problem is that students cannot conceptualize the definite integral comprehensively. Hence, the components of the Fundamental Theorem of Calculus should also be analyzed in a detailed way to find why students have difficulty understanding integral and what their deficiencies in conceptualizing integral are.

Studies about integral focused on the teaching or learning the integral (Bressoud, 2011; Carlson et al., 2003; Davidson, 1990; Pantozzi, 2009; Prenowitz,1953; Thomas, 2003; Thompson, 2008; Thompson et al., 2013); however, they neither focused on constructing the integral concept nor reasons behind the learning or teaching. Mostly they depicted what students learnt or not and they focused on the results of their intervention, not the process which means how students learn.

However, it is essential to detect the reason for the proper construction of the integral. Moreover, focusing on the process will provide richer information about the deficient points in the students learning or intervention. That provide the researchers to manipulate the problematic issues during the process.

Various studies also showed that one of the main reasons for lack of conceptual understanding of the integral is that students do not understand functions very well. Thus, it is important that before teaching the Fundamental Theorem of Calculus, students' conceptions of functions checked and reconstructed if necessary, so that there should be a hierarchical sequence to prevent the departmentalization of students' understandings of the integral concept (Kouropatov & Dreyfus, 2013; Radmehr & Drake, 2017; Swidan, 2020)

In addition to students' difficulties, teachers also have difficulty teaching integral to the students. They cannot know how to begin teaching integral and what can be the first step (Oberg, 2000; Marron). Also, they do not know how to connect the concepts in integral and the other science areas (Bezuindenhout & Oliver, 2000, Contreas, 2002; Davidson, 1990). Moreover, these research studies emphasize that instruction affects the conceptualization of integral and offers guidance to researchers and teachers in studying and teaching the integral concepts in a proper sequence. Furthermore, understanding the nature of students' conceptions is an essential first step in developing well-planned pedagogical approaches that may provide students with opportunities to construct a robust understanding of the concept (Bezuindenhout & Oliver, 2000). As a design-based study involves hypothetical planning (Cobb & Gravemijer, 2008), an instructional sequence should be prepared to provide a structured understanding.

Kolata (1988) claims that knowledge construction is a cumulative process, Educators at universities should know that students' prior knowledge may be inadequate to learn the new concepts. In this respect, Steen (1988) advocates that although students learn about the limit, derivative and integral through high school, they have many deficiencies in calculus concepts at the university level. In the first semester of the university, when students take calculus courses, educators assume that students have accurate knowledge about all subjects in calculus, and new knowledge is added to previous ones. Students relate new information to their existing concept images and adopt them to create new ones, often forming misconceptions based on incomplete or inaccurate knowledge (Redish, 1994; diSessa, 1993). Because the students' concept images may differ from that of the instructor of the course. Hence, to provide qualified learning in integration concepts, students must have an accurate concept image on the definite integral. Moreover, educators need to understand the interaction between students' learning process and their own constructed concept image.

Moreover, research indicates that using interdisciplinary approaches provides less fragmented, more related experiences (Frykholm & Glasson, 2005; Koirala & Bowman, 2003). However, as students learn major subjects separately in the current education system, they cannot relate those subjects in a context or implement what they know to a new situation. Hence, they have difficulties solving problems as they do not understand the relationship between the embedded context and the concepts they learn. Katehi et al. (2009) suggest that to prevent such issues and provide a stronger connection between the disciplines and facilitate a deeper understanding of concepts, engineering contexts should be used. Since engineering is "a catalyst for integrated STEM education" (NAE et al., 2009, p. 150) and it is also a 'vehicle' that provides a real-world context for learning science and mathematics, promote communication problem-solving skills and teamwork, (Hirsch, Carpinelli, Kimmel, Rockland, & Bloom, 2007; Lachapelle & Cunningham, 2014; Mann, Mann, Strutz, Duncan, & Yoon, 2011). Lachapelle and Cunningham (2014) stated that engineering is a tool that contributes to the motivation of students and the integration of science and mathematics.

To achieve these goals, teachers should be aware of the importance of STEM, and also, they should know how to integrate a course with another. National Council of Teachers of Mathematics (NCTM) emphasizes the importance of merging interdisciplinary knowledge with students' experience with real-life problems and states that "thinking mathematically involves looking for connections and making connections builds mathematical understanding" (NCTM, 2000, p.274). Berlin (1991) also asserts that education with integrated disciplines can enhance students' understanding of and attitudes toward both mathematics and science.

To understand whether the intended objectives are gained, the distinction between the mathematical concepts and their formal definition and the cognitive processes by which they are conceived should be formulated. In other words, students' total cognitive structure associated with the integral concept, which includes all the mental pictures, constructing processes, and conceptions that students have during this process, should be defined. Also, their concept definition of integral should be stated as mathematical terms to specify that concept. Finally, students' concept images about integral should be sought to provide the researchers a general picture of students' conception of integral.

With this scope, the study aimed to investigate how students' concept images regarding the definite integral change when STEM activities are used through engineering design-based instruction. The following research questions guided the study:

1) What is the nature of pre-service mathematics teachers' existing concept images of definite integral before exposed to a design experiment involving engineering design-based instruction in a STEM context?
2) How do pre-service mathematics teachers' knowledge and transition from Riemann integral to Fundamental Theorem of Calculus can be fostered through STEM activities including engineering design practices?
    - What understanding do pre-service mathematics teachers develop during engineering design-based instruction in a STEM context?
    - What is the nature of pre-service mathematics teachers constructing the Fundamental Theorem of Calculus on the Riemann Sum?

3) What is the nature of pre-service mathematics teachers' concept image of definite integral after working on STEM activities within engineering design-based instruction?

1.1 Significance of the Study

The significance of the current study can be discussed in four different dimensions: 1) contribution to literature, filling an essential gap in the literature contributing integral and design-based research literature, 2) practical contribution through the instructional design and materials developed as part of the study, 3) theoretical contribution through clarification and extension of Sfard's three-phase theory and classification the concept image and definition theory of Tall and Vinner (1981), and finally, 4) methodical contribution through adopting a new analytical method, namely linkography, for analyzing group-work and pre-service mathematics teachers' thinking.

First of all, despite there is an increasing number of studies focusing on definite integral (see, e.g., Abdul-Rahman, 2005; Contreas, 2002; Bressoud, 2011; Camacho, Depool & Santos-Trigo, 2004; Delice & Sevimli, 2010; Ely, 2017; Hall, 2010; Kaplan, 2005; Jones, 2013; Orton,1983; Salleh & Zakaria, 2012; Sealey, 2008; Serhan, 2015; Tall, 1993; Yasin & Enver, 2007), there is limited research or reporting about the nature of students' understanding. Also, a limited number of studies reported reasons for student's difficulties (Bressoud, 2011; Sealey, 2008; Smith, 2008; Thompson, 1994) in learning definite integral. Moreover, some of the studies focused on teaching the Riemann Sum (Jones, 2013, 2015a, 2015b; Sealey, 2008) some of them focused on teaching the Fundamental Theorem of calculus (Kouropatov & Dreyfus, 2013; Radmehr & Drake, 2017; Swidan, 2020; Thompson et al., 2013). In some studies, even though the teaching of the Fundamental Theorem of Calculus started from the Riemann Sum (Kouropatov & Dreyfus, 2013a; 2013b; Swidan, 2020), the Riemann sum and its items such as accumulation, approximation, covariation were not mentioned in sufficient detail which hinders students’ conceptions of definite integral. Thus, the current study aims to fill this gap by combining the teaching of both Riemann sum and Fundamental Theorem of Calculus in detail and provide a holistic picture about students’ understanding of integral

Furthermore, this study is important in terms of simultaneously considering the Riemann sum and the Fundamental Theorem of Calculus, which are fundamental in integral teaching and ideas such as covariation, accumulation, approximation, limit summation, and multiplication. In other words, the conceptual understanding of the integral is all of these ideas equally crucial for connecting the Riemann sum and the Fundamental Theorem of Calculus. Therefore, unlike the studies, a set of activities in a sequence containing these ideas has been developed in this study. Furthermore, while developing this sequence of activities, unlike other studies, it was focused on the transitions between the integral and its sub-concepts and the nature of these transitions. Thus, more prosperous and more detailed information was obtained about why students had difficulties while learning the integral, and the reasons that hinder their learning were determined. In this respect, this study made a significant contribution to integral teaching.

Moreover, the results of this study may contribute to the literature for teaching integral. On the other hand, different from other studies, the construction processes of the students were examined in detail, and the difficulties the students faced while learning the integral, how they overcome these difficulties, and how the students were provided with assistance in this process were explained in detail. In this context, the theoretical arguments in teaching integral concluded from this study may guide educators and researchers.

Secondly, studies conducted about definite integral emphasize that understanding a concept in a different context would facilitate students’ richer understanding and provide them to interpret the concept in different contexts (Awang & Zakariya, 2012; Delice & Sevimli, 2011; Ely, 2017; Hunter, 2011; Jones, 2013; Marongelle, 2001; Wagner, 2018). Studies about how to teach mathematics from an interdisciplinary approach is limited (English, Hudson, & Dawes, 2013; English & King, & Smeed, 2015; English & King, 2015, 2017; Jones, 2013). The studies in which the engineering design-based approach is used generally focus on mathematical skills (English, Hudson, & Dawes, 2013; English & King, & Smeed, 2015; English & King, 2015, 2017). However, this study has focused on teaching the subject of integral by using engineering design-based instruction with a transdisciplinary approach. Therefore, in this study, a new intervention has been tried and developed for integral teaching. While this study may significantly contribute to the engineering design process studies in STEM, it may also contribute to the design-based studies on mathematics teaching. Moreover, as a result of this study, engineering design activities and the teacher guide of these activities were developed. Therefore, this study provides educators an instructional tool and materials which can be used while teaching integral with STEM activities.

Thirdly, in this study, Sfard (1991)' three-phase theory, the theory of concept image and definition (Tall & Vinner, 1981), and the engineering design process were used to develop the activities for teaching the definite integral and analyze the data. This study aims to contribute how these three theoretical perspectives can be used in a way to complement each other when analyzing students' thinking process during learning the definite integral. Moreover, some weaknesses and vague points of Sfard's three-phase and concept image and concept definition were determined in the study, and new additions and explanations were made to these theories. Therefore, this study can guide researchers in developing these theories or be an opportunity for them to test the added arguments.

Finally, a recent analysis method, linkography (Goldschmidt, 2014), was used in the current study. This method is used in engineering and architecture to depict students' ideas and how they produce something while designing something. It has been used in various research; evaluation of cartoons (Chou, Chou, & Chen, 2013), architecture idea generation process (Goldschmidt, 2014), students' thinking process (Blom & Bogaers, 2018). The method was improved by different researchers (Kan & Gero, 2008; van der Lugt, 2005), and it was used in qualitative and quantitative studies investigating thinking processes or cognition of the designers and their teams (Kan & Gero, 2008; Hatcher, Maclachlan, Marlow, Simpson, Wilson, & Wodehouse, 2018; Blom & Bogaers, 2018). In the current study, it is used, for the first time in the field of education to the best of my knowledge, as methodology to analyze and show how students construe a subject (i.e., integral) when they learn through integrating various other concepts. Thus, the current study's methodology might offers a new perspective for educational researchers in analyzing dynamics involved in students' development of concepts when they work in groups In addition, it is a beneficial analysis method in analyzing students and groups, especially in studies where students work as a group. Hence, it may be helpful for researchers to examine group dynamics in their research involving group work. Thanks to Linkography, it was determined which thinking processes each student went through while learning the integral and how they constructed it. In addition, while this method is used in the analysis of group discourses, it also allows analysis of students' progress within the group and apart from the group. Thus, it was determined where the students had difficulty understanding the subject, where the group had difficulties, and which concepts the group and individuals focused on.

1.2 Definition of Important Terms

This section involves definitions of the important terms that are used in the entire study.

Concept Definition:

Concept definition is decsribed as "to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole" (Tall & Vinner, 1981, p.152). In this research, the theory concept definition is used to determine the existing definitions about definite integral of pre-service mathematics.

Concept-Image:

Concept-Image is described as “the entire cognitive structure associated with the concept, including all the mental pictures and associated properties and processes, including graphs, symbolism, verbal representations, or numerical data” (Tall & Vinner, 1981, p.152). In this research, the theory concept image is used to determine the existing definitions about definite integral of pre-service mathematics teachers and depict their thinking process while they were engaging with the given tasks.

STEM Activities:

A STEM activity is an ill-defined task with a well-defined outcome situated within a contextually rich task requiring students to solve several problems, which, when considered in their entirety, showcase student mastery of several concepts of various STEM subjects. (Capraro, Capraro & Slough, 2011, p.2). In the current study, special designed transdisciplinary STEM activities about definite integral was used to depict the pre-service mathematics teachers thinking and construction process.

Engineering Design Process Based Instruction:

Engineering design has different form consisting of vairous numbers of steps. The research model often depends on “its intended purpose, teaching, and learning that rely heavily on internalizing or learning new content” (Morgan, Moon, & Barroso, 2013, p.4). In general, steps of the design is identifying the problem, research for a solution, ideate, analyzing the ideas, test and refine, communicating, and reflect (Morgan, Moon, & Barroso, 2013). In this study, engineering design In the current study, engineering design based instruction refers to an instruction in which pre-service mathematics teachers make discussion in a group and express their ideas about definite integral by working on a STEM activity in a reals classroom environment.

Sfards Three-phase Theory:

To explain the conceptions of students Sfard (1991) proposed three- phase theory and she separated this process into three hierarchical stages; interiorization, condensation, reification to explain conception evolves into operational to structural. In the first stage of the hierarchy in Sfard's Three Phase Theory, in the interiorization phase, students are in the lowest stages and become familiar with the concept, and have limited skill about the concepts. Sfard (1991) explains this stage as "gets acquainted with the process will eventually give rise to a new concept (p. 18)."In the condensation stage, the student can combine other processes more efficiently, and "the operation of the process is squeezed into more manageable units." In this stage, the student can deal with various representations of the concept. Moreover, the student is capable of making reasoning about the whole concept without dealing with details. Thus, students can handle the whole process within awareness. In the reification processes, students can see the whole process and straightforwardly explain the whole process. Moreover, students can transform information through the steps and interpret the concepts in different aspects. In the current study, this theory was used to examine students' definite integral construction processes under the light of the definitions of each step.

Linkography:

Linkography, introduced by Goldschmidt (1995), analyzes the designers' cognitive process. Linkography is a protocol analysis method that codes and visualizes the links between the 'design moves' expressed during an ideation session (Hatcher, Ion, MacLauchlan, Marlow, Simpson, Wilson, & Wodehouse, 2018, p.129). In the current study linkograph is used for to analyze the pre-service mathematics teachers' construction process of the definite integral during the STEM activities while they work with group.

CHAPTER 2

LITERATURE REVIEW

This chapter of the study includes a review of the relevant research studies on which this study is grounded and the theoretical perspectives that the study used. Integral is an essential calculus concept; however, the literature indicates that learning integral is difficult for students, and teachers have problems teaching the concept. This review of the literature about integral will be given in four subheadings by classifying the research in terms of their primary focus on integral, Fundamental Theorem of Calculus (FTC), Riemann Sum, concept image, and definition, misconceptions, and STEM education, including engineering design process.

2.1 Studies about Integral

The literature review has shown that students at different grade levels learn integral based on procedural and rote learning relying on rules, and they have a lack of conceptual knowledge (Fisher & Samuels, 2016; Foley, 1994; Jones, 2010, 2013; 2015; Kiat, 2005; Orton, 1983; Thompson et al., 2013). One of the studies about students' understanding of definite integral is introduced by Orton (1983) to identify the students' misconception about the concept. He studied with 110 students enrolled in a calculus course and investigated students' problems behind integration methods, particularly when calculating areas under a curve. He found that students had no idea of integration if the algebra involved summing rectangular areas, creating confusion. Also, he found that students have difficulties with the signs of definite integrals and emphasized that students made the procedures by memorizing them. They could not explain the reasons when asked. This situation shows that students have a procedural rather than a conceptual understanding of integral and a lack of conceptual understanding.

A study was carried out by Foley (1994) in the following years to determine students' conceptual and procedural knowledge of definite integral. The data was collected from 97 engineering calculus students. After the students were asked open-ended questions, interviews were conducted with the students. It was reported that while students' procedural knowledge increased, problem-solving anxiety affected their performance. Conceptual questions revealed the missing knowledge of the students and the connections between the concepts and the conceptual information created by the students. Accordingly, the strongest conceptual bonds were between integral, area, and inverse derivatives. The weakest connections were established between integral and Riemann. However, Orton (1983) and Foley (1994) have depicted the existing situations only, and neither of them reported the deficiencies in students' understanding of integral in their studies.

Oberg (2000) looked at the subject of integral from a different perspective and examined it in more detail by dividing it into its sub-concepts. Although the study's focus was on the students' understanding of integral as the previous studies, it is evident that it presented a more detailed and more precise picture than previous studies. Oberg (2000) investigated mathematics students' understanding of the definite integral. He also explored how students perceive integral. In this context, the definite integral in the closed interval $[a, b]$ is displayed as follows: calculation, area, accumulation or sum, abstract object, or function. The results of the study revealed that the students saw the most integrals as area or work. The study revealed that students above the average see the integral as a total change; on the contrary, the students below the average see the integral as the limit of a summing process or total change. The critical and guiding part of Oberg's (2000) study was that low-achiever students focus on the Riemann sum and limit part while learning the integral, while high-achiever students focus on the part of the area under the curve. The following conclusion can be reached from this result: To teach the integral more conceptually and construct the integral properly, the Riemann Sum should be taught first in integral instruction. However, this judgment has not been put forward firmly in the study, but rather the concept of approximation has been focused on. In addition, the study did not give information about why low and high-achiever students focus on different situations while learning the integral. Awang-Salleh, and Zakaria (2012), also support this study; in an experimental study, they tried to teach the students the integral by using technology. They reported that progress was observed with the high and moderate students, while they did not with the low-level students.

Similar to Oberg (2000), Contreas (2002) did a similar study and examined the connections between concepts while learning the integral. Contreras (2002) found the most vital conceptual link between the definite integral and area of a region or the derivative's inverse. As a result, students explain the integral as an area Riemann sum, limits, and the Fundamental Theorem of Calculus. Although some studies used context in integral teaching, most studies focused on students' learning difficulties and the relationship between derivative and integral. In his study, Abdul-Rahman (2005) argued that the difficulty in teaching integrals stems from the derivative inverse derivative relationship. Abdul-Rahman (2005) focused more on the connection between integral derivatives and investigated students' difficulties using examples. These difficulties are of two types, building relation between derivative and integral. They had fewer difficulties passing from derivative to integral, while they had more difficulties when passing from integral to derivative because they saw it as an inverse process. They found that the reason for the strain in the integral is that the integral is both the inverse of the derivative and used in area volume calculations. Another study found similar results by Rasslan and Tall (2002), and they explained the reasons for these misconceptions by not having a conceptual understanding of integration. In their study, Ferrini-Mundy and Graham (1994) reported inconsistencies of students' achievement on procedural items and conceptual understanding, and they mentioned conflicting conceptions related to the sign of the function and area.

Fisher, Samuels, and Wangberg (2016) determined the concepts that students conceptualized while learning integral and reported their prevalence for the students' difficulty. As a result, four primary categories emerged: inverse derivative, area, the infinite sum of one-dimensional parts, and approximation limit. In addition, when the student responses about the accumulation function were examined, three basic categories were determined: those based on the process of calculating a single definite integral, those based on the result of the calculation of a particular integral, and the input and output variables of the accumulation function. Li, Julaihi, and Eng (2017) also examined the difficulties, mistakes, and misconceptions students encounter while learning integrals. While 63% of students had difficulty with indefinite integrals, 43% of students made mistakes in trigonometric and fractional functions in integration techniques. Students' conceptual errors stem from recognizing the function and determining the integration technique; on the contrary, procedural errors stem from students' poor mathematical skills.

Studies after focused on relation between integral and derivative (Delice & Sevimli, 2010; Hall, 2010; Kaplan, 2005; Salleh & Zakaria, 2012; Serhan, 2015; Yasin & Enver, 2007). When the studies have done so far are examined, it is seen that the integral is given without context. This situation retains students from learning conceptually. It also causes students to see integral mainly as an operation. Therefore, in the current study, the concept of integration is given in physics contexts. Marrongelle (2001) investigated how students taking the mathematical methods course in physics use their knowledge of physics to understand mathematics concepts. This research shows that students use the concepts of physics to understand the mean rate of change conceptually. However, students use physics less frequently to understand derivatives and integrals conceptually. Cui (2006) found that although the students have the necessary mathematical skills, they cannot transfer their knowledge in physics. In addition, they experienced problems in the context of physics during this process. These problems were; not being able to decide the appropriate variable and the limits of the integral and the inability to determine what kind of mathematical method to use in problem-solving.

Moreover, it has been suggested that multiple representations can be used to overcome the students' difficulties (Christensen &Thompson, 2010; Pantozzi, 2009; Nguyen & Rebello,2 011). Pantozzi (2009) found that students learned more conceptually when using multiple representations while learning the Fundamental Theorem of Calculus. Christensen and Thompson (2010) also explored students' understanding of the mathematics required for productive reasoning about physics. In addition to mathematics in the answers to the students' conceptual questions about physics, the analysis of the mathematics questions found that the students attended high-level physics courses lacking the assumed prerequisite mathematics knowledge. Christensen and Thompson (2010) found that students could not learn mathematics and physics together and some mathematical difficulties observed among physics students were due to the difficulties in transferring mathematical knowledge to physics contexts and problems in understanding mathematical concepts.

Nguyen and Rebello (2011) also explored how students understand the area under the curve and its relation to integration in physics problems. The study was conducted with 35 students in the first semester of the mathematical methods in the physics course. As a result of the study, they found that only a few students could solve the questions. Also, they could not apply the concepts of the area under the curve, integral, and accumulation to physics contexts in the given problems. As a result of the study, it was suggested that the instructors solve physics problems and do more scaffolding. Bajracharya (2012) explored students' conceptualization of definite integrals in physics contexts. At the end of the study, they identified students' difficulties in graphic representations of definite integral and focused on how students think about integrals resulted in negative results. In this study, seven students who took introductory physics and multivariate mathematics courses and studied in the physics department were interviewed. Students were asked to identify the signs of the given integrals and compare the magnitudes. In the interviews, students' in-depth understanding of integrals was examined with the different representation features of the graphics in the written questionnaire. Students have difficulties applying the Fundamental Theorem of Calculus into graphical representations, determining the signs of integrals, and applying to physical contexts in terms of definite integrals. Moreover, students predominantly use the "area under the curve" in solving questions, and their understanding of the Riemann sum is not that they are deep enough to help consider integrals (Nguyen & Rebello, 2011). It was seen that the students could not apply the integral in different contexts, and the reasons behind it were not explained. Students cannot transfer their knowledge of integral into different contexts because they cannot learn the integral conceptually and do not make sense of the given notations (Jones, 2013). In this sense, Jones (2013) explored how students apply their integral knowledge to physics and engineering. In this study, nine entry-level physics and engineering students were interviewed. Symbolic forms of integral were found in students' minds. It turned out that they interpret the integral with these symbolic forms, and the field of these interpretations consists of inverse derivative and addition. Symbolic forms related to domains and antiderivatives were common. While the students solved the mathematics questions more efficiently, they had more difficulty in the physics context. Students need to solve questions in different contexts in order to understand mathematical issues better. Thus, they understand that integrals have different uses.

Kouropatov and Dreyfus (2013) discussed a curriculum based on teaching the definite integral basis of accumulation. They claim that students learn the integral more meaningfully. They administered the questionnaire to 250 high-level 12th-grade students. According to the result of the study, they claim that teaching accumulation is more manageable than the Riemann Sum since it is intuitive and concrete. On the other hand, they claim that the Riemann sum is a formal and well-defined object. Moreover, through the idea of accumulation, both integral and the Fundamental Theorem of Calculus can be taught simultaneously. On this claim, Kouropatov ve Dreyfus (2014) developed a course and tested their conjecture.

They proposed an approach that focuses on the accumulation concept to teach the integral for advanced high school students, and they argue that teaching integral as accumulation contributes to students' conceptual and more in-depth learning of the integral. Based on teaching integral as accumulation, they prepared ten-unit course content. Students' learning processes were examined with the Abstract in Context framework. The results of the study showed that the students have a proceptual understanding which means they have both conceptual and procedural understanding. He also stated that the construction processes of the students were not linear; they went through different construction processes due to personal and social differences.

Sauerheber and Muñoz (2020) explored the students' understanding of certain concepts, such as the integral limit and the change in the students' ways of thinking and understanding, while developing their understanding of these concepts. In order to reveal the algorithm in students' understanding, the students who took the introductory calculus course were interviewed. Transformative and integrative aspects of concepts such as limit and integral affect the understanding of these concepts. Hence teaching integral by using the different contexts is important. In a second study they also explored the second theorem of the FTC by comparing integrals (Sauerheber & Muñoz, 2021). In other words, the $F(x)$ s were compared over the places they trace between $f\,'(x)$ and the horizontal axis. In addition, the change in $dF(x)$ with $f(x)dx$ has been compared, and these comparisons have been viewed at tiny changes. They showed, both arithmetically and graphically, for trigonometric, polynomial, and transcendental functions that the area of a function under the curve is proportional to the ordered value of the function and that changes in height along an integral function are equal to the area followed by the derivative from the x-axis.

Noteworthy Point :In the literature review presented above, there are clearly studies about integral focused on the teaching or learning the integral; however, they neither focused on constructing the integral nor reasons behind the learning or teaching. Mostly they depicted the existed situation and focused on the results of their intervention, not the process. However, it is essential to detect the reason for the proper construction of the integral. Moreover, focusing on the process will provide richer information about the deficient points in the students learning or intervention. Thus provide the researchers to manipulate the problematic issues during the process.

Another essential point is introducing the integral in the context that provides students to construct the integral conceptually. Since in the given context, they can engage with the application of integral and determine which variable corresponded in the real-life. Hence, they can interpret the integral within the context.

2.1.1 Studies About Riemann Sum

In the textbooks, the Riemann Sum is usually given at the beginning of the definite integral chapter. Studies assert that using the Riemann Sum is an effective way to develop a conceptual understanding of the integral (Chhetri & Martin, 2013; Jones, 2015; Sealey, 2008; 2014; Sealey & Oehrtman, 2005; Thomas, 1996). In a previous study Orton (1983) identified students' poor understanding of differential. Sixty high school students and 50 mathematics teacher candidates participated in the study. Orton (1983) pointed out that failure in understanding integration as the limit of a sum is blocking structural understanding of definite integral. In this sense, Orton (1983) proposes an approach to mathematics teaching that includes numerical and graphical explorations using real life as a starting point. Bezuindenhout and Oliver (2000) also focused on the limit part of the Riemann Sum, and they conducted a study with first-year university students; written tests and interviews were used to reveal students' understanding of calculus concepts. Their interviews showed that the students had problems with the Riemann Sum due to the deficiency of the limit concept. In addition, since they applied the rule that the limit of sum in limit is the sum of limits to Riemann Sum, it was revealed that they had problems when applying Riemann Sum in questions. Bezuindenhout and

Oliver (2000) also defend that students' misconceptions in Riemann Sum will affect understanding the Fundamental Theorem of Calculus.

Thompson and Silverman (2008) defend that accumulation is at the core of the integration, and Riemann sum also consists of the accumulations. In this sense, to understand the integration, Riemann Sum should be taught on the accumulation idea. They claim that without trying to construct, represent and understand Riemann Sums, students cannot understand that accumulation and its importance in the Fundamental Theorem of Calculus. Moreover, they defend that students have difficulty in the multiplication step in the Riemann Sum because $f(c)$ and Δx quantities, and the product of these two quantities indicates a new quantity.

Brijlall and Bansilal (2010) investigated the genetic decomposition of the Riemann Sum. They examined the development process of the Reimann Sum understanding of third-grade students. In their analysis, they found that Riemann Sum consists of area, limit, and function schema. They found that students had partial learning about Riemann Sum and were at lower area and function schemas. They also found that students were more successful in solving lower-level questions. They reported that when the students went through graphical representations, they could interpret an advanced function concept, albeit indirectly. In the same sense, Chhetri (2013) investigated students' modeling process in the Riemann Sum in questions in "real life" context focuses on students' mental difficulties and how they solve these difficulties. She conducted multiple interviews with two students. She reported that students tend to use the Riemann Sum graphical representation; besides, she also reported that students had difficulty in the first task to model the Riemann Sum, and she could not observe a transition from a model of to model for.

Sealey and Oehrtman (2005) examined the preliminary research to develop a strong understanding of conceptual background in mathematics students. Seven students participated in this study. Students were asked to question the concepts of limit and accumulation. In the study results, it was observed that the students learned the concept of limit, although the word limit was not given explicitly. Sealey (2014) examined the obstacles mathematics students face when solving definite integral problems without associating them with the area under a curve and how to overcome these obstacles. Students have had trouble with contextual problems involving definite integrals, even if the context was familiar. A total of 22 students participated in the data collection process. In the study, the students were asked three activities to understand the Riemann integral structure. The purpose of these questions is to make students understand how Riemann Sum and its structure are used to solve these problems. Moreover, Sealey (2014) proposed a framework. According to the results of the study, the most challenging part was the multiplicative part. Their results showed that in contextual definite integrals questions, students "knew" that the area should represent the integral under a curve, but some could not determine which curve should be graphed. Sealey (2014) suggests that during the teaching of Riemann Sum and definite integrals, it should be given opportunities to participate in activities that require them to make sense of the terms, and students should be encouraged to use the Riemann Sum effectively in integrals. The students must discover the connection between the Fundamental Theorem of the Riemann Sum and Calculus and interpret the Riemann Sum's notations.

Thomas and Hong (1996) explored the connections established by students between the integral and the Riemann integral and their understanding of the Riemann integral. Data was collected by giving questionnaires to 47 students. The results of the study revealed that the students saw the Riemann Sum as an operation. It was revealed that they did not understand conceptually. Technology has been used to involve students in the process. They suggest that there should be processes in which students can actively play a role in the process. Jones (2015a) focused on teaching the Riemann Sum since it is advantageous to calculate the pressure on the surface area, volume. However, a relatively tiny number of students used the Riemann sum, and also the instruction made sense for them. He explained why the students had difficulty using the Riemann sum, and they did not expose the Riemann Sum instruction and its application. In the follow-up study, Jones

(2015b) investigated that students primarily explain the integral over fields and inverse derivatives while not explaining it over Riemann sum. He reported that MBS (Multiplication Based Summation) was caused by the teachers not spending enough time explaining the Riemann sum and quickly go over to FTC. Therefore, he revealed that his students were directed to develop predisposition according to the field definition and inverse derivative without being aware of it. In addition, it was found that the instructors' explaining the Riemann sum only by making calculations rather than conceptually explaining caused the students to perceive the Riemann Sum as a technique requiring only difficult calculations. In that study, Jones (2015b) emphasized that the Riemann Sum has an essential role in interpreting the definite integrals.

Moreover, in another study by Jones, Lim, and Chandler (2017), they also defend the same argument and discuss why the students undermined the importance of the Riemann sum. Moreover, they also discussed which kind of instructions may reveal the importance of the Riemann Sum and how to make students pay attention to the Riemann Sum. They suggested that it is not enough to help students understand Riemann Sum. However, they must learn the Riemann sum structure conceptually. Therefore, when describing the integral, it should start with the Riemann Sum, and the benefit of the Riemann Sum in integral interpretations should be prioritized.

While teaching Riemann Sum, different contexts from the area concept should also be used. Ely (2017) proposed a new theoretical model for integral representation and compared his work with the multiplicatively-based summation (MBS) by Jones (2015b). Although Jones (2015b) is limit-based, this study is based on infinitesimals, and the notation and construction of the integral informally are based on infinitesimals. In this way, Ely (2017) argues that students can easily connect between derivative and integral and connect between antiderivative and field more efficiently. In line with the previous studies, Wagner (2018) explored the difficulties that beginner and senior physics students face in learning the integral and persisting even after learning integral concepts. While the basic level of these students were those who have taken calculus and physics was advanced than students were students who had taken differential multivariate analysis linear algebra. The students did not establish a relation between the Riemann Sum and the integral. As a result, it has been revealed that while students establish a connection between Riemann Sum and integral, there is no dichotomy between algebra and calculus in the background, which prevents students from learning RiemannSum. Simmons and Oehrtman (2019) worked with six students who had developed the Riemann Sum understanding in groups of two in their studies. The students were presented with teaching that emulated quantitative relationships in the Riemann Sum. Students developed different kinds of reasoning. Moreover, right, or wrong steps are taken while solving the problem affected the students' reasoning. Their work emphasized that emphasis should be on quantitative relations while teaching Riemann Sum the symbolic representation of integral and differential.

Czarnocha, Dubinsky, Loch, Prabhu, and Vidakovic (2001) examined students' understanding of Riemann Sum using historical methods. During the research process, they observed that the way students calculate the area in their mind may be consistently different from the one in the lesson and that students can produce different understandings. Throughout the course, students were given the relationships and historical processes of the Archimedean and Riemann methods used in the course and how indivisibles were formed throughout history. They argued that the physical analogies given in the research affect the students' mathematical understanding and are essential for understanding the thinking structures of the students. Therefore, they suggested that since the students have different thoughts, this should be considered while planning the teaching. Hamdan (2019) also investigated integral in three aspects; one of them is the Riemann Sum. She proposed a teaching module to promote and evoke students' conceptual understanding of the definite integral in three aspects. In the study, it is found that students rarely applied to Riemann Sum. Moreover, she defended that deficiencies in the Riemann Sum affect evaluating the definite integral, volumes, and other applications of the integral.

In the recents years studies focused on teaching the Riemann Sum with the help of ICT. In this sense, Bricio-Barrios, Arceo-Díaz, and Maravillas (2020) explored the processes of students learning integrals through Riemann sums with flowcharts used in ICT. This study was carried out with 24 university students continuing to study in the industrial engineering department. The students were given instruction using flowcharts and GeoGebra. The study results revealed that the students who were taught integral and Riemann subjects in High School could not approach the Riemann sum and the area under the slope. The qualitative analysis results showed that the use of GeoGebra made it easier for students to establish the relation between the Riemann sum and the Fundamental Theorem of Calculus. Maravillas, Arceo, Salazar, Andrade, Montes (2020) investigate that engineering students understand Riemann sum. A series of contextual problems were asked to two groups of students and asked them to solve the problems with the GeoGebra help and helped them visualize the problems and the other group solved the questions in traditional ways. The study showed that the group that used GeoGebra successfully solved the Riemann sum questions and transferred the problem into a definite integral. The same results were also found from Hanifah (2021) and Nisa, Waluya, and Mariani (2020). Both studies suggested that GeoGebra is a valuable tool for teaching the Riemann Sum.

From this point of view, to teach integral in an appropriate way, the first important feature of the task should be as follows:

It is noteworthy in the literature review that the Riemann sum is a prerequisite concept in conceptualizing integral. Teaching integral using under a curve representation for integral is not enough, and it involves deficiencies. Thus to compensate for those gaps, Riemann sum should be taught. This idea is valid for three main reasons a) not all functions have antiderivatives, so FTC cannot be used b) Riemann sum is the basis for understanding the numerical methods c) understanding the Riemann sum will provide students an intuition when they are setting up appropriate definite integral (Sealey, 2008). Two additional vital points are for teaching the Riemann Sum is introducing the concept in a context and quantitively by providing them enough appropriate scaffolding. In this way, students can engage the Riemann sum on their own and construct more concretely.

2.1.2 Studies about the Fundamental Theorem of Calculus

2.1.2.1 Studies about teaching the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is crucial as it enables the transition by establishing a connection between the two essential concepts of calculus, the derivative and integral. Studies have shown that traditionally teaching the Fundamental Theorem of Calculus is not effective in students' learning, so many students have difficulty in learning (Bressoud, 2011; Carlson et al., 2003; Davidson, 1990; Pantozzi, 2009; Prenowitz,1953; Thomas, 2003; Thompson, 2008; Thompson et al., 2013). On the other hand, studies deal with teaching the Fundamental Theorem of Calculus differently and straightforwardly as the Fundamental Theorem of Calculus involves integral and differentiation. It is a milestone, so the students should conceptualize it. By this concern, Prenowitz (1953) conducted a study. He claims that students should gain ideas in calculus in terms of certain impressions in his study. For instance, the essential idea in integral should be the area under a curve, and students should be aware of this. However, he did not consider in his study that students constructed ideas about the Fundamental Theorem of Calculus and teaching the Fundamental Theorem of Calculus. In addition, the general picture is not given in an evident and detailed way in this study. Cunnigham (1965; cited in Oberg, 2000) pointed out a similar issue, and he argued that as the Fundamental Theorem of Calculus has two theorems having a different meaning, one of them should be proven. He recommends in this study that in the lesson instructor should relate the Fundamental Theorem of Calculus with the application of the definite integral. Even if the Fundamental Theorem of Calculus consists of two theorems, these subtwo parts complement each other, proving one of the Fundamental Theorem of Calculus will lead to incomplete learning in students.

Davidson (1990) conducted a study about the Fundamental Theorem of Calculus. Davidson (1990) used the discovery learning method to teach students the Fundamental Theorem of Calculus. Working in small groups, students solved specific integral problems using the Riemann sum. The students were asked the particular case of some functions, and when they solved the Riemann sum questions, the students were asked to predict what they would get the integral of. In this way, they were expected to establish a link between the Riemann sum and the Fundamental Theorem of Calculus and to understand the Fundamental Theorem of Calculus. However, in the study, no information was given about how many students were successful with this teaching method or whether the students learned the Fundamental Theorem of Calculus conceptually. Although teaching the Fundamental Theorem of Calculus starting from the Riemann sum is a correct step in the study since the Riemann sum is only used to verify the results obtained with the Fundamental Theorem of Calculus, it seems unlikely that students will learn the Fundamental Theorem of Calculus conceptually, or students are more likely to see Riemann sum as a series of cumbersome calculations.

Similarly, Strang (1990) used a different method from the traditional method for teaching the Fundamental Theorem of Calculus. Strang (1990) indicates that using the context-based application of the Fundamental Theorem of Calculus will help make the theory more understandable. These studies claim that teachers should help students build bridges between theory and the Fundamental Theorem of Calculus application. However, this study did not mention in-class situations as in the previous study. It did not give information on whether the method they use works.

Bressoud (1992) advocated that teaching the Fundamental Theorem of Calculus via simple proof and differentiation will help students construct the theory in their minds. However, none of these studies mention the initial steps about making students gain insight into the Fundamental Theorem of Calculus. As a result of this, studies did not go further simply interpret the Fundamental Theorem of Calculus.

With the development in technology, Upshaw (1993) used graphing calculators in his study. It was an experimental study, and this study and the experiment group did not show significant performance according to students taught in the traditional group. Thompson, Byerley, and Hatfield (2013) designed a lesson focused on teaching accumulation and the rate of change by using technology. Their focus was on integral as accumulation, and they advocated that the Fundamental Theorem of Calculus should be taught based on the idea of accumulation. In this sense, they designed a lesson. The lesson aimed to make students establish a relationship between the concepts of accumulation and rate of change, interpret this relationship, and make a notation. The course is designed in two stages. In the first stage, students developed the concept of the accumulation function from exchange rate functions. In the second phase, they focused on the concept of the rate of change of the accumulation function. Thus, students learned about covariation, variation accumulation, and the invariant of the function. However, this study is a course that includes a lot of theoretical and notation, and while the concept of accumulation is being taught, more than one different concept is taught. This situation may cause students to experience confusion.

Blasjo, Dalgamoni, and Robertson (2010) studied teachers' approaches to teaching the Fundamental Theorem of Calculus. This study used a pedagogical content knowledge framework and studied the seven instructors teaching Calculus II during the spring semester. They also questioned other instructors that possessed experience teaching Calculus II. They found that these past experiences influenced participants' approach to teaching the Fundamental Theorem Calculus.

In this section, the research studies conducted about teaching the Fundamental Theorem Calculus are reviewed. When these studies are examined, no guidance can be obtained on how to teach the Fundamental Theorem Calculus to students. The studies in this section focused only on the Fundamental Theorem Calculus without combining the Riemann sum concept and the Fundamental Theorem Calculus. However, definite integral includes the Riemann sum concept. Moreover, there is no clue how comprehensive understanding should be and about students' mental construction of the Fundamental Theorem Calculus.

Noteworthy Point: When the studies conducted in the Fundamental Theorem Calculus are examined, there is a need for a document that can guide teachers and solve the difficulties they encounter.

2.1.2.2 Studies about learning the Fundamental Theorem of Calculus

To provide proficient teaching technology, it is not enough to know the pedagogical approaches, but also it is needed to know the students' ways of learning. Thus, In 1991, Dubinsky and Schwingendorf presented a study indicating the students' performance on the given assignments. In this study, the authors obtained some significant improvements in students' understanding of the FTC. However, they also claim that they could not observe the improvements in a direct way. It was one of the weaknesses of the study.

Thompson (1994) carried out a study to determine students' learning process and difficulties learning the Fundamental Theorem. He conducted a teaching experiment with 19 senior students. The study revealed that the students' poor understanding of the Fundamental Theorem of Calculus was knowledge deficiency on the rate of change and the accumulation. They conclude that students could not grasp how quantities change. Moreover, students also have problems with the construction of function and covariation. Another critical issue that Thompson (1994) emphasized is that understanding the Riemann Sums affects understanding the Fundamental Theorem of Calculus. Hence, students have difficulty interpreting the theorem and could not explain their reasoning on the theorem or the Riemann Sum. Besides, the students also have problems interpreting the notations without reasoning why they read how it was written. Thompson (1994) suggests that to construct the Fundamental Theorem of calculus, accumulation, rate of change, and this coordination since the synthesis of these constitute the Fundamental Theorem of Calculus should be taught. In addition, before teaching students the Fundamental Theorem, instructors should teach the infinitesimal concepts well.

In another study, Thompson and Silverman (2008) found consistent results and tried to overcome this difficulty by using multiple representations. This study focused on the students' difficulties, such as understanding the accumulation and the accumulation functions. Moreover, they focused on how to introduce the accumulation functions to students. They suggest a further study examining the students' understanding of the Fundamental Theorem of calculus and its mental representation in their minds. Students have difficulty in the multiplication step because $f(c)$ and Δx are quantities, and the product of these two quantities indicates a new quantity. Moreover, to understand this relation, it is necessary to know covariation. Hence in the study, they also focused on the foundation of the covariation. In short, they suggested that the Fundamental Theorem of calculus should be based on three critical concepts: accumulation, Riemann sum, and covariation. These findings were supported in the study of Smith (2008). She improved the framework developed by Carlson, Smith, and Persson in 2003. In this study, Smith (2008) worked with three students and conducted a teaching experiment. The study results revealed that this framework shows the reasoning used in learning the FTC, covariational reasoning, and students' mental states and notational notes. She found that this framework is beneficial for understanding the students' mental process through learning the Fundamental Theorem of Calculus.

Another component was determined by Thomas (1995), which is function. She investigated the effects of the Fundamental Theorem of Calculus learning process and technology-based activities on the learning process. Data were collected from three students. Genetic decomposition was used during the analysis. The results of the study showed that there are misconceptions in the students' function charts. Students believe that the independent variable is an essential feature of the function. Thomas (1995) suggests that integrals given by functions, instructors should pay attention to students' function schema, and before teaching the Fundamental Theorem of Calculus, function schema should be reconstructed.

As Thompson (1994) did, Carlson, Smith, and Persson (2003) focused on the students' reasoning about the rate of change, accumulation, and covariation while learning the Fundamental Theorem of Calculus. Their main focus was on covariational reasoning. In this sense, to develop their understanding and reasoning that developed curricular materials. They also developed a framework for evaluating the learning processes of students. They studied with 24 students who enrolled in the calculus course. The students were able to coordinate the accumulation of the instantaneous change of the input variable with the accumulation of its function but still found that some students had difficulty establishing relationships and expressing theory. They found that the materials developed at the end of the study contributed significantly to the students' failure to understand the Fundamental Theorem of Calculus.

Unlike Carlson, Smith, and Persson (2003), Ely (2012) focused on the limit part of the theorem. Ely (2012) claimed that due to the nature of the limit concept, students has a problem understanding the Fundamental Theorem of Calculus. Ely (2012) proposed that to get a qualified understanding of the Fundamental Theorem of Calculus, students should have built accurate insight and focused on the limit part in the theorem. Understanding the fundamental theorem depends on coordinating the change in the area accumulated under a curve with the underlying variable. However, students experience a loss of focus while doing this coordination because calculus students who learn the Fundamental Theorem of Calculus always lose focus totally when they take the limit. This image serves as an obstacle to their understanding of the Fundamental Theorem of Calculus.

Similarly, "indivisibles" negatively affect the Fundamental Theorem of Calculus understanding, such as in history. Hence, to provide an accurate understanding and build more meaningful concepts, the Fundamental Theorem of Calculus should be given depending on its historical development stages. Besides, Ely (2012) suggests that instructors should consider teaching ways that will be more useful to understand the fundamental theorem of mathematics and prevent this loss.

Thompson and Dreyfus (2016) designed a course that combines integral and derivative conception with the Fundamental Theorem of Calculus. They built the course on two essential concepts. The first one focuses on the variable that students think of as fixed. The second is to explain the Fundamental Theorem of Calculus by combining the integral and the derivative. In order to achieve this goal, the concept of differentiation as focused on, and the Fundamental Theorem of Calculus was explained by establishing a connection between the changes of dy and dx relative to each other and their rate of change. A controlled experimental was made with a total of 294 students. As a result of the experiment, a significant difference was observed.

Kouropatov and Dreyfus (2013) designed an instructional intervention. As a result of the instructional intervention, students analyzed the construction process of the Fundamental Theorem of Calculus. While developing the intervention, accumulation was adopted as the main idea and accumulation function, and the Fundamental Theorem of Calculus was put on this concept. The intervention consisted of 10 units and was applied to advanced mathematics students consisting of 5 small groups. Abstraction in context was used as the theoretical framework. The study showed that the students learned the concepts related to the subject before, and they built new knowledge on these concepts. Besides, thanks to this developed intervention, the students have established the integral concept of accumulation. However, they found that this structure was fragile. Despite the rather impressive overall performance of the students, we consider the rate of change structures to be inappropriate or too fragile.

Radmehr and Drake (2017a) investigated the understanding of the Fundamental Theorem of Calculus in terms of metacognitive aspects. They focused on the students' metacognitive experiences when solving the problems about the Fundamental Theorem of Calculus. They conducted interviews with 13 students.

The questions which measure the metacognitive experiences showed that students understood the Fundamental Theorem of Calculus. However, students had difficulty interpreting the Fundamental Theorem of Calculus and its notations. They had a limited understanding of the accumulation function. In another study, Radmehr and Drake (2017b) unpacked the knowledge dimension of Revised Bloom's taxonomy for definite integral. They conducted the document analysis, and they defined 11 subtypes of knowledge dimensions were defined. They suggested that instructors and researchers determine the objectives and plan teaching activities according to this knowledge dimension. Moreover, they also emphasized metacognitive knowledge.

Swidan (2020) created a learning trajectory for students to learn the Fundamental Theorem of Calculus with a digital educational tool. Students were asked to explain the connections between mathematical relations embedded in a digital educational environment and to establish a conjecture between the relationships they realized. As a result of the study, it was found that students focused on nine points while learning the Fundamental Theorem of Calculus. These foci are Δx,$f(x)\Delta x$, $\sum f(x)\Delta x$, accumulation function properties, Riemann function, etc. They defined the Fundamental Theorem of Calculus' learning trajectory by collecting these focuses under three separate headings.

Noteworthy Point: To sum up, researchers showed that integral is not an easy concept to understand, as it contains limit, Riemann sum, antiderivatives, and FTC. Even though students can calculate algebraic operations in integral, they do not know why they were doing that (Orton, 1983). Moreover, they have difficulty interpreting these symbolic expressions (Jones, 2010), have fragmented conceptualizing between Riemann sum, definite integral, FTC, and area under a curve (Sealey, 2008; Bressoud, 2011; Orton,1983; Rasslan & Tall, 2002) and have misconceptions about the area under a curve (Ferrini-Mundy & Graham, 1994). In this context, in order to teach the integral conceptually, it is first necessary to identify the cornerstones that make up the integral. It is evident that before teaching the students the Fundamental Theorem of Calculus, students should know limit, accumulation, accumulation function,function, Riemann sum, rate of change and covariation (Carlson, Smith, & Persson; 2003; Ely, 2012; Kouropatov & Dreyfus, 2013; Radmehr & Drake, 2017; Swidan, 2020; Smith, 2008; Thompson & Silverman, 2007; Thomas, 1995; Thompson, 1994; Thompson & Dreyfus; 2016). In this sense to teach integral, accumulation should be the first step. However, it can be difficult fot the students to understand approximation and this difficulty undermine the conceptual understanding of the Riemann Sum. Hence to make the emphasize both the Riemann sum and the accumulation, the Riemann Sum should be introduced with using the approximation. Then with increasing the increment and make studetns to realize the infinitely small increments, accumulation should be introduced. By this way students can get insight about conceptual meaning of the limit. Hence they can learn the Riemann function more easily. Furthermore, this understanding help them to understand and construct the Fundamental Theorem of Calculus accurately. Another difficulty that students confront while teaching FTC is that students do not fully understand the concept of function (Oberg, 2000; Ferrini-Mundy & Graham, 1994; Fisher, Samuels, & Wangberg ;2016; Raaslan & Tall , 2002; Thompson, 1994). The main reason behind this difficulty is that students do not understand function very well. Their lack of understanding of the concept of function prevents them from switching between concepts. Before teaching the Fundamental Theorem of Calculus, the concept of function needs to be reconstructed. Thus, students will be able to construct Fundamental Theorem of Calculus on the Riemann Sum. Besides accumulation, rate of change, and covariation are essential components for teaching the Fundamental Theorem of Calculus (Carlson, Smith, & Persson; 2003; Thompson & Silverman, 2007; Thompson, 1994; Thompson & Dreyfus; 2016). After giving the concept of function in the construction process of the the Fundamental Theorem of Calculus, it should be focused on these concepts. Covariational reasoning should be taught to evaluate the change of dy and dx with respect to each other. so they will better understand the conceptual meaning of the Fundamental Theorem of Calculus and why it is a reverse process with differentiation. Finally, students have difficulty in notation and interpretation (Jones, 2015; Radmehr & Drake, 2017; Thompson, 1994; Thompson, Byerley & Hatfield, 2013) In this context, when a teaching is carried out in the light of the above-mentioned, students will learn the notations and make correct interpretations.

In addition to these, teachers are also having problems with teaching integral to the students. When teaching integral, teachers cannot know how to begin to teach integral and which can be the first step (Oberg, 2000), and also they do not know how to make connections between the concepts in integral (Davidson, 1990; Bezuindenhout and Oliver, 2000, Contreas, 2002). Moreover, these researches emphasize that detailed effective conceptual instruction research explains how instructional effects the mean should be to assist other researchers or teachers in teaching concepts in a proper sequence and guide them about the levels of understanding these concepts. There should be a hierarchical sequence among the integral and its sub-concepts to prevent departmentalization of students' understandings of the integral concept. Furthermore, understanding the nature of students' conceptions is an essential first step in developing well-planned pedagogical approaches that may provide students with opportunities to construct a robust understanding of the concept (Bezuindenhout and Oliver, 2000). As a design-based study involves hypothetical planning (Cobb & Gravemijer, 2008), the study's questions should be prepared based on it to provide a structured understanding.

2.2 Theoretical Perspective

A theoretical framework will guide the current study in two ways: conceptions about integral and designing the instruction in terms of STEM context. For this aim, concept image and definition, concept development process, and FTC framework will be used to conceive the integral.

2.2.1 Science Technology Engineering and Mathematics (STEM)

The rapid development of technology in recent years led to change in the education field and also needs in the workforce. Hence, to get qualified students, a recent study focused on teaching subject matter, and teaching technology use, skill development, and teaching interdisciplinary in educational research (Lacey & Wright, 2009), and researchers developed multidisciplinary approaches (English & King, 2015; Lin & Williams, 2016; Marra et al., 2016; Pucha & Utschig, 2012; Wells, 2016). STEM is one of those that focus on integrated education (Zollman, 2012). It aims to prepare students according to the needs of the 21st century (Kuenzi, 2008). To achieve these goals, teachers should be aware of the importance of STEM. Also, they should know how to integrate a course with another one and be aware of the importance of using STEM while teaching their subjects. National Council of Teachers of Mathematics (NCTM, 2000) emphasizes the importance of merging interdisciplinary knowledge with students' experience with real-life problems and states that "thinking mathematically involves looking for connections and making connections builds mathematical understanding" (p.274). Berlin (1991) also asserts that education with integrated disciplines can enhance students' understanding of and attitudes toward both mathematics and science.

Moreover, research indicates that using interdisciplinary approaches provides less fragmented, more related experiences (Frykholm & Glasson, 2005; Koirala & Bowman, 2003). However, in the current education system in Turkey, as they learn major field separately, students cannot relate the subjects in a different context and implement their knowledge to the new situation and also they cannot solve the problem as they did not understand the embedded context as they construct discipline in their mind separately. To prevent this kind of problem and provide a stronger connection between the disciplines mentioned above and facilitate a deeper understanding of engineering contexts should be used (Katehi, Pearson, & Feder, 2009). Since engineering is "a catalyst for integrated STEM education" (NAE et al., 2009, p. 150) and it is also a bridge that provides a real life context to students while they learn science and mathematics, and also it enhances the students' problem-solving and communication skills, and the teamwork between them. (Hirsch, Carpinelli, Kimmel, Rockland & Bloom, 2007; Lachapelle & Cunningham, 2014; Mann, Mann, Strutz, Duncan, & Yoon, 2011). Lachapelle and Cunningham (2014) stated that engineering is a tool that contributes to the motivation of students and the integration of science and mathematics. Therefore, it argues that engineering practices and experiences should be included in the curriculum. They examined the engineering curricula of countries such as America, Australia, and New Zealand identified core concepts on implementing the engineering curriculum. They suggested that in engineering, the focus should be on generating knowledge, not just on communication or using the information they have memorized. Daly, Yilmaz, Christian, Seifert, & Gonzalez (2012) focused on a study that investigated the idea generation process of engineering students. They focused on improving the product and what kind of solutions students offered to examine the idea generation process and how engineers and designers evaluate the products and their resources. They defined design heuristics as adjusting function through movement, contextualizing and imposing the hierarchy on functions, attaching independent functional components, and distinguishing functions visually. They also suggested by using this design heuristics; it can be ensured that students understand the subjects easily while explaining the subjects. Moreover, the engineering design processes can be applied more smoothly.

2.2.1.1 Engineering Design Process

There are many different uses of the engineering design process, which is widely used in engineering and STEM education. Although the basic step steps are the same in these different uses, there is no consensus on the used model. In this sense, some engineering design processes are completed in three steps, while others go up to 10 levels (Cunningham, 2009; Daly, Yilmaz, Christian, Seifert, & Gonzalez 2012; English & King, 2017; Fan & Yu, 2017; LaChapelle & Cunningham, 2014;

Morgan, Moon & Barroso, 2013; Siev, 2017). In the lower-grades consisting of fewer steps engineering design processes are commonly used. However, consisting of more step engineering design processes are used in complex situations that is higher-grades or in the situations which require in detail. While the generally accepted engineering design process has a cyclic structure (English & King, 2017; Morgan, Moon & Barroso, 2013; LaChapelle & Cunningham, 2014), some design processes have an iterative, hierarchical, or linear structure (Fan & Yu, 2017; Siev, 2017).

The engineering design process is defined as "The design process is a systematic approach followed when developing a solution for a problem with a well-defined outcome" (Morgan, Moon & Barroso, 2013, p.29). A well-designed design process is crucial since it provides a structure to the researcher to get the best solution and requires high-order thinking and problem-solving skills. The engineering design process can be represented in seven steps: Identify the problem and constraints, research, ideate, analyze ideas, build, test and refine and communicate and reflect (see Figure 2.1).

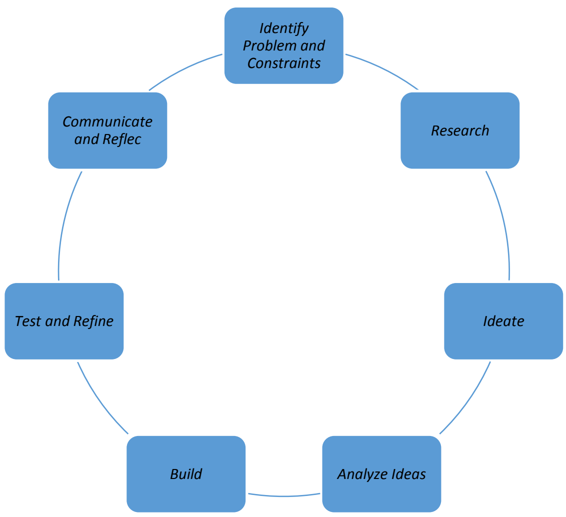

Figure 2.1 Engineering Design Process (Capraro, Capraro & Morgan, 2013).

In the first step, engineers define the problems clearly and plan the production process (Morgan, Moon, & Barroso, 2013). Hence they define the scope of the design process and the goals of the project. In this process, the demands of the customers and society are determined. Besides all supplies, limitations and time arrangements are determined (Morgan, Moon, & Barroso, 2013). In the second step, research, necessary information is obtained to formulate the design process and produce ideas. In this process, engineers investigate the laws, local customs, regulations, and related industry standards (Morgan, Moon, & Barroso, 2013). To satisfy the customers' demands and determine the constraints and criteria in the design process, they investigate the feature of the materials (Morgan, Moon, & Barroso, 2013). In the ideate steps, based on the research step, engineers produce different and creative ideas. Mostly they use brainstorming and identify the preliminary ideas used. In the analysis step, engineers refine the ideas and apply mathematics, science, and appropriate technology to produce a prototype. In this step, engineers do not limit themselves with a solution; on the contrary, they also have different solutions for different possibilities (Morgan, Moon, & Barroso, 2013). In the fifth step, build, engineers develop the prototype using the calculation and the determined materials in previous steps. In the test and refine, engineers perform the prototype, test, and evaluate it. During the testing process, engineers record deviations. In this step, it is essential to go back and reanalyze and prototype and retest it. The final step is to communicate and reflect where engineers use four communication types: oral, written, visual, and interpersonal. These are important and necessary for engineers. Because the engineering design process is carried in teams, and it consists of sharing works, brainstorming, sketches, and other visual and written documents.

2.2.1.2 Studies Related with Engineering Design

There are different types of engineering design. One of them is reported in a study by English, King and Smeed (2017) where fourth-grade students attempt to solve an engineering problem based on a tower's design, which has constraints such as stability. The engineering design process famework is consisting of five steps. It showed that the students iteratively applied the engineering steps. Students used four of the design steps. While spatial reasoning is examined mathematically in students, the distribution and positioning of loads are examined as physics subjects. Other findings show that the students use the engineering design steps iteratively and spontaneously use their gestures to convey their ideas. In an another stud, English, Hudson, and Dawes (2013) investigated how students can design, build, test, and evaluate a catapult. In this study they used the same engineering design process. In the first framwerok they illustrated the design process as cyclic. However, in this study they illustrated their study as a linear process. Descriptions of simple machines used in catapult designs and their evaluation of engineering designs were examined. The findings of a STEM-based unit in which 8th-grade students discovered engineering concepts related to the operation of simple machines were reported.

Feedback from teacher interviews has been proposed as a framework for promoting engineering education, as more information is added about the development of students and the professional learning of teachers. This framework focuses on students' problem-solving. As in other engineering studies, cascading has been made, but only the design process is iterative in the others. English and King (2015) proposed the framework consisting of five engineering design processes (problem scoping, idea generation, design and construction, design evaluation, redesign) based on design thinking in children. They investigated the progress of fourth-grade students while working on an aviation problem. It has shown that students can be successful in engineering fields early and that STEM fields can be taught. However, the teacher's scaffolding process is essential to guide students in the right direction. Because the instructions given by the teacher can help improve students' use of STEM knowledge. They have developed a framework that aims to integrate STEM content know-how into the engineering process. Fan and Yu (2017) examined the effectiveness of a holistic STEM approach in engineering design applications in high school technology education with a quasi-experimental study. Students' conceptual knowledge, higher-order thinking skills, and performance in the engineering design process were evaluated in this study. Siew (2017) investigated STEM students' learning experiences and higher-order thinking skills in an Engineering Design Process (STEM-EDP) with 89 students. The study results revealed a statistically significant difference in students' STEM-related knowledge, skills, attitudes, and practices. EDP enabled students to gain awareness in teamwork, thinking, and problem-solving.

Noteworthy Point: The studies mentioned above are about STEM integration and its results.Neverthelses studies have not been able to present an activity example on mathematics subjects. Studies involving the engineering design process, on the other hand, focused on general mathematical skills rather than mathematics lecturing. In addition, while the studies carried out primarily focus on existing knowledge, in this study, integral teaching will be done to the students by using the engineering design-based process. Also, it will be ensured that the students produce new knowledge as well as their existing knowledge.

As mentioned above, it can be said that preparing students, teachers, and practitioners in STEM fields is of increasing concern. Moreover, as a group, these four content areas are crucial to a nation's economic competitiveness and youth's ability to survive and succeed in modern society. Thus, preparing students for the future and making them tomorrow's successful and qualified individuals requires qualified teachers. STEM framework relies on an integrated curriculum. The integrated curriculum connects whether to real-life, different disciplines, or about all knowledge (Drake & Burns, 2004). This perspective combines different subject areas, past experiences in order or making more fulfilling learning environments. Although students are motivated by interdisciplinary instruction and enhance students' achievement significantly (Perkins, Goodrich, Tishman & Owen, 1994), there are obstacles to implementing STEM education in lessons. Some of the reasons are teachers' deficiency in adopting different disciplines in their teaching (Schleigh, Bossé, & Lee, 2011) and lack of studies in the field. Çorlu (2014) proposed a model for integrated STEM education. In that model, while the subject areas do not lose their characteristics, they interact.

Combining this STEM literature with Sealey's (2006) and Bressoud's (2011) theory remark that integral concepts should be taught in real-life situations. Also, Hestenes (2014) emphasizes that the divorce of mathematics from physics is one of the most severe deficiencies in the current educational system. They should be taught integrated to help students construct accurate and intended concept images. STEM-based activities will help students' contextual understanding and help them to transfer their knowledge from one discipline to another. Carefully designed activities can enrich students' learning context and provide them meaningful experiences (Furner & Kumar, 2007). Moreover, STEM activities should be project-based work on open-ended problems, clearly defined content in each STEM discipline (without delimiting it to one discipline), and provide students practical discussion through problem-solving. Also, an integrated STEM education curriculum should include applying and exploring real-world problems that require students to think and reason about the problem (Carter, 2013).

2.2.2 Concept-Image and Definition

The terms ''concept definition'' and ''concept image'' were first introduced by Tall and Vinner (1981). They suggest that concept definition and image are when we think of something evoked in our mind. Tall and Vinner (1981) claim individuals have different features in the scope epistemologically and psychologically. Therefore, the same concepts can be interpreted in different ways. Based on these, Tall and Vinner (1981) developed concept image and concept definition models in their study. According to this model, all mathematical concepts except primitive ones have formal or non-formal definitions. This process also defines the constructivist approach (Schoenfeld, 1998) and the socio-cultural approach (Renshaw, 1996).

Tall and Vinner (1981) defined the concept definition as ''the form of words used to specify the concept'' (p.152). Also, they classified the concept definition into two classes: a personal concept definition consisting of individuals' own words to define the concept and the formal concept definition, which is the formal definition used in mathematics. Besides, they explained that concept definition produces its own concept images, which is defined by concept definition image.

Tall and Vinner (1981) define concept image as the whole cognitive structure which includes all the thoughts, mental pictures such as symbols, pictures, graphics, processes experiences related to to the concept in the mind of the individual. Concept image is shaped by the individual's all experiences and thinking style. Therefore an individual's concept image may be accurate in terms of containing all the features of the concept and established accurate connections between all the knowledge structures or may be inaccurate or deficient, which may contain only the relations established by the individuals themselves. However, it does not fully meet the concept features (Rösken & Rolka, 2007). Mathematical concepts are defined with definitions except for primitive ones, and many of them are introduced in high school or college students. Hence, even though students did not use the idea explicitly, they behave according to their concept image and mental pictures in their minds. That means learning and application of the knowledge are affected by the concept images (Raaslan &Tall, 2002; Rösken & Rolka, 2002). In addition, since the concept image and definition depend on people's experiences, it may cause difficulties in students' learning as the intended information and the information learned may differ from each other. Hence, they found that if students cannot inform accurate concept images about a concept and have difficulty on the learned subject with the concept, learning occurs in connecting previously existing knowledge structures with new ones (Tall & Vinner, 1981). That means concepts images can be changed over time when encountered new situations (Tall & Vinner, 1981) and in the concept development process, having rich concept images is essential to know the concept's definition (Vinner, 1983).

Another important concept in the concept image and definition theory is coherency (Viholainen, 2008) which is defined as "how well the cognitive structure concerning the concept is organized" (Viholainen, 2008, p. 235), and the lack of consistency in the concept images prevented the student from learning the subject or would learn the subject incorrectly (Viholainen, 2008). In addition, he stated that all understandings, representations, and images are interdependent. According to him, the student who has a coherent concept image has a clear understanding of the concept taught, and there are no contradictions within the conceptual understanding. In addition, the coherent concept of mathematics does not contradict its axiomatic structure. That means there should be coherence between the students' answers in different contexts. On the contrary to Viholainen (2008), Dahl (2017) focused the fragmented part of the concept image and asserted that concept images might have a defective part or the concepts may be constructed involving inconsistent parts. At the end of the study, it was found that the students had a partial understanding; they had the correct information on some parts of the definition or concept image, but there were deficiencies or inconsistencies in the essential parts. That means there can be a different combination through the knowledge structures such as accurate and inaccurate parts can be structured or constructed as departmentalized part by part.

2.2.2.1 Studies Related Integral and Concept Image and Definition

Raaslan and Tall (2002) studied integral and tried to define the students' concept image and definition to check this idea. They argue that all mathematical concepts are defined with definitions except primitive ones, and many of them are introduced in high school or college students. They researched 41 students in their four final years of high school students and found that only 3 out of 41 students can give the intended integral definition and 27 did not seem to handle the concept of definite integral with an infinite discontinuity. Moreover, these students also had a problem with the limit and the function concept. This causes them difficulties in understanding the integral concept. From these findings, it can be inferred that students' concept image and definition influence their learning. Even though students did not use the idea explicitly, they behave according to their concept image and mental pictures in their minds. Based on these, we know that the students' concept image and the definition are limit and integral. In this point of view, if the mental representation of the students about integral can be defined, the relation can occur between their concept image. Furthermore, they stated that the definition of the concept is a part of the process, and its formation is the result of the individual's experiences

Moreover, a study by Rösken and Rolka also supports this idea. Rösken and Rolka (2002) examined whether concept image influences students' learning. To conduct the study, they revisited the concept definition and image comprehensively. According to the results of this study conducted with 24 students, concept image and definition play an essential role in students' learning. Also, intuitive learning is dominant due to the nature of the concept image. In addition, since the concept image and definition depend on people's experiences, it may cause difficulties in students' learning as the intended information and the information learned may differ. In this sense, they found that students' understanding of integral is insufficient. They also found that the students did not grasp the meaning of integral. Moreover, their evoked concept image was not built properly. Hence, they found that if students do not have accurate concept images and have difficulty with the learned subject, they connect new knowledge with thwir previously existing knowledge. They found that students' understanding of integral is insufficient. They also found that the students did not grasp the meaning of integral. Moreover, their evoked concept image was not built properly. As a result, they analyzed students' cognitive processes according to their concept images, including their associations, making it possible to grasp the ideas behind a mathematical notion. Hence, they found that if students cannot inform accurate concept images about a concept and have difficulty with the concept as learning, learning. Hence, it can be said that the model of concept image and concept definition influences the learning of mathematics. In addition, the results indicated that concept image plays an essential role in students' learning, whereby dominating concept images on the conceptual learning.

Vinner (1991) points out that having rich concept images is essential to know the concept's definition in learning a concept. Based on this argument, Oberg (2000) conducted a study about students' concept image about definite integral; in this study, he found that average understanding students represent deeper understanding than those above the average. He explains this situation as their understanding of definite integral and constructed part of their knowledge is still incomplete.

Different from Vinner(1991); Viholainen, (2008) focused on the concept of coherency in concept image theory and defined this concept as "how well the cognitive structure concerning the concept is organized." In his study, he argued that the lack of consistency in the concept images prevented the student from learning the subject or would learn the subject incorrectly. In addition, he stated that all understandings, representations, and images are interdependent. According to him, the student who has a coherent concept image has a clear understanding of the concept taught, and there are no contradictions within the conceptual understanding. In addition, the coherent concept of mathematics does not contradict its axiomatic structure.

On the contrary to Viholainen (2008) and Dahl (2017) focused the fragmented part of the concept image and asserted that concept images might have a defective part or the concepts may be constructed involving inconsistent parts. This study was conducted a study with six first-year university students, who were asked to solve four continuity and asymptotes questions. The study aimed to investigate students' concept image and definition of derivative, limit, and continuity. At the end of the study, they found that the students had a partial understanding; they had the correct information on some parts of the definition or concept image, but there were deficiencies or inconsistencies in the essential parts.

In brief, based on the literature, traditional methods are not effective in teaching. Moreover, even FTC is taught by a unique method. It is a particular topic that students have difficulty with. Indeed, this difficulty is derived from the nature of FTC. Because it includes limit, derivative and integral concepts, even some of these show themselves explicitly. For example, if a student does not have enough conceptual knowledge about derivatives, he/ she may have difficulty grasping FTC. While this theorem has these kinds of compounds, it is necessary to understand the limit, integral, and accurate concept images. Moreover, to overcome difficulty in understanding it, students' concept image has to be identified. In this way, problems like understanding or computation can be identified and solved by developing appropriate lesson plans based on common deficiencies in students' concept images. When the literature is examined, it can be seen that most of them focused on a different method to teach FTC. However, none of them focuses on concept image and definition. That means those studies did not fouces on the source of the problem which is which factors influence or prevents students' learning. To find the reasons of this problem, students' concept image and definition should be identified about FTC. This study will be conducted for two reasons. The first one is to decrease the gap between theory and implementation, and the second is to make a good example to guide them on implementing the STEM education model in classroom settings. As the main focus of the study is to teach integral conceptually and there is much application of integral in engineering sciences, there will be an interaction between engineering, mathematics, and science.

2.2.3 Sfard's Three Phase Theory

In her theory, Sfard divides the conceptions into two parts, operational conception, which deals with mathematical processes, and structural conceptions, which deal with mathematical objects. According to her theory, these two components are complete with each other. However, since the transition between these two concepts is challenging, the transition needs much effort. Once they are constructed accurately, one can understand the mathematical notions deeply. Sfard (1991) explains two concepts as follow:

> Seeing a mathematical entity as an object means being capable of referring to it as if it were a real thing—a static structure existing somewhere in space and time. It also means being able to recognize the idea “at a glance” and to manipulate it as a whole, without going into details. . . . In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential and detailed, (p. 4)

Sfard (1991) separates this process into three hierarchical stages; interiorization, condensation, reification to explain conception evolves into operational to structural.

In the first stage of the hierarchy, interiorization, students are in the lowest stages and become familiar with the concept and have limited skill about the concepts. Sfard (1991) explains this stage as "gets acquainted with the process will eventually give rise to a new concept" (p. 18).

In the condensation stage, the student can combine other processes more efficiently, and "the operation or the process is squeezed into more manageable units." In this stage, the student can deal with various representations of the concept. Moreover, the student is capable of making reasoning about the whole concept without dealing with details. Thus, students can handle the whole process with awareness.

In the reification processes, students can see the whole process and straightforwardly explain the whole process. Moreover, students can transform information through the steps and interpret the concepts in different aspects in this process.

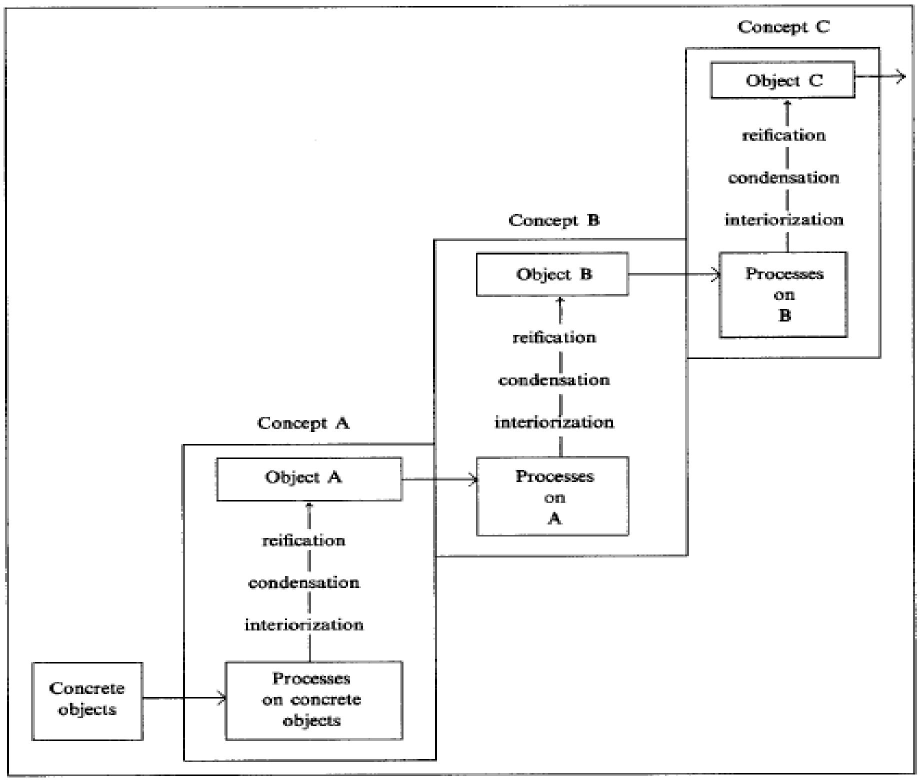

Figure 2.2 Sfard's general model of concept formation (Sfard, 1991, p.21)

According to Sfard (1991), "it should already be clear that our three-phase schema is to be understood as a hierarchy, which implies that one stage cannot be reached before all the former steps are taken" (p.21). That means the learning process is hierarchical and without achieving one step, the learner cannot go over the next step. Hence concept development does not evolve from operational to structural. After the reification stage, learned concepts evolve from operational to structural, and reified concepts become an input for constructing a new concept.

In this study, engineering design process activities were used for the pre-service mathematics teachers to teach the integral. These activities allow the pre-service mathematics teachers to both see the application of the integral in real life and learn it more conceptually. In this sense, the pre-service mathematics teachers' initial and final concept images were examined. From this perspective, the frameworks mentioned above were used to evaluate the pre-service mathematics teachers' concept images about integral. However, the concept image definition theory is not enough to evaluate the process. Because concept image theory gives information about the current situation of the the pre-service mathematics teachers. In this sense Sfard’s three phase theory was used to evaluate and analyze the pre-service mathematics teachers’ developmental stages during the process. In this way, the development processes of the the pre-service mathematics teachers and which concepts they should learn while constructing the integral were more clearly revealed. In other words, by using these frameworks the knowledge of how they construct or reconstruct the definite ingtegral is obtained.

CHAPTER 3

METHODOLOGY

The study investigated how students' concept images regarding definite integral change when STEM-based activities are used through engineering design-based instruction. An undergraduate course focusing on the concept of integral was designed, experimented with, and evaluated from a design-based research perspective. In this chapter, the research method of the study is presented. Throughout the chapter, the context of the study, participants, procedures of planning activities, implementation, and evaluation of the activities are reported. Finally, data analysis procedures, the trustworthiness of the study are presented.

3.1 Research Design: Design-Based Research

The term "design-based research" is first used by Ann Brown and Allan Collins in 1992 (Gravemeijer & Cobb, 2006). Through the development of the phenomena, design-based research is defined as "the study of learning in context through the systematic design and study of instructional strategies and tools" (Design-Based Research Collective, 2003). In the literature, design-based research is used in different names: design and development research (Seel & Klein, 2007), development research (Van den Anker, 1999), formative research (Newman, 1990), and developmental research (Richey, Klein, & Nelson, 2003; Richey & Nelson, 1996; Seels & Richey, 1994).

Design-based research is generally used for producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings. From this point of view, it can be said that design-based research contributes to the theory and provides an opportunity to put in practice the theory appropriately (Gravemeijer & Cobb, 2006). Moreover, the research has a cyclical nature: Analysis, design, evaluation, and revision of activities are iterated until to get an appropriate balance between the intended and realization. Reeves (2006) illustrated the design-based research as follows:

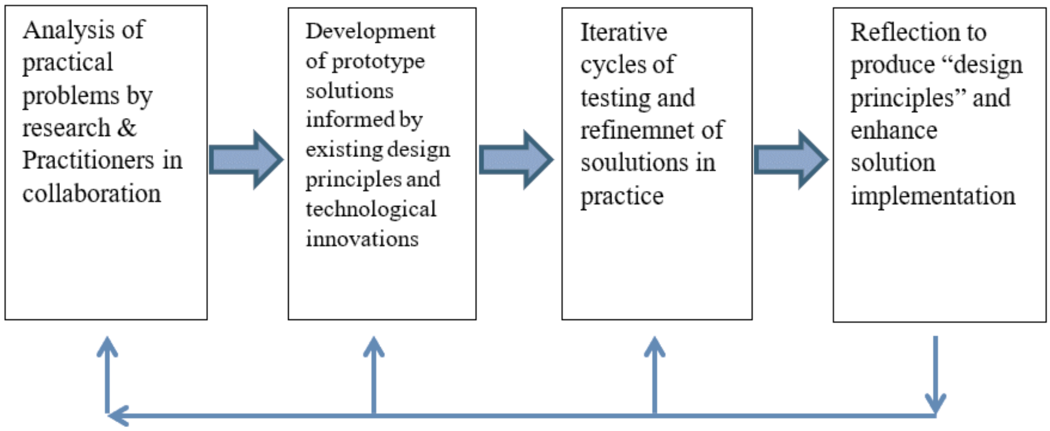

Figure 3.1. Refinement of Problems, Solutions, Methods, and Design Principles (Reeves,2006, p.59)

In summary, design research is iterative, based on designing, and theory-oriented. From this point of view, design-based research decreases the gap between theory and practice. Moreover, design-based research follows a holistic approach instead of focusing on determining variables (Cobb et al., 2003).

From this perspective, design-based research is suitable for this study because of design-based research's perspective and reflective components (Bakker & Van Eerde, 2015, p.15). That means that while implementing a hypothesized trajectory, the researchers can evaluate the actual learning. Thus researchers produce conjectures and have a chance to test those conjectures. Finally, revise the lesson plan since the current study aims to examine the changes in concept images of the pre-service mathematics teachers about integral throughout the engineering design process. As aforementioned before, concept images are based on knowledge development. In this sense, design-based research provides the researcher with immediate feedback on whether the activities served their aim due to their interventionist nature. Moreover, if they do not, provide the researcher to revise them. Consequently, design-based research was employed to examine pre-service mathematics teachers' constructions of the integral concept

3.1.1 Phases of Design Research

Although various researchers made different classifications (see, e.g., Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Gravemeijer & Cobb, 2006; McKenney & Reeves, 2012; Plomp & Nieveen, 2013), there is an agreement on the number of phases involved in a design-based research: a preparation (e.g., design of instructional materials), conducting design experiments, and retrospective analysis (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Gravemeijer & Cobb, 2006). In the current study, the phases in Gravemeijer and Cobb (2006) were used. In the following sections, general characteristics of the phases, and in line with these phases, general characteristics of the study are given. More detailed information will be given in the data collection section to avoid confusion.

3.1.1.1 Preparation Phase

This phase consists of defining needs, reviewing the literature, and developing a conceptual or theoretical framework for the study. At this stage, a roadmap is determined based on the problem statement. This road map is created using all the resources we have. These resources that can be used are a wide range of topics, including the concept to be taught, the characteristics of the students, the characteristics of the learning environment, and the difficulties students may encounter or its possible solutions, sequence of the activities (Bakker & Gravemeijer, 2006). In this phase, initial design principles or HLT are also formulated.

The first step is in the first phase is to determine how to design the knowledge generation process. To do this, all sources about the taught concept should be determined. These sources can be historical development of the concept, literature about how to teach, how students learn the concept, and literature about the theoretical aspect. To gather all of these sources, the researcher should decide what to teach and how to teach. In other words, the researcher and the design team should determine the learning goals by asking themselves, “What are the core ideas in this domain?” (Gravemeijer, & Cobb, 2006, p.76)

After the specification of the learning goals, the researcher and the design team should consider introducing the concept to the students; in this step, the aim is not simply to describe the students’ level but instead to develop a local instruction theory. Developing the theory, the researcher and the team should consider students’ thinking, reasoning, possible difficulties, or understanding to develop reasonable conjectures. In this regard, previous experiences, related literature, and, if available, instruments or tests can be used to determine the starting point and appropriate approaches. Moreover, the design team also considers the classroom culture, teacher’s role, school facility. To gather all of these team and the researcher should frame the study.

Moreover, the design research aims not to analyze the classroom; above that. It should enable the researcher more detailed information about the theory. In this sense, the researcher and the design team should also formulate the theoretical intent, including interpretative framework (Cobb & Yackel, 1996), or other theories; quantitative reasoning (Thompson, 1994, 1996), Realistic Mathematics Education (Gravemeijer, 1994).

3.1.1.2 Conducting Design Experiment

This phase’s primary goal is to improve the initial design by testing and revising of the conjectures and understanding how the formulated local instructional theory works (Gravemeijer & Cobb, 2006). To achieve this, iterated cycles are vital components. While each lesson is a cycle in design-based research, there can be a mini cycle in each lesson. During the cycles, through the thought experiments, the researcher and the design team evaluate the activities, students thinking process, interaction with the activities, and the teacher. Moreover, they also evaluate their participation, reactions to the activities, learning, or difficulties that the students or the teachers confronted. Different kinds of data can be used during these processes, such as field notes, video and audio records, observation notes, students' written materials, pre and post interviews, and reflection papers. It allows the research and the design team to see the whole picture, helps them analyze the instructional activity, and enables the researcher and design team to revise goals, settings, or activities. Hence, according to this reflective process, the local instructional theory is revised, and new conjectures are developed. As a result, the subsequent activities are reorganized and applied according to the conjectured local instructional theory. This process is called the micro cycle (Figure 3.2).

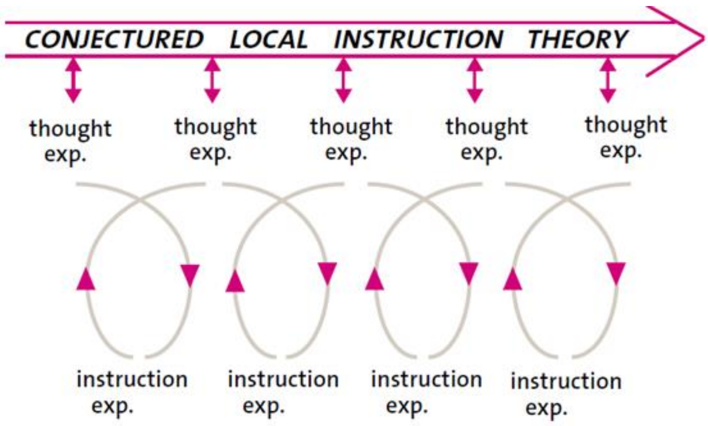

Figure 3.2. Reflexive relation between theory and experiment (Gravemeijer, & Cobb, 2006, p.85)

During these cycles, HLTs guide the teacher. HLTs contributions can be defined as "to guide the enactment of the trial or teaching experiment and guide the data collection about phenomena in which you are most interested" (Bakker, 2018, p. 60).

Consequently, the phase consists of iterative steps, which are the microcycle of research. In each iteration, revision and modifications should be done, and these iterations should be tested in each cycle. Therefore, there can be multiple iterations which including the overall process in the design experiment.

As with the microcycle, each macrocycle contributes to the development of the local instructional theory, and also each micro cycles feeds the macrocycles (Figure 3.3).

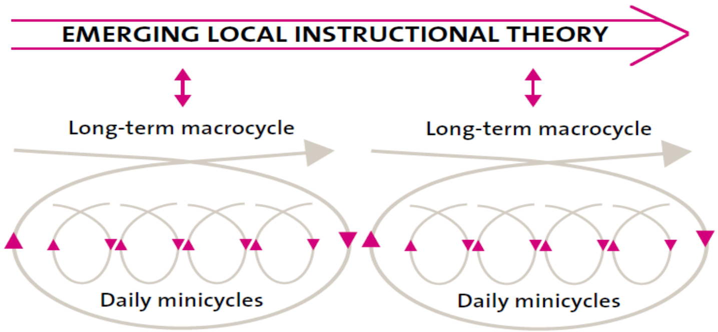

Figure 3.3 Micro and macro design cycles (Gravemeijer, & Cobb,2006, p.85)

Another essential key element for the design experiment is the interpretative framework. The design experiment mentioned above provides us knowledge of the students learning and construction process in the classroom environment. In this sense, the interpretative framework is necessary during the analysis to enable the researcher and the design team to provide a big integrated picture with all its components by helping them interpret the process. This framework can be used both during the ongoing process in the design experiments and during the retrospective analysis (Gravemeijer & Cobb, 2006).

3.1.1.3 Retrospective Analysis Phase

The final phase of the design experiment is the retrospective analysis aiming to develop the local instructional theory. In this phase, the whole data set, such as fieldnotes, pre-post interviews, students' written papers, audio/ video records, are analyzed (Gravemeijer & Cobb, 2006). Since much different material is analyzed depending on the aim of the design experiment, different kinds of analysis are used, such as task-oriented analysis, cyclic approach, and the constant comparative method (Bakker, 2018). According to the chronological order, all the video records are watched, and all the transcripts are read during the constant comparative method. Considering the HLT (see in Appendix G) and the research questions, conjectures are developed and compared with the other parts of the design experiment or other material such as filed notes, reflection papers, or any evidence supporting our conjecture or counterexamples falsify it (Bakker, 2018). this process is conducted for all data set. It is suggested that qualitative data analysis software can be helpful while conducting such/this analysis.

3.2 Context of the study

For this study, an elective undergraduate course, "SSME 305 Teaching and learning mathematics in STEM context," is designed for both pre-service mathematics and elementary mathematics students in the fall and spring semesters of the 2017-2018 academic years. The course focused on "teaching and learning of mathematics in an interdisciplinary context and emphasizes the role of mathematics in STEM education.

This course aimed to provide pre-service mathematics teachers with theoretical knowledge in STEM education and applications of STEM activities in classroom settings. Moreover, at the end of the course, pre-service mathematics teachers were also expected to develop STEM activities and implement in the classroom to their peers, and write a report. Thus, through this course, students would gain practical experience in developing, assessing STEM activities focused on mathematics topics. Morever, it is aimed that they would learn how to use engineering design method into teaching and learning mathematics and implement engineering design processes and mathematical content for approaching and solving real-world problems and learn how to develop critical thinking through the design process. The course comprised three components. In the first component, theoretical knowledge of STEM education was explained to pre-service mathematics teachers.

In the second component, pre-service mathematics teachers experienced the STEM activities on the focus of definite integral. In this regard, they started to design their own STEM activities. Moreover, they presented their tentative activities in the classroom and received feedback from the researcher as instructor and physics education research. In the third component of the course, pre-service mathematics teachers implemented the activities that they developed in the classroom. Based on the feedback received from their peers and the researcher, as instructor they revised the activities and prepared a final report about the results of their activities when they implement their activities in the classroom.

In the second component of the course, it was aimed to examine how pre-service mathematics teachers concept images about definite integral change within an engineering design process by using the design experiment. In this sense, I used the data of the second part of the course.

3.3 Participants

Research indicates that methods courses profoundly impact how a teacher will teach (Haigh, 1985); therefore, it is essential to introduce pre-service teachers to a contextual way of understanding the curriculum when learning how to teach mathematics and science (Frykholm & Glasson, 2005). The current study investigated pre-service mathematics teachers' concept images of definite integral and changes in it. Hence, three pre-service mathematics teachers were selected to obtain more profound information about the pre-service mathematics teachers constructing the integral. To get richer data, purposive sampling was used in the study (Patton, 1990).

Nine students enrolled in the "SSME 305: Teaching and learning mathematics in STEM context" course in 2017-2018 spring term: five fourth-year pre-service elementary mathematics teachers, two third-year pre-service elementary mathematics education teachers, one third-year pre-service elementary science teacher, and one second-year pre-service secondary mathematics teacher. Before taking this course, all of the students completed courses on differential equations, physics, and calculus, which were considered prerequisites for taking SSME305. Hence, the study was conducted with nine prospective teachers in a public university in Ankara, Turkey. Since integral was the selected subject, to interpret their initial level accurately and make a correct interpretation, students' academic background about courses consists integral for both programs is asked from the students.

In the beginning of the course, a test of integral conceptions was given to students. In the third week of the course, an interview was conducted with all pre-service mathematics teachers to understand their thinking processes and initial knowledge about integral. During the interviews, previous answers of the students' on the test of integral conceptions and their own concept map were asked to them. Participants' concept images on integrals were identified. Since the study's primary focus was to determine how their conceptions were changed through the study, pre-service teachers with a fragmented structure of the definite integral and potential conflict factors were purposefully selected for the study. However, students are influenced by each other because they work in groups. Therefore, this influence affects the individual definite integral construction process of the students and, ultimately, the concept images. For this reason, both student and group characteristics were taken into account when choosing students. Groups were allowed to form naturally at the beginning of the course. Because the students would do group work, and they would have to discuss while developing the activities constantly. For this reason, they had to form groups in a way that would enable them to express themselves well and carry out their work with high efficiency. In addition, another aim was to ensure that activities are carried out in the usual flow without disturbing the natural environment of the classroom. In this context, the students and groups needed to be selected to have a fragmented concept image and express themselves well within the group. For example, in one group, one group member was so dominant that the other two group members could not express their ideas, and that causes them not to clarify their construction process. However, in the group which was chosen for analysis, every group member could express their ideas. Another criterion for the selection was their participation in the activities as a group. Since they worked in groups, they affected each other's thinking, and accordingly, the group's construction of concepts. The purpose of this selection was to allow the researcher to investigate the role of the instructional process in terms of building a coherent concept image on the integral concept and how mathematical understanding regarding definite integral grows. Therefore I decided to focus this group on analyzing and report.

3.4 Data Collection Tools

In this study, different data collection tools were used to collect data for determining students' initial and post concept images and how they construct the definite integral through STEM activities. These data collection tools were open ended tests on integral conception, concept map, video and audio recordings, interviews, written solution of participants, follow-up questions, reflection papers, field notes.

3.4.1 Initial and post interview questions

To understand the participants' initial knowledge of definite integral, pre-service mathematics teachers' pre- test of integral conceptions were given to them, and the same tests were asked at the end of the course for understanding their development through the course as post-test of integral conceptions (Appendix B). Pre- test questions were administered beginning of the course before the activities were given, and post-tests of integral conceptions were administered on the 14th of the week when all activities were conducted. The questions were constructed by the researcher based on the basic ideas of the integral. The aim of the test was to get an accurate picture of students' understanding and the existing knowledge of the students' on the definite integral. The questions were prepared as open-ended questions to analyze the students' solutions and ways of thinking in detail. The questions have been prepared by considering the fundamental concepts of definite integral that students should know while learning integral and the big ideas of the definite integral. Moreover, questions are modified according to the results of the designing interview process. In Table 3.1, big ideas and the purpose of the questions are given.

Table 3.1 Table of specification

Q	*Mathematical content*	*The purpose of the question*	*Source*
1	Integral	Participants definition about integral (definite /indefinite Area Oriented/ Analytic	
2	Riemann sum	Students understanding of Riemann sum / define students' interpretation about graphics. Determine students' transformation in graphical to Algebraic interpretation	Adopted from Sealy (2000)
3	FTC	Concept image for FTC (Riemann sum and integral part) Relation between Riemann Sum Interpreting the symbolic meaning	
4	Application of FTC	Applying FTC in contextual problems.	

3.4.2 Concept Map

A concept map is a useful graphical tool to activate prior learning and organize meaningful learning (Novak & Gowin, 1984). Concept maps help determine the students' knowledge and reflect their cognitive structure (Daley & Torre, 2010). To determine students' existing knowledge about integral, they are asked to draw a concept map about integral at the beginning and the end of the course. Concept maps were asked from pre-service mathematics teachers in the classroom before integral related activities were held. Pre-service teachers were not given a limited time to remember their knowledge better and express themselves well. Moreover, at the end of the interview, another concept map was asked pre-service mathematics to draw. However, in that concept map, integral related terms were given to them. Since in the first concept map, the aim was to see what they remember and which concepts they were related with the integral. In the first concept map, teachers only talked about rules or common situations, so it was not clear whether they knew the sub-concepts or not. The second concept image aimed to clarify the relations and focus on extracting their knowledge. For this reason, sub-concepts are given in the second concept map. The purpose of giving the concepts in the second concept map is to reveal whether the teachers know the relationships between the sub-concepts and whether they know the sub-concepts. Again, no specific time limit was applied. After the students made a concept map, they were asked to explain how they established the connections and why they established them.

3.4.3 Video and Audio Recordings of class sessions and group work

In this study, two researchers observed the classroom. During the lesson, researchers observed the classroom environment, group discussions, group reactions. During the class sessions, pre-service mathematics teachers' discussion and their processes during the activity were video and audio recorded. To ensure that all participant discussions are clearly documented, there was also an audio recorder in each table to get precise information from the discussions. Moreover, researchers also controlled the cameras to be sure to record the pre-service mathematics teachers' work. There were five cameras, four of them recording each group, and one of them recorded the whole class, and there were five audio records in the same as cameras. During the analysis process, only the videos and interview records of the selected group were examined. In this sense, 50 hours of interview and class video recordings were analyzed. Besides, class and interview records were transcribed to be used in the analysis. Hence these recordings and the transcriptions helped to clarify other data sources. I constantly returned and watched the video records to understand how and when pre-service mathematics teachers construct the integral concepts. Moreover, these records helped clarify the vague points I confronted during the pre- interview and reflection papers.

3.4.4 Interview Process

Task-based interviews were conducted to determine the pre-service mathematics teachers' concept images on the definite integral. The researcher examined their concept map and test of integral conceptions questions carefully analyzed before the task-based interview. Hence missing parts and vague points of the concept map and test of integral conceptions were asked to each pre-service mathematics teacher. It took approximately 80 minutes and was conducted by the researcher.

At the beginning of the interviews, consent was taken from the pre-service mathematics teachers to take audio and video record. Interviews were executed in the seminar room of the SSME department. During interviews, only the researcher and the participant were in the room. Every interview began with a general conversation, and a camera focused only on the participants' writing area. Moreover, during the interviewing researcher took notes about the critical points about the pre-service mathematics teachers. At the beginning of the interview, a pencil and booklet were given to students, and they were asked to solve the questions think aloud. In this way, pre-service mathematics teachers' all responses, all thinking processes in the formation of a solution can be seen step by step in the data analysis. The interview data were transcribed in detail by the researcher. Analysis of the interview questions was presented in Chapter 4.

Interview tasks were composed of 15 questions prepared according to central topics in the integral concept (See Appendix C). At the beginning of the course, the pre-interviews were conducted to learn about the participants' existing knowledge about integral and the relations between the integral related concepts.

Table 3.2 Table of Specifications of the interview questions

Q	*Mathematical content*	*Purpose of the question*	*Source*
1	Definite integral	Definition of integral Graphical Interpretation	Adopted from (Smith, 2008)
2	Riemann sum	Understanding of Riemann sum Graphical to Algebraic interpretation	Taken from (Özdemir, 2014,p.39)
3	Riemann Sums and FTC	Explaining the relation between Riemann Sum and FTC Explain the notations	
4	FTC	Function interpretation FTC	Adopted from (Smith, 2008)
5	Definite integral	Interpreting the integral constant C Relation between definite and indefinite integral	
6	Definite integral	Interpreting area under the x-axis	
7	Definite integral	Explain the relationship between the integral related terms.	

3.4.5 Students' Written Solutions

Pre-service mathematics teachers' activity sheets and posters were an essential tool for the study. The participants' written work, including individual notes, during the group work, was used as data in this study. At the end of some activities, they prepared a poster to share their solutions, also used as data. Moreover, posters gave an idea to the design team about how pre-service mathematics teachers perceived the activity. In this sense students' scribble papers while they solve the problem; their final solution, which they present and discuss in the class, a poster, was ualso sed as a data.

3.4.6 Reflection Papers

After each lesson, pre-service mathematics teachers were asked to write a reflection paper about the activity they had worked on before. Writing a reflection paper was crucial because the researcher could have an idea about their construction process. Moreover, according to their reflection paper, the design team could get an idea about the activities and the instructional design. For this purpose, guidance consists of four main questions and sub-questions (see Appendix C). In the reflection papers, the participants were asked to reflect upon their solution processes strategies during the lesson. They were also asked to write about their opinion and emotions during solutions and ideas about their solutions from the point of view of a teacher.

By asking them to write reflection papers, I had two aims. The first one was to get more information about their development process, building the concept in their mind, and understanding their STEM perspective. Furthermore, I aimed to collect further data to check the accuracy of the conclusion drawn from their behaviors in the group and their written material through the lesson.

3.4.1 Follow-up questions

Follow-up questions were also asked after each activity. These questions were composed of pure mathematical contexts, and they were also related to the activity implemented that week. The purpose of asking follow-up questions was to learn more about whether they determine the definite integral and meet the expected objectives. Moreover, determine whether they could transform their definite integral knowledge in the contextual questions into non-contextual questions.

3.4.2 Field Notes

Field note is defined as the researchers' written account of what they hear, see, experience, and think in the course of collecting and reflecting on their data" (Fraenkell & Wallen, 2006, p.516). The researcher took notes during the study to help the data analysis process. Field notes were taken during the participant selection process and were used during the interview process, and also observation notes were taken during the class. The observation aimed to observe the instructional design and also constructions of the pre-service mathematics teachers. Observation notes were also taken during the interviews. In each interview, observations and thoughts were being noted to analyze the interview more accurately. Moreover, notes were taken in order to document related conversations before and after the interviews. The design team also took notes about noteworthy points of students' group discussions and solution processes during the course. Those notes were both used for revising the next lesson and also were used in the analysis.

3.5 Sitting Arrangement

All the activities were conducted in the classroom, and pre-service mathematics teachers worked in groups. The researcher arranged the tables every week before the course to work in a group with three members. The tables were placed in such a way that interaction between the groups was minimal. Thus, the groups were not affected by each other, and they could move while designing. The material desk was in the middle of the classroom to get the materials quickly without disturbing other groups or distracting their focus to work without interruption.

3.6 Data Collection Procedures

The data was gathered from interviews and students' activities implemented during the course. The data collection process consisted of videotaping and audiotaping of interviews and classroom group work and discussions, taking field notes. The written part consists of participants' written material through the lesson, written tasks, and their writings and scrawling during the interviews. Reflections about the topic after each activity were also collected. This procedure was reported in line with the design experiment phases.

3.6.1 Phase 1- Preparing for the experiment

Under this heading, a literature review, the development of a conceptual and theoretical framework for the study, and the determination of the HLT were presented. At the beginning of the study, the design team was created according to their expertise. This team was composed of one faculty member in mathematics education, one mathematics education doctoral student with a BS degree in mathematics, one research assistant who was a doctoral student in the physics education department, an engineer, and the researcher. There was also an advisory group that assisted the development of the STEM activities. The advisory group was composed of five people: Four of them were doctoral students who were also research assistants, two of them from engineering department, one from the mathematics department, and the other is from the physics education department and a mathematics teacher. This group's difference from the research team was that they could not attend the whole process; however, they gave feedback or advice to the researcher regarding the nature of the topics, activities, etc. In preparing the design experiment, I followed three steps, and all details of the step were given in the following figures (Figure 3.4, Figure 3.5, Figure 3.6).

3.6.1.1 Step 1

In this phase, for the first step, a literature review was conducted. The focus of this review had three main titles; (1) the concept taught, integral, (2) the instructional theory, engineering design process, and (3) the interpretive framework that included concept image, Sfard’s three-phase theory, and STEM education. After examining the literature, aspects of the design experiment are formulated. This process was held in four central aspects.

The first central aspect was the concept that is intended to be taught. To determine the big ideas and the sequence of the topics, calculus boks (e.g., Adams & Essex,1999; Balcı, 2002; Stewart, 2003; Thomas, Finney, Weir & Giordano, 2003) were examined. Moreover, objectives in the Turkish Middle School Mathematics Curriculum (MoNE, 2013) were also reviewed. After a literature review, studies were divided into subsections: learning of integral and teaching of integral. Learning integral refers to the student context from a design research perspective, while teaching integral refers to the instructional theory. In this context, students’ difficulties, understanding, ways of thinking, and how they structured the integral under which environment and conditions were investigated and determined.

On the other hand, in the teaching of definite integral compon, teachers’ difficulties, in which environments and with which instruments they work, and whether the instruments used were helpful or not were determined. Moreover, a professor from mathematics is interviewed to get more information about the sequence of the tasks, introduce the integral and the problems that students and they confronted. Consequently, in a sequence, big ideas were determined.

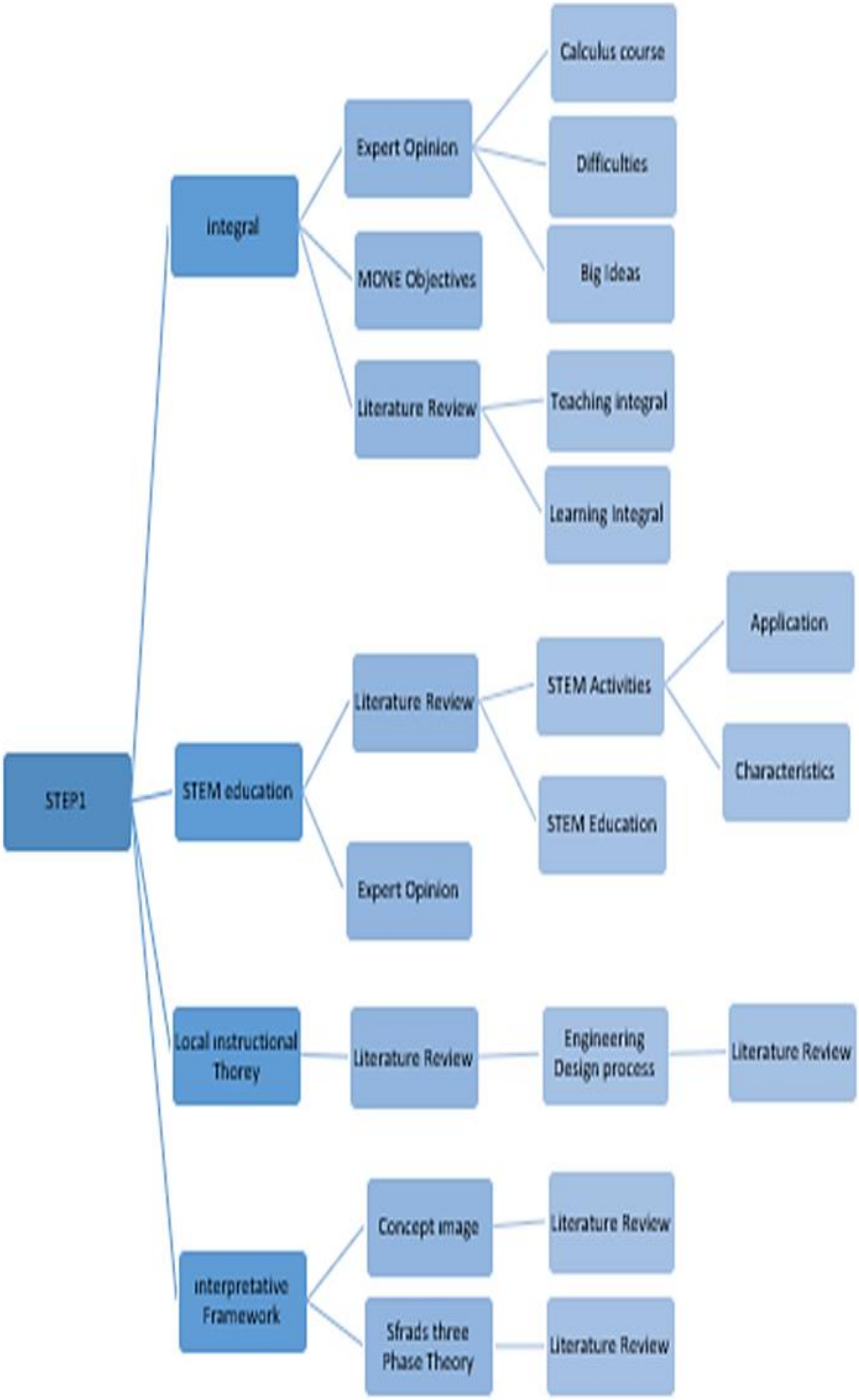

Figure 3.4 First step of preparing design experiment.

Another aspect of the literature review was STEM education. Since this field is comparatively new, there was a limited number of examples of STEM activities. First of all, readymade activities about integral were searched. Besides the application of STEM activities, how to apply them in the classroom environment was investigated. Moreover, expert opinions from whose doctoral dissertation were about STEM education were received. In this regard, the general properties of the activities and instructional theory were determined. Based on the expert opinion and the literature, the engineering design process was selected as instructional theory.

Finally, in Step 1, appropriate interpretative frameworks were investigated. Based on the research questions and the nature of the tasks, the framework had to show the participants' level of development and how this progress was. Sfrad's three-phase theory (1991) was chosen to meet these criteria.

3.6.1.2 Step 2

Analyzing the data in Step 1, the researcher focused on the three main issues; a starting point, activity/ pedagogy, and interpretative framework (Figure 5). As mentioned, determining the starting point was essential in design-based research (Gravemeijer & Cobb, 2006). Also, researchers themselves should conduct formative assessments (e.g., written tests, interviews, whole class performance assessments) before initiating a design research study (Gravemeijer & Cobb, 2006). Moreover, it is also essential to have information about the student's characteristics, thought process. In this regard, pre-interview questions were prepared to formulate the HLT more accurately and have an idea about the students' knowledge about the integral.

3.6.1.2.1 Preparation procedures of interview tasks

In order to prepare integral conception test and the interview questions, I examined theses, articles, and calculus books about integral in the literature. Then I decided to form three subtopics of integral, which were Riemann sum (Jones, 2013; Sealey, 2004, 2008, 2014), fundamental theorem of calculus (Bressoud, 1992, 2011), and definite integral (Bezuidenhout & Olivier, 2000; Delice & Sevimli, 2011; Oberg, 2000; Orton, 1983; Rasslan & Tall, 2002; Schleigh, Bossé, & Lee, 2011). Table 3.3 determined the mathematical content and the purpose of the questions is given.

The subject distribution of the questions about the definite integral was determined as in Table 3.3. Afterward, the questions were discussed with a doctoral student in mathematics education and the research team, and sub-questions were added to some questions. The questions were tested by doing a pilot interview with two preservice teachers. The pilot interview took an average of 70 minutes with each student. As a result of the students' statements and the questions asked during the interview, additions were made to the questions prepared previously. The pre-interview aimed to see the participants' existing knowledge about integral and the relations between the integral related concepts. In this regard, it was determined that the pre- interview questions did not sufficiently reveal the connection between the concepts. For this reason, sub-questions were added to reveal the connections between the concepts more clearly.

Table 3.3 Table of specification of open ended integral conception test

Q	*Mathematical content*	*Purpose of the question*	*Source*
1	Definite integral	Perception of definite integral/ Area-Integral connection	Adopted from Smith, (2008).
2	Definite integral /Riemann sum	Perception of the integral	Adopted from Çorlu (2010)
3	Riemann sum	Understanding of Riemann sum / Graphical to Algebraic interpretation	

Table 3.3 (continued)

4	FTC	Concept image for FTC (Riemann sum and integral part) Misconception
5	FTC	Notation interpretation
6	Definite integral	Interpreting the integral
7	Indefinite integral	Misconception

While revising interview questions, subconcepts were included. The reason for that was to determine at which points students might have deficient constructions because students have both correct and incorrect understanding about integral. In this regard, accumulation, approximation, variable, boundary, and interpretation of the area under the x-axis are included in the interview. After taking expert opinion, sub- questions were added, revised, and then the revised interview was piloted again. The interview questions were revised, and this question was piloted with six preservice teachers. Consequently, the revised questions were successful in exhibiting how students constructed the integral and what kind of connections they made between concepts. This gave the researcher clues on how to shape the HLT like determining which subconcepts should be focused on, the sequence of the sub-concepts, and decided the big ideas.

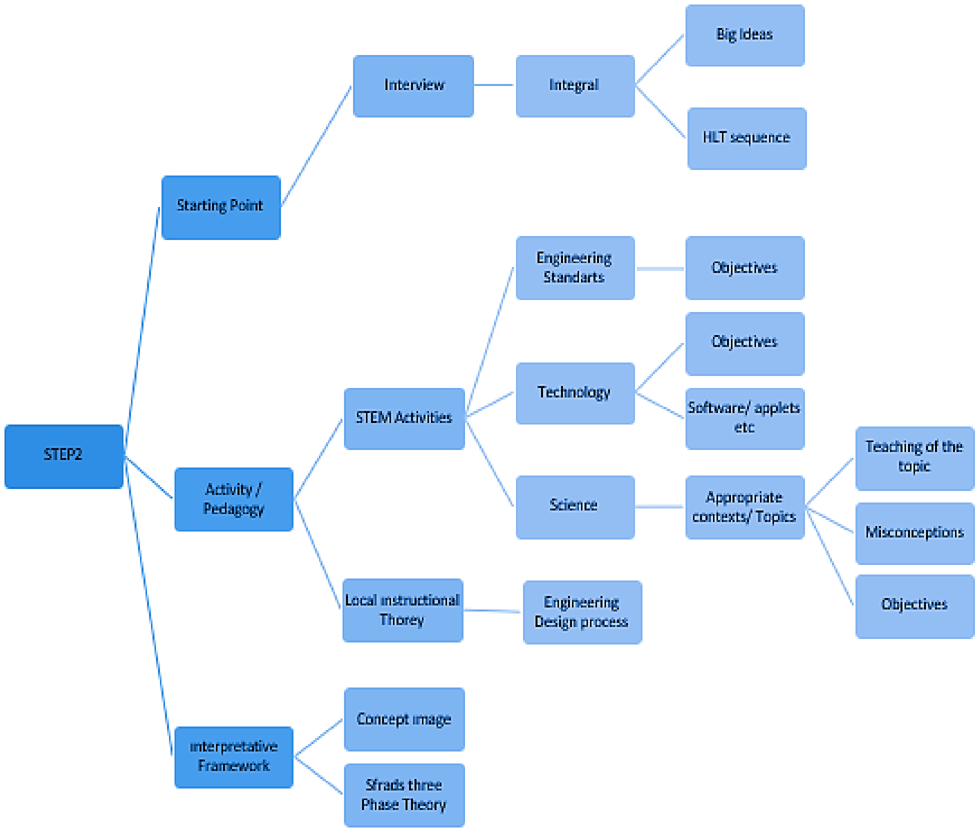

Figure 3.5 Second step of preparing design experiment.

In the meantime, determining the big ideas and the sequence, appropriate instructional theory, and STEM activities were investigated. Questions such as what to pay attention to, characteristics of the activities, task structure, how to introduce the task to students, and what should be the boundary of the used majors in STEM activities were answered. In addition, researching the application areas of integral in engineering and a literature review on engineering objectives to be promoted. In the same way, the subjects in which integral is used for science were determined. The objectives of science subjects were examined, a literature review was conducted to investigate student difficulties, misconceptions, and how to teach the determined science subjects. Finally, the objectives to be gained by the students were determined with a similar process for technology. All these data were analyzed later, and previous research was considered on designing a STEM activity aiming to teach the concept of integral and how it should be applied. Then various criteria were determined as a result of this review. Within the framework of these criteria, to be able to follow the students' way of thinking and how they construct the integral more clearly and efficiently, and to obtain richer data on whether the activities work or not, the engineering design process was determined as the local instruction theory and produce conjecture about the process. In order to help the data analysis process in terms of assessing whether the produced conjectures are true or not, interpretative frameworks are determined, and the next mini-cycle called Step 3 started..

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3.6.1.3 Step 3

Analyzing the data in Step 1, the researcher focused on the three main issues; a starting point, activity/ pedagogy, and interpretative framework (Figure 3.6). In Step 3, HLT is formulated, and interpretative framework analysis details are clarified.

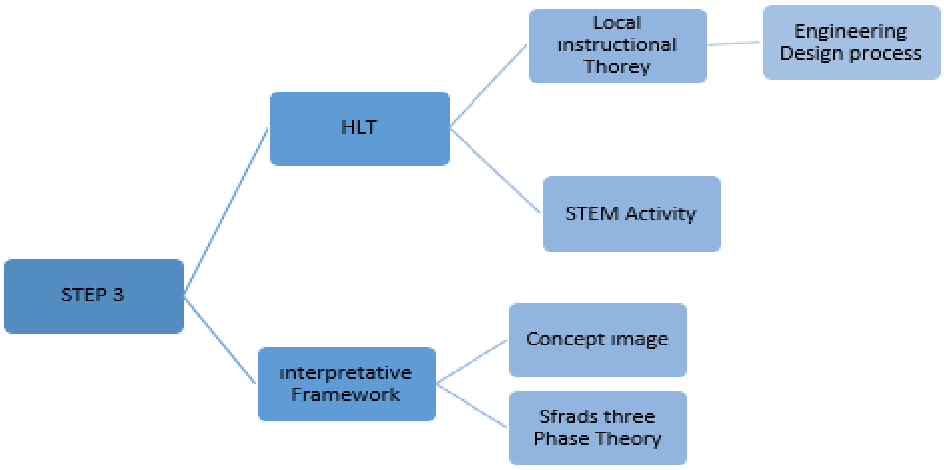


Figure 3.6 Third Step of preparing of engineering design experiment

Gravemeijer and Cobb (2006) suggested that by thinking on students' reasoning and ways of thinking, the research team conjectures a local instructional theory and analyses its consequences via the instructions made earlier. Hence, the team may make inferences about the theory's efficiency whether it supports the students learning. In this sense, it can be concluded that as a whole, HLT, local instructional theory, and instructional activities should support and foster students' understanding. Moreover, these items are parts of a whole that complement each other and serve the same purpose. The key, the interpretative framework, is needed to interpret and understand the big picture in this sense to understand how students constructed the integral concepts and what was changed during the instructions. Two frameworks are used, concept image and Sfard's three-phase theory.

3.6.1.4 Local Instructional Theory

During the design experiment, the engineering design process was used as an instructional theory. Activities were developed according to this theory, and during the experimentation process, activities and the processes were revised.

3.6.1.4.1 Engineering Design

The engineering design process is defined as a “systematic approach followed when developing a solution for a problem with a well-defined outcome” (Capraro, Capraro & Morgan, 2013, p.29). The engineering design process is consisting of seven steps. This process has iterations due to its nature. A well-designed design process is crucial since it provides a structure to the researcher to get the best solution and requires high-order thinking and problem-solving skills. The engineering design process can be represented in seven steps which are illustrated in Figure 3.7.(Capraro, Capraro, & Morgan, 2013). This design was selected because, in this design, the design process was given in detailed steps; hence it was provided to the researcher to see the students’ transition processes more clearly.

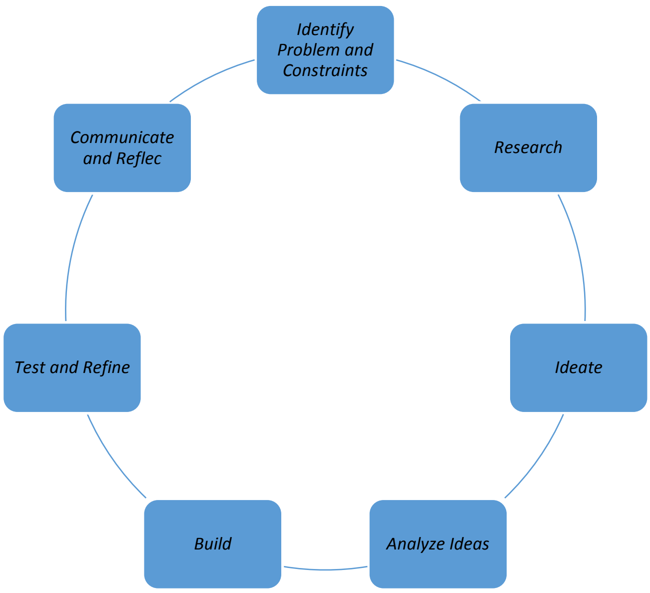


Figure 3.7 Engineering Design Process (Capraro, Capraro & Morgan, 2013)

Step 1. Identify Problem and Constraints

In this step, pre-service mathematics teachers have identified the problem and described the problem clearly (Capraro, Capraro, & Morgan, 2013). In this process, they understood the problem and described the goal of the planned design work. They examined the supplies that they had and decided how they would use those materials. In this respect, they identified the constraints and criteria of design, problem, and supplies.

Step 2: Research

In this step, pre-service mathematics teachers did research that provided them with new ideas or information necessary for their design problem. During the research step, pre-service mathematics teachers obtained more information about the materials and started to clarify the constraints and design ideas. Hence they could minimize the problems during the design step.

Step 3: Ideate

Considering the first two steps in this step, pre-service mathematics teachers started to produce ideas about their design and solution to the problem. Pre-service mathematics teachers made brainstorming about how they would build their design or how they would make calculations. At the begging of the semester, they had difficulty in this step because they could not produce the ideas, so in this step, it was necessary to encourage them to write all ideas even it was not very sensible for the design. Because sometimes they are inspired by that idea and decide their design.

Step 4: Analyse Ideas

In this step, pre-service mathematics teachers reviewed the ideas they produced and differentiated the ideas they could use in the problem-solving or design process. It was checked whether these selected ideas met the criteria of the problem. They determined and tried various strategies for the solution of the problem. This step is the stage where the ideas mature and give their final form. Mathematical calculations are made at this step. Dimensions are determined for the design to be made. This step is the part where pre-service mathematics teachers discuss the concept taught.

Step 5: Build

In this step, pre-service mathematics teachers brought together all the materials required for the prototype to determine all the design blueprint parameters. All materials were measured and cut according to their plans, and they were ready for prototyping. Then a prototype was produced.

Step 6: Test and Refine

In this step, pre-service mathematics teachers tested their prototype under the same conditions in the given problem. They evaluated their product and determined the problematic points that did not work or satisfy the conditions. Finally, they refine their prototype. During the semester, in some activities, pre-service mathematics teachers utilized the build and test steps since, during the building of the prototype, they both tested their designs. Without finishing the process immediately, they refined their plan and rebuilt it. Moreover, in some cases, while they were testing their prototype, they realized new points, so they had to go over the whole process and improve their design.

Step 7: Communicate and Reflect

In this process, pre-service mathematics teachers prepared their posters which illustrate the whole process in summary and their mathematical calculations. After all groups completed their designs, each presented their posters in turn. While making their presentations, they explained how they accepted the solutions, which parameters the pre-service mathematics teachers accepted, how they encountered problems, and how they solved them. After the presentation, each group and the researcher asked questions about their design and the concept taught related to the activity. As a whole class, they discussed the concept taught. Finally, each group member gave feedback on the design to improve it more.

3.6.1.4.2 Interpretative Framework

In this chapter, it was presented that how the interpretative framework was adapted to the study. In the current study, two different frameworks were used concept image (Tall and Vinner, 1981) and three-phase theory (Sfard, 1991). Moreover, to see pre-service mathematics teachers initial and the final situations concept image theory, how they developed or constructed the definite integral thorough the process, the three-phase theory was used.

3.6.1.4.2.1 Developing a Structures of the Concept Image

The study aimed to see the developments of pre-service mathematics teachers. It was necessary to determine their initial and final situations; hence concept image theory (Tall and Vinner, 1981) was used. However, this theory was insufficient to illustrate the pre-service mathematics teachers' initial constructions related to the definite integral concept. For instance, there were no classifications between the concept images, so to illustrate the concept images, it was necessary to explain all the construct properties that the learners have. This was time-consuming and confusing. Hence, to generalize the classification of different concept images and describe the characteristics of each concept image, the mental model theory shows the same features as the concept image. In the following, general definitions of the mental model and the reasons for why the mental model and the concept image might be the same were given.

A mental model is a term that has wide usage in physics education whose roots are based on various areas like phycology and physics. Thus, there are various definitions about mental models stressing different viewpoints;

According to Hrepic, Zollman, and Rebello (2010), a mental model is "A mental structure built of more fundamental cognitive and knowledge elements, e.g., p-prims or conceptual resources" (p.1). Greca and Moreira (2002) defined a mental model as

> an internal representation which acts out as a structural analog of situations or processes. Its role is to account for the individuals' reasoning both when they try to understand discourse and when they try to explain and predict the physical world behavior (p.108-109).

From the definitions above, it can be seen that in the first one, while its dynamic structure is emphasized in the second internal representation is the critical point. Other researchers who study mental models referred to them as the elements of constructing knowledge with working models (der Veer, Kok, & Bajo,1999; van Johnson-Laird, 1983), which has "robust and coherent characteristics" (Bao & Redish, 2006) and may contain conflicted, erroneous substances, which can be coherent scientific or unscientific all kinds of internal representations (Chi, 2009; Norman 1983). Namely, mental models are working cognitive structures which include all kinds of representations in the mind. In this point of view, it can be inferred that the notion of mental models has similar aspects with concept images which are defined as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes including graphs symbolism, verbal representations, or numerical data" (Tall & Vinner, 1981, p. 5). In mental model literature, researchers have detailed its characteristics. In this regard, I uploaded the literature related to the concept image and mental model into the MAXQDA software. Then, I coded them according to the term that they define and characteristics. Then I make a code table in the MAXQDA, which shows the codes, document name, the file name that documents were in, concept image or mental model, and the coded segment. Then, I export this document to MS Excel. Then I grouped the same codes and then compared and utilized the same codes, representing the same characteristics. Based on these characteristics, I selected the key terms differing from the mental model's primary definition. Table 3.4 identified characteristics of the mental model and studies that point out the same characteristics of concept image.

Table 3.4 Overlapping Characteristics of Mental models with concept Image

Characteristics	*Mental Model*	*Concept-Image*
Dynamic	It can be said that mental models are dynamic, evolving systems, and they can be changed (Holland et al., 1989) Via predictions or explanations mental models they can be changed, in this sense they are dynamic *and generative representations* (Greca & Moreira, 2000, 2002; Johnson-Laird, 1983)	During acquiring and mastering a concept learner's concept images are exposed to an ongoing process and evolve (Vinner & Dreyfus, 1989) When students encounter with an idea their concept images constantly evolve (infant, Sealey, 2018). Concept image as in coherency changes over time due to mental activities (Viholainen 2008) Due to its cumulative structure concept images changes over time hence it has a dynamic structure not a static item stored in memory. (Wavro,2011)
Inconsistent	People can have different variety of "*inconsistent"* mental models together at the same time (Gentner, 2002)	"...when one aspect of a *student's concept image conflicts with another aspect of that same concept* image or definition." (Tall & Vinner, 1981, p.153)
Synthetic	People may have *synthesized* mental models in their mind (Vosniadou, 1994)	"…single pieces of knowledge, as such, were correct, but they were connected in an erroneous way…" (Viholainen, 2008, p.245)
Incomplete ./unstable	"Mental models are *incomplete.* Mental models are *unstable:* People forget the details of the system they are using, especially when those details have not been used for some period" (Norman; 1983, p. 8)	"…. some peculiar behaviors are likely to occur. Several cognitive schemes, some *even conflicting with each other, may act in the same person in different situations that are closely related in time.* (Vinner& Dreyfus, 1989, p.365) "*portion of a person's concept image* that is activated at a particular time" (Tall and Vinner,1983, p. 152).
No boundary	"Mental models *do not have firm boundaries*: similar devices and operations get confused with one another". (Norman; 1983, p.8)	"…*different portions of the concept image can be activated* in different situations, but as a whole, the concept image of one concept is an entity". (Viholanen, 2018, p.233)

Table 3.4 (continued)

Mental Representation	Mental models underlie *visual images*, but they can also be abstract, representing situations that cannot be visualized (Johnson-Laird & Byrne, 2002). They consist of propositions, images, rules of procedure,and statements as to when and how they are to be used" (Gentnert, 2002, p. 797).	Representations and mental images2 may be, for example, verbal, symbolic, visual, spatial or kinesthetic. It is also important for the concept image to be well ordered. (Viholanen, 2008, p.235) "the total cognitive structure that is associated with the concept, which includes *all the mental pictures* and associated properties and processes including graphs symbolism, verbal representations, or numerical data" (Tall & Vinner, 1981, p. 152).
Substrate knowledge system Knowledge system	"Mental models should (1) involve a strong well developed 'substrate' knowledge system, such as spatial reasoning, (2) allow explicit hypothetical reasoning, and (3) involve only a small, well-defined class of causal inferences" (Hrepic, Zollman, & Rebello, 2010, 53-54)	Vinner (1991) defines as "person's concept image can be a visual representation, as well as *a collection of impressions or experiences associated with that concept*" (p.68) "A concept image is *cumulative,* changes over time, and is not simply a static item stored in memory
Coherent	mental model: a robust and coherent knowledge element or strongly associated set of knowledge elements (Bao and Redish, 2016, p.3)	coherence of a concept image" refers to the internal organisation of the knowledge structure concerning a certain concept" (Viholanen, 2008, p.233).
Experience	Mental models can be formed by experiences	Concept images are based on our previous experience, including both mental pictures and symbolic processes (Tall, 2013,p.284) "…learning of concepts from formally defined ways of them and *through experiences* in appropriate contexts" (Tall & Vinner, 1981) .."is built up through experience during one's lifetime" (Viholanen, 2008, p.233)
Context dependency	"Students' mental models about the quantization of physical observables are *context dependent*" (Didiş, 2012, p.22)	"concept images as developing differentially over students through a multitude of experiences is essentially a contextual viewpoint" (Bingolbali & Monangham, 2008, p.21) "…learning of concepts from

Table 3.4 (continued)

		formally defined ways of them and *through experiences* in appropriate contexts" (Tall & Vinner, 1981,p.1)
Variation of person to person	Mental model varies person to person every entities due to own reasoning of each person (Black, Freeman & Johnsan-Laird, 1986),	Concept image can *vary from person to person* due to its experiences (Tall & Vinner, 1981; Vinner, 1983; Vinner, 1991)

As seen from Table 3.4, both concepts have the most common characteristics. Hence these characteristics can be valid for concept images. In that sense, the same classification used for the mental model theory can be used for concept image such as coherent scientific, unscientific, etc. Here, coherency is defined as in every different context; the students use the concept in the same way, either correctly or incorrectly.

Another conclusion can be drawn from the table that students who have inconsistent structures can give conflicting answers since concepts are fragmented or incoherent in their minds. Thus, students may hold two or more insistent models, which may be scientific departmentalized, or they somehow combined scientific and unscientific knowledge. That means students' constructions are either bifurcated (Greca & Moreira, 2000) or synthesized (Vosniadou,1994). Tall and Vinner (1981) called this situation a potential conflict factor. In mathematics education literature, the same patterns are called the "intuitive model" (Fiscbhein, 1982).

Briefly, fragmented structures are built up when partial understandings appeared, and they may have correct parts, but essential elements are missing or erroneously connected (Viholainen, 2018), or incoherencies occur as they do not reflect accurate mathematical structures. To sum up all associations and provide a general classification for a common language, the mental model's general classification following Figure 3.8 can be drawn. Since it depends on context, new classifications may occur.

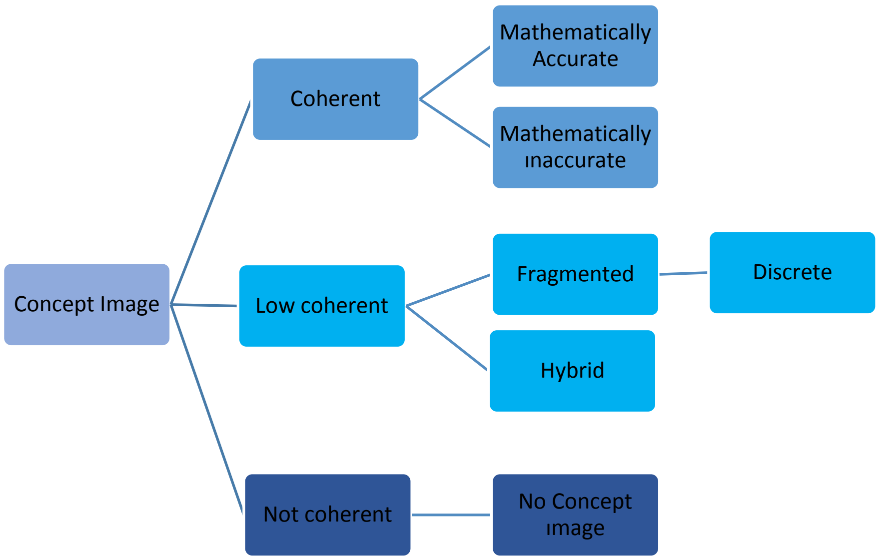


Figure 3.8 General Classification of Mental Models

Since there is no precise classification in concept images and to provide a common language, the same terms will be used. However, not to cause any confusion, instead of using the term mental model, the concept image term will be used in the thesis.

Based on the literature and the data, this dissertation's aim can be restated as students who have fragmented concept images will be identified and classified. According to this classification and with the help of STEM activities, their concept image will be reconstructed to have coherent and mathematically accurate concept images.

To make a comparison and be more understandable, first of all, a mathematically accurate concept image is defined. Students who have this kind of concept image explain the concepts accurately and describe the relations correctly. The operational definition of this concept image can be given as;

- The student who has this kind of concept image can explain integral accurately in terms of geometric definition ("oriented area of a region under a curve" and analytic definition ("the limit of a sum of areas of rectangles) and relate them.

- The student who has this kind of concept image can explain Riemann Sum, its relation with the Fundamental Theorem of Calculus (FTC), and make transitions easily.
- The student who has this kind of concept image can determine the independent and dependent variable in a data and set up the integral
- The students who have this kind of concept images can explain FTC
- The students who have this kind of concept images can model integral in different contexts and explain every symbol in the integral form.

Students who have a scientific concept image can be visualized as Figure 3.9.

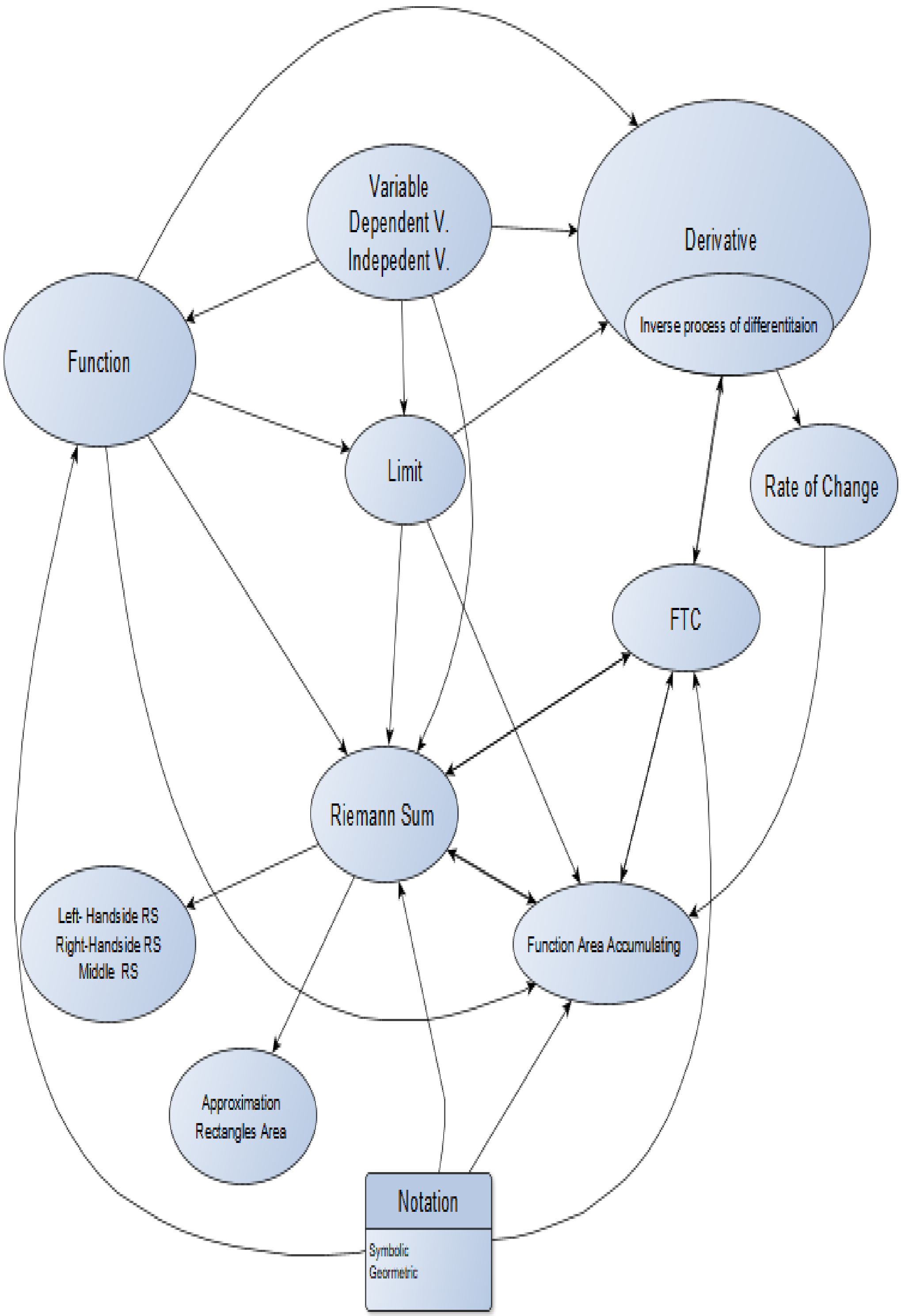


Figure 3.9 Mathematically Accurate Concept Image

3.6.1.4.3 Concept Development Process

To analyze the pre-service mathematics teachers' concept development process, Sfard's (1991, 1992) theory of reification was used because classifying is made clear and detailed. With this classification's help, more accurate learning strategies and treatment can be developed for students at different steps. Since their conception characteristics and main points in each step are defined, practical learning strategies will be implemented.

In her theory, Sfard divides the conceptions into two parts, operational conceptions, which deal with mathematical processes, and structural conceptions, which deal with mathematical objects. According to her theory, these two components complete each other. However, since the transition between these two concepts is challenging, the transition needs much effort. Once they are constructed accurately, one can understand the mathematical notions deeply. Sfard (1991) explains two concepts as follow:

> Seeing a mathematical entity as an object means being capable of referring to it as if it were a real thing—a static structure existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details. . . . In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential and detailed, (p. 4)

Sfard (1991) separates this process into three hierarchical stages, interiorization, condensation, and reification, to explain how conceptions evolve from operational to structural. In the first stage of the hierarchy, interiorization, students are in the lowest stages, become familiar with the concept, and have limited skills. Sfard (1991) explains this stage as "gets acquainted with the process will eventually give rise to a new concept (p. 18)."

In the condensation stage, students can combine other processes more efficiently, and “the operation or the process is squeezed into more manageable units” (Sfard,1991, p. 18). In this stage, the student can deal with various representations of the concept. Moreover, the student is capable of making reasoning about the whole concept without dealing with details. Thus, students can handle the whole process with awareness.

In the reification processes, students can see the whole process and straightforwardly explain the entire process. Moreover, students can transform information through the steps and interpret the concepts in different aspects. According to Sfard (1991), “it should already be clear that our three-phase schema is to be understood as a hierarchy, which implies that one stage cannot be reached before all the former steps are taken” (p.21). After the reification stage, learned concepts evolve from operational to structural, and reified concepts become an input for constructing a new concept.

3.6.1.4.3.1 Learning Process of Definite Integral According to Sfard’s “Three-Phase Model.”

In this section, an adaptation of Sfard’s three-phase theory to explain learning of definite integral is presented.

3.6.1.4.3.1.1 Conception of Riemann Sum

In this study, Sfard’s “Three-Phase Model” is used to examine the formation process of the integral through the cognitive development of pre-service mathematics teachers. In the presented study, there are two main concepts Riemann sum and FTC. According to the process in Figure 2.2, the Riemann sum is the first concept that should be constructed. Hence, the Riemann Sum components will be process on the concrete object, and through the three-phase model, pre-service mathematics teachers will be reached to object phase to construct the Riemann Sum as an object.

While preparing the objectives showing pre-service mathematics teachers' development, the books used in Calculus lessons were first examined (Spivak, 1994; Steward, 2009; Thomas, 2004), and the big ideas were determined. Later, theses and articles related to the Riemann Sum (Jones, 2010; Oberg, 2000; Sealey, 2014) and the places where students had difficulties teaching the subject were determined in the articles. Also, theses and articles on the FTC (Jones, 2010; Thompson, 2013) were also examined. Besides, the instructors who offer the Calculus course were interviewed. Objectives were determined based on the definitions made for each step in theory. Later, with the interviews, conducted objectives were revised.

The conception of Riemann Sum at Interiorization Stage

In the interiorization process, pre-service mathematics teachers can

- Determine independent and dependent variables from the given data.
- Calculate the unknown irregular area by dividing the given interval into rectangles, calculating these rectangles areas, and adding them.
- Start to realize that the accumulation is incomplete in the area calculated with a limited number of rectangles, and it is approaching to the exact solution when the number of rectangles is increased by adding the "next piece."
- Be aware of generalization of the process
- Connect with the real world.

The conception of Riemann Sum at Condensation Stage

During the condensation process, pre-service mathematics teachers can write the algebraic form of Riemann Sum by using their graphical representation or vice versa. For example, in this process, when the pre-service mathematics teachers draw rectangles on the function's graphs, they consider the variable of the functions and type of Riemann Sum. In this process, pre-service mathematics teachers can

- Write and interpret the product of $f(x_i)\Delta x_i$
- Determinate the accumulation of the changes according to the independent variable.
- Grasp that the quantity which is accumulating has a multiplicative nature.
- Flexibly use various representations.
- Differentiate left-hand side and right-hand side Riemann sum
- Comprehend that accumulation is a process of adding new increments.
- The smaller shifts in the independent variable, the more accurate estimates of the change in the accumulation one can obtain
- Be aware left-hand side and right-hand side Riemann sum

The conception of Riemann Sum at Reification Stage

Pre-serve mathematics teachers reach the reification stage when they are capable of operating the Riemann sum as an object. Thus, treat Riemann sum as a whole and translate it into graphical representations uses algebraic methods to find the solution.

In this process, pre-service mathematics teachers are able to

- Write and interpret the $\lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)\Delta x)$
- Can easily translate between representation
- Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.
- The accumulation of any function can be generalized by means of the graph surrounded by change independent variable, and the value of the function.

3.6.1.4.3.1.2 Conception of the Fundamental Theorem of Calculus

In the second process, the Riemann sum turned into the object from the process to construct FTC. By constructing FTC on the Riemann Sum, students were taught definite integral as a whole. At the end of these two stages, integral will be an object from them, and in the following learning process, integral will be processed on learning differential equations

The conception of FTC at Interiorization Stage

In this process, pre-service mathematics teachers can

- To find the exact value of accumulation in a range, it calculates the area of each piece by dividing the given range into smaller pieces and uses limit in this process.
- Coordinate different changes in a function’s input variable with the average change of the function for the same range.
- The accumulation of any function can be generalized by the graph’s area bounded by the change in the independent variable and the function value.
- The value of $F(x)$ represents the accumulated area under the curve of f from a to x
- F is a function that can be found from the accumulation of the function f, the original function which is a rate-of-change of F, with independent variable t, and x represents the independent variable of a second "pass" through the function f with a view for coordinating the rate-of-change $f(x)$ with the change of the independent variable to determine the multiplicative structre of the accumulation, F.

Conception of Riemann Sum at Condensation Stage

In this process, pre-service mathematics teachers can

- Coordinate the accumulation of smaller ranges of a function’s input variable multiplied by the average function change in range.

- Understands that the rate of change in total accumulation is determined by the average rate of change of accumulation in small intervals
- Understands the instantaneous rate of change of the accumulation that is equal to the function value at a given value of the independent variable. (Carlson, 2003, p.165).

Conception of FTC at Reification Stage

Pre-serve mathematics teachers reach to the reification stage when they can operate the Riemann sum as an object. Thus, treating Riemann sum as a whole and translate into graphical representations uses algebraic methods to find the solution. In this stage, they can

- Write and interpret the expression $\lim_{n \to \infty} \sum_{i=1}^{n} (f(x_i)\Delta x)$
- Can easily translate between representation
- Understands that the change of a cumulative function equlas to multiplication of constant rate of change in the interval and the independent variable's change in a given range is equal to the sum of any function
- The accumulation of any function can be generalized by the area of the graph bounded by the change I the independent variable and the function.

3.6.2 Phase 2- Design experiment

This step's primary goal is to improve the initial design by testing and revising conjectures and understanding how the formulated local instructional theory works (Gravemeijer, & Cobb, 2006). To achieve this, iterated cycles are vital components. While each lesson is a cycle in design-based research, there can be a micro cycle in each lesson. During the cycles, through the thought experiments, the researcher and the design team evaluate the proposed activities, students thinking process, interaction with the activities, and the teacher. Moreover, they also evaluate their participation, reactions to the activities, learning, or difficulties that the students or the teachers confronted.

3.6.2.1 Conducting Design Experiment

In line with design research steps, the design team and the researcher formulate the instructional sequence and its implementation in the class. As mentioned before, with the interview data and the instructional framework, an instructional sequence is determined. All the materials, such as the teacher guide, materials used in the class, were determined and supplied. After the completion of the preparation, the researcher implemented the instructional sequence.

One week before each lesson, the researcher implemented the activity with a researcher from the design team, and the parts of the activity that will not work were determined and revised. Later, the design team came together, the teacher guide is corrected, and the researcher implements the activities to the design team as if she were implementing the activity in the classroom. Meanwhile, the design team points out the problematic parts of the related activity. For example, what should we pay attention to while explaining the science part, what are the possible misconceptions that may occur in students and how to overcome them or how to teach the lesson in order to avoid misconception, parts such as teaching the math part are discussed, the activity is rearranged and reconstructed. In the classroom, tasks are shared among the people in the design team.

Just before starting the lesson, we briefly reviewed what we should pay attention to again. Later, a person from the design team wandered around the class and noted the places where the students were studying. Besides, the researcher took field notes about what she paid attention to while implementing the lesson and which points should be paid more attention. The other person from the design team checked that the cameras were not capturing the students and their activities, paying attention to the students' discussions in the science sections. During the lesson, after the activity was introduced, students were asked to work individually for an average of five minutes, and then they were asked to discuss the given problem in the activity with their group mates. Discussions were held with the whole class, and explanations of the difficulties were made at the points where all groups had problems. “Why did you think that to students in the classroom? How did you do?” questions were asked. By asking example case questions similar to the situation that each of them did to reveal the situations in their minds more clearly, it was examined how to solve it.

At the end of the lesson, each person’s opinions in the design team were taken, and the questions to be asked to the students at the end of the following week’s activity and activity were revised. Moreover, the students’ reflection paper was examined, and the videos and the audio recordings were listened. Revisions in the activities were presented in the following chapter.

3.6.2.2 Revisions in the design experiment

The instructional sequence was conducted in the “STEM context in Mathematics Education” course during the 2017-2018 fall semester with 16 pre-service mathematics teachers. In the first four weeks of the course, theoretical issues about STEM education involving characteristics of STEM education, STEM activities, implication of sample STEM activities, and engineering design processes. In the following weeks, six STEM activities were implemented. Pre-service mathematics teachers worked as a group consisting of four-person in each group. Before the lesson, activity sheets were distributed to all students. The pre-service mathematics teachers’ activity sheets and posters were collected at the end of each session.

Moreover, they wrote a reflection paper after each session. In these reflection papers, pre-service mathematics teachers were asked to solve questions in pure mathematics on the subject learned that day. The structure of the questions is conceptual and interpretative. Also, students criticize the course in their reflection papers. These criticisms explain how the lesson is handled, the handling of the topic, the nature of the activity, or the points where they have difficulty or did not understand. Besides, pre-service mathematics teachers explained how they thought, explain their thinking processes while solving the activity in reflection papers. After each lesson, the opinions of each person in the design team were taken.

Moreover, classroom videos were viewed, and audio recordings were listened to quickly, and notes were taken. Each student's reflection paper was read. After these, the design team met, and according to these data, necessary revisions were made in the activities. Some significant modifications were made in Activity 2, Activity 3, Activity 5, and Activity 6. The initial version of Activity 2 was about constructing an earthquake-resistant building composed of three different shapes in each layer. However, it was observed that students had severe problems understanding the Riemann Sum and the relation between the real world and mathematics. Hence the activity was simplified, and it was decided to be only a semi-circle for each floor.

Moreover, in the text of the activity sheet, some values were given, which caused students to focus on numbers only, so numbers in all activities were removed, and more general expressions were used. Likewise, the relationship between the force and pressure affecting the dam will be tested in Activity 3, designed as the dam construction. Since the physics subject knowledge level was so high for pre-service mathematics teachers, it was only focused on the force that can collapse the dam. The sequence of Activity 5 and Activity 6 was changed. Because Activity 6 required higher-level skills. Thus students may not focus on the Fundamental Theorem of Calculus.

Moreover, according to Sfard's three-phase theory, it was aimed that the last activity should be the basis for differential equations. As a result, the sequence of the two activities was changed. Moreover, in Activity 5, accumulation and covariation were clear, so Activity 5 was implemented before Activity 6.

3.6.2.3 Conducting Design Experiment 2

The second cycle of the design experiment was conducted during the 2017-2018 spring in the STEM in Mathematics Context course. As mentioned previously, the course aimed to introduce STEM education to pre-service mathematics teachers. As in the first cycle in the fifth week of the course, the implementation of the ınstrcutional sequence started. Moreover, the previous cycle pre-service mathematics teachers experienced the STEM and the engineering design process activities before the actual implementation began. Their works and the group discussion were video and audio recorded. Hence they got used to working with cameras and their group friends. Furthermore, they also got used to giving feedback to their friends, discussions, and also presentations.

Overview of Data collecting Process

Before implementing the activities, students were asked to draw concept images about integral and answered the initial conceptual test. After that, interviews were conducted based on those answers to determine the students' initial concept image. This means that in the first weeks, there were general activities to determine the students' level. Then students who were representative of the class were selected.

Implementation of the activities;

Through the lesson:

Activities were implemented as the group works with three members, and participants work on with the same group throughout the semester. During the course, the researcher assisted the students and gave reflections if necessary. In the lesson, the problem was given to students and let them study individually and then work as a group. They solved the given problem according to engineering design steps, and finally, they presented their solutions as a poster and discussed their and other groups' solution.

Through the semester :

There were two focus :

The first focus of the course was to teach prospective teachers about what STEM education is and how they develop and implement STEM activities. Course outline and used methods through the course are prepared. In this process, for the first four weeks' information about STEM (its origins, different aspects of STEM, latest trends in STEM in Turkey and around the world, appropriate teaching and learning methods for STEM) is given. Through 5 weeks, STEM activities were conducted. To decrease the threats about engineering design, the researcher warmed the students up to the course and made them gain experience for the actual implementation of the study. A sample STEM activity was implemented, and the engineering design and its steps were explained comprehensively. The 10th theoretical information about designing and developing STEM activities and general rules and principles about them and their teaching was given. In the 11th week, the characteristics and principles required in developing a STEM and engineering design process were discussed. After discussing and determining the principles, sample STEM activities were given and asked them to write a criticizing paper according to predetermined principles and present in the class. At the end of the course, they presented their draft STEM term projects submitted. 12th and 13th weeks of the terms last two activities were conducted. For the last two weeks of the course, students implemented their revised projects in the class and took feedback from the project team and their classmates. According to this feedbacks, they revised their projects.

In the second focus of the course, the conception of the integral with STEM activities was investigated. Each week's sequence of activities determined objectives, and the big ideas about integral are given to students. After each activity, students were asked to write a reflection paper and answer the follow-up integral question. Detailed information about the reflection paper will be given in the data collection tools section. At the end of the course, students were expected to learn integral concepts coherently and conceptually.

Each week;

- Before the lesson, the researcher presented the activity to the project team, gave feedback, and the activity was revised and presented to the project team. Taking their approval, activity was implemented in the class.
- According to the students' levels, the activity was revised immediately, and those revisions are recorded. Moreover, some misunderstanding points which arose from the text were revised. According to students' feedback in the class and reflection papers, activities are examined, and necessary revisions were made. Besides, these observations were made and recorded.
- After the course, the project team meets, and their observation notes and working and not working points of the activity are discussed, and next week's activity is revised.

In a broad perspective, the structure of the first and the second cycle is presented. More specifically, all details about the first and second cycle structure will be given in the cycles section.

3.7 Data Analysis

3.7.1 Analysis of pre/post-interviews

Merriam (2015) emphasized six different strategies to analyze the data: phenomenological analysis, ethnographic analysis, narrative inquiry, constant comparative method, and content analysis. The constant comparative method (Bakker, 2018; Glaser & Strauss, 1967) was used in this study. In this method, different or the same data sets are compared in terms of incidents. This method aims to find the evidence to confirm or disconfirm the conjecture by comparing or contrasting the incidents in different data set or in the same data (Bakker, 2018; Glaser & Strauss, 1967).

Pre-interviews were conducted to determine the students' current situation before starting the design experiment and making the necessary changes in the first activity. Also, conceptual questions were asked to the students, and they were asked to make concept maps. According to the chronological order, all the video records were watched during the constant comparative method, and all the transcripts are read. The researcher first separated the data into units and code them, then by comparing the codes in the recurring patterns constructed the categories (Glaser & Strauss, 1967; Strauss & Corbin, 1998). As a first step, the researcher started to read a data set, generate tentative codes, and try to capture the themes among codes. After capturing the themes, she reviewed themes and formulated the themes. Then the researcher started to examine the second data set with the same procedure by considering the first themes in mind (Merriam, 2015).

In this study, interviews, concept maps, and the conceptual test were different data sets. Different data types were collected from the student's constant comparative method in analyzing the data. Before analyzing the pre-and post-tests, I transcribed and uploaded them to the Maxqda program (VERBI Software, 2019). Then I examined the literature about integral and concept image theory and re-examined the HLT and the integral concept's big ideas. Then I examined the conceptual task of the students. I grouped the questions which measure the same objective in each data set and examined these questions one by one in groups. In the interview and conceptual questions, if the question asked the definition of integral, these questions were put into the same group. Then each student was examined one by one for the same questions in the first data set. Through this process, specific codes were obtained. The second data set, an interview question related to the same objective, was examined with these codes. If the newly released codes exist in conceptual questions, the previous data set was re-coded. Then categories finally themes were generated. Codes for the pre-service mathematics teachers' concept definition was given in Table 3.5. In the table, themes, categories, and codes were given.

Table 3.5 Coding preservice mathematics teachers' concept definition.

Themes	*Categories*	*Codes*	*Content*
Concept Definition	Appropriate Definition	CDA- Code 1	Area- Oriented
		CDA- Code 2	Partition- Oriented
	Inappropriate Definition	CDIA-Code 3	Incomplete Definition -area-oriented -derivative relation -analytic definition
		CDIA-Code 4	Only Visual Drawing
		CDIA-Code 5	Incorrect
		CD- Code 6	No Definition
Conception of Riemann Sum	Approximation	RSAp- Code 1	Determining /Coordinating Variables
		RSAp- Code 2	Multiplication of average change in dependent variable with average change in independent variable
		RSAp- Code 3	Definition of Integral oriented
		RSAp- Code 4	Partition- Oriented -left hand side - right hand side - middle - superior sum - inferior sum
		RSAp- Code 5	Algebraic- Oriented
		RSAp- Code 6	Procedural Calculation
		RSAp- Code 7	Partial correct answer
		RSAp- Code 8	Incorrect Answer
		RSAp- Code 9	No Answer
	Accumulation	RSAc-Code 1	Geometric Interpretation -summation - accumulation RS
		RSAc -Code 2	Algebraic Interpretation -summation - accumulation RS
		RSAc -Code 3	Partial correct answer
		RSAc -Code 4	Incorrect Answer
		RSAc -Code 5	No answer

Table 3.5 (continued)

Transition to FTC	Accumulation	TFTCAc-Code 1	Accumulation function
		TFTCAc-Code 2	Limit of accumulation function
	Covariation	TFTCCov-Code 1	Variation in two variables simultaneously
		TFTCCov-Code 2	Variation of the variables constant rate of change with respect to each other by increments
		TFTCCov-Code 3	Rate of change of the accumulations equals to the rate of change of variables by the increments
	Riemann Sum Relation	TFTCRSR_Code 1	Riemann Sum and integral Relation
FTC	Rate of Change	FTCRoC Code 1	Rate of change of the accumulation function
	Application of FTC	FTCAp Code 1	Derivative Relation
		FTCAp Code 2	Difficulty in application
		FTCAp Code 3	Boundaries
		FTCAp Code 4	Procedural
		FTCAp Code 5	Conceptual
		FTCAp Code 6	Partial Conceptual
		FTCAp Code 7	Incorrect answer
		FTCAp Code 8	No answer
Notation	$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$	NRS- Code 1	Correct Explanation
		NRS- Code 2	Partial Correct Explanation
		NRS- Code 3	Incorrect Explanation
		NRS- Code 4	No Explanation
	$F = \int_a^b f(x)dx$	NDI Code 1	Correct Explanation
		NDI Code 2	Partial Correct Explanation
		NDI Code 3	Incorrect Explanation
		NDI Code 4	No Explanation
	$\int_a^b f(x)dx = F(a) - F(b)$	NFTC1 Code 1	Correct Explanation
		NFTC1 Code 2	Partial Correct Explanation
		NFTC1 Code 3	Incorrect Explanation
		NFTC1 Code 4	No Explanation

Table 3.5 (continued)

$\frac{d}{dx}\left[\int_a^b f(x)dx\right] = f(x)$	NFTC2 Code 1	Correct Explanation
	NFTC2 Code 2	Partial Correct Explanation
	NFTC2 Code 3	Incorrect Explanation
	NFTC2 Code 4	No Explanation

The concept image theme was consists of two categories. These categories were appropriate and inappropriate definitions. In the appropriate definition, the pre-service mathematics teachers were expected to define the integral wholly and correctly. The definition of "Inappropriate" includes the missing definitions. Under these categories, the codes are defined. Even the students were usually provided with formal integral definition by instructors in the previous courses that they enrolled in, they tended to learn the integral definition as an area-oriented form. The reasons for this selection were discussed in Chapter 5. In this sense, instead of giving the students integral definition based on the partition, the area-oriented definition was accepted as appropriate, and CDA- Code 1 was given as

> The symbol $\int_a^b f(x)dx$ denotes the *precise* value of the area under the graph of f between $x = a$ and $x = b$. It is known as the integral of y with respect to x over the interval from a to b. (SMP, 1997, p.143, cited in Rasslan & Tall, 2002).

This definition was "precise" since with numerical methods and the Riemann sum approximate area was calculated. In this sense, CDA- Code 2 was given as

> If f is a continuous function defined for $a \leq x \leq b$ we divide the $[a, b]$ into n subintervals of equal width Δx . We let $x_0 = a, \dots, x_n = b$ be the endpoints of these subintervals. We let $x_1^*, x_2^*, \dots, x_n^*$ any sample points in these subintervals, so x_i^*lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f form from a to b is
>
> $\int_b^b f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ (Stewart, 2009, p.381)

For CDIA-Code 3, “area under a curve” can be an example. It is an area-oriented definition; however, it is an incomplete definition. Because students did not mention the function f and also its boundaries. CDIA-Code 4 is evident since pre-service mathematics teachers only draw graphs without making any explanations. Finally, writing integration rules can be an example for CDIA-Code 5. In terms of mathematical accuracy, written integration rules are correct; it does not define the definite integral.

For the other codes, the same process was conducted. First, I determined deficiencies in the test and examined their mistakes. I asked reasons for the explanations and took notes of them. Next, I coded the data and generated codes by considering the big ideas, literature, the HLT, and the conceptual tests. Then, a second coder coded the data. She coded two pre- interview and two post-interview data. Inter-rater reliability of pre-interview was found 86% and post interview reliabitiy was found 92%. After completing the coding, we met with the second coder again and discussed the codes we disagreed on.

Furthermore, since the explanation of the codes was not detailed and sufficient, there were discrepancies between my codings and those of the second coder. Through discussing those discrepancies, we reached a consensus on the disagreements. RSAp- Code 1 which is determining variable code and FTCRoC Code 1 which is Rate of change of the accumulation function codes were emerged. Then, I re-examined the data and tried to classify the themes. After that, I followed the same procedure in the interview data. Finally, I re-coded the interview data after four months, a second time independent of my first coding. I found the intra-rater reliability %97.

In the light of these encodings, the initial and final concept images that the students have and the missing points about the integral topic were determined. Moreover, during the design experiment, it was ensured that students have a mathematically correct concept image.

3.7.2 Analysis of design experiment

I analyzed the design experiment in three main parts. In the first part first, I organized the transcripts of the design experiment sessions in chronological order and the reflection papers written after each session. Then I watched the videos of each week and read their transcripts. On the other hand, I noted down what students thought during their thinking process on the papers they used in the lesson. Besides, I read the reflection papers of that week. While I was reading and watching, I took notes about each student. Every week, I studied the notes we took in class. I reviewed the design team's suggestions and overflows. I generally had an idea of how the pre-service mathematics teachers' progress week by week.

As a second step, I added the transcripts of each week's discussion to the excel file. Since the initial situations and concept images of the students were different, the analyzes were made in the form of case by case and also results presented as in cases. Using the linkography method (Goldschmidt, 1995), I made links between pre-service mathematics teachers' ideas, as explained in the following section.

3.7.2.1 Linkography

Linkography, introduced by Goldschmidt (1995), analyzes the designers' cognitive processes. It has been used in various research; evaluation of cartoons (Chou, Chou, & Chen, 2013), architecture idea generation process (Goldschmidt, 2014), students' thinking process (Blom & Bogaers, 2018). The method was improved by different researchers (e.g., Kan & Gero, 2008; van der Lugt, 2005), and it was used in qualitative and quantitative studies investigating thinking processes or cognition of the designers and their teams (Kan & Gero, 2008; Hatcher et al., 2018; Blom & Bogaers, 2018).

Linkography's main area of use is not educational research. In fact, its intended use is to portray how architects or designers construct a design in their minds while designing, through which mental processes it goes through. In this context, as the purpose of this study was to examine the mental processes of pre-service mathematics teachers and to reveal how they construct the integral in their minds, it coincided with the usage areas of the linkography. Thus, it has been deemed appropriate to linkography to examine the design experiment sessions. Linkography gives the researcher access to which mental stages the individual goes through while learning, at which points the individual has difficulty, which points prevent learning, and when and what the individual has learned the concept taught.

Several terms are essential for a linkograph as shown in Figure 3.11. In the linkograph, in order to produce links, utterances should be in chronological order to produce links. The link between the utterances is called a “design move,” which is defined as “a step, an act, an operation which transforms the design situation relative to the state in which it was before that move” (Goldschmidt, 1995, p. 195. The researcher determines whether there is a link between each sentence by matching it with the previous move to establish a connection between the moves (Goldschmidt, 2016).

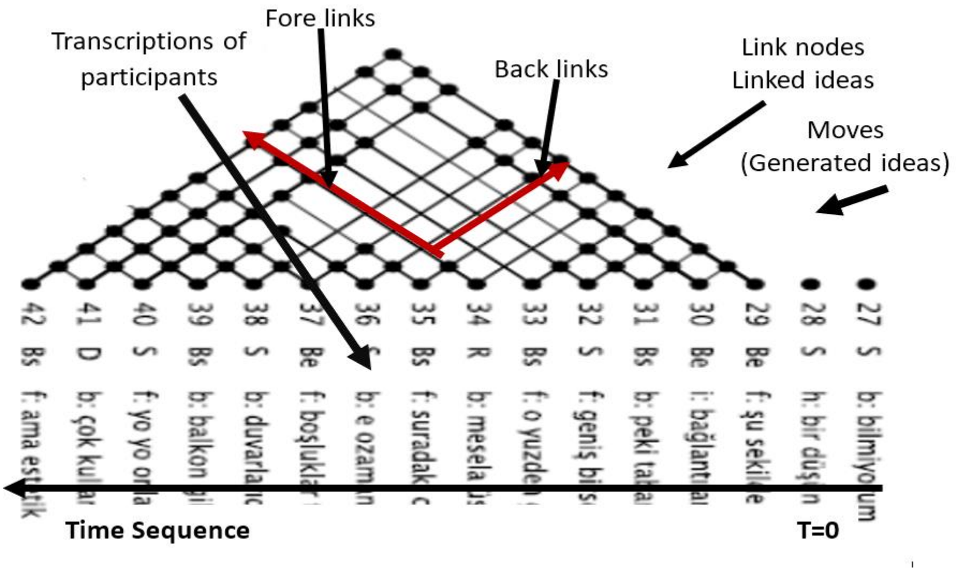


Figure 3.10 Terms used in Linkograph.

If this link is established, it is called a backlink because it was created based on the previous move (Goldschmidt, 2016). According to Goldschmidt (2016), forward links refer to the future with different thoughts, while backlinks are links built on past conversations. The nodes between them are checked to see if the two movements are connected with each other. If there is a knot at the intersection of two movements, it means that the two movements are linked. If there is no knot, these two movements are disconnected from each other (Goldschmidt, 2016). In this sense, a linkograph is the graphical visualization of the movements that follow each other sequentially in the design process and the connections between these movements.

Interpretation of the Linkograph

Interpretation is made according to the nodes formed in linkographs and the directions of the links. There are different types of links and different interpretations, but they are not included in this thesis because they are specific to the design field. According to Hatcher et al. (2018), the dynamics of the group's thought vary according to the numbers and shapes of the links formed in the linkographs. For example, if the nodes formed in linkographs are too scattered, the design process is disorganized and not strongly structured. In Table 4, possible interpretations were given.

Table 3.6 Possible Interpretations in Linkograph (adopted from Goldschmidt ,1996; Kan & Gero, 2005, El- Khouly & Penn, 2013; 2014; Schön & Wiggins, 1992; Wiesberg, 1993)

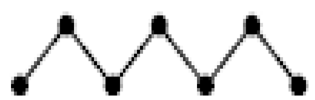
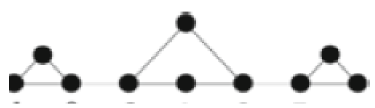
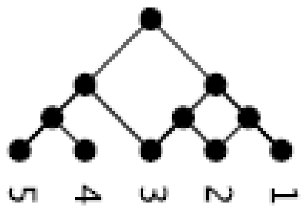
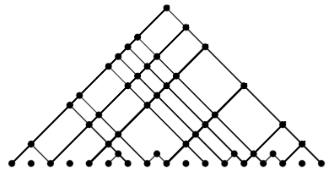
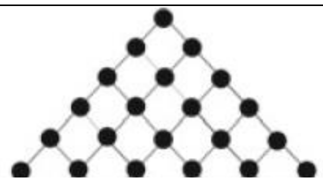
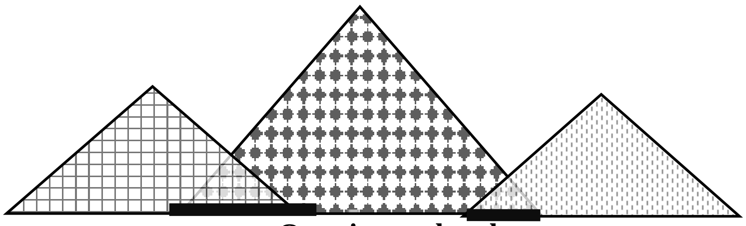
This type of moves shows that all the produces ideas are unrelated to each other, the discussion process is poor	
In this linkograph, moves mean produced ideas are related, and the idea process is ongoing but not rich.	
This move shows three different ideas and this kind of process designed may lose its focus, and there may be confusion.	

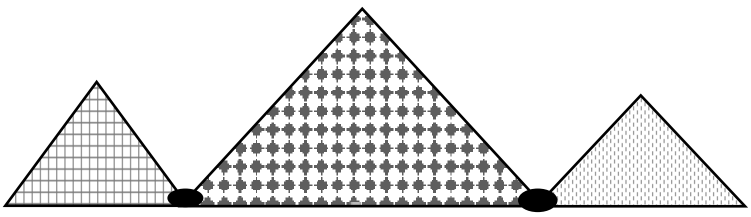
Table 3.6 (continued)

Move 4 is not related to Move 3,2,1. It is just related to Move 5. Move 2 is related to Move 3 and Move 1 but not related to Move 4 and Move 5. There is an idea production process. It can be considered there is a rich generation process.	
İdeas are produced on each other. There are some different ideas. There is a transition to a structured design process, but different design ideas may arise within the process.	
There are a structured design process and a design process based on an idea. It is an indication that the designer has worked on the same idea so far, developed that idea, and matured his idea.	

Another interpretation of the linkograph is about the sub-networks. Sometimes if different ideas are discussed, sub-networks can occur. These sub-networks also need to be interpreted. These networks are named according to the shapes they create overlaid interrelated, sparse, and intersected (El- Khouly & Penn, 2013).



Over interrelated



Intersected

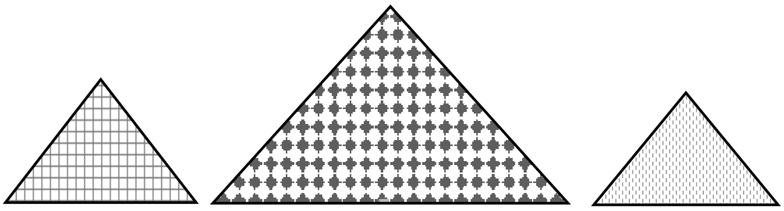


Figure 3.11 Possible relations between sub-networks Adopted from (El- Khouly & Penn, 2013, p.6)

The configuration between subnetworks in the linkograph may be in the following forms: “overlaid interrelation, intersection, or sparse” (El- Khouly & Penn, 2013, p.6). In Figure 3.12, possible relations are illustrated. In the figure, each discussed idea is shown in the form of triangles. That means there are three different ideas were discussed in the figure. To discuss each transformation of ideas and the pace of its emergence of new ideas in discussions may cause paradigm shifts. That means that without completing the discussion on an idea, the discussion may evolve into another idea. If transitions of ideas occur suddenly, the design process’s structure, the design problem, and the conceptual idea undergo a significant change that can be called over-interrelated. In this type of discussion, ideas are intersected partially. Suppose the ideas’ transition process is reconfigurable and can cause disconnection or bridge nodes in the model. In this sense, fast disconnects reflect a dispersed process differences between the ideas, while connecting them is a synthesis process.

In this study, what is required in the linkograph is that there are links between the design moves, and the ideas overlap in a single triangle. This situation shows that the concept to be taught is constructed in the current study linkograph was modified according to the learning process.

The current study segments are pre-service mathematics teachers’ discourses during the discussion, and their ideas about the mathematical discussion are linked. Since the duration of a lesson takes an average of four hours and the discussion contains not related parts about the integral, the relevant parts were taken while editing the segments for the linkograph. As mentioned before, the speeches were first transcribed for linkographs and created in segments in the excel file. Unlike the guidelines in Goldschmidt (2014, p. 42), I included the utterances such as “yeah,” “OK,” “emm,.” Although not in the sense of design, sentences like these would give me information about the student’s learning process and thoughts.

For this reason, I added such discourses without deleting them. In addition, unrelated discussions or conversations during the design process, off-topic discussions, repetitive speeches were removed from segments (Hatcher et al., 2018). While creating segments, the speeches’ order was used as in the original (Kan & Gero 2008; Goldschmidt 2014). The linking started from whomever the idea came from, and design moves were made depending on whether it is related to a prefix idea or not. Unlike design, this situation was taken into account in linking because learning is a process, and the student structures the subject in his mind in a group discussion. Considering the words used by the student, linking was made in the speeches that did not belong to the student in constructing the concept if they were related to the studetns’s learning process and if the student expressed this. In short, both the individuals and the content of the speech were considered. In this process, every word sentence has been examined to the finest detail. The incomprehensible or unclear parts were made by watching the video or by examining the reflection papers. Because generating links on the utterances is subjective (Goldschmidt 2014). Hence, a guideline was prepared considering the guideline made by some researchers to provide consistency (Kan & Gero, 2017; Goldschmidt 2014; Hatcher et al. 2018; van der Lugt 2005).

In order to be more objective, some criteria were determined and followed according to previous guidelines (Blom & Bogaers, 2020; Goldschmidt 2014; van der Lugt, 2000).

- Pre-service mathematics teachers relate to previous ideas while stating their own opinions.
- Pre-service mathematics teachers relate hand gestures, sketching, or scribbles with the previous utterances.

- Design moves happened sequentially, in the same chain of thought while developing a solution.
- The concepts can be used in the same meaning in different contexts.
- The links between moves involving the researcher’s multiple interventions can be encoded if the related idea between these actions is not related. However, the researcher’s utterances are not coded.
- In cases, an idea reappears multiple times in the transcript, unless a new element is added to create a new link in a later move, the chapters are linked only to the time the idea was first presented in that pre-service mathematics teacher.
- If there is confusion about connecting non-sequential links, make sure which idea is being constructed and link according to it.
- Since it is an educational linkography and learning construction linkography, related concept moves should be linked with the previous utterances.

	A	B	D	E	F	G	H
1	NUMBER	UTTERANCE	LINKS				
2	1	İlke: How about making cross-link connections?					
3	2	Funda:Well, would it be advantageous for us to do so?	1				
4	3	Betül: It should be something broad because if we make the straws longer than twice if the base is narrow, it will come out of the support point and collapse. (Funda drew a new shape)	2	1			
5	4	Funda: How do we proceed when we go second *floor (talking about the drawing of Funda)*	3				
6	5	Funda: We will cut the straws over there like this	4	3			
7	6	Betül: Then, how will it stand?	5	4	3		
8	7	Funda:Gaps base gaps base	6	5	4	3	
9	8	Betül: Assume that we have built the walls, extra loads will stay outside, I mean, think like a balcony, very unusable	7	6	5	4	3

Figure 3.12 A sample of Linkograph

In Figure 3.13, example moves can be seen. Excel file is generated as in the figure and linked. The first three utterances are related to each other asFunda and Betül responded to İlke. In the third line, İlke draws a new shape for the building, and the conversation evolves on this shape. While Betül talks about transitions rather than the shape of the building, Funda talks about shapes and transitions, and the focus of the discussion shifts to Funda’s drawing. Thus, while the first three lines are linked, with Betül’s question, the 4th line is no longer related to the first two lines. Since the speeches made after here are always based on each other, the lines 4-8th are related to each other.

As in this example, in this study, each line is linked by looking at the speech's content and who said it. Then this excel file was uploaded to the linkograph software, and the linkograph was obtained (Figure 3.14).

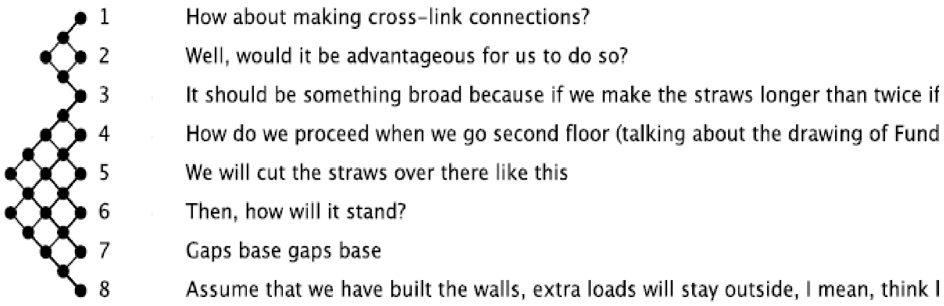


Figure 3.13 A sample of Linkograph produced in this study

The links were examined again in the linkograph. Linkographs were created separately for groups and individuals. At the same time, the focus of group discussions and individual speeches was determined. The linkograph was interpreted according to the results and the Sfard three-phase theory.

To ensure the consistency of building links, the second coder builds links between the utterances. At first, it was explained the linkographs and their usage. Moreover, a sample coding was done. A guideline was given. It was asked not to finish linking so a long time. Because in that way, she may forget data or the main ideas about the linkography to affect the reliability. Inter-rater reliability was calculated the same as the pre-post interview and found %80. We met online with the second coder and discussed the unclear points. At the end of the discussion, we reached a consensus on the disagreement links. Then I re-examined the data and built links two months twice. I found the intra-rater reliability % 94

3.8 Validity of the Activities

In the phase of development of activites, experts' judgment was taken. The researcher worked with two instructors in the mathematics department at METU whose undergraduate degrees were from engineering. Their Ph.D.'s were from the mathematics department, and the researcher also worked with one instructor whose Ph.D. was from the mathematics department and worked engineering department. While ensuring the trustworthiness of the activities, two situations, fidelity and verification, were taken into consideration. For Fidelity, the conformity of the activity to the nature of STEM education and its authenticity in the fields of science, technology, engineering and mathematics were examined. In other words, in the light of the STEM literature, it was examined whether the developed activity was suitable for the nature of STEM and whether it met the conditions of being a STEM activity. Also discussed is whether the context is developed appropriately. For example, while the canoeing activity was being developed, the activity was evaluated according to the conditions of being a STEM activity, on the other hand, the activity was sent to a research assistant from the Department of Naval Architecture and Marine Mechanical Engineering, and this research assistant was asked to evaluate the truthfulness and accuracy of the activity. The verification of the activity was provided by observing the person performing the activity during the lesson by the design team and filling and evaluating the observation form created by the design team watching the lesson. Throughout the development process, the researcher also sought opinions from different faculty working in different engineering departments (mechanical engineering, civil engineering etc.). According to those evaluations, researcher revised the activity.

3.9 Trustworthiness

In qualitative research, trustworthiness means determining the accuracy of the findings from the researcher, the participant, or the reader of a study (Creswell &

Miller, 2000). The terms for validity and reliability, in qualitative research in general researchers, refer to use credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985; Yıldırım & Şimşek, 2006). For establishing the trustworthiness of the study, the following was provided.

Lincoln and Guba (1985) emphasized that credibility refers to internal validity, which is the most important component to establish the study's trustworthiness and suggested some strategies to ensure credibility—triangulation of sources member checks, "prolonged engagement, persistent observation, triangulation (data sources, methods, investigators), peer debriefing, member checking, the reflexive journal" (p. 328). In this study, I used most of these strategies.

The first strategy that I used was prolonged engagement and persistent observer. I attended the course the whole semester and observed the pre-service mathematics teachers when they were working as in groups group and helped them when they asked for. Hence a rapport occurred between the participants and me. The persistent observation process provided me with insight into the nature of the students and classroom environment. Hence, I quickly understood the discussions and how pre-service mathematics teachers thought and its reasons. This observation helped me identify the pre-service mathematics teachers' general characteristics, which was very useful for me throughout the analyzing process. Another strategy was triangulation that I used. There are four types of triangulations; data sources, multiple investigators, theory, and collecting data (Cohen et al., 2000; Lincoln & Guba, 1985). In this study, I used multiple data sources triangulations such as interviews, field notes, observation of video and audio records, concept maps, reflection papers. These were also different ways of collecting data. That means I used data triangulation. Investigator triangulation was also used for data because there were two researchers during the whole process. These researchers observed the classroom and took field notes about the participants of the study. They share valuable information and ideas about the instructional design, pre-service mathematics teachers' thinking ways, their reasoning. The design team also support and lead me with their valuable ideas.

Moreover, I used member checks to enhance the credibility of the conclusions I draw in the interviews. After the interviews, I gave participants their written solutions and reflection papers. Moreover, I asked the participants whether they agreed with the ideas or not in those papers. To provide theory triangulation, I examined the pre-service mathematics teachers' concept images and constructed the integral by considering the theories of concept image and three-phase theory.

Another concept to increases the current study's trustworthiness is transferability, which refers to the study's external validity in quantitative research. In the design research classroom environment, activities are the generalizability components; characteristics of the classroom characteristics only belong to that (Gravemeijer & Cobb, 2006). Namely, "what is generalized is a way of interpreting and understanding specific cases that preserve their individual characteristics" (Gravemeijer & Cobb, 2006, p. 47). To ensure transferability, thick description and purposive sampling strategies were used. The selection process of participants, the purpose of the study, used instrument in the study was explained in the study. Hence, readers of the study could easily understand the data and had the opportunity to compare it with their studies.

Dependability is the third item to ensure trustworthiness, and it refers to reliability which is replicability and consistency in the findings (Lincoln & Guba, 1985). To ensure dependability, three research assistants who were also Ph.D. students from the engineering department, two research assistant and Ph.D. students in physics education, one research assistant and Ph.D. student from the mathematics department, and one Ph.D. student in mathematics education helped me during the development of the data collection tools, activities, and the design experiment process. I first send them activities and took feedback and revise the activity. I constantly consult them during the process. Moreover, I gave detailed information about the data collection process, reflection papers.

Moreover, how to code and link the data and instructions given to the second coder. This instruction involves linking the data and how long it should be ended since it can affect the dependability. The second coder coded linked one randomly selected activity and coded the pre-and post-interviews. After the process, and second coder came together and compared the codes. Coding and the linking process were nearly the same. Different links and the codes were discussed until we reach an agreement.

The last item to establish trustworthiness is confirmability which refers to objectivity. Confirmability means that by referring the raw data, making logical explanations about the findings of the study (Yıldırım & Şimşek, 2006) to decrease the researcher' bias. Lincoln and Guba (1985) suggest two strategies to ensure confirmability, confirmability audits, and reflective journals. In this sense, a Ph.D. student in mathematics education examined the data collection process and my inferences, and the consistency between conclusions drawn from the data. Moreover, during the study, I wrote a reflective journal in the whole process that represents how the process is going, what points did not work well, and what needed to be revised.

3.10 Researcher Role

During the design- experiment, the researcher was in the classroom and observed the pre-service mathematics teachers. Hence, during the classroom observations, researchers may take different roles "complete participant, observer as a participant, participant as an observer, and complete observer" (Creswell, 2002, p.213). Moreover, during the design experiment to provide consensus with the design team and conclude the classroom environment (Gravemeijer, & Cobb, 2006) researcher play an active role. Hence, the researcher was the participant-observer in the current study. That means the researcher participated in the group discussion at a minimum level. Mostly observed the pre-service mathematics teachers' discussion. The researcher participated in the group's discussion when the pre-service mathematics teachers asked to assist or clarify their construction process, which they could not understand from the video records.

3.11 Ethical issues

For ethical issues, necessary permission was taken from the Ethical Committee at METU (see Appendix E). Data was collected in an elective course so, in the first meeting of the course, students were informed about the study, data collection process, and the nature of the data to be collected. In this sense, detailed information was given to them, from the video camera's usage to interview questions, reflection papers. After explaining this process, all the students who were twelve people registered for the course. They participated in the study voluntarily. Therefore, they were reminded that they could withdraw from the study whenever they want during the process. After one week, a consent form explaining the whole process in detail (See Appendix D) was taken. Moreover, none of the pre-service teachers' names were used anywhere, and pseudonyms were used to analyze and report the data. All the questionnaires' results, interviews were confidential, and without the researcher, no one in the study had access to the data. Moreover, data collection tools were not used for grading pre-service mathematics teachers.

CHAPTER 4

FINDINGS

In this chapter, pre-service mathematics teachers' results were given case by case in the main sections. Before the engineering design activities were reported, their initial concept images were presented. Their developments through the activities and the final concept images after the activities were reported in subsections. In the first subsection, each participants' existing concept images were analyzed using conceptual questions, and their concept maps to assess the participants' existing knowledge about integral. Moreover, their verbal and written responses on integral during the pre-interviews for each participant were analyzed. In the second subsection, their discussions during engineering design activities were examined using linkographs for each individual. Moreover, using obtained linkographs, changes in their concept images about integral is analyzed and classified according to Sfard's Object-Process theory and supported by their reflection papers written after each lesson. In the last subsection, the participants' final concept images were reported using their verbal and written responses from the post-interviews.

4.1 Developmental Phases of Betül

In the analysis process of the pre-service mathematics teachers' initial concept images, pre-interviews were the primary data source, and the data were supported by conceptual questions and concept maps created by the participants on integral. After examining these data sources, concept images of the participants were classified according to characteristics given in detail in Chapter 2. It is found that pre-service mathematics teachers have three different types of concept images: partial-primitive concept image (PPCI), discrete concept image (DCI), and hybrid concept image (HCI). In this section, PPCI was presented, and other concept images were given in the related participant's cases.

4.1.1 Betül's Initial Concept Image

To characterize the concept image of Betül, using her initial conceptual test, her concept map, and the initial interview data, which was conducted before the course started. Based on the analysis of this data set and the literature about the concept image and mental model, Betül had a partial-primitive concept image (PPCI). That means Betül had partially developed a knowledge scheme about definite integral at a basic knowledge level separately. Namely, she divided the whole definite integral concept into sub-concepts, and she either ignored some parts or learned them as basic level knowledge or basic sub-concepts about integral. In case sub-concepts were inter-related, she mentaly formed weak relations and constructed those sub-concepts around the definite integral separately. Related to the definite integral concept, she composed her knowledge around "integral term," based on the area-oriented definition, and she had fundamental problems with determining variables.

The characteristics of her concept image are presented as follows:

1) *She explained integral in terms of geometric description "area under a curve" with a standard and straightforward definition.*

In her conceptual test, at Question 1(Q1), Betül defined integral as

The area bounded by graph of f and x-axis when $x \in [a,b]$

which is an area-oriented definition. Namely, Betül constructed the geometric interpretation of integral on a simple and common area representation. Besides, she drew its geometrical illustration and its symbolic representations, and see in Figure 4.1.

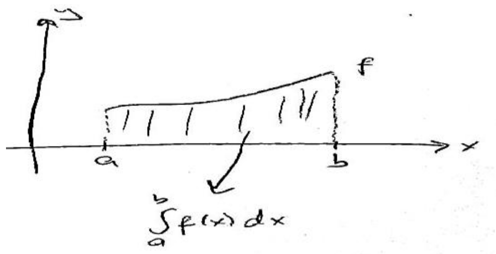


Figure 4.1 Betül's representation of integral

She applied her integral definition in all questions consistently. Question 2 (Q2) is a contextual problem about a stuffed gorilla that was dropped down from the top of the building. In the question, a speed vs. time graph was given. Q2 was asked to determine whether students know Riemann sum (a). In Q2, it was proposed that students find how long the gorilla traveled (b) and explain their solution strategy (c). It seems a simple question, however, unlike the usual questions, the function of the curve was not given, so it was not easy to solve the problem in the usual way and find a precise answer. She answered b and c items by referring to the area-oriented definition and expressed that *"... find the area under the graph (b); it is integral if we find the function of the graph (c)*". Such an answer is the multiplication of the speed and time gives the distance, and in their physics courses, they learned that area under graph gives a distance, and this kind of problem is solved by using integral. Betül was not able to explain it.

2) *She believed that the graph's function should be given to calculate the area under a curve. Otherwise, the area cannot be computed.*

In Q2, when Betül tried to solve part (b), she assumed the graph of the function was linear and found the answer by calculating the area of the triangle

I assumed that velocity increase linearly
and find the area under the graph. $\frac{11.\cancel{2}}{\cancel{2}} = 11\,m$ (about)

During her solution process, Betül did not mention Riemann Sum or partition; she only talked about finding the area under the graph. In the Q2 part (c), students were asked to explain if there is a way to find a more precise solution. Firstly, there are two purposes in this question: whether pre-service mathematics teachers know the Riemann sum, enhancing the partition gives a more precise result, and there should be a function for the integral. In this regard, in line with the aim of pre-service mathematics teachers' knowledge was investigated. She answered this question: *"yes, we can calculate its integral and find the function of the graph."* To clarify Betül's thought process, I asked this question in the interview, and she said that to find the area under the graph, *"I should find or know its function; otherwise, its area cannot be calculated."*

3) She set up the integral for prototypical examples correctly, and she calculated the integral quickly. However, she had difficulty in determining variables.

In the interview, it was observed that when a regular graph was given, whose variables were speed and time, and it is asked to find the position of car A at $t = 1$. Betül quickly answered it by stating, "area under the graph of speed vs. time gives us displacement so we can find it by integral." Then, Betül was asked to write a general form of integral by assuming that she knows the speed function of car A. Eventhough she could determine the boundaries of the integral, she had trouble in determining the variables and writing the integral form (see in Figure 4.2).

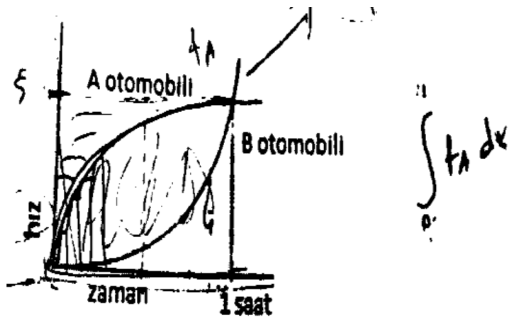


Figure 4.2 Betül's answer

Researcher	Knowing the function, how can we calculate the area?
Betül	Well, let us say this is $f\ (a)$. Let us say zero to one *(thinks a moment about writing dv or dt*). Hmm, *t* will be variable then.
Researcher	What does the function depend on?
Betül	I guess I do not know. This is a very meaningless thing for me

Researcher	Why?
Betül	I did not know because I am not sure about the variable. However, the time *(thinks.)* I said t I write 0-1 *(showing the boundaries)* is the time (*by speaking slowly and thinking*). We get the variable t . That is probably the variable. I guess it is time.
Researcher	ok, what if we want to find this area (*assume that transpose the graph and want to find (0, €ξ)*
Betül	I guess not taking time (refers to boundaries) I should write what the speed is there.
Researcher	ok, what will your function depend on?
Betül	What would it be then? *(thinks)* I think again, then the speed of the thing. I do not know.

In the interview Question 4 (Q4), $F(\_) = \int_a^x f(t)dt$ is asked to interpret the boundaries and what should be written in the parenthesis. Betül answered that function should depend on integrand's variable, and the boundaries should be constant. Moreover, when she is asked to explain dx, she said that it shows just according to what the given function is integrated and related to the derivative.

Betül	What is the variable? It is t
Researcher	Why?
Betül	Because the function's variable is $t,$when I integrate this, the variable does not change. But I think a and x are t- constant numbers
Researcher	Do you think they can be variables?
Betül	Maybe, I do not know it goes from x. Why can it be variable? I will write t
Researcher	Ok, I will ask another question if $\int \; xtdt$ is given, what can you say?
Betül	I will accept this x as constant and write $tx + c$ and dx show as variable
Researcher	Well. You said this dx is pointing to the variable. If I write this question like, Integrate the following question according to x and never write dx . Is it OKAY?
Betül	I think it is.
Researcher	Why?
Betül	So, because this means to get the integral according to x, they both say the same thing, so we do not need to write it.
Researcher	Do you think does dx is related to the derivative?
Betül	Yes
Researcher	How?
Betül	I will think it is a function. I think that inside of it *(refers to the integrand)* is a derivative of something. So, we take its derivative and do something like that. However, I cannot explain it right now.

4) Betül had constructed antiderivative and area under the x −axis concepts on procedural knowledge without understanding the reasons lying under them.

Another deficiency in her concept image is the interpreting area under the x-axis. She calculated the given integral quickly. Betül made explanations with her procedural knowledge. She knows that the area cannot be "negative," but she cannot explain why the minus sign is used in the negative area.

Researcher	Why did you write minus there?
Betül	Well, with that, I did it "positive." However, that minus confuses me
Researcher	Where does that minus come from?
Betül	Well, when the area is here *(under x-axis),* we multiply it by "minus." I mean, we take it with a minus.
Researcher	Why?
Betül	Because the area cannot be negative, however, I do not know why.

5) Betül could not transfer the integral knowledge into contextual problems.

In the integral questionnaire to identify whether the participants could apply definite integral knowledge in a contextual problem, they were asked to find how much work is done when an elastic spring, whose spring constant is k, is extended 4 cm. In this problem, Betül substituted the spring extension to the force formula, and then she applied it directly to the work formula directly.

$f(4) = k.4$ $\qquad$ $W = F.x$

$W = 4k.4 = 16k$

Similarly, in the interview, it was asked whether the work could be found by using integral. Betül stated that they are related, but she was not able to explain it. Although Betül said how to find the replacement by using integral, she could not explain in this question and could not transfer the same knowledge from Q1 to this question.

Researcher	What do you think about work and force?
Betül	Work. I guess they are related, but I cannot relate to them right now. *(Thinks)*
Researcher	OK, you think that they are somehow connected with derivative, or integral. Am I right?
Betül	Well, maybe *(try to link this question with the time vs. speed problem).* Well in that problem, I think we should find the area *(doubted)* work I am trying to use Force x Distance, however, I do not know.

Finally, in her concept map, which is created independently, she only wrote the pure mathematical terms and did not give any example about the usage of integral in real life.

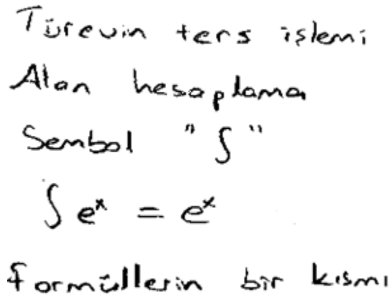


Figure 4.3 Concept Map of Betül

In a general sense, it was observed that Betül had trouble interpreting contextual problems in terms of integral. Moreover, she could not combine real-world situations with mathematics.

6) *She could not relate Riemann Sum and Fundamental Theorem of Calculus*

In the interview, integral related concepts were given, and if there is a relation among the concepts, Betül was asked to show and explain the relationships. She articulated that she did not know about the terms "*accumulation of average rate of change, infinitely small number, FTC and Riemann Sum*." She explained that upper and lower limits are the boundaries of the integral and combined them with integral. She related integral and derivative as the opposite of each other. Besides, she explained that the variable is the function's component, and its derivative and integral could be taken. However, most of Betül'statements of were trivial and lower-level knowledge sentences, which are typically used in Calculus courses.

Betül	What infinitely small? I have never heard of this term before. For the function, we take its integral, so we found the area, so they are related. Derivative gives us the rate of change. The derivative of the function at a given point. Hmm. Yes. It gives the rate of change, and I do not know the accumulation of rate of change. Integral is antiderivative, so they are related in this way.
Researcher	What about variables?
Betül	Well, a variable is the things that we use to express the functions.

It is seen that Betül constructed her concept image on integral and had no idea about Riemann Sum and FTC. She used her concept definition of integral consistently in every problem that she dealt with, and she did not mention the analytic definition of integral. Briefly, she has problems at setting up integral like determining the integrand of the integral and its variable and upon these problems in determining the boundaries of the integral, antiderivative, negative area, interpreting the symbolic representation of integral, dx and she does not know Riemann Sum.

Moreover, Betül's concept image was composed of sub-concepts of the definite integral, including calculation, antiderivative, areas below the x-axis, geometric definition of integral, geometric, and symbolic representation. Procedural knowledge based on applying the integral's formula and geometric description of definite integral has a dominant role in her concept image.

Considering this point of view, to provide a general view to the reader, Figure 4.4 presents a visual summary of Betül's initial concept image.

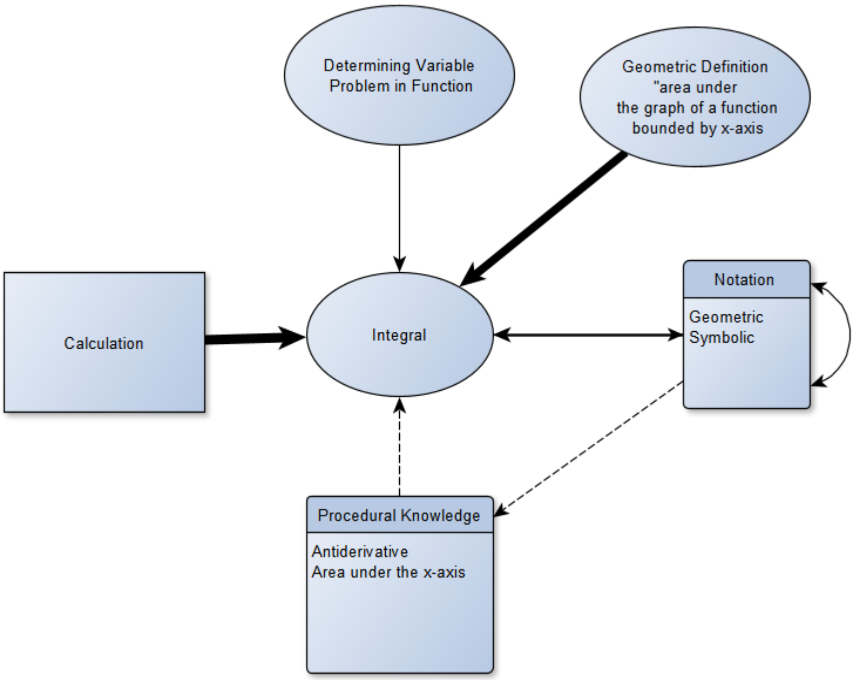


Figure 4.4 Betül's initial concept image

In Figure 4.4, while concepts in the circle are sub-concept of the integral or interrelated by themselves with integral, the written ideas represent smaller sub-concepts in any of the sub-concepts of the integral, and the rectangular shapes are related concepts that are necessary for the integral knowledge construction. Arrows show the relationship between the concepts, which concept feeds another one. The thickness of the arrows indicates the strength of the relationship between ideas. The relationship's strength is determined according to the frequency of their reasoning, their written and verbal explanations on the participants' related concepts when they solve the given task. Hence, being dashed or not shows that the concept is wholly or partially learned. In other words, the concept is represented by dashed lines if it is not fully mastered.

4.1.1 Concept-Image development of Betül about integral through engineering Activities

Developments in prospective mathematics teachers' knowledge about integral were presented into two main sections: developments in prospective teachers' knowledge about (i) Riemann sum (ii) Fundamental Theorem of Calculus. These two main sections are also categorized according to Sfard's "Three-Phase Theory," namely Object-Process Perspective, and it has three main phases: interiorization, condensation, and reification. According to gained pre-determined objectives for each phase, students' development process is categorized. Whether the objective for the related big idea about integral is gained or not is determined from each participants' linkographs obtained from group discussion discourse.

In the following sections, developmental phases of the pre-service mathematics teachers' concept images were presented through illustrative segments and individual linkographs from the group discussions engaged during engineering design activities. Linkographs were formed in three parts; links created based on each participants' contributions to the group discussion, themes formed by group discussion focus, and the codes created by the participant's contributions. These contributions can be a new idea or explanation about the given problem's design or solution through each lesson. The data from their reflection papers also supported the findings.

Their conceptual test results and their concept map, which is constructed on their own, written and verbal answers during the post-interview were used and presented in the next section to identify participants' final development.

4.1.1.1 Developments in Betül's knowledge about Riemann Sum in terms of Object -Process Perspective

As presented in the previous section, Betül's initial concept image about integral was problematic in three aspects. First, Betül had no idea about the Riemann sum.

She has procedural knowledge about functions, but she could not determine variables of the function when its domain and range are changed. Moreover, she could not interpret the given function or the function's graph in terms of its variables. The second major problem was that she constructed her concept image around the integral concept but had no idea about FTC. Hence, she was not able to understand the relationship between the derivative and the integral. Finally, to find the area under a curve, she needs to know the curve's function, and she could not connect mathematics with real-world applications. In this regard, Betül was not at the interiorization phase. During the application of engineering design activities, these difficulties that Betül confronted are taken into consideration, and appropriate feedback and scaffolding were given to Betül individually. Furthermore, Betül's knowledge about Riemann Sum was developed through the engineering design activities and reached the reification process since she started to interpret and explain the Riemann sum as a whole and built a connection between Riemann Sum and integral. All the development processes of Betül in interiorization, condensation, and reification process are considered remarkable events, and under that headings, noteworthy points were given.

4.1.1.2 Interiorization Phase

Betül's pre-interview data showed that she had problems determining the independent and dependent variables from given data, and she had trouble interpreting the variables when the region and the range of the given function are changed. Moreover, when the variables that can be less familiar were changed according to the problem's context, she treated the mathematical content of the problem as very different from the dealt one. For instance, let us assume that there are two functions; one of them represents speedas a function of time f(t), and the other represents the pressure as a function of height g(h). When she was asked to interpret and write the integral form of the problem, while she was able to interpret f(t) easily and write its integral form, she was not able to interpret g(h), in terms of when the height is changed and could not write its definite integral since she could not be able to identify the variables of the function. Moreover, Betül could not calculate the integral unless the function was given. She could not calculate the approximate area. using the area of the rectangles

Furthermore, she did not know the Riemann sum, and the most critical factor that hinders Betül's progression was difficulty in connecting mathematics with real life. In this regard, she is not in the interiorization phase. Before the first stage of the hierarchy, which is interiorization, students are in the lowest stages, become familiar with the concept, and have limited skills. In this regard, she did not know how to calculate an unknown irregular area by dividing a given interval into rectangles and calculating the area of these rectangles, adding them up, and knowing that calculating the area with the limited number of rectangles will not give an exact solution. However, by subdividing the interval to finer intervals and thus calculating the area of the rectangles a more accurate calculation can be gotten. There will continue to be a "next piece," that is, the accumulation is not complete. Throughout the engineering design activities, Betül began to conceptualize how to calculate the unknown irregular area with the rectangles.

Moreover, she realized how approximation changes the area and how to identify the variables from actual data in Activity 1 and Activity 2. In this regard, how and when Betül realized, conceptualized, and engaging through Activity 1 and Activity 2, how she met the interiorization phase's objectives was presented under the remarkable events heading. In order to be clear, the general group discussion was presented first.

4.1.1.2.1 Developments of Betül in Activity

4.1.1.2.1.1 Group Discussion in Activity 1

Activity 1 was about designing a canoe. In this activity, the participants learned about building the canoes from an engineering point-of-view and discovered the principle of buoyancy. Throughout this activity, the participants did not learn about Archimedes' principles and the engineering solutions components, which are shaped by the availability of resources. Students must calculate the best solution according to the data collected through the design activity and find the best solution; they should interpret the data correctly and find the best approximation for the solution. In that way, they can provide the consumer demands: their own constructed canoe should carry an acceptable proposed load without sinking, and the best canoe is the one that carries the most massive load, so they should make some assumptions as engineers do. To propose a load, they must calculate the buoyancy and use volume formulas for their irregular shape. They will also learn what the meanings of variables are and what dependent independent variables mean. Students will be able to calculate the area under a curve by using rectangles. Moreover, they must analyze the characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships. Students use this information to design, construct, and test their canoe model.

Activity 1 was for the participants to built their prototype first.Then they made the calculations, so they dealt with the actual data, and they move the concrete object to mathematical representation. In this way, they could build a connection between real-world and mathematical representation. Since all the participants have common difficulties relating to real-world application and finding an unknown irregular shape, Activity 1 is designed to overcome these problems. In this regard, at the beginning of each lesson, activity sheets are given to each participant and asked them to work five minutes individually. After that, they started to discuss what they understand from the problem and what they should do to reach asolution. Figure 2 indicated a variant of sparse and segregated ideas, like purple triangles and yellow ones. That means the participants may be lost during the process or focused on different ideas, which hinders their progress. The group discussion linkograph in Figure 2 showed that they had two types of discussion; design discussion was about designing the canoe, and mathematical discussion was about finding the maximum load the canoe could carry and its calculations. In the following linkograph, links were formed from the beginning of Activity 1 to the end of the activity and included all the participants' contributions until they finish the activity. Lines represented the links that the participants built during the activity. Triangles represented the discussion's focus. For instance, the yellow triangle represents that in that time interval, the participants' discussion centered around understanding the problem (UP), etc. Arrows showed the direction of the links. While design discussion means that the participants discussed the design, mathematical discussion means that they started to discuss their design's mathematical aspects. The codes under two main parts were the focuses that the participants dominantly discussed.

In Figure 4.5, it can be seen that during the design discussion, group focused on understanding the problem (UP). UP, shown as a yellow triangle, is separated from other links, and the others are intersected. That means the participants did not understand the problem well enough, such as what they will do or how they will do it, and they do not produce any ideas. Namely, during the UP part, they just read the problem and repeated what was asked. In this part of the activity, they mentioned: "they did not conceptualize the problem and did not know what to do." After that, they discussed the shape of the canoe. When deciding the canoe's shape, they had to consider the maximum volume so that it should carry the maximum load. It is seen that there was still a limited number of links that are showed with purple triangles, and they did not produce an idea, and all the produced ideas are separated from each other. For example, while Betül proposed it can be like a parabola, Funda proposed a rectangular shape, and İlke proposed a typical canoe shape with straight sides that she found through her internet search.

After deciding the canoe's shape, they discussed how to provide balance when they put the load in the canoe. From Figure 4.5, the focus of the group was buoyancy. As aforementioned before and seen from Figure 4.5, until they started to discuss the buoyancy, they were not able to link the ideas since the produced ideas showed purple, red, and yellow are separated from each other.

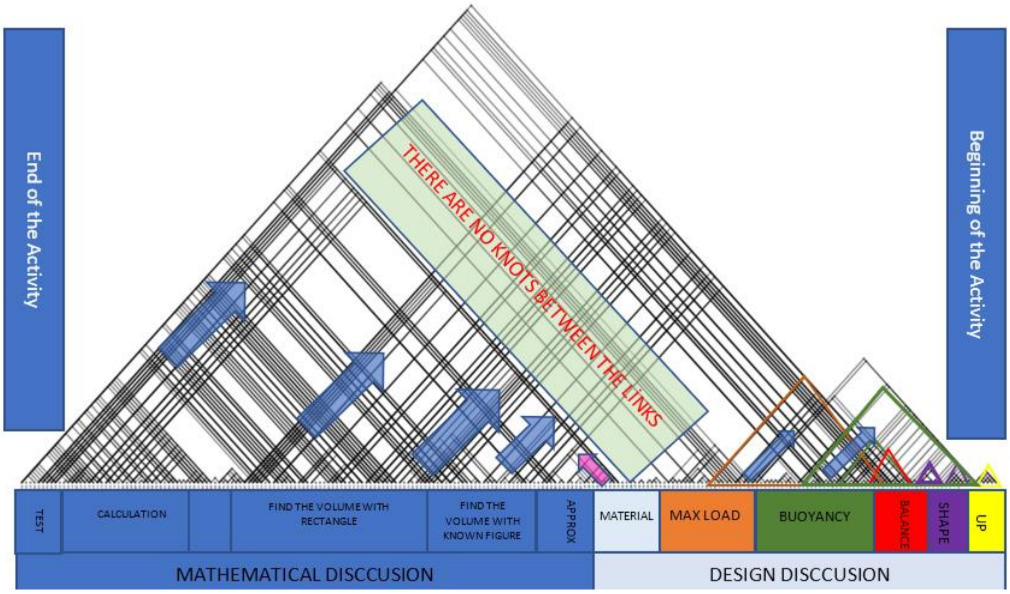


Figure 4.5 Class Discussion Linkograph (Linkograph 1) *UP: Understand the Problem; APPROX: Calculate the area approximately.

With the buoyancy discussion, they started to build links between ideas. It can be seen from Figure 4.5, green triangle represented the buoyancy discussion overlaps the other triangles, and the number of the link is more than the other link until here. Moreover, it can be seen that there are two green triangles and the links under them that means the participants discussed the affect of the different shapes of the canoes that they proposed on buoyancy (Figure 4.6)

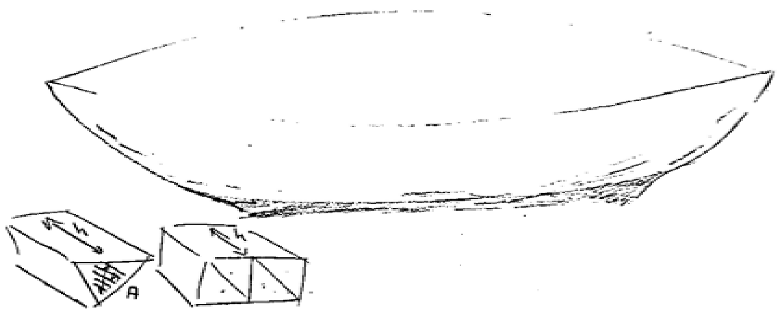


Figure 4.6 Proposed Canoe Shapes from İlke, Betül, and Funda

For example, in Figure 4.7, represented as green, red, and purple rectangles in the linkograph in Figure 4.5, Betül and İlke discussed the shape, buoyancy, and canoe's balance. After this process, they decided on the maximum load of the canoe. From Figure 4.5, since there are no links through the material code, they just took the material and designed their canoe. From Figure 2, it can be seen that they did not connect the whole process even though there seems to be a link between the mathematical discussion part and the material.

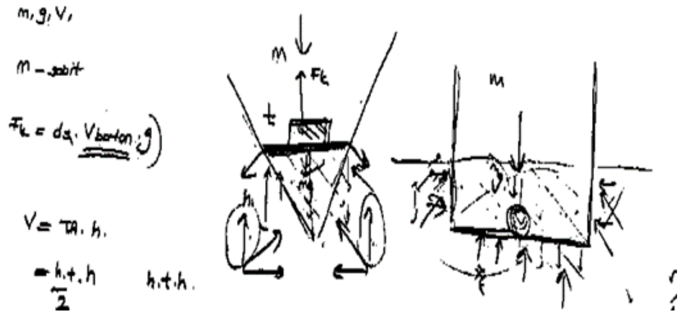


Figure 4.7 Canoe Shape and The Buoyancy Relation

In Figure 4.5, there is a remarkable increment in the mathematical discussion part that coincides with the number of links, which means preservice mathematics teachers built their ideas upon each others'. In Figure 4.5, while the blue arrows represent the backward links, pink arrows represent that they built the code's connection. For instance, in pink arrow after the idea of finding the area with approximation, the participants took into consideration that code and produced ideas based on approximation, or for blue arrows, in finding the volume of the canoe, the conceptualize that they will use the volume in calculating the buoyancy or they conceptualize that by finding the maximum volume they can calculate the maximum load, etc. In Figure 4.5, in the mathematical part finding the volume of known figures, calculations, the test is the other focuses of the group discussion . In the following section, how Betül met the interiorization phase's objectives, how and when she contributed to the discussion and conceptualized the objectives were presented.

4.1.1.2.1.1.1 Remarkable Events for Betül in Activity

In this chapter, important events that contribute to Betül's development were presented briefly for the following two paragraphs, then how she went through the activity was presented in each event was presented in a particular way.

In Figure 4.8, the amount of Betül's contribution to the discussion was represented by the number of lines throughout the activity. As seen in Figure 4.8, the number of links were minimal relatively to the links during the mathematical discussion part, and it can be seen that the number of overlapping links increased. That means, while she did not participate in the design discussion, and she had trouble understanding the group mates' discussions, she can contribute to the mathematical discussion and conceptualize what they were discussing. In the second part of the activity in Figure 4.8, by experiencing and connecting the activity with the physics, Betül started to build a connection between the concepts. Finally, until the experience code, Betül had backward links that mean she contributed to the process in a sequence. However, after experience code, she started to build backward links and red lines and produced forward links, yellow lines. Namely, this was the breaking point for Betül since she both understood the concept and produced ideas that helped solve.

Finally, at the end of the activity, they had to propose an acceptable load that the group's constructed canoe can carry. In this phase of the activity, which is coded as a physics connection and represented with purple lines, the participants had to calculate the buoyancy related to their canoe's parameters. This process was the second crucial moment for Betül, when she made a connection.

Before, the amount of her contribution to the discussion is represented by the number of lines extremely limited through design discussion. Her initial concept image revealed that Betül was having difficulty connecting real life with mathematics, which was dominant in her thinking process. At the beginning of the activity, she showed deep resistance to relating the real world with mathematics, and she could not to visualize the problem in her mind. She was not able to transfer mathematical language to the problem with mathematics. In Figure 4.8, there is a minimal number of links in the design discussion part. Moreover, Betül's discourse through the classroom discussion, which creates the links in the linkograph, is analyzed, and it revealed that Betül always approaches the problem mathematically. Namely, she discussed primarily mathematical points of view, and she consistently stated *she did know and did not understand what she should do to solve the problem in terms of design*.

İlke drew her canoe shape. Betül looked at her drawing:

Betül	I think something like a parabola
İlke	Where should it be like parabola?
Betül	Both sides of the canoe
İlke	Do you think the bottom of the canoe should be sharp?
Betül	Hmm, is it necessary? I do not know.
İlke	I think we can even do loads like this when we cut here, and I wonder if we can put the weight on the canoe's edges, can we balance it?
Betül	Hmm, is it necessary? I do not know. I do not understand it. Maybe the bottom should be wider.
İlke	I think I should be sharp as in the ships.
Betül	I do not know. By doing that, do you want to increase the surface area of the canoe? I cannot connect it with mathematics

İlke explains what the difference between prism-shaped and sharp-shaped canoe is.

Betül	I understand what you said, but this is not like a prism if we open this side as I do not know how much will sink. I do not understand it

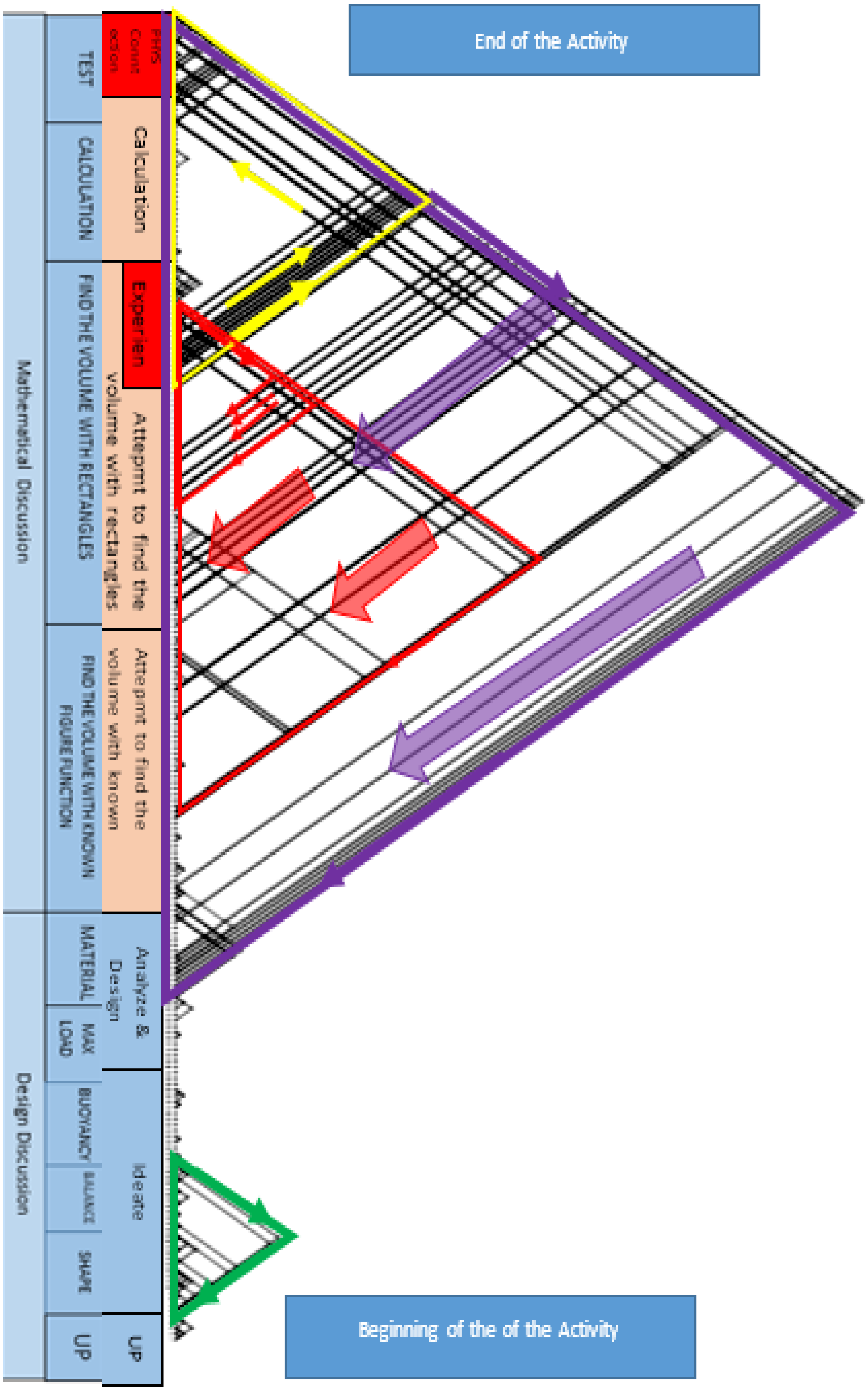


Figure 4.8 Betül’s Linkograph in Activity 1

As in the conversation, while İlke tried to decide the shape of the canoe, she constantly asked mathematical questions. In the design discussion parts, she tried to support İlke's mathematics ideas and wanted precise explanations. While İlke made general assumptions and directly wanted to try the prototype, Betül wanted to do it step by step using mathematical strategies. For example, when they were testing the prisms:

İlke	I think that both sank for 1 cm, for example, I can put weight here where buoyancy force will affect?
Betül	Why? How do you know that? We should calculate first. We should find two prisms whose surface area is the same, but the sides are different, then we should decide which is better.

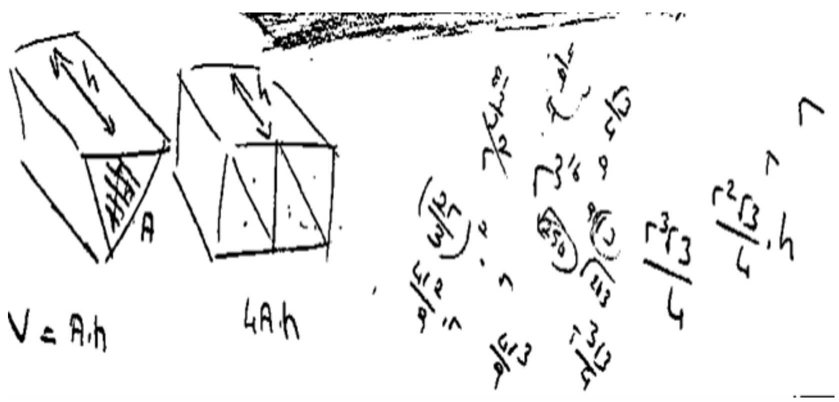


Figure 4.9 Betül's Calculations

Although Betül tried to prove her ideas as mathematically, she failed. What prevented Betül was that she thought the real canoe's measurement and its drawing were different, even if their measures were the same. Although the group friends said that the canoe measurements are 1:1, she strongly objected to her friends and insisted that they were different, and she stated that "they do not have the same size." Until here, Betül neither has any idea about calculating canoe volume nor linking ideas forwardly (See in Figure 4.8). In Figure 4.8, it can be seen that Betül participated in the discussion about finding the volume only four times, which is represented with green lines in the linkograph. Because their canoe is an irregular shape and its function was unknown, she had difficulty understanding Funda and

İlke when they talked about dividing the blueprint into rectangles. By calculating their areas and participation, she stated that (by referring the finding the volume with known functions discussion) “we know what the function is in there, but we do not know how we calculate that? I do not understand…”. This statement indicated that even though she understood calculating the area under a curve of a known function with rectangles, she could not apply it to the the unknown function situation. The reason behind that, when she drew rectangles under the curve, she knew that she could find the height and width of the rectangle by using the function’s equation. However, in their canoe, she could not recognize the picture and the measurements of the real canoe as 1:1. Hence, she could not conceptualize that she can calculate the area of rectangles by using the blueprint’s measurements.

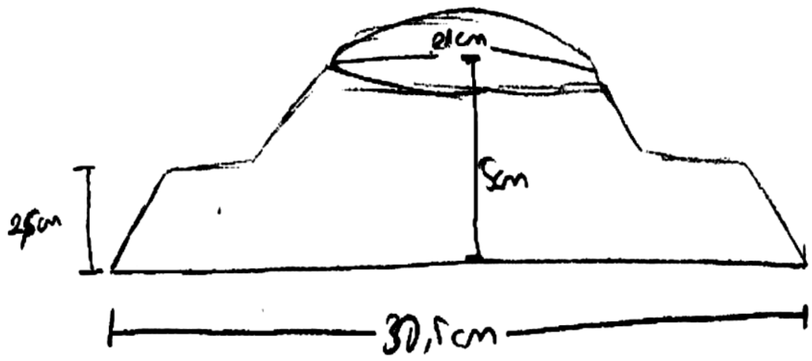


Figure 4.10 Blueprint and the measurements of the canoe

The breaking point for Betül was when she took canoe’s measurements herself, which is coded as “experience” in Figure 4.8. She visualized the canoe’s blueprint in her mind, and she understood how group mates drew the rectangles and how they placed the canoe into the coordinate system. The figure also indicates there are two types of links: forward and backward, respectively. Backward links, which were red lines, means that Betül made an explicit connection between discussions and conceptualized that she could find the areas of the rectangles drawn on the blueprint with the measurements. When Betül surrounded the canoe with iron wire and transferred its shape to a paper, she understood that they were working on a 1:1

scale. She stated that "*yes, OK, they are our real measurements…so we can find the area of the rectangles*," which indicates that she produced a backward link and understand the mathematical discussions about finding the canoe's volume. After this experience, she participated in the discussion process and made a forward link to determine how much load the canoe carries; she put ideas forward. In Figure 5, from the yellow triangle and the yellow lines, it can be seen that Betül produced more ideas to find out the final solution. That means she used existing knowledge to calculate the volume of the canoe. The following statements clearly showed that with the existing knowledge bases, she produced ideas represented as links in the related figure.

Funda	Let us measure from here to here; it is 5cm
İlke	If you measure like this, you are replacing the rectangles like this (in here, Funda thinks she is slicing the canoes vertically to the baset). The calculations will take time
Betül	we can divide the canoe into four and find the area under a curve ¼ then multiplying it with 4. We can find the whole area; then, we can find the canoe's volume by multiplying it with the rectangle's height if we assume that these four parts are approximately equal.

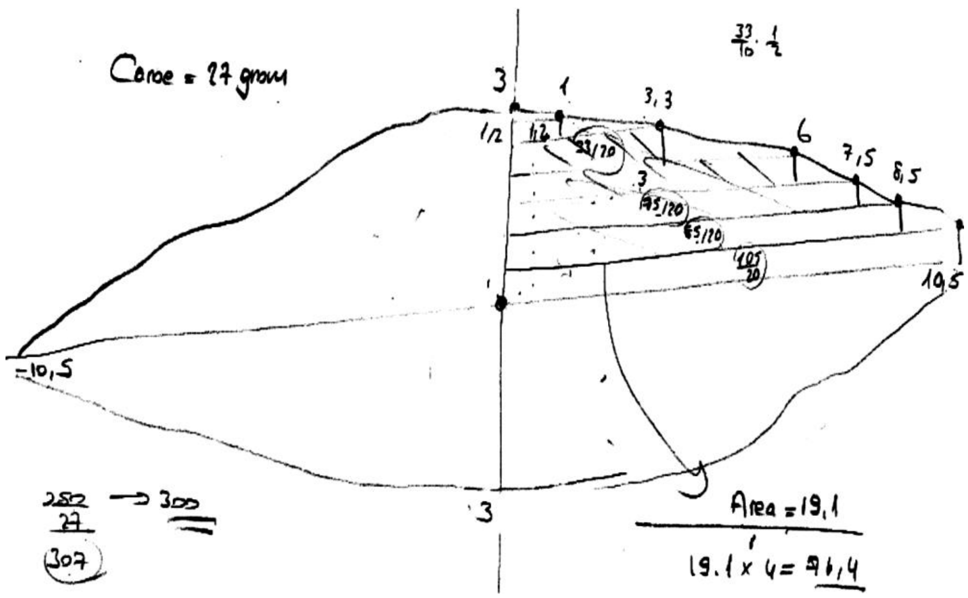


Figure 4.11 Betül's Solution to Finding the Volume of the Canoe.

In Figure 4.8, the purple triangle and the lines represent that Betül linked the mathematical discussion with analysis and codes, which is under the design discussion. The purple link in Figure 5 means that during the physics content discussion part, which took place at the end of the lesson, she made an explicit connection to issues discussed during the "*analysis and discussion*" phase. At the end of the lesson, the group calculated the buoyancy force and decided how many grams they will propose.

Betül stated that *"I think at first how much we want to make it sink then determine that height and calculate the submerged volume and then by multiplying it with the density of the water and g we can find the buoyant force. Hence this $F(k)$ equals the total weight of the canoe (She refers to total weight canoes own load and the load that they put on the canoe) if we subtract the weight of the canoe we can find the load that we will propose…*" which indicates she was making a connection to why they had to find the volume of the canoe and why shape and volume were necessary to solve the problem. In Figure 4.12, their calculations can be seen. In the figure below, their calculation of the volume of their canoe as 382 cm^3 and even though they found that their canoe can carry a max of 380 grams, they claimed that their canoe could carry 300 g. They proposed this weight because they did not want to make their canoe sink. Finally, their canoe carried 380 g in the test part, which is their canoes' max load. This result affected Betül because she experienced that real-world application of their mathematical calculations is worked, and she stated that "*wow, that is amazing, we found 380 g and the canoe carried that*".

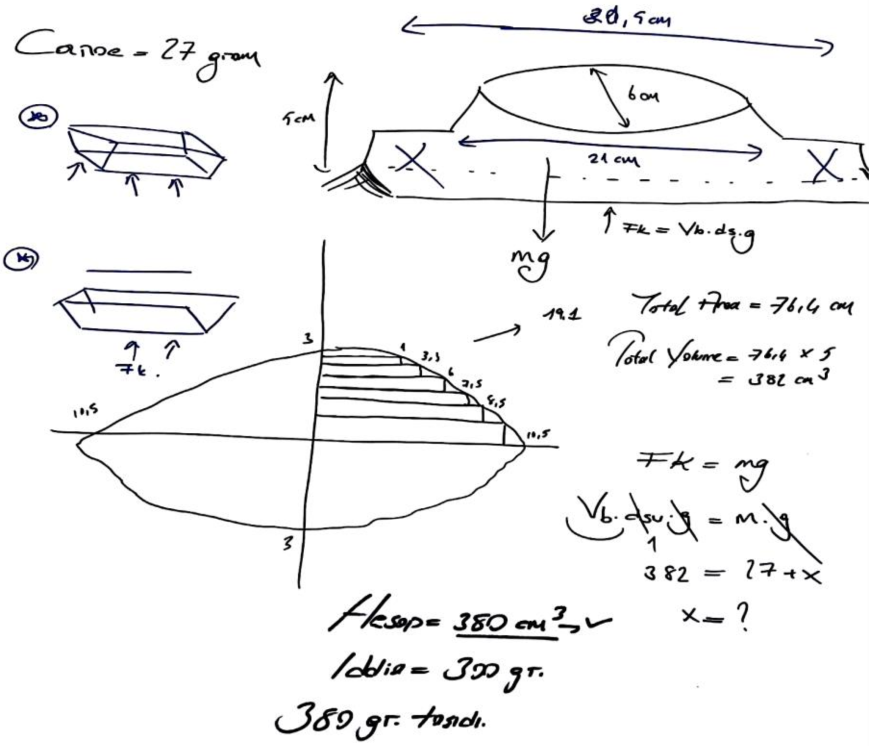


Figure 4.12 Storyboard of the group

By this last calculation, Betül’s visualization and connections between the real-world and mathematics were made more robust, and this helped her make the transformation between real-world and mathematics. She also stated this in her reflection paper as “this was the first time [I] used Riemann sum in a real situation. That is why it was hard to integrate the Riemann sum concept to the real situation.”

In brief, through the activity, Betül was able to calculate unknown irregular areas by dividing a given interval into rectangles, calculating the area of these rectangles , and adding them up and without knowing the function, with the help of the rectangles, thus approximated the area to be calculated. Moreover, she conceptualized that calculating the area with the limited number of rectangles will not give an exact solution, but by increasing the number of rectangular regions, a more accurate approximation can be calculated.

Since her difficulty in interpretation real-world situations such dominant items in the below could not be detected or hindered by this difficulty:

- Determine the independent and dependent variable from given data
- Be aware of the generalization of the process.

Betül's improvement process can be seen table below.

Table 4.1 Betül's Improvement in Interiozation Process

Interiozation Stage Objectives	***Reached***
Determine the independent and dependent variable from given data	-
Calculate unknown irregular areas by dividing a given interval into rectangles, calculating these rectangles' areas, and adding them up.	+
Without knowing the function, with the help of the rectangles approximated area can be calculated.	+
Start to be aware of calculating the area with a limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten there will continue to be a "next piece," that is, the accumulation is not complete.	+
Be aware of the generalization of the process.	-
Be aware left-hand side and right-hand side Riemann sum	-
Connect with the real world.	+

4.1.1.2.1.2 Developments of Betül in Activity 2

4.1.1.2.1.2.1 Group Discussion in Activity 2

In this activity, students are asked to design a new two-story building to be used as an art museum that would be one of the city's signature buildings. Since the building will be mainly used for art education and exhibits, they are also asked to design each story's base in the shape of a semicircle. The towers must remain standing for the simulated earthquake. Through Activity 2, students must build earthquake-resistant structures. While building the tower, they should know how to deal with seismic forces coming from the ground. Moreover, they should calculate the center of mass and the center of gravity to make their building resistant to earthquakes. Moreover, through the process, they learned how to transform their ideas into practical solutions by testing, evaluating, and modifying. The mathematical objectives were notating the Riemann Sum algebraically, setting up definite integral explain the relation between the Riemann Sum notation and definite integral

In this regard, at the beginning of each lesson, activity sheets are given to each participant and asked them to work five minutes individually. After that, they started to discuss what they understand and what they should do to reach the desired solution. The group discussion linkograph in Figure 4.14 showed that they had three types of discussions; design discussion was about how to design the building, the mathematical discussion was about finding the Center of Gravity (CoG) of the building and its calculations, and the design part was about the how to design the building. The codes indicate that the discussion part's focus was UP, appearance, mezzanine floor construction, second-floor design, the whole appearance of the building, making the prototype of their building.

In Figure 4.14, links were more overlaid and interrelated, contrary to Activity 1. More specifically, in Figure 4.14, while from the beginning of the activity through the first part of the mathematical discussion, the lines generated from the group were either intersected or sparse, most parts of the mathematical discussion and the design part were segregated.

That means participants started to associate the ideas, and the ideas were more segregated than Activity 1. Namely, their ideas started to overlap. Moreover, the participants made more efficient brainstorming, and during the mathematical discussion part, they constructed their knowledge on each other, and they affected each other. In Figure 4.14, it can be seen that the red triangle refers to the mezzanine floor construction discussion, the yellow triangle belongs to the appearance discussion, and the blue triangle refers to UP are overlaid interrelated that means they started to produce the ideas on the base of the problem. After that, they discussed the building's appearance since it was to be construct from semicircles, and the building should be eye-pleasing so, they tried to decide how to use those semi-circles. Inside the yellow triangle, different links mean the group produced different ideas like just putting all the semi-circles on each other facing in the same direction or different directions or putting them like a flower, (see Figure 4.13).

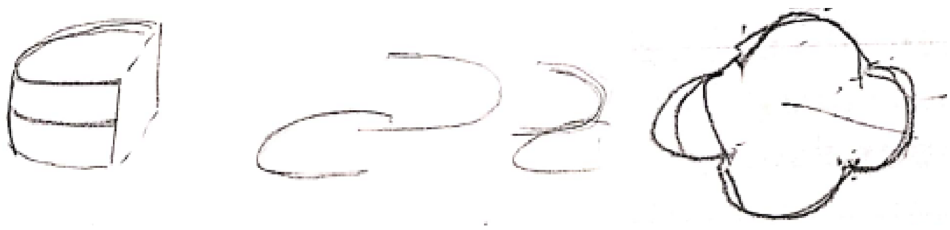


Figure 4.13 FBI group discussion about the appearance of the building

Then, they discussed how to construct the mezzanine floor, how to put the straws and how many straws they should use. In Figure 4.14, many links mean the participants contributed to the discussion based on the previous idea and proposed a new contribution to the existing one. For example, in this time interval of the discussion, while İlke focused on providing the balance of the building with straw, Betül focused on the building's strength with straws. Moreover, in Figure 4.14 red triangle intersects with UP and appearance. The participants also considered criteria in the problem and the mezzanine floor and constructed their ideas on each other.

Finally, to have more ideas, they constructed a prototype and decided how and where to put the straws. There was a slight transfer from to design part to the mathematical part. In Figure 4.14, the links from triangles to rectangles refer to this process. Because they started to discuss CoG to get the more durable building to the earthquake and discussed changes in placement of CoG during the earthquake (see Figure 4.15), and discussed how the displacement affects the building. Hence, they conclude how to construct their building based on CoG.

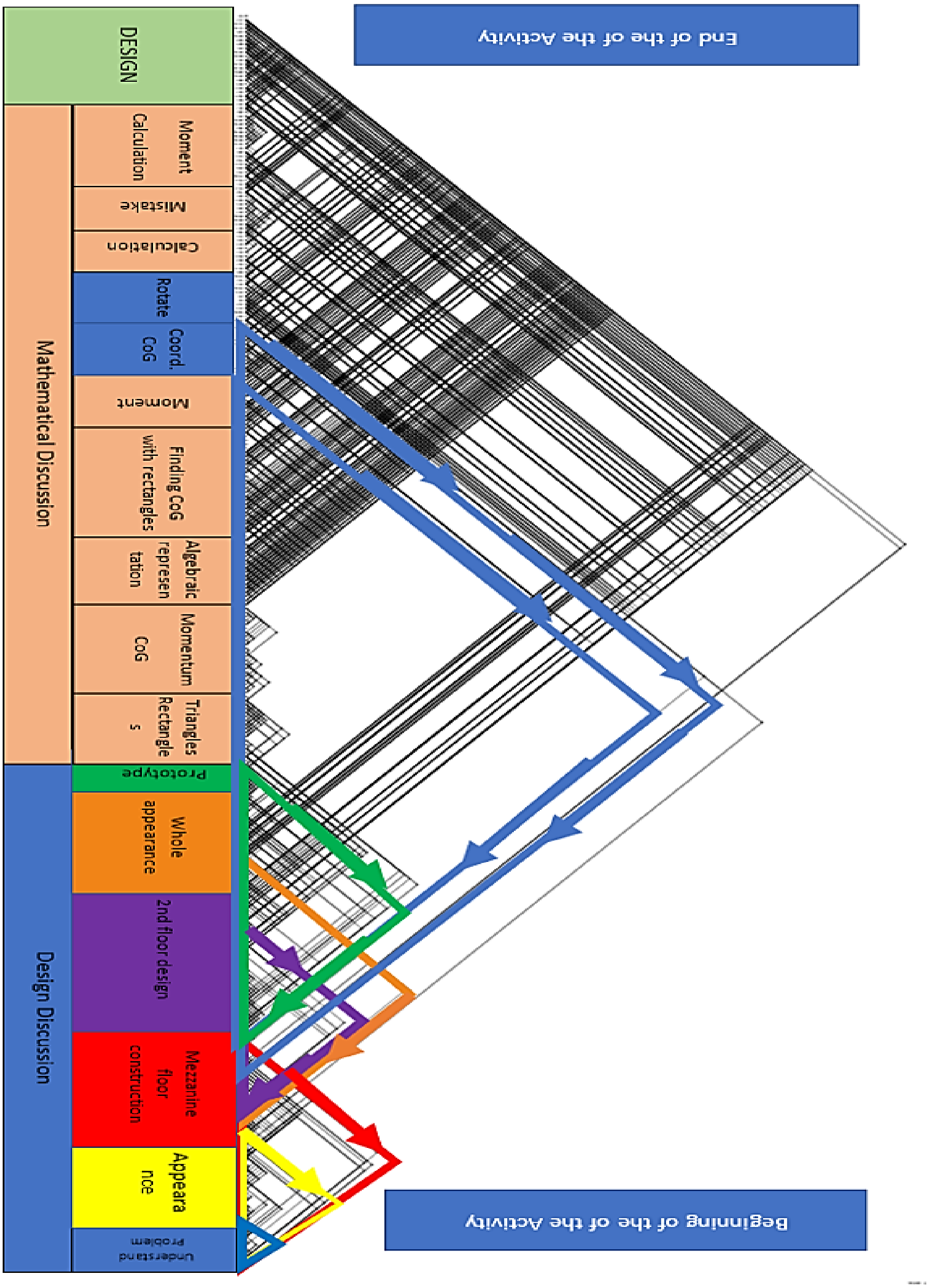


Figure 4.14 Group Discussion Linkograph in Activity 2

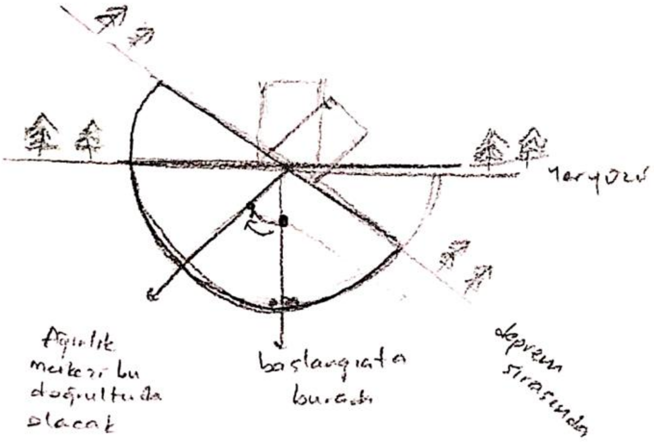


Figure 4.15 Group’s discussion about CoG

In Figure 4.14, even though there is a slight transfer, the participants’ contribution is limited and intersected. Namely, they could not successfully express the process mathematically since they could not connect the process with a mathematical aspect. As can be seen from Figure 4.14, the links between the triangles & rectangles codes and momentum & CoG codes just intersected, but were not overlapping. Hence the participants as a whole group did not connect physic and mathematics; however, after the algebraic representation, they understood how to notate, and they started to connect physics, design, and mathematics. After they decided how to calculate the moment and the CoG of each semi-circle, they realized how to put the second floor to provide the projection of each CoG coordinate. Moreover, in this way, they produced links from mathematical discussions to design discussion.

In Figure 4.14, generally, in the mathematical discussion part, there is a remarkable increment and coincident in the number of links; that means in this part of the discussion, the participants built their ideas upon each other since the time elapsed from left-hand side to the right. In the following section, how Betül met the interiorization phase’s objectives, how and when she contributed to the discussion and conceptualized the objectives were presented.

4.1.1.2.1.2.2 Remarkable Event in Activity 2 for Interiorization Phase

In this chapter, the summary of important events that contribute to Betül's development was presented then for the interiorization stage, what she achieved was presented in a particular way.

According to developments and the critical concepts that students could not conceptualize, Activity 2 was redesigned. Since the nature of the task did not consist of just calculating area under a curve, they could not use their previous abstracted knowledge. It was aimed in this activity that they should construct their learning instead of using prior constructed experience. Thus, all activities were designed to help pre-service mathematics teachers to think in different ways. Betül discussed this situation in her reflection paper as "*I have already used the method Riemann sum, but it was used in a different situation in this problem. They have the same logic because, in both, we are finding the area, but I had not seen Riemann sum being used to find the total moment.*"

In Figure 4.16, the amount and nature of her contribution to the discussion are represented by the number of lines throughout the activity. As seen in Figure 14, the number of links increased in Activity 2, relative to Activity 1. As in the previous activity, the number of links in the mathematical discussion part is higher than in the design discussion part. Also, the number of overlapping links increased in both parts in comparison with Activity 1. That means she started to participate in the design discussion, and she was able to produce ideas and construct the ideas on each other. However, there are missing areas in Figure 4.16 during the design discussion part. Those missing areas mean that either Betül did not understand what the other participants discussed or constructed the concept in those gap intervals.

In the second part of the activity in Figure 4.14, by experiencing and connecting the activity with the physics, Betül started to build a connection between the concepts. Before applying the formula in their semicircle, which will be the floor in their design, she could not conceptualize what the formula meant and why they had to find the CoG. Then she started to interpret the formula and realized what each notation in the formula corresponded in their semicircles. As a result, she realized that they miscalculated the CoG, which was coded as a misunderstanding. However, after the mistake code, she started to build backward links, which are red lines, that indicate she contributes to the process in a sequence, but she also produced forward links, which are green lines. Namely, this was the breaking point for Betül since she both understood the concept and produced ideas that helped solve the problem. In general, during Activity 2, the development process of Betül was presented. In this regard, after a brief reminder of what Betül achieved in Activity 1, how she made progress during Activity 2 in terms of the interiorization phase is presented next.

At the end of Activity 1, she was able to calculate unknown irregular area by dividing a given interval into rectangles and calculating these rectangles areas and adding them up and without knowing the function of a curve, calculating approximated area with the help of the rectangles and finally, she was able to build a connection between real-world and mathematics. From her initial concept image, it was known that she had problems determining the independent and dependent variables from given data. However, in Activity 1, whether she achieved this objective could not be clearly detected. Moreover, since her difficulty in relating the real world with mathematics was so dominant, she could not connect two processes that they discussed during the activity until the end of the mathematical discussion part. She could not realize calculating the area with the limited number of rectangles will not give an exact solution. However, by increasing it, the more accurate calculation can be gotten; there will continue to be a “next piece,” that is, the accumulation is not complete. Hence, she could not be aware of the generalization of the process. As a result, she did not achieve all the determined objectives of the interiorization phase, and she could not pass the condensation process.

Even after overcaming the difficulty in connecting real life with mathematics, she was still so naïve and needed some practice to strengthen the connections since there is still a limited number of links in Figure 14. Both the idea and the situation was new; she was able to talk about balance, finding the center of gravity. As seen from Figure 14, at the beginning of the activity, her participation process synchronized during the UP; however, she could not contribute much. She was still trying to understand how to construct a different building with semicircles and could not produce any ideas, and she stated this as "all of the layers are semicircle, how can we change the design of the building?" After other group members gave examples like a flower or different kinds of shapes and started to discuss the appearance of the building or constructing a mezzanine floor, her focus shifted to the building's strength. This shift is represented as yellow triangles and blue triangles in Figure 4.16. As seen, they are overlaid, which means Betül started thinking about strengthening the building to fulfill the problem's criteria and learning how she should approach the problem.

	Participants were talking about the mezzanine floor
İlke	How about making cross-link connections?
Betül	Well, would it be advantageous for us to do so?
Funda	It should be something broad because if we make the straws longer than twice if the base is narrow, it will come out of the support point and collapse.
Betül	How do we proceed when we go to second *floor (talking about the drawing of Funda)*
Funda	We will cut the straws over there like this
Betül	Then, how will it stand?
Funda	Gaps base gaps base
Betül	Assume that we have built the walls, extra loads will stay outside, I mean, think like a balcony, very unusable
Funda	But aesthetic
Betül	There are gaps here. How do we support those gaps? Most of the load is on them?
İlke	We will support them with straws as follows (draws on the figure) to distribute the building's weight equal to the floor. As we will construct something like this (Figure 4.13)
Betül	For example, tell me that one will come from here and one from here will come here, the flat part of the other will come to the rounded part of the other so that it will not overlap, or there will be gaps at the bottom of the top. It makes sense when one floor will come from here, and one from here will come here, the flat part of the other

	will come to the rounded part of the other so that it will not overlap, or there will be gaps at the bottom of the top, It does not make sense when you do. For example, if the flat part is more massive in the case of balance load, one balances here and one balances here, that make sense, but does the rest of the space cause trouble?
İlke	We support those with straws in those space
Betül	Think about a building in space. Why would there be a column in space?
İlke	We can draw triangles inside the semi-circles. Can it be related to CoG?

This sentence of İlke affected Betül's ideas, and while Betül group members were trying to decide how to put the second semicircle, she focused on finding the CoG. She stated this situation "*May the CoG be in the center? If this were a circle, CoG would be here (pointing out the center of the circle); however, we omitted half of it, then it will shift here. So, I am saying that when we place the mezzanine floor so that the third floor is superimposed when it is like this, all of them come to equilibrium*."

As a result, this process helped Betül to build a connection between the real-world and mathematics. In Figure 4.16, it can be inferred that Betül could not connect design and mathematics in the middle of the discussion. At the beginning of the mathematical discussion part, she started to construct the Riemann Sum and tried to connect it with the previous one. She stated this process as "*we should find the CoG of the semi-circles, but we should find it using rectangles, but I do not know how we can do it*."

In Figure 4.16, it can be seen that she started to link different ideas. Betül construction process formed step by step. At first, she tried to relate mathematics and design naively. Because when the group mates were supposed to find the exact coordinate of the CoG, she was not able to produce any ideas. Even she overcame the difficulty in connecting the real-world and mathematics; it was still preventing Betül's ideation.

Moreover, she was also affected by Funda, whose concept image constructed on assimilating previous knowledge into the new situation rather than seek a different solution. Hence, Betül could not break the cycle in the ideate part. She thought she could find it from rectangles, but she could not connect it with CoG.

Funda	Can we use the diagonals to find the CoG? Oh, this has no diagonals, right?
Betül	You cannot fold it. There is one axis of symmetry. Can we use rectangles like in the previous lesson?
Funda	Let us find the center of gravity of this one from the triangles we talked about first.
Betül	We are supposed to find the most accurate coordinates
Funda	We can find their CoG then finds semi-circle's.
Betül:	OK, try it then.

As seen from the conversation, even she started to produce intuitive ideas about the Riemann Sum, and Funda prevented her.

Between the codes Riemann Sum Math (RS/Math) and relate RS with CoG, Betül could not connect two processes and conceptualize what she should do to solve the problem. It was difficult for Betül to establish that connection from practice to theory because aforementioned in the first activity. This activity was a little more intertwined in terms of practical and theoretical aspects. Thus, a smooth transition is aimed from practice to theory. In this regard, it is also aimed that it was intended for Betül to visualize the building, imagine where the center of gravity would be in their building, and then express the center of gravity mathematically. In this sense, it was seen that while she was thinking about design, Betül also made a connection between design and physics at first, and then she made a connection between physics and mathematics due to the calculation of the CoG.

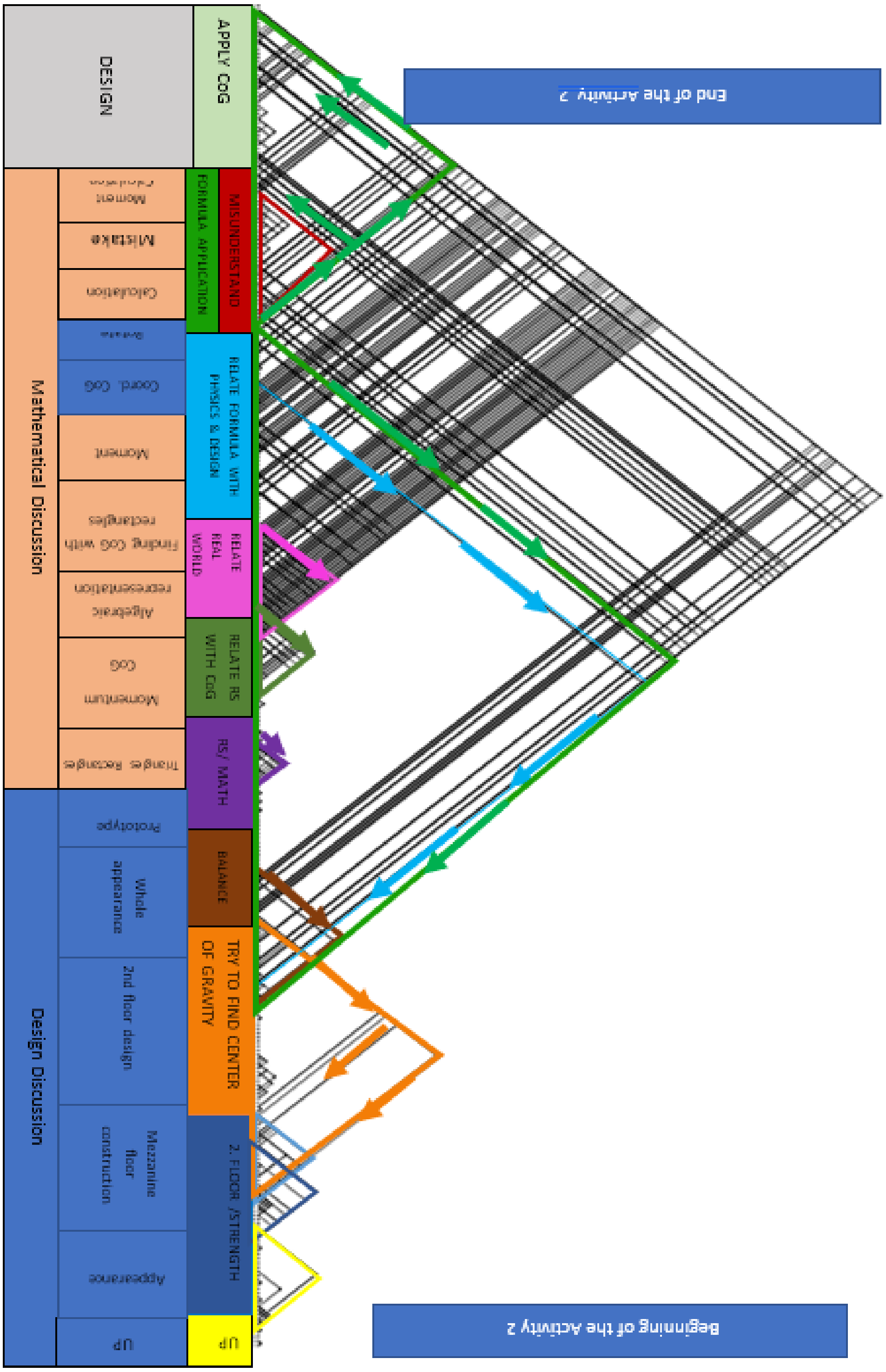


Figure 4.16 Betül’s Linkograph in Activity 2

This process is represented in Figure 4.16, as orange triangles, which are the links indicating Betül thought about trying to find CoG, intersecting with the dark blue triangles, which are the links about the second floor and the strength of the building that means Betül made a connection between design and physics at first then light blue triangles, green triangles, and the orange triangles are also intersected that means Betül build a connection between mathematics and the physics (see in Figure 4.16.) In this connection process, Betül started to determine the independent and dependent variables. Because she built a connection between design physics and mathematics, then she realize which variable in the design part is corresponded to mathematical representation. Thus, she was able to start to interpret the mathematical formula according to the design point of view. As a result, she was able to determine the variables in the activity.

Funda	We need to find the center of gravity so that it does not rotate.
İlke	Yes exactly
Funda	However, where will it return from?
İlke	Our issue now is to find the exact location of the center of gravity in our semi-circle.
Betül	I think it will come out of this *(examining the drawings and CoG formula)*
İlke	Just have this as an idea *(rotating each semi-circle on each floor)*
Funda	I still do not understand how to place a semi-circle?
Betül	I think, when a line cross here, it seems to balance it from this place.
İlke	I also think it is symmetrical from here
Betül	It does not matter; even if we think that it will balance each other as if it went from the middle of the semi-circle. *Haaa (understand aha moment),* we will place the straws according to CoG.
İlke	How do we find it?
Betül	From rectangles.

She understood which variable depends on the other and how that affects the design when it changes. As seen from Figure 4.16, the triangles in her thinking process started to overlap, which means she began to think about design and produced ideas about applying to solve their problem. On the one hand, she put forward ideas; on the other hand, she produced new supporting ideas supporting as well as problems she percieved with the design or mathematics. For example, in the dark blue triangle, while she talked about designing the second floor and designing a sturdy building with a red triangle, she tried to find solutions to the first arguments, such as using triangle connection between levels or dividing semi-circle into triangles. In addition to this, she also tried to find the coordinate of the Center of Gravity (CoG) by using triangles. That means she started to connect different ideas with mathematics; however, it also can be seen that these connections are still fewer and need to be improved.

When all these are considered together, it is seen that Betül is making progress. She reached all the objectives in the interiorization stage and passed through the condensation process. The contributions and the links during the mathematical discussion part were related to the condensation process. Their comprehensive explanation was given in the following section.

4.1.1.2.2 Condensation Phase

- Sfard (1991) described this phase as "squeezing lengthy sequences of operations into more manageable units." Moreover, she emphasized that the learner can be more skillful in thinking about the given concept in this stage. She resembled this process to a computer program that works with input-output relations instead of showing all detailed processes. That means learners do not go into details during the processes. In this step, a new concept is "officially" born when the learner confronts any difficulty that provides a new entity. In this step, learners may also connect the current process with another process and compare it within old and new knowledge and start to generalize it. Also, learners started to use different representations. In this regard, the study participants should be able to deal with various representations of the Riemann Sum, such as writing algebraic form of Riemann Sum by using their graphical representation or vice versa. They also should be able to handle the whole process with an awareness. For example, in this process, while the pre-service mathematics teachers are drawing rectangles on graphs, taking into consideration the variable of the functions, they should also explain the type of Riemann Sum accurately, differences between different types, and the margin of the error. In this process, pre-service mathematics teachers should also be able to:

- Understand that adding new increments continuously yields accumulation
- Determine the accumulation of the changes according to the independent variable.
- Understand the reasons for estimating the exact accumulation of changes that occur from tiny changes in the independent variable.
- Grasp that the quantity which is accumulating has a multiplicative nature.
- Write and interpret the product of $f(x_i)\Delta x_i$
- Flexibly use various representations.
- Differentiate and apply left-hand side and right-hand side Riemann sum

Taking into consideration these objectives, Betül achieved all of them at the end of Activity 2. For example, in this process, she was able to draw rectangles on the graph of the function by taking into consideration the variable of the functions, and according to left, middle, or right-hand side Riemann Sum, she was flexible at transferring graphical representation to algebraic representation or vice versa, etc. In the following section, how Betül met the condensation phase's objectives through Activity 2 was given, how and when she contributed to the discussion and conceptualized the objectives were presented.

4.1.1.2.3 Remarkable Events in Activity 2 for Condensation Phase

The remarkable points for Betül were coded in Figure 4.16as RS/Math, relate with CoG, relate with the real world, relate formula with physics and design, misunderstanding, and formula application.

Betül's development process through RS/Math, relates to CoG, relates to real-world codes that can be assembled to a preparation phase. In that time interval, Betül's focus was not exactly on the Riemann Sum. She was trying to understand the process and seek a solution to the problem. Since her links in the linkograph (Figure 14) do not overlay interrelated, she built a limited number of links; however, after the relationship with real-world codes that overlap with scaffolding with the researcher, she started to get familiar with the mathematical concept. She constructed the small pieces, then combined them during the formula with physics and design code; but could not build forward links (see Figure 14). After experiencing the preparation and familiarity phases, she was able to move to the condensation process. These two processes were like a bridge that involves the characteristics of the interiorization and the condensation phase. Betül was able to move to the condensation phase towards the end of the mathematical discussion. Until that time interval, Betül connected to design, mathematics, and design. After those connections, Betül was ready to learn the new concept.

The key points that help Betül to learn the concept were the discussion made while formulating the formula with the researcher and the mistake they made when implementing the extracted formula in their designs. The first one is that the phase before, which is the familiarity phase, and the second one is the condensation phase. The indicator of which Betül has passed to the condensation phase is finding the calculations and interpreting the formula by comparing the extracted formula, the formula that they applied to their designs, and their designs.

The first key point for Betül was that she needed some scaffolding about connecting Riemann sum with CoG. At that point, rather than teaching Riemann Sum at first and combining it with the CoG, the instructor started to scaffold CoG first and started to ask why it is essential for the design and make the participants think about using Riemann Sum to solve their problem. Beginning like this was essential to help Betül break her difficulty in connecting mathematics and the real world. Thus, she could relate the Riemann Sum with the real-world and start thinking about its applications. This process in Figure 4.14 is represented with dark green, pink, and light blue triangles and links. The scaffolding of the researcher to Betül was given in the following:

Betül	How to find the center of gravity from here? I cannot understand
Researcher	Ok, if I draw a circle, where is its the center of gravity?
Betül & İlke	Right here center of it
Researcher	So if I cut it in half, where does the center of gravity move?
Betül & İlke	Right up here, this line
Researcher	Well, then my center of gravity is on this line, and the problem is "what are its exact coordinates," right?
Betül, İlke & Funda	Yes
Researcher	Well, I assume that here is the origin so that the center will be (0,0), but what will be the coordinate of the center of gravity?
Betül, İlke & Funda	$(0, y)$
Researcher	Telling just on the y-axis is enough, I think?
Betül	No
Researcher	So why do I have to find it?
Betül	Because in that way, we can also support our building from its CoG, and it will stand the earthquake, so I need to find the y coordinate.
Researcher	So what should I find first?
Betül	Its moment
Researcher	Yes, we have to find the moments of the system. What else?
İlke	Its mass
Researcher	Yes. So how can I do it? How can I find its mass?
İlke	We can measure it
Researcher	Yes, let us start over here. Let us assume that we know the mass, so how can we find the moment? *(Waits a bit for an answer, but theyt did not answer).* Ok. I saw you drew some rectangles, Betül. Do you have any idea how to do it? *(Waits a bit longer).* Ok, let us start from here. If we think of each circle as a plate, each will be its mass center. Another variation would be to put a weight on the center of each plate's mass. I can find the moment based on the equilibrium point of these weights and find the coordinates of the

	equilibrium point dividing the total momentum found in total. How do you calculate the center of gravity of just this rectangle?
Betül	I can accept the weight of the given object as if all the weights were collected.
Ilke and Funda	Yes
Researcher	Let us divide the semi-circle into rectangles, so can you show me just one of the rectangles' weight and where its CoG is?
Betül	Yes, here. Now, if we say m_1
Researcher	Weight of this m_1
Betül	Think of m_1
Researcher	So how do I write the coordinate of m_1 here? I know it is the middle of it (gives students time to think). Now you know the length of the piece so far, right? So you can find the length of this piece
Betül	Through what we drew?
Researcher	Of course
Betül	yeees, we can find
Researcher	So mathematically, how can you express it?
Betül	here x_i and here x_{i+1} is, and the center is the middle of these two
Researcher	I said $\tilde{x}$ Here. What is the formula for this?
Betül and İlke:	$(\mathrm{x}_i + \mathrm{x}_{i+1})/2$
Researcher	That is x coordinate, right?
Betül	If we substitute, put it in f
Betül and İlke	We can find and replace them instead
Researcher	So what do I do because these lengths are equal? Moreover, can I do these to all rectangles here?
Funda	Do we draw rectangles like this?
Researcher	Yes
Funda	Let us find this rectangle
Researcher	Can you tell the weight and coordinate of each part for the center of gravity?
Betül	Yes, and if we sum them up, we can find the system's moment and divide it by its mass; we can find it. Teacher, then we do not need to get it according to y, we know it is on the y-axis?
Researcher	Yes, do we need to take a moment according to y or x according to y?
Betül	Need to get by x

Researcher	So?
Betül	We have to look at their distance according to x
İlke	Teacher, how many rectangles should we use because there are empty spaces here?
Betül	We can use as many rectangles as you want. The more rectangles we use, the better we get results, but drawing too many rectangles makes it more challenging. Look, we can draw rectangles like this, then our error will be less

From her last sentence in the conversation, she understood that accumulation is a process, and the more minor changes in the independent variable give more approximate results of the changes in accumulation. Another point that she started to connect arithmetical calculation with algebraic calculation. Due to the nature of the activity, to find CoG accurately, students have to define the independent variable and take the moment according to it. From the conversation above, Betül can determine the independent variable correctly and moment according to it. Since she uses the Riemann sum in the moment process, she determined the accumulation of the changes according to the independent variable. Although she achieved these objectives, she did not fully conceptualize the Riemann Sum. The process that she experienced was being familiar with the concept. It was completed when she made a mistake and started to monitor and interpret the whole elapsed time since they started to extract the formula. Because when they calculate the moment of their design, she realized that the area of the rectangles had to become smaller and smaller. However, in their calculations, it was not, so she rechecked the formula. At that time, she compared each variable in the formula and the computation that they did. It was the breaking point for Betül since she had to think reversible processes from theory to design and vice versa. Then she combined all the processes in the activity. It is represented in Figure 4.16 as red and green triangles. It can be seen that she combined the design and mathematical discussion.

Moreover, she produced both back and forward links. While she was calculating the moment, she also thought about how she would use it in the design. How she experienced this is also presented in the following conversation.

Betül:	Now we know this length is half cm long.
İlke:	Let us write here.
Betül	Can you tell me what the teacher did? (Revisits her notes) instead of substituting 1/2 in f hmm teacher, do we have to write the equation for it, right?
Researcher:	No, you calculate the center of gravity, then tell us how you calculated it
Betül:	However, we write in f, hmm OK, İlke can you find the middle of that rectangle?
İlke:	Yes, here 6
Betül:	No, it would not be 6; the total length is six it should be a little less than 6
Funda:	Should we find for all m?
Betül:	Hmm?
Funda:	Find all m for all of them, and then it will be a little above our center of gravity. Are you going to average those two?

İlke says, and Betül calculates the weights of each plane.

İlke:	One will slide down 0.1 cm; the other will slide down 0.1 then.
Betül:	How do you know? Is there any pattern?
İlke:	Approximately. Should we measure them one by one?
Betül:	Of course, it does not always shorten at the same rate.

They did calculations, and their calculations were wrong. However, they were not able to find the mistake

Funda:	Can you write what you do one by one? So we can find the mistake more easily
İlke:	Since it is symmetrical now, we will write accordingly to it *(Still trying to find the mistake)*
Betül:	OK, we write according to the formula
İlke:	Let us write like this
Betül:	Now we will write y instead of it I guess something like this will happen
İlke:	What is this?
Betül:	We need to find their areas in the future.

İlke: Area? Why?

Betül: We said we would take area instead of weight. all the variables are the same, so they cancel out each other

İlke: OK, then this is always $one\ cm$.

Betül: Heights

İlke: Exactly. Wish we could write to what we found at the beginning

Betül: Didn't we write the formula?

İlke: No, we did, but we multiply by 2

Betül: Yes, we add them up then multiply by 2

İlke: Exactly.

When interpreting the formula, İlke realizes that it must be multiplied by x_i

Betül: No, we had to multiply them with what we found

İlke: Yes, we should always multiply when we find them. That is why we have to find areas separately

Betül: Twice it is square

İlke: I do not understand you right now

Betül: Twice this is the height of these two we have taken the height of this as the area, or then the areas have to multiply this distance, the height is already twice that two times this, and we have to find the side of m^2x_2, but, now I will tell you twice these, you write

İlke: Yes

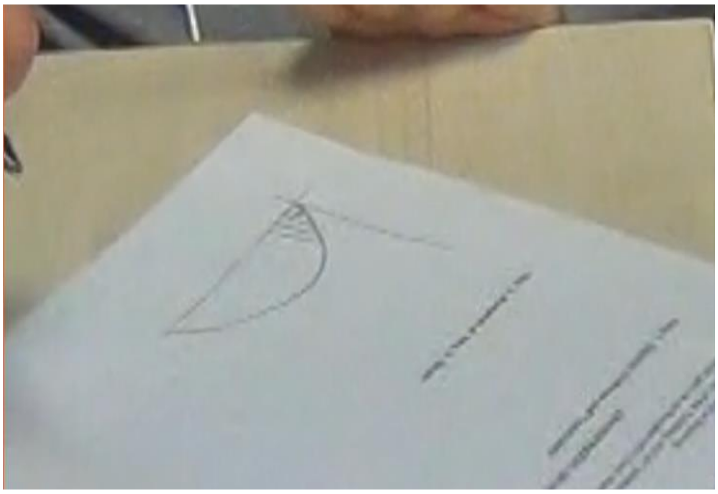


Figure 4.17 Betül construction of Riemann Sum

From the conversation, it can be concluded that Betül was able to write and interpret the product of $f(x_i)D(x_i)$. Another aspect that supports the development of Betül is the structure of activity, which is design, mathematics, and the use of mathematics with design. Namely, after establishing a connection with real-life and Riemann sum, Betül's application of the formula to her prototype has enabled her to see how the method works and match each variable's meaning in real-life. Thus, the discussions at the beginning of the course were meaningful for Betül and enabled her to the reason the solution of the problem. For example, in dark red triangles, at first, she misunderstood the formula and miscalculated CoG. When she realized that that calculation did not match with the design, she had to reinterpret the method and correct it. In this way, at the end of the lesson, she connected the whole process.

To verify Betül's development, I also examined her reflection paper belonging to the related lesson examined. In the reflection paper, they are supposed to discuss the approximation process and show that accumulating has a multiplicative structure and interpret the product $f(x_i)\Delta x_i$. Moreover, she stated algebraically and wrote the general formula. In this activity, it was not expected from Betül that she expressed the accumulation process as a limit form. Her paper shows that she did not write it, although she expressed that in words. Moreover, it was asked that express those different representations in the follow up questions. She combined the arithmetical and algebraic, and geometric parts. Thus, she was able to move between different representations flexibly. Hence, she passed through the reification process.

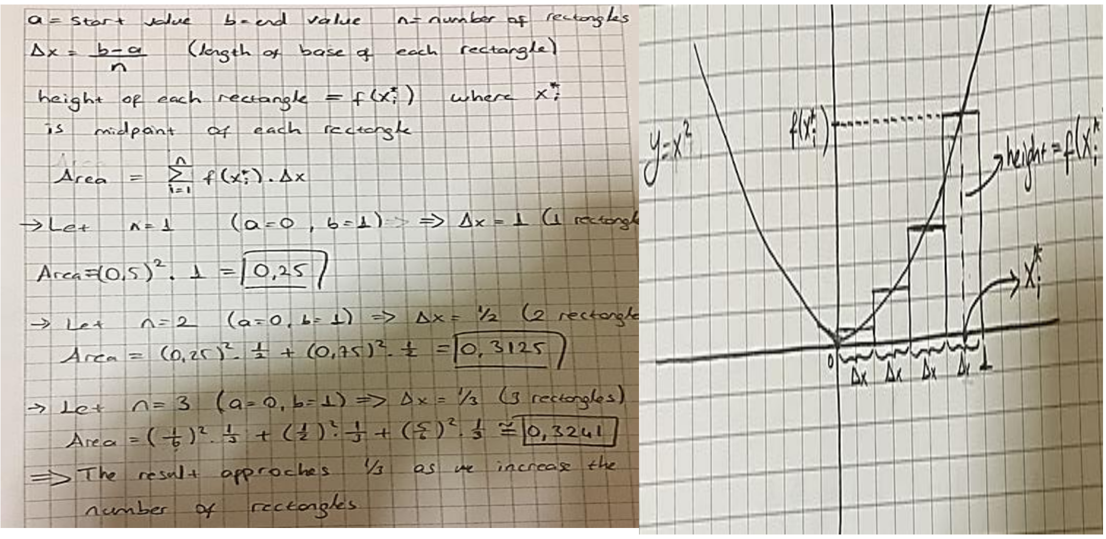


Figure 4.18 Betül's solution in a reflection paper

In the table (Table 4.2) below, Betül's development is shown. In the next chapter, it was presented how Betül achieved the reification process objectives.

Table 4.2 Betül's Improvement in Condensation Process

Condensation Stage	*Reached*
Write and interpret the product of $f(x_i)\Delta x_i$	+
Determines the change in accumulation depending on the independent variable.	+
Grasp that the quantity which is accumulating has a multiplicative nature.	+
Flexibly use various representations.	+
Differentiate and apply left-hand side and right-hand side Riemann sum	+
Comprehend that accumulation is a process of adding new increments.	+
The smaller the shifts in the independent variable, the more accurate estimates of the change in the accumulation are obtained.	+

4.1.1.2.4 Reification Process

Sfard (1991) defines the reification process as a shift. Students interpret familiar context with a new perspective. That means "reification is an instantaneous quantum leap." Students become more skillful. Moreover, in this process, students use different representations and try to generalize the concept.

In this process, students transform the concept into a more concrete and static object. Students utilized different representations of the taught concept meaningfully. In the end, students detach newly learned concepts from the step. This detached concept becomes a member of a class and becomes a basis for another new entity called "object-process" (Sfard, 1991).

Pre-serve mathematics teachers reach the reification stage when they are capable of operating the Riemann sum as an object. Thus, treat Riemann sum as a whole and translate it into graphical representations uses algebraic methods to find the solution.

In this process, preservice mathematics teachers can

- Write and interpret the $\lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)\Delta x)$
- Can easily transfer between representation.
- Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.
- The accumulation of any function can be generalized by means of the graph, independent variable, and the value of the function.

4.1.1.2.4.1 Group Discussion in Activity 3

Activity 3 was about designing an arch dam. In this activity, students will build a dam with given conditions. The conditions are that surface area of the dam should be optimum, and it should be persistent. By considering these conditions, the participants of the study should satisfy the persistency of their dam. Thus, throughout the design process, they have to think critically, do the proper calculation, then design the model. At the end of the activity, the study participants should be able to explain hydrostatic pressure, and students will be able to explain hydrostatic forces. Besides, students will be able to explain the relationship between Riemann sum notation and integral form. Students will be able to set up integral, students will be able to determine the boundary of the integral, and finally, students will be able to explain Riemann sum in polar coordinates.

Activity 3, designed as the participants, should have determined the variables, formulate the collapsing force, and apply that formula in their design. In this activity, the theoretical aspect was also dominant as in the second one. Moreover, there were several different variables that students have to think about together and decide which variables' effect is more on the durability of the dam. In this way, it was aimed that students were made to conceptualize the variable concept, and also, they were able to see how and which variables affect their design. Moreover, they learned how to interpret the formula to express variables in theory. Unlike previous activity, theory and practice were intertwined entirely in this one. The critical point of the activity was determining the boundary of the integral in polar coordinates. The situated abstraction of the defining boundary was prevented, and the participants could see how the theoretical calculation affected their design directly. In this regard, at the beginning of the lesson, activity sheets are given to each participant and they were asked to work five minutes individually. After that, they started to discuss what they understand and what they should do to reach an asked solution. Figure 4.19 indicated that the ideas in Activity 3 were overlaid. This means that the participants started to contribute to each other's ideas and construct their ideas on the previous one. The group discussion linkograph in Figure 2 showed that they made two types of discussion; a mathematical discussion about constructing the formula and a design discussion about the dam and deciding the θ.

In the following linkograph, during the mathematical discussion part, they focused on understanding the problem, such as what is an arch dam, how it works, etc. Then the pre-service mathematics teachers discussed the forces on the dam and how these forces affect the dam. Through the process, they found that the center angle of the dam affects the collapsing force. Hence, they tried to formulate the collapsing force by using Riemann Sum. On the contrary to the previous activity, they connect physics and mathematics at first. After that, the group discussed the summation notation of the formula. Then they explained that by adding new increments, they could decrease the error, and they expressed this process as taking a limit of the summed rectangles. At the end of the mathematical discussion, they related the Riemann sum to the integral form by explaining the relations between two notations. After constructing the formula, they began to apply the formula to their design. In this process, at first, they tried to understand which variable is dominant in the design. At first, they thought that the radius of the arch dam is essential for the design. After a while, they realized that the center angle of the dam is essential.

Aforementioned before, the critical point was boundaries given as polar coordinates, which meants that the participants could not annot directly determine the boundaries. There are two reasons for this kind of design; firstly, to prevent the situated abstraction, and secondly, the students' difficulty that they have difficulty in determining boundaries. In this regard, students focused on θ,r and the line of the dam. At the end of the discussion, they decide the optimum angle, and by using that, they designed their arch dam.

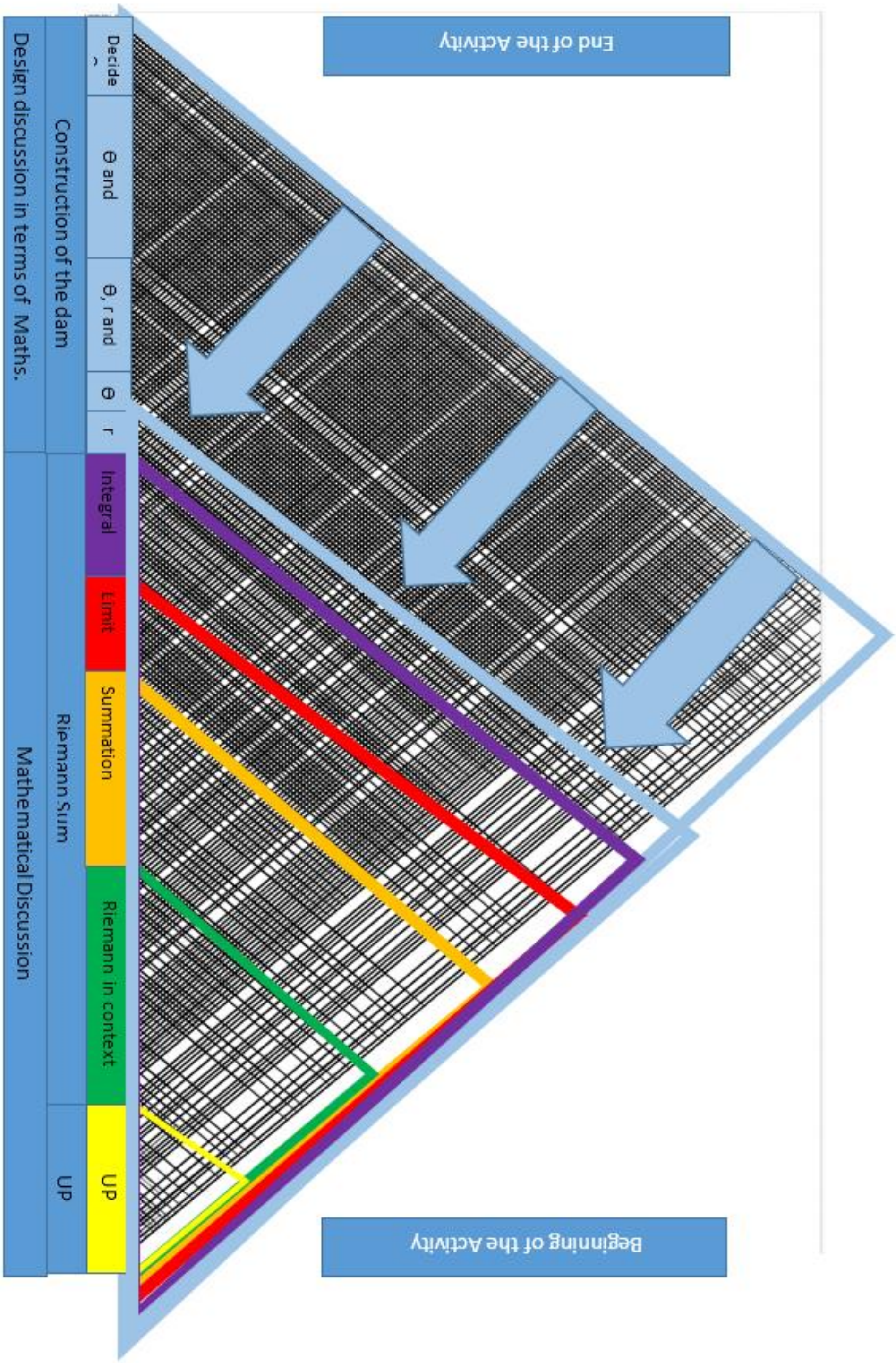


Figure 4.19 Group Discussion in activity 3 UP*: Understand the problem

4.1.1.2.4.2 Remarkable Events in Activity 3 for Reification Phase

The remarkable points for Betül were coded in Figure 4.20 as UP, Riemann in context, finding a general formula, writing integral form, trying to apply the formula, based on the formula trying the construct, r, Θ and line, Θ and design and finally relating the whole process. Unlike the other linkographs, in this linkograph, Betül, in some of the time intervals, constructed the concept by herself then contributed to the discussion. This construction process is coded as self-construction in the linkograph.

In Figure 4.20, the amount and nature of her contribution to the discussion were represented by the number of lines throughout the activity. As seen in Figure 4.20**Hata! Başvuru kaynağı bulunamadı.**, the number of links increased, and here is an almost balanced distribution between the discussion parts. That means she started to think about the design and the solution to the problem. Namely, while she was participating in discussions, she conceptualized what her group mates were discussing. It can be seen from Figure 4.20**Hata! Başvuru kaynağı bulunamadı.** hat her discussion contribution codes and the group discussion codes were started to synchronize. Moreover, Betül started to overcome the real-world and mathematics connection difficulty and started to think physics and mathematics nested and apply that knowledge to the design. The critical point here is that while her group friends were trying to construct the formula in a limit form, she constructed the formula, and even went further by trying to apply the formula to their design, which is shown as purple triangles and the arrows. She stated to combine the whole process until that moment. Thus, she started to build two-way connections that are back and forward links. Furthermore, she start to build two-way links until the end of the activity.

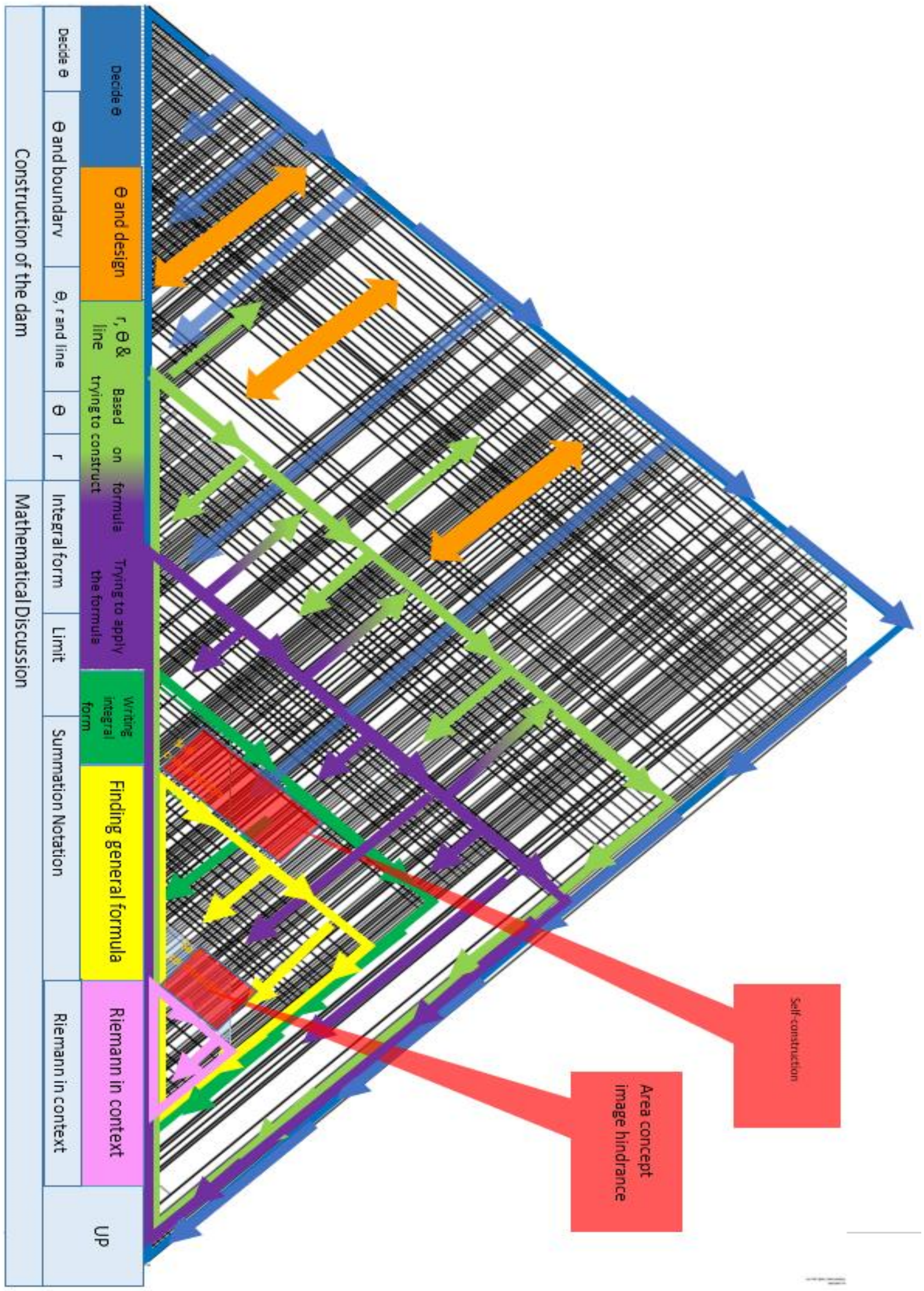


Figure 4.20 Betül’s linkograph in Activity 3

During the UP code, she tried to understand how the dam functions and its shape, and she also tried to understand what she should do to solve the problem. After explaining that liquid pressure is exerted equally in all directions, pressure force which is force causing pressure, that is always vertical to the surface, the researcher asked where the extension of the pressure force would pass and after this moment group began to try to find the solution of the problem. In the Riemann ın context, with the researcher's guidance, she and her group friends discussed the forces on the dam and its components. End of the discussion, they decided that the Fy component of the force will collapse the dam, and Betül began to understand that they can also build their dam according to that force if they can find it. In this regard, she started to be aware of that by manipulating the collapsing force. However, her area-oriented concept image caused hindrance in writing the Riemann sum in terms of angle. Since she always tried to find the area and was not able to pass through the length. Through the end of the discussion, she conceptualizes that she should write the formula for every pressure force, then she conceptualized the process and was able to write the Riemann Sum. This hindrance process is shown in the linkograph Figure 18; at the beginning of the finding, general formula discussion, an incubation process shows that she was trying to construct or conceptualize the process. In the following excerpts, Betül construction is given.

Researcher	For your problem, do not forget that the pressure will be the same for the same depth. Also, the liquid pressure is exerted equally in all directions. Where does the extension of the pressure force pass?
Betül	From center
Researcher	What component do you think is the force that will destroy this dam?
İlke	I think those who are vertical
Betül	Fy will collapse the dam. Are we trying to find the pressure on the center-right now?
Researcher	Why did I separate them as vertical and horizontal?
İlke	Because the forces in the horizontal have no

	effect
Betül	Hmm
Researcher	What do you mean by horizontal?
Betül	x component
İlke	x component
Researcher	Yes, they will cancel each other
Betül	x component will transfer the force to the mountain.
Researcher	So what will the y component will do to my dam?
İlke	Collapse the dam
Researcher	How do I write this collapsing effect for F_1
İlke	$pcos\theta_1$
Betül	p divided by what?
Researcher	What is the pressure here?
Betül	$psin\theta_1$
İlke	$pcos\theta_1$ (*thinks*), and it was wrong. Yes, it will be $psin\theta_1$
Betül	How can we use the area here? We will divide the angles, but I cannot use the area anywhere.
İlke	Teacher, let us say theta in the arch or let us say something or beta, for example $\beta/360$ is going to be, won't we divide it by total Δy?
Researcher	is $\beta/360$?
İlke	now $\Delta\theta$ is too tiny I mean its infinitesimal
Researcher	ok, you are going right, think a little more, and how we express it depends on y We see the $\Delta\theta$, what about each a?
İlke	a yes, it is the sheer force on the dam
Funda:	Is the average of P here is the distribution of pressure on the arc in that circle?
Betül	uh, we should find the area of the circle slice right now, but we found the length of the arch, think of the arch part, then if we take something like $2\pi rh$ instead of πr^2. Teacher, it is not an area. It is a length. *(The breaking point for Betül)*
Researcher	Why? Can you explain it?
Funda	I got it at a certain level, and then we have to write the pressures differently. If we are going to write a lot of different pressures, it said a single surface. Think of it as a surface

Betül	Exactly, I understood. If I write πr^2 it will be a circle slice, but there is no water there. However, we should consider the length.

There were two main points here; the first one is that Betül tried to find the formula with the area, so she could not catch the discussion and the second one is that Betül could not think pressure (P) was not the same. After realizing that since the depth is the same for that circle P is the same, and the pressure only affects the surface of the wall. Thus, she realized that she also calculates the dam's inner area when she took the area where there is no water. By considering these, she conceptualized that she should use the length for every single F. Then she wrote the summation formula of the total force.

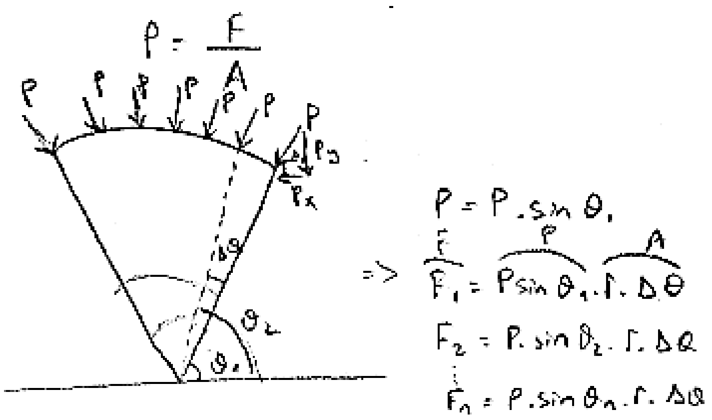


Figure 4.21 Betül's formulization of Riemann Sum

Figure 4.20 can be seen at the beginning of the integral writing form; there is a gap that is represented as self-construction. Since that time interval, the researcher asked students how to write the Riemann Sum in the integral form. At that point, Betül did not contribute to the group discussion. Instead, she revises what she wrote and matches every expression in the Riemann Sum representation with the integral form, especially the boundaries of the integral, which were one of that lesson's objectives. Then she wrote the formula correctly and contributed the discussions by explaining her process to İlke as "*From Riemann Sum, we can write. Think about that we divided the θ into small parts and they will go to infinity so we* *can write it as in the limit form*". She also explained the writing boundaries as "*we will write the boundaries from θ_1 to θ_n*"

$$\sum_{i=1}^{n} P\sin\theta_i . r\,\Delta\theta$$

$$\lim_{\Delta\theta \to 0} \sum_{i=1}^{n} P\sin\theta_i . r.\Delta\theta = \int_{\theta_1}^{\theta_n} P.r.\sin\theta \, d\theta$$

Figure 4.22 : Writing integral form

Until here, Betül builds backlinks, and still, her mathematical construction process was limited and separated from the formula's solution. That means she conceptualized the integral in a specific context and still did not think about using the integral in their design. This moment was important for Betül since she was having a problem applying mathematic to real life.

Pre-service mathematics teachers needed to determine the necessary variables and use them to apply the formula to construct their dam. In this way, they thought as engineers do, combined the practice and the theory to their design and developed their variable concept. Another property of the current activity was about replacing the coordinates into the pre-prototype dam. Since replacing the coordinate system on to the dam correctly, they were able to determine the boundaries of the integral and force on the dam. Thus, satisfy the necessity of the problem in the activity. Betül achieved this process while trying to apply the formula into their design. Through this process, Betül learned to determine the constants and variables. Thus, in the current activity, p and r are the constant, and according to the size of the dam, they will determine. Betül realized that the only variable that determined the dam's demolition was the angle, and she realized that they had to determine the angle first. Because while practicing, Betül realized that small changes in the significant angle, which they divide into an equally smaller angle in the formula, would change the dam's forces. The understanding process of Betül is given in the following dialogue.

Betül Now I will say something when we take the integral of this and get this one, and in it, we will write θ_1 instead of θ now hmmm no

Funda In conclusion, we need to determine that angle, so we have to conclude

Betül Hmm so let us do something like that (thinks) when we write θ_n - θ_1 , we have to choose r as big, hmm no we have to say r smaller

Funda No, if this grows, p grows. We need to choose r small because we try to decrease p, so let us say f here because r is increasing pressure decreases because of f right?

Betül no, this is the total f for us, and we should reduce the force so that our dam become sustainable

Funda Do we have to reduce p, right?

İlke I understood and came

Funda We are trying to understand how we need to choose our values

Betül I think we can not reduce p, so we should manipulate these (r and θ)

Funda p and f are inversely proportional, but p and f are the same things, aren't they?

İlke They are not the same

Betül They are not the same

İlke Think of it as a total force, and this is a which force acting per unit area,

Betül yes $p = f / a$

İlke It is a very small unit

Funda How do we solve the problem?

Betül I think we should manipulate r and θ

İlke How do we decide on r?

Betül We cannot decide p right now, it depends on water, right?

İlke Yes

Betül So we have to play with r and θ? Wait, r is also constant because we should determine the boundaries according to angle.

Funda Yes

Betül We should decrease r but how about θ?

Funda But if r decreases, f decreases

Betül I think our θ as much as big

.

.

.

İlke	The shape will be like this if we keep the angle large and r small (*showing that the circle segment grows by opening the hands to the side*)
Betül:	Yes, we need to keep the angle large because $cos\theta$ gets smaller as it approaches 90^0 or the $sine$ grows, or the $cosine$ gets smaller
İlke:	We assume that angle is bigger than 90^0 .
Funda:	Increases at 90^0, be one at 90^0, is it going to be the same at 0.
İlke:	The value of cos 60 is 1/2, the value of cos 0 is 1, what is the value of cos 90?
Betül:	then the cos gets smaller as you get closer to 90
Funda:	So r should be like this?
Betül:	r changes according to angle and our r is this (refers to the width of the container). Let me say something what does happen if we assume the angle as 90? Is that being the smallest value?
İlke:	No, the smallest value is -1/2

As seen from the conversation, Betül was able to understand how to determine the integral boundary. Moreover, she connected the dam prototype with the coordinate systems and helped her groupmates determine the boundary of the integral. Here, it was seen that she could determine the variables and match them in the integral form. Moreover, she was able to move from the Riemann sum to the integral form.

The process of establishing advanced links starting with the purple triangles continued in the later parts of the discussion, and at the end of the discussion, she combined all the processes. In the purple triangles, Betül's Riemann sum grasp takes place, while in part consisting of the green triangle, the connections he has made regarding the combination of real-life and mathematics have been strengthened. Moreover, Betül was able to conceptualize and determine the boundaries of the integral.

Betül’s developments are shown in Table 4.3 below. In the next section, how Betül achieved object- process phase is presented.

Table 4.3 Betül's Development Through Reification Process

Objectives	***Reached***
Write and interpret the $\lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)\Delta x)$	+
Can easily transfer between representation	+
Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.	+
The accumulation of any function can be generalized by means of the graph surrounded by change I, independent variable, and the value of the function.	+

4.1.1.2.4.3 Group Discussion in Activity 4

Activity 4 was a reverse engineering activity. The context of the activity was finding the volume of a cooling tower. In this activity, a cooling tower in a power plant has collapsed. Hence the tower should be repaired. In this context, the students' task is to calculate volume of the tower by using only the photos and the tower's height. However, they are supposed the find the exact volume of the tower in a limited time. Thus, the size of the new tower should be the same as the previous one. They do not have enough information even to estimate the volume of the cooling tower, so throughout the design process, they have to think critically, do the proper calculation, then design the model. At the end of the activity, the participants will be able to solve the problems related to solids of revolution and explain the Bernoulli principle. Besides, students will be able to explain the relationship between Riemann sum notation and integral form. Students will be able to apply FTC to problems.

Activity 4 was designed as the participants should have determined the variables, write the volume formula, and then have to calculate the tower's volume. In this activity, the theoretical aspect was also dominant as in the second one. This activity is purely theory-oriented and designed with reverse engineering; subsequent activities are theoretically based. Moreover, there were several different variables that pre- service mathematics teachers have to think about together and decide which variables' effect is more on the durability of the dam. In this way, it was aimed that students were made to conceptualize the variable concept, and also, they were able to see how and which variables affect their design.

Moreover, they learned how to interpret the formula to express variables in theory. Unlike previous activity, theory and practice were intertwined entirely in this one. The critical point of the activity was determining the boundary of the integral in polar coordinates. Because both situation abstraction of the defining boundary was prevented and the participants could see how the theoretical calculation affected their design directly. In this regard, at the beginning of the lesson, activity sheets are given to each participant and asked them to work five minutes individually. After then, they started to discuss as a group what they understand the problem and what they should do in order to reach an asked solution. urna.

4.1.1.2.4.4 Object-Process for Betül

In this process, students transform the concept into a more concrete and static object. Students utilized different representations of the taught concept meaningfully. In the end, students detach newly learned concepts from the step. This detached concept becomes a member of a class and becomes a basis for another new entity called "object-process" (Sfard, 1991). Including this activity until here, Betül was able to explain various representations, and she was able to use different representations of Riemann sum easily. Namely, she expressed and explained the geometrical representation in different coordinate systems of the Riemann Sum into algebraic representations. Moreover, she could write and interpret the algebraic form of the Riemann sum and represent it geometrically.

As a result, the Riemann Sum became an object for Betül, and she constructed the Fundamental Theorem of Calculus on it. Hence the Riemann Sum turned into a "process" for Betül.

4.1.1.3 Developments in Betül's knowledge about Fundamental Theorem of Calculus in terms of Object -Process Perspective

Until this section, how Betül constructed the Riemann sum in terms of the Object-Process perspective was presented. According to Sfard's Theory, after learned knowledge was detached, it was categorized, and this new concrete object became a base for the new entity. In other words, new knowledge was constructed on this object, and that was performed as an input. Namely, it became a new process for a new knowledge series. In this regard, Riemann sum was an object which is the ultimate phase of the theory concurrently it was a first phase process for the Fundamental Theorem of Calculus. As in the Riemann Sum Object-Process, Betül went through the interiorization, condensation, and reification process. Finally, the Fundamental Theorem of Calculus became an object for Betül, and this objectification category was prepared to be the base for differential equations.

4.1.1.3.1 Interiorization Phase

Sfard (1991) described this phase as the students started to be familiar with the new concepts and had limited skills about the concept, however, through the process, the students became skillful. In this regard, in the interiorization process for the Fundamental Theorem of Calculus, students got acquainted with the Riemann sum's integral representation. Moreover, they were expected to be able to

- Understand that variation in an accumulation occur with increments in it (Thompson, 2017). Organizes the shift of the independent variable and the shift in function value on small intervals with the amount of accumulation resulting from the multiplicative nature of these two values.

- Write and interpret the $\lim_{\Delta x \to 0} \sum_{i=0}^{n} f(c_i)\Delta x = \int_a^b f(x)dx$
- Grasp that a multiplicative relationship involving instantaneous realization on an interval can be generalized and represented by an area.
- Define exact accumulation functions and represent them as $y = \int_b^a f(x)dx$ where $f(x)$ is a function and dx is the size of "an infinitesimal change." (Thomspon, Byerley, Hatfield, 2013)

In the next chapter, Betül's construction of the Fundamental Theorem of Calculus is presented. Through the activities, the essential points that Betül achieved, pre-determined objectives for the phases were presented.

4.1.1.3.1.1 Remarkable Events in Activity 4 for Interiorization Phase

As mentioned before, Activity 4 was a transition activity, so in this activity, students were expected to express the Riemann sum, and by relating the expression with the integral, they were expected to set up integral. In this regard, Activity 4 was a base for teaching the Fundamental Theorem of Calculus. It was aimed to make students construct variation, and the accumulation is an incremental process. In the following, how Betül found the solution using the accumulation process and set up integral is presented.

Betül	We did the same thing while finding the volume of the canoe
Funda	Yeah, we measured the canoe
Betül	I think we split height into pieces. For example, if we divide it into pieces, it is almost the same here and up to here, and if we take it like this until here *(referring to the boundary of the integral)*
Funda	Yes, and we will draw a rectangle. We just did not put that paper. Here we draw the curve.
Betül	Do you think like this (perpendicular or parallel the plane)
Funda	Like this. However, I did not know if this would be in that model.
Betül:	However, this pole is already a cross-section. So how?

Funda:	For example, we are thinking about the area here, now in 2D; for example, if we find the area under the curve
Betül:	How about volume?
Funda:	We can solve this find the line in this then we should find the area under the curve as we did in Activity1
Betül:	How did we solve Activity 1? Let me remember
Funda:	We found the basement area of the canoe then we multiplied it by its height.
Betül:	Yes, I remember
Funda:	So, what do we do now?
Betül	But we do not have to calculate the base area with the integral
Funda:	Yes, it is correct.
Betül:	First of all, there was trouble as the radii were constantly changing.
Funda:	Well, what is the height currently? We will find it with integral. But everything changes. The diameter of the structure is also changing.
Betül:	Yes. Do you remember that we rotate some of the shapes, then we get 3D shapes? How about we rotate it around the y axis?
Funda:	You mean a rectangular area like this, right? An area like this?
Betül:	Not area
Funda:	This plane, I mean
Betül:	Yes, I mean we should rotate this curve around the origin we can get what we want. Our function depends on the radii so that we can integrate it and find the volume

It can be understood from the conversation that she applied the Riemann Sum and mentioned splitting the tower's height into pieces. The reason for not reaching the solution immediately in problem-solving was, not being able to transfer a situation in 2D to 3D in the first place. Although she noticed the change of r, she hesitated because it was a volume calculation. Then she remembered solids of revolution and was able to solve the problem. She followed a similar way to the Disc method, first formed the Riemann Sum in her mind, and was able to write in the form of a direct integral without performing the operations in the Riemann Sum. This shows us that she can both interpret and write $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(ci)\Delta x = \int_{b}^{a} f(x)dx$. In addition, it is understood that by multiplying the areas of circles with varying radii and heights, each increment added to this multiplicative structure can be generalized, and the volume of the part with definite boundaries can be calculated its integral

4.1.1.3.1.2 Group Discussion in Activity 5

Activity 5 was about designing a cat house, and students built a cat house for alley cats thinking of a city with severe winter conditions. In this regard, they had to build an insulated house and think about temperature changes and maximum volume, and surface area. They have to decide the thickness of the insulation material and how and where to use it. Moreover, they also had to design the most efficient insulated house. Thus, throughout the design process, they have to think critically, do the proper calculation, then design the model. The engineering design process is understanding the problem, gathering data, and testing those results.

Furthermore, after constructing the house, evaluating energy loss, they revised the structures according to those results. Moreover, they should be creative in order to find the most efficient design. Through this process, they will learn how to model the problem and set up the integral. By designing the activity, they understood the mathematical structure in the activity conceptually and learned the integral for the solution. Moreover, it is aimed to reinforce the integrant boundaries of the integral and their meanings. Thus students were able to explain the second type of FTC and able to explain antiderivative and integral. Furthermore, at the end of the activity, they explained heat and temperature, the difference between them formulate heat transfer equation. Finally, they were able to design an insulated building for energy saving.

Activity 5, designed as the participants, should determine the variables, formulate the heat transfer formula, and apply that formula in their design. In this activity, engineering, physics, and mathematics parts were dominant, and they were progressing simultaneously. Namely, while students were thinking about the design, they also have to think about its physics and mathematics. While they were deciding its shape, they had to think of maximum volume minimum surface area related to mathematics. Besides, they had to think these properties affect the heat transfer, so students had to consider the physics part in the previous ones in this activity; they also learned how to interpret the formula to express variables in theory and how it affects the practice. The critical point of the activity was thinking about two variables simultaneously since those are covariate.

In this regard, at the beginning of the lesson, activity sheets are given to each participant and asked them to work five minutes individually. After then, they started to discuss what they understand the problem and what they should do to reach a solution. Figure 4.23 indicated that the ideas in Activity 5 were overlaid intersected.

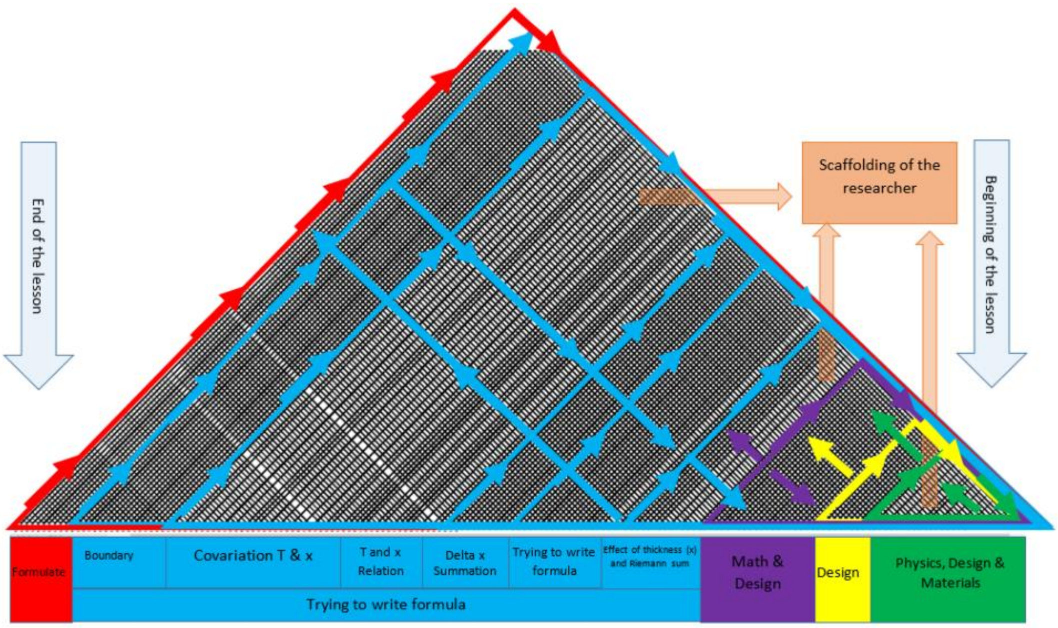


Figure 4.23 Group Discussion Linkograph in Activity 5

At the beginning of the activity, students construct their ideas on each other, thinking about solving the problem. It can be seen that mainly they discussed together and construct their ideas by effecting the group mates' ideas. Some comparatively rare links show the researcher scaffolding. . In those areas, which are also shown as orange arrows in Figure 4.23, students had difficulty or needed to be informed. For instance, while students had a lack of knowledge about heat and how it is transferred in physics, design, and materials code, the researcher assisted them by asking questions about heat and temperature in order to lead them to think about what they need to solve the problem, the effect of thickness and the covariation t and x codes students had difficulty in understanding their relation. Hence researchers are involved in their discussion process.

In Figure 4.23, in their discussion, physics, design, and materials, design, mathematics, and design, trying to write the formula, formulate were the main focus during the whole discussion. Besides, when they were trying to write the formula effect of the thickness $(x), RS\ /division\ of\ x,$ summation of x, temperature (T) and x relation, covariation T and $x,$ finally, boundary codes and finally boundary the observed codes.

During the physics, design, and material code, students, focused on how heat transfers through the wall, how the type of the materials affects heat transfer, which material they should use, which shape they should use to have maximum volume and minimum surface area. Since they had not got enough knowledge about heat and temperature during this part of the discussion, researchers supported them. After they decided on how and which materials they would use in their design, they started to think about which shape they would use. At first, they thought semi-cylinder, then they thought it would be challenging to use styrofoam. By calculating the surface areas and the volumes of some shapes, they decided to construct their house as a regular tetrahedron (Figure 4.24) due to easiness of using styrofoam.

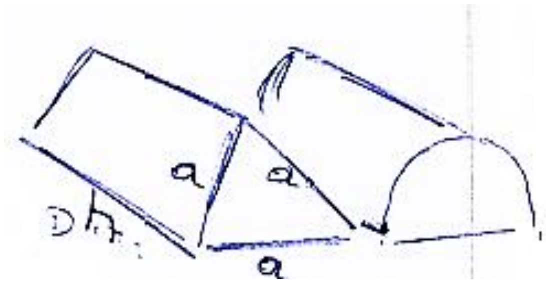


Figure 4.24 Some mathematical shapes for cat house

After that, to make calculations, they started to obtain the formula. When they were trying to determine the formula's variables, İlke had a problem with how the wall's thickness or styrofoam would affect heat transfer. While Betül and Funda defend that direct proportion, İlke defends that they should ignore the thickness since it was impossible to construct a very thick wall in real life. After a while, she realized that heat flows from high to low temperature, so the wall thickness was needed for the formula. The most crucial discussion part was for the students was to see that t and x are changing at the same time since there was a covariation between them. It was also the most critical part of the activity because when they realized that covariation, they would also discuss their rate of change and the accumulation of the heat loss. In this way, they would understand the second part of the Fundamental Theorem of Calculus. As can be seen from the linkograph in Figure 4.23, they had difficulty, and the researcher scaffolded for them. It was a though process for them to see the relationship between T and x and their covariation. However, in the end, they understood and achieved to get the formula and build a link between covariation and the whole process.

4.1.1.3.2 Condensation Phase

Sfard (1991) described this phase as "squeezing lengthy sequences of operations into more manageable units." Moreover, she emphasized that the learner can be more skillful in thinking about the given concept in this stage. She resembled this process, a computer program that works with input-output relations instead of showing all detailed processes. That means learners do not go into details during the processes. In this step, a new concept is “officially” born when the learner confronts any difficulty that provides a new entity. Namely, that difficulty may be a trigger for learning. In this step, learners may also connect the current process with another process and compare it within old and new knowledge and start to generalize it. Also, learners started to use different representations. In this regard, the participants of the current study should be able to set up the integral and be aware of the relation between the accumulation and the integral. That means while $\Delta x \to 0$, area accumulates under the f curve in $[a, b]$ and it will equal to the total change in primitive function F.

As mentioned in the previous paragraph, the difficulty is a primary trigger for a new mathematical idea. This point was the students’ starting point since this difficulty was presented to the students as dealing with two variables that vary simultaneously. In this way, the Fundamental Theorem of Calculus had been presented to them, and their construction process started officially. Moreover, they started to understand the relation between the antiderivative and the integral. In a very general point of view, during the condensation process, preservice mathematics teachers should be able to :

- Comprehend and explain that two quantities differ smoothly and simultaneously.
- The increments can be small enough, therefore no matter how much the accumulations change, they covary in relation to each other with increases at a fundamentally constant rate of change.
- The rate of change of accumulations related to each other is the rate of change in their increase.
- Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.

- Explain $\int_a^b f(t)dt = F(b) - F(a)$ as the accumulated area under the curve of f from a to b is equal to the total change in F, the accumulation function, from a to b.

Taking into consideration these objectives, Betül achieved all of them at the end of Activity 5. In the following section, how Betül met the condensation phase's objectives through Activity 5 is given and how and when she contributed to the discussion and conceptualized the objectives were presented.

4.1.1.3.2.1 Remarkable Events in Activity 5 for Condensation Phase

The remarkable points for Betül were coded in Figure 4.25 as physics design and material, design, math and design, the effect of a thickness (x), RS (Riemann Sum) division of x, T and x relation, covariation T and x, boundary, formulate.

As in the previous linkograph in Activity 3, also in this linkograph, Betül constructed the concept by herself then contributed to the discussion. This construction process is coded as self-construction in the linkograph. Although Betül builds a forward link during all sessions, her links were not as frequent as in the previous one. Because she had not enough knowledge about heat and temperature, she treated them as the same concept. Thus, she needed some assistance at the beginning of the lesson on physics, design, and material codes.

Moreover, Betül participated in the group discussion actively and her construction process progressed, syncing as the same codes with the group. In this process, Betül learned different variables by linking them, used them as to find the solution.

From the beginning of the lesson until the effect of thickness code, she both thought about finding the formula, designing the house, and providing the most insulated house. That means she learned to combine the real world with mathematics.

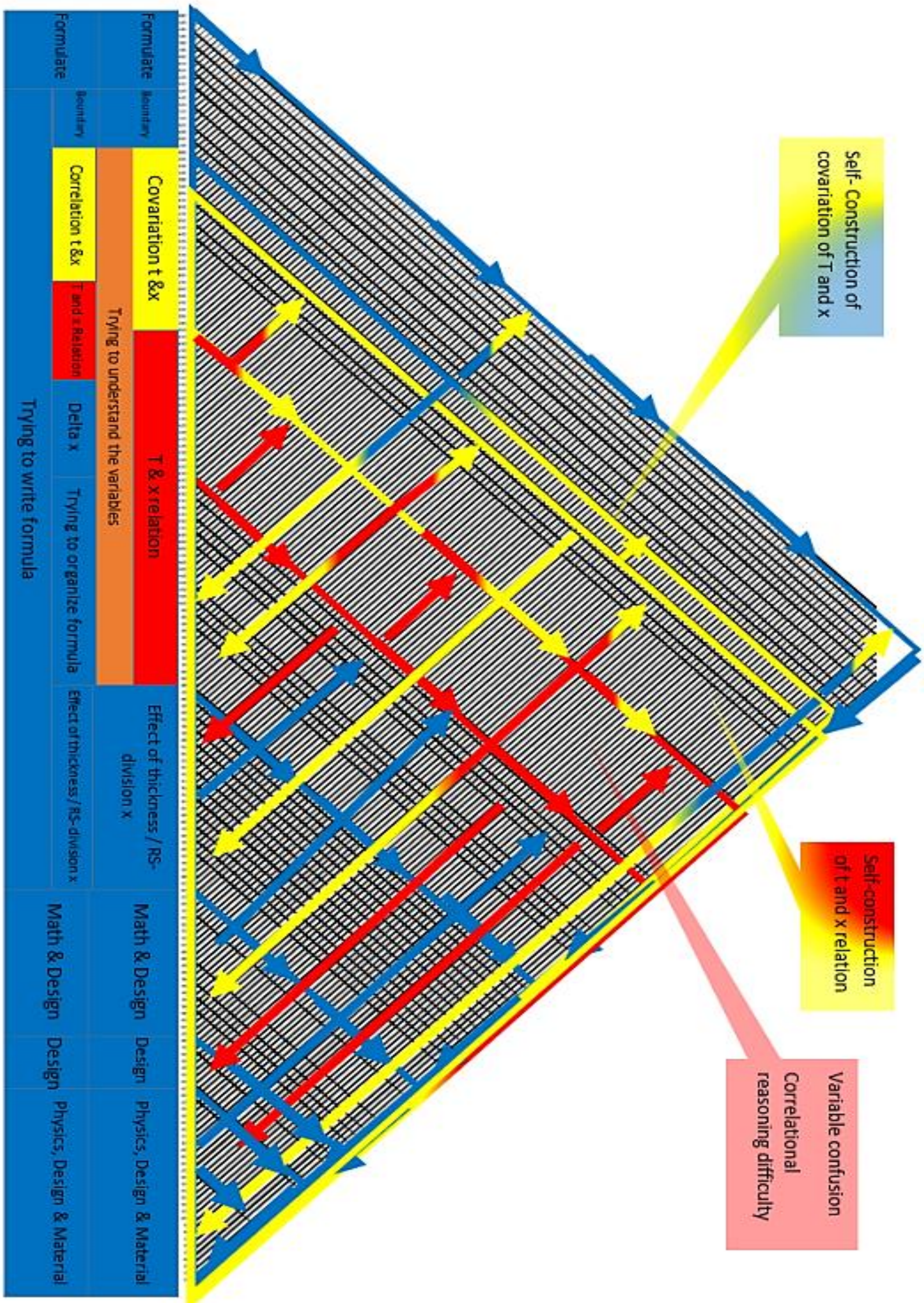


Figure 4.25Betül’s Linkograph in Activity 5

İlke	I thought we could lay it on the rectangular side, not the triangle base. This topside will be a triangle, and this part will be metal so that the snow slips down from there. We can already use foam inside it. Even though we do not have metal, it could be something smooth like that.
Betül	The part you call metal can be made with cardboard.
Funda	It would be better if we have it.
Betül	I said that what good is it for us to be like this?
Funda	I think we should make one more layer with cardboard.
Betül	We can increase its space. We put foam together; it becomes cardboard inside and outside.
Funda	We can use the same things on another side of the triangle.
Betül	What will its shape be like?
Funda	Does the order of the materials from outside to inside make a difference?
Betül	It does. Because it changes the surface area
Funda	OK. Let us do it like a curve.
Betül	This is the surface area of this parabolic one. It looks lie as if this one would be smaller. Because in the other one there are already six faces. One on the floor, Five on the surface.
Funda	Exactly.
İlke	I am stuck between a triangle and a parabola, and the other quadrilateral ones already do not have minimum surface area maximum volume. I think this is a wrong choice too.
Betül	Yes, I am thinking about it right now. How could it be?
İlke	Let us think like this when this is the case, consider two rectangles, axb
Betül	Well, if it were a pyramid rather than a triangular prism. So we can ignore one of these faces since they are on the floor. Three faces remain.
Funda	How will it be?
Betül	We have two choices tetrahedron and prism. If we make a pyramid, three faces will remain.
Funda	does the cat fit in?
Betül	If we make a triangular pyramid?
Funda	Yes
Betül	The base is also a triangle. I do not know right now; I am just thinking. Either we reduce the surface area compared to the prism.
Funda	it is like it is not for one cat
İlke	Well, for one cat.
Funda	then it fits

As seen from the excerpt, she thought the physics, design, and mathematics simultaneously. Moreover, she was also reasoning how mathematics affects design and how design affects its mathematics. That means she overcame her real-world difficulty during the whole process from the beginning.

Another critical point was that she could determine the vital variable from the physic, which is the thickness of the wall, which is very important for heat transfer. Moreover, even she did not write the Riemann sum, she expressed it and tried to write the formula.

Betül	Let us assume here is our wall. (D*raw the wall on the paper*)
İlke	Ok, he is all Δx. When we break it down into minor intervals.
Betül	Exactly then, we split this x into smaller ones.
İlke	Ok, so one of them is Δx.
Betül	We can write the thing that we call as Δx instead of the thickness of the wall.
İlke	Yes.
Betül	But we have got many variables.
İlke	This is thickness which we write in the summation formula, right?
Betül	Yes, then we will integrate it. Aaa, we do not have any formula right now? So how can we do it, and where do we use the Δx? What do we find when we substitute T?

However, she could not realize that different variables are covaried simultaneously until determining all the variables to write the heat formula. There are two main problems for Betül. She could not see the relationship between temperature (T) and the x, and she treated heat and temperature as if they were the same or their representation was the same. She could not realize that by using the decline in the temperature, she could find the heat that they had to calculate to decide whether their design is efficient. This confusion was represented in the linkograph in Figure 4.25**Hata! Başvuru kaynağı bulunamadı.** with the variable confusion and the ovariational reasoning difficulty.

In the linkograph until the red codes, blue triangles, and the arrows showed that Betül actively participated in the discussion, and within the red arrows, she started to think separately from the group. While they were discussing about how to organize formulas or to try to write the x as Δx, she was trying to understand the x and T relation. Even though she built backward and forwarded links in the linkograph, she could not go beyond the writing Δx or the Riemann Sum. She still could not realize that she had to find the heat, another reason was she still treated the heat, and the temperature as the same. Through the discussion and examining the styrofoam by imagining how T behaves through it, she started to realize the relation between two variables. However, while x and T were accumulating simultaneously, there was reciprocal proportion between x and T, which hindered Betül from realizing the process. This transition process was also represented with red and yellow arrows in the linkograph in Figure 4.25 which means Betül began to realize the relation.

Moreover, she realized that she had to divide T into small intervals. Hence, she talked about t as "ΔT." However, Betül lost her focus and forgot that x was also to be divided into small pieces and build the relation between x and ΔT. When Funda asked Betül about finding the heat with the temperature and showed her written relations at the beginning of the lesson, Betül realized her mistake, of treating heat and temperature as the same. Then she focused on comparing the proportions with the variables. In the following conversation, how Betül went through this process is given.

İlke	I divided it into 50 pieces.
Funda	I guess we assume that if it is $L = x$, it will lose $1T$ for $2x$, $2T$. I think it goes on direct proportional. I mean at the same rate.
İlke	So why do we need integral?
Funda	Write it down directly then we can find the answer
İlke	However, we thought that this wall was plain. How do we do it is a curve?
Funda:	Hmm, so do we use integral later? (Thinks) hmm, ok, I understood. In integral, we measured the variables which are constantly changing. Remember, in the previous activity; the radius was changing constantly
Betül:	When the thickness increase T changes

İlke:	I think we try to find the change with the help of the formula because the formula should give us change
Betül:	Yes, but what did it happen?
İlke:	So, we have to write things like that. It really should give me a bigger value of Q when I increase them.
Funda:	Just like this. We will say Q_1, Q_2
Betül:	between ΔT and thickness. As the thickness increases, the ΔT decreases.
Funda:	Are we going to find the energy it lost in each x? Are we going to find the heat? Will the variables be $x_{1,}, x_{2,}, x_3$?
Betül:	I do not know we can get the answer?
Funda:	I guess both of them are increasing. We will find the area under that curve.
Betül:	I understood
Funda:	the formula will be like $QxLxN$ this we will replace those x's. The change of Q with $x.Q$ shifts. We think the temperature will increase.
Betül:	Does the temperature increase? No, we are trying to find Q
Funda:	Yes, because Q changes according to the temperature, we wrote these proportions because of that.
Betül:	Ok, I thought Q like T , but they are different.

After this conversation, Betül started comparing the variables, and she realized that T and the x were divided into small pieces and talked about they should go to zero. However, she was not able to conceptualize those two variables are covarying. She stated this process as *"Δx goes to zero, so according to it ΔT also goes to zero, so what does it mean to have both dt and dx in an integral*? "This situation is represented in the linkograph with the yellow arrows. It is seen that the frequency of the arrows was decreased. As can be understood from the excerpts, she mostly focused on organizing the formula and the variables. The whole connection of the process occurred when the instructor gave Betül a known example. Because Betül was not able to interpret both dt and the dx at the same time. Hence instructor asked Betül to interpret the time and the distance at a tiny interval. In this example, Betül stated that "*time and the distance are changing simultaneously so these to can be changed as them"* After that, Betül realized that T and x are varying simultaneously. This connection process is represented in the linkograph Figure

4.25, with the blue and yellow graded arrows. It can be seen that after this connection, the whole process was linked and the number of links was increased, and all the ideas were constructed on each other. That means Betül connects the whole process. The following conversation also supports this connection process.

İlke:	The more I divide my x; the smaller my ΔT will become
Researcher:	yes
Betül:	we divide this interval into n parts. For example, while $\lim_{n\to\infty}$ a, my temperature change is ΔT changes. Both are changing. For example, when $n = 1$, $T_2 - T_1$ is my temperature value in x that I first divide. If I do this if, my temperature value at x that I divide when the first n = 1. So, if I make $(T_{n+1} - T_n)\Delta x$ and the limits go to infinity and integrate this sum symbol; I find the total heat loss.
Researcher:	You consider the limit according to x; how about ΔT?
Funda:	It will also get smaller
Betül:	We found changes in very, very small intervals like points. We must calculate the sum of these changes. To find out how much change there is. Because integral means sum.
Researcher:	So?
Betül:	We must integrate both sides. However, how do we ensure that equivalence continues?
İlke:	T and x are changing at the same and the same rate.
Betül:	Aa yes, and this is a proportion. So, we can write like this. Because when $\Delta x \to 0$, ΔT also goes to 0.

During the conversation, at first, she thought that there is a change after she explained that the increment could be tiny and both $\Delta x \to 0$ and $\Delta T \to 0$ she wrote as in Figure 4.26. Moreover, she also realized that the rate of change in the Δx and ΔT is the same. That mean she conceptualize accumulation's rate of change and accretion's rate of change with respect to each other is the same.

$$Q = k \cdot S \cdot \frac{\Delta T}{\Delta x}$$

$$Q = k \cdot S \cdot \frac{dT}{dx}$$

Figure 4.26 Betül's representation of the formula

After that, Betül wrote the following expression.

$$Q = \frac{(T_n - T_o)\, k \cdot S}{L}$$

$$\int_0^L Q\, dx = \int_{T_o}^{T_n} k \cdot S\, dT \Rightarrow Qx \Big|_0^L = kSt \Big|_{T_o}^{T_n}$$

Figure 4.27 Betül's integration process

That means she understood via limiting process that incremental changes in the independent variable cause instantaneous change in accumulation which equals to that function's rate-of-change in that point and could write the expression of the Fundamental Theorem of Calculus.

Betül's development is shown in the Table 4.4 below. In the next section, how Betül achieved the reification process objectives' objectives is presented.

Table 4.4 Betül's development through Condensation process

Condensation Stage	***Reached***
Grasp and express that two quantities accumulate) at the same time	+
The increments can be small enough, therefore no matter how much the accumulations change, they covary in relation to each other with increases at a fundamentally constant rate of change Thompson, 2019, p.38)	+
The rate of change of the accumulations with respect to each other is the rate of change of their increments with respect to each other (Thompson, 2019, p.38)	+
Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.	+

Table 4.4 (continued)

At a point piling up of infinitely smaller changes in the independent variable with limit results with an instantaneous change in accumulation which equals to the rate-of-change function	+
Explain $\int_a^b f(t)dt = F(b) - F(a)$ as the accumulated area under the curve of f from a to b is equal to the total change in F, the accumulation function, from a to b.	+

4.1.1.3.3 Reification Phase

Sfard (1991) defines the reification process as a shift. Students interpret familiar context with a new perspective. That means "reification is an instantaneous quantum leap." In this process, students transform the concept into a more concrete and static object. Students utilized different representations of the taught concept meaningfully. In the end, students detach newly learned concepts from the step. This detached concept becomes a member of a class and becomes a basis for another new entity. Moreover, in this process, students use different representations and try to generalize the concept.

Pre-serve mathematics teachers reach the reification stage when they can operate the Riemann sum as an object. Thus, treat Riemann sum as a whole and translate it into graphical representations uses algebraic methods to find the solution.

- Write and express the relationship between accumulation and rate of change by writing the formula.
- Explain $F(x) = \int_a^x f(t)dt$ As "the value of $F(x)$ represents the accumulated area under the curve from a to x and F is a function that can be found from the accumulation of the rate of change of F, with independent variable t and x represents the independent variable of a second "pass" through the function f with a view for coordinating the rate of change ($f(x)$) with the change of the independent variable to determine the multiplicative structure of the accumulation, F.

- State the expression of $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ as "the values of the function f at x equals to the instantaneous rate of change of the accumulation function f at x" (Radmehr & Drake, 2017, p.1057).
- Express the $F(x) = \int f(x)$ as "the antiderivative of f is F, F is an accumulation function, and f is the function that describes the rate of change of F"

4.1.1.3.3.1 Group Discussion in Activity 6

Activity 6 was about building a bridge, and pre-service mathematics built a suspension bridge that can carry the most weight (cars), and their model bridge must meet the determined requirements. Pre-service mathematics explored the forces on the bridge during the activity and tried to understand how beams and the columns of the bridge transferred the loads on the bridge. They discussed the shape of the rope. In this regard, while constructing the bridge, they had to consider the bridge's material, by how much, and how to use the rope. Moreover, they tried to construct a bridge that must carry the maximum load. In the engineering part, they should decide the design like the length of the bridge and the number of the ropes tied to the bridge, and they had to distribute the load equally to the bridge; in physics, they learned the external and the internal forces, the tension of the cable.

This activity is designed as a complete application of the Fundamental Theory of Calculus into the real world, so at first, students had to find the cable's shape conceptually and learned the integral for the solution. Moreover, it is aimed that students formalize the relationship between accumulation and rate of change. Thus, students were able to explain the second type of FTC and able to explain antiderivative and integral. This activity had two layers first one is realizing the covariation and understanding the constant "c." In other words, this activity was designed as a base for differential equations. Hence this activity was a simple example of an initial value problem. As a result, it was expected from students to solve the differential equation and then substitute the initial value into the equation and obtain the bridge's formula.

In this activity, engineering, physics, and mathematics parts were dominant, and they were progressing simultaneously as in the previous activity, namely, while students were thinking about design, they also must think its physics and mathematics. Moreover, in this activity, they also learned how to interpret the formula to express variables in theory and how it affects the practice. The critical point of the activity was that students should express "the instantaneous rate of change of the accumulation function at x is equal to the values of the function at x"(Radmehr & Drake, 2017, p.1057) and connect the whole process.

In this regard, at the beginning of the lesson, activity sheets were given to each participant and they were asked to work five minutes individually. After that, they started to discuss what they understood and what they should do to reach a desired solution. Figure 4.28 indicated that the ideas in Activity 6 were overlaid intersected.

In general, the discussion comprises four main parts: understand the problem, organize the equation, and obtain the equation. During the understand the problem part, students tried to understand what a suspension bridge is. As it can be seen from Figure 4.28, during understand the problem, the researcher assisted them since they did not know the suspension bridge in detail. During the activity, in trying to organize the equation, students discussed the bridge's properties like the type of forces, the shape of the bridge such as symmetrical or not, forces on the bridge, tension, and discussion about the distribution of the forces on the bridge by drawing free body diagram. During the trying to organize the equation part, based on the tensions in the ropes, they discussed the slope of the curve formed by the ropes, and they tried to find the equation of the rope using known three points then they tried to connect it with integral. Finally, they discussed the tension, load, angle, and slope. When they were discussing, they tried to find what is changing and what they should know. Thinking in this fashion, they determined that t

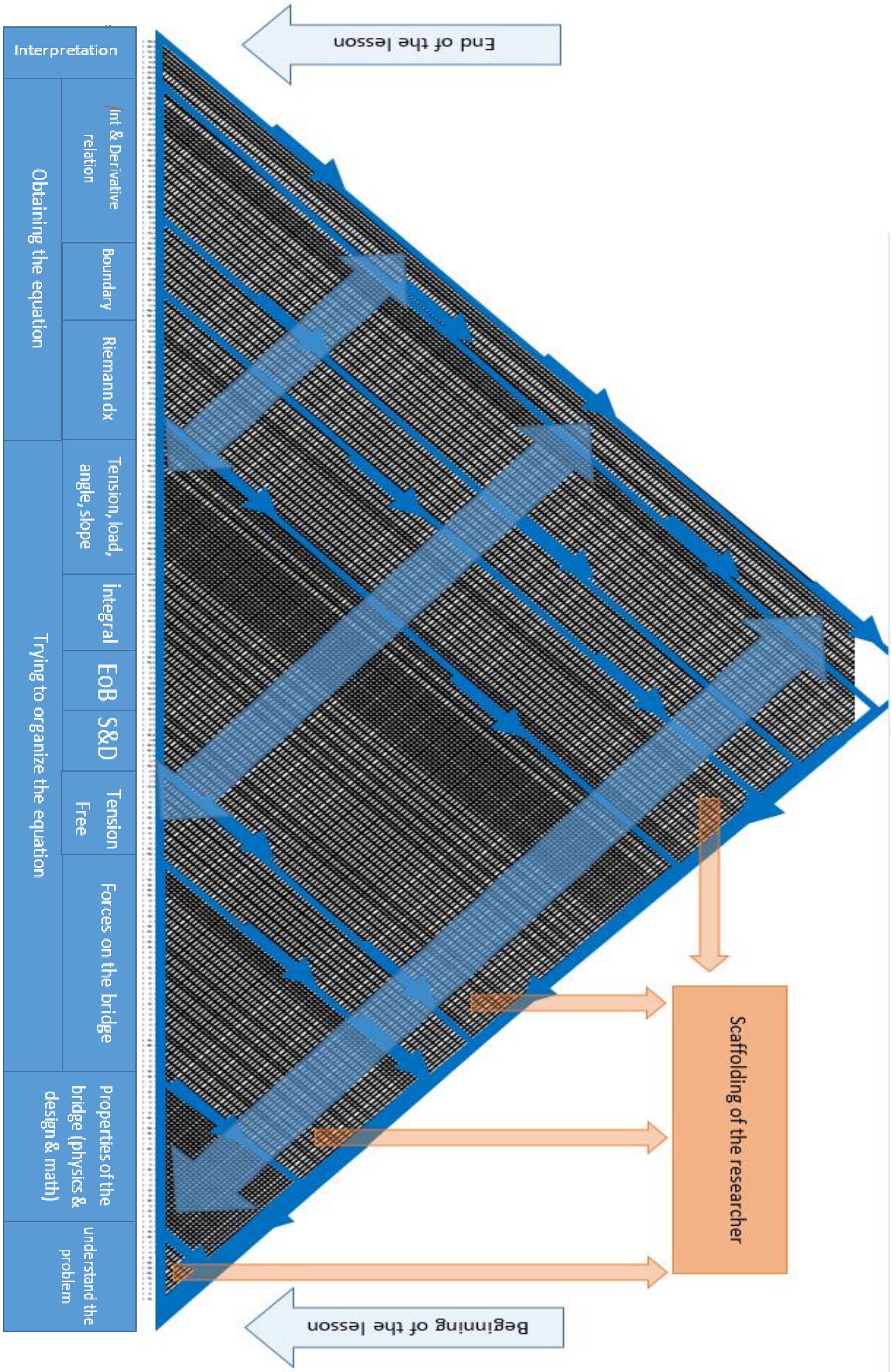


Figure 4.28 Group Discussion Linkograph in Activity 6 *EoB: Equation of Bridge, S&D: Slope & Derivative

(tension) was changing and wrote the formula. During the equation part, the two variables and their change were challenging for them since they focused on just one variable and tried to write it based on tension without thinking about the angle. During the discussion, when they tried to understand how they tied the ropes to the bridge, they realized that they must add the rope's depth, etc. With the researcher's leading, they finally obtained the formula and interpreted it.

In the following section, how Betül met the reification phase's objectives were presented and how and when she contributed to the discussion and conceptualized the objectives were presented.

4.1.1.3.3.2 Remarkable Events in Activity 6 for Reification Phase

Lorem Pre-serve mathematics teachers reach the reification stage when they can operate the Fundamental Theorem of Calculus skillfully and meet the reifications' objectives. Thus, they had to treat the Fundamental Theorem of Calculus as a whole and translate it into two types of theorem. In this regard, it was expected from Betül to:

- Write and express the relationship between accumulation and rate of change by writing the formula.
- Explain $F(x) = \int_a^x f(t)dt$ as "the value of $F(x)$ represents the accumulated area under the curve from a to x and F is a function that can be found from the accumulation of the rate of change of F, with independent variable t and x represents the independent variable of a second "pass" through the function f with a view for coordinating the rate of change ($f(x)$) with the change of the independent variable to determine the multiplicative structure of the accumulation, F.
- Express the $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ as "the instantaneous rate of change of the accumulation function at x is equal to the values of the function at x"(Radmehr & Drake, 2017, p.1057)

- Express the $F(x) = \int f(x)$ as "the antiderivative of f is F, F is an accumulation function and f is the function that describes the rate of change of F"

The focus of her discussion was coded in Figure 4.29 as design and understand the problem tension, S&D (slope & derivative), equation of the slope, tension, load, and angle, organizing the variables, boundary, writing the equation, and finally interpretation. As in the previous linkograph, Betül produced both backward and forward links, and her discussion focus was mainly synchronized with the group discussion. As in the previous linkograph in Activity 5, also in this linkograph, Betül constructed the concept by herself then contributed the discussion. This construction process is coded as self-construction in the linkograph. Unlike the other linkographs in this activity, Betül primarily constructed the concept by herself then contributed the discussion. Although she linked all the processes, she did not contribute so much. Hence, the number of links decreased compared with the previous one. The most important reason she had difficulty understanding the second type of the Fundamental Theorem of Calculus since she lacked knowledge of a covariational change in the variables.

Moreover, she had not enough knowledge about suspension bridges. Thus, she needed some assistance at the beginning of the lesson. Moreover, it can be seen that Betül participated in the group discussion actively and her construction process progress, syncing as the same codes with the group. In this process, Betül learned different variables by linking them, serving as to find the solution. During the design and the understanding of the problem code, she tried to understand the suspension bridge structure, such as tensions on the ropes, and she tried to determine how to get the equation of the bridge and its design. In general, Betül focused on finding the equation of the bridge by using slope and its integral. She was able to write the equation. There were two reasons for this; the first one, Betül, could not define the variables; hence, she could not organize the formula,

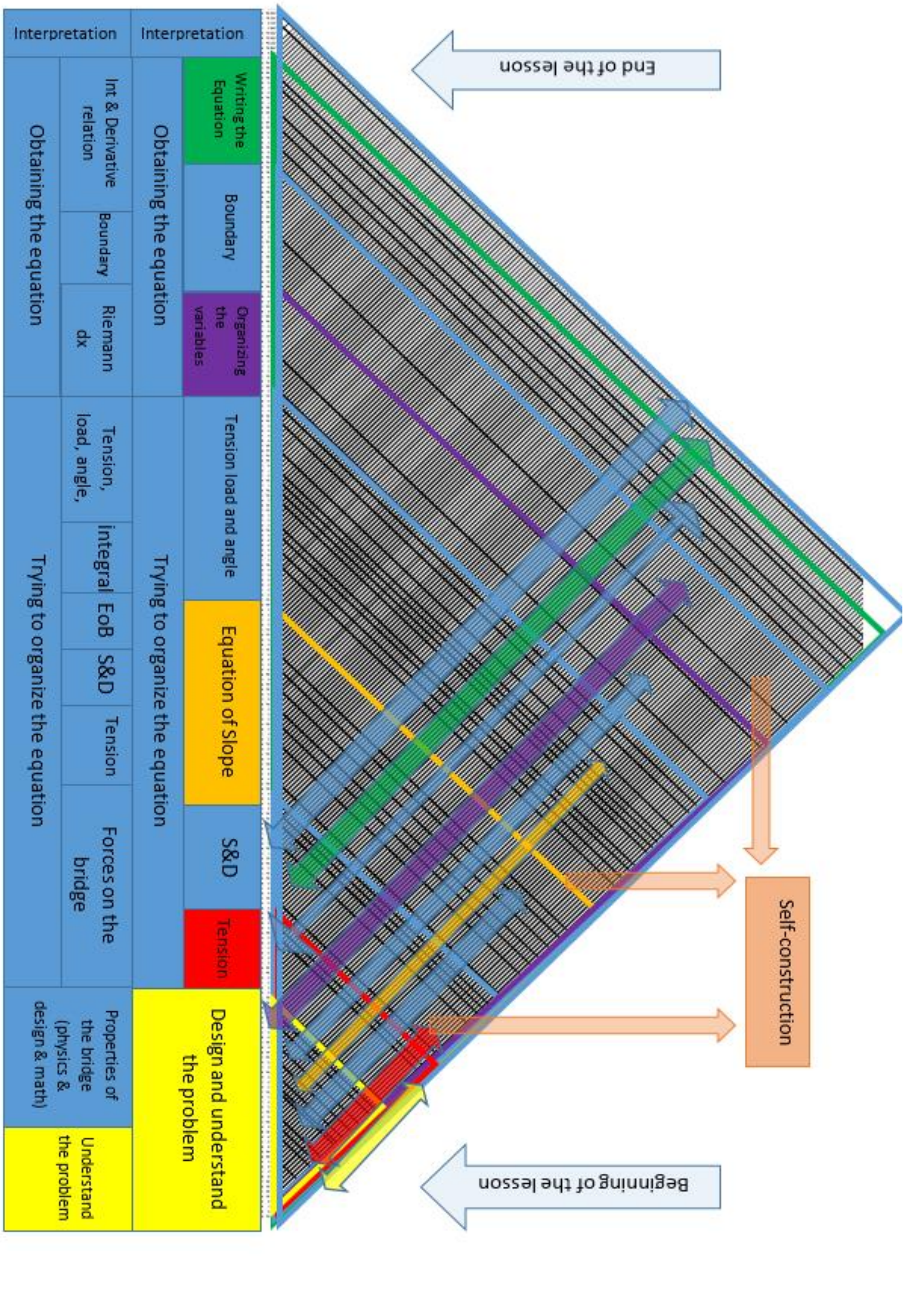


Figure 4.29 Betül’s Linkograph in activity 6

and the second one was having a lack of knowledge about covariational reasoning. Thus, she was not able to organize the accumulation and their changing simultaneously. This situation continued until the organizing of the organizing the variables' codes.

Mostly she tried to use the equation of slope to solve the problem, however, she was not able to write the equation. There were two reasons for this; the first one, Betül, could not define the variables; hence, she could not organize the formula, and the second one was lacking of knowledge about covariational reasoning. Thus, she was not able to organize the accumulation and their changing simultaneously. This situation continued until the organizing of the variable's codes.

Realizing that the tension changes according to the angle and obtaining the formula with constant c and substituting the conditions into the equation were the benchmarking for Betül to relate the whole process through determining the variables. Also, she realized that tensions in the ropes change according to tensions Tx and the bridge's weight.

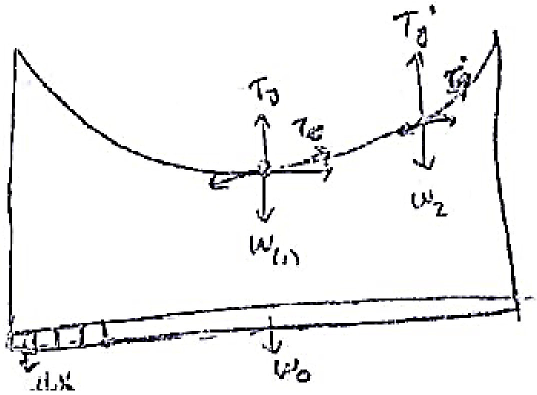


Figure 4.30 Betül's drawing about forces on the bridge

By drawing the forces' free body diagram, Betül tried to determine the variables that she needed. As seen from Figure 4.30, she could not realize that a change in the angle affected the rope's tension. A question was directed by the studetns in the class about how vehicles affect the bridge when they go through the bridge. Since the weight of the vehicle is distributed via ropes to the towers, she realized that the number of the ropes and the angle affect the bridge's shape. She expressed this process through the discussion as "Of two points. This is our suspension bridge; let us have ropes like this downwards, and let us call the downward component bx. There is also a force over there. Moreover, this breaks down into its components. There is a cosine component right there. It leaves here as well. How is t over there? Here again, let us say a vx, I am writing these vx in general. Again, there is an α angle here. T is here again, (thinks) here is equal to the slope of this here, so what does this give me (thinks) its derivative at that point." After this, Betül started to listen to her groupmates' discussion. Then she realized that she knew that she knows that the derivative at a point, then with the researcher's assistance.

Researcher	If you know the derivative, how can you find the equation?
She answered this question as taking its integral	
Researcher	Why do you need integral?
Betül	Hmm actually, this derivative is the slope of T, and it depends on the W and T_x
Researcher	Ok, how can you write it
Betül	The slope of T equals this (Writes the figure 4.30)
Researcher	OK, so what do you need?
Betül	Actually, this is the instantaneous rate of change, but I need integral them.
Researcher	Why?
Betül	Because slope T is the antiderivative of slope T and T is an accumulation function and $W/T_x$is the function that describes the rate of change of slope T."
Researcher	How can you say this?
Betül	From Fundamental Theorem of Calculus.
Researcher	OK
Betül	We said that integral is an accumulation and derivative rate of change, accumulation's instantaneous rate of change function gives us the function's integral. We can say $F(b) - F(a)$, but here it is indefinite integral, so I wrote constant C.
Researcher	What does C means in there?
Betül	Hmm, there is a lot of different equations, and they do not have constant and by definite integral define it
Researcher	But here, we have to find the specific equation of the bridge
Betül	Hmm, yes
Researcher	So, how do you find it?

Betül: The general equation of the slope of $2x + 3$ is if I take its integral, I find its function. We know that we call the antiderivative the inverse of the derivative, which is integral, and if I take the integral from both sides, the slope at this point is 3. I can find the slope that point at that point. If I take the one here, I can find the whole curve with the integral rather than adding all the tys one by one. If I integrate the slope of T and that from both sides here, I would find the equation for that curve with the initial conditions. We assume this in the center of the coordinate system, so I know the initial and the boundary conditions. When I substitute these into the equation, I can find it (see in Figure 4.31)

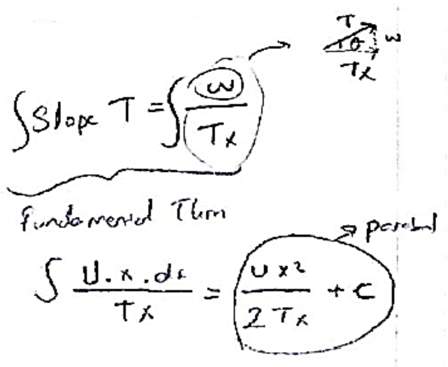


Figure 4.31 Writing Equation.

It can be understood from Betül's explanation that she explained the relationship between the derivative and the integral and interpret the Fundamental Theorem of Calculus correctly.

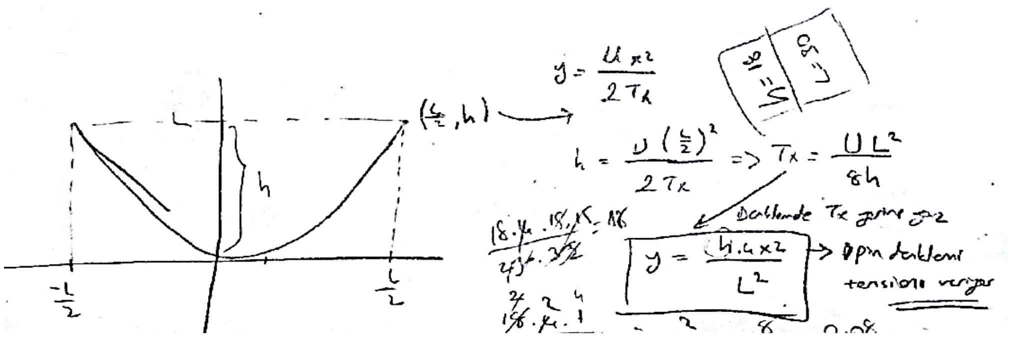


Figure 4.32 Boundary conditions

Taking into consideration these objectives, Betül achieved all of them at the end of Activity 6. Moreover, she was able to explain the various representation, and she was flexible during the transformations. Although she knew the Fundamental Theorem of Calculus and the Riemann Sum, she did not deal with some details and treat the integral as an object. Moreover, this activity was a simple initial value differential problem, and she solved it. That means the Fundamental Theorem of Calculus as an object for her ad and the process for the differential equations. In the next chapter, the final concept image of Betül was discussed.

4.1.2 Betül's Final Concept Image

At the beginning of the study, Betül had a partial-primitive concept image (PPCI). Activities are conducted to modify her concept image. Data analysis showed that Betül has a mathematically correct concept image at the end of the study.

1) *She enriched the personal definition of integral and gave its analytic definition as well.*

In her conceptual test, Betül defined integral considering all-inclusive relations between Riemann sum and FTC. She also refers to the analytic definition of integral. Moreover, she defined the integral as the instantaneous rate-of-change of accumulation of function based on the FTC, and she stated that the definition of the area under the curve is a limited definition considering the examples in the course.

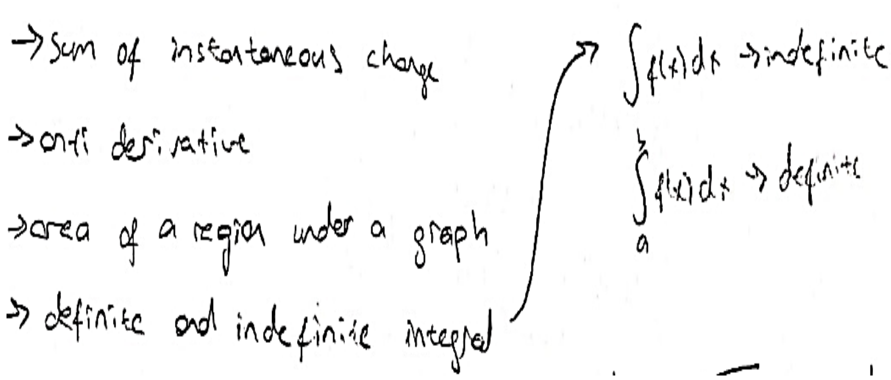


Figure 4.33 Betül's Definitions

In Figure 4.33**Hata! Başvuru kaynağı bulunamadı.**, the “sum of instantaneous hange” can be perceived as incorrect, so Betül was asked to explain what she meant by that. She explained that “*in an interval, we divide it so tiny rectangles like thinking as if they were lines and we sum them up. I mean it is a sum of accumulated change*”. She also could write to them algebraically and explain them.

2) She learned Riemann’s sum comprehensively.

A contextual problem about a stuffed gorilla was dropped down from the top of the building to determine whether the students know Riemann sum, and a speed vs. time graph is given. Betül solved this question by using Riemann Sum, and she also mentioned the type of Riemann sum and reducing the error with smaller intervals (Figure 4.34).

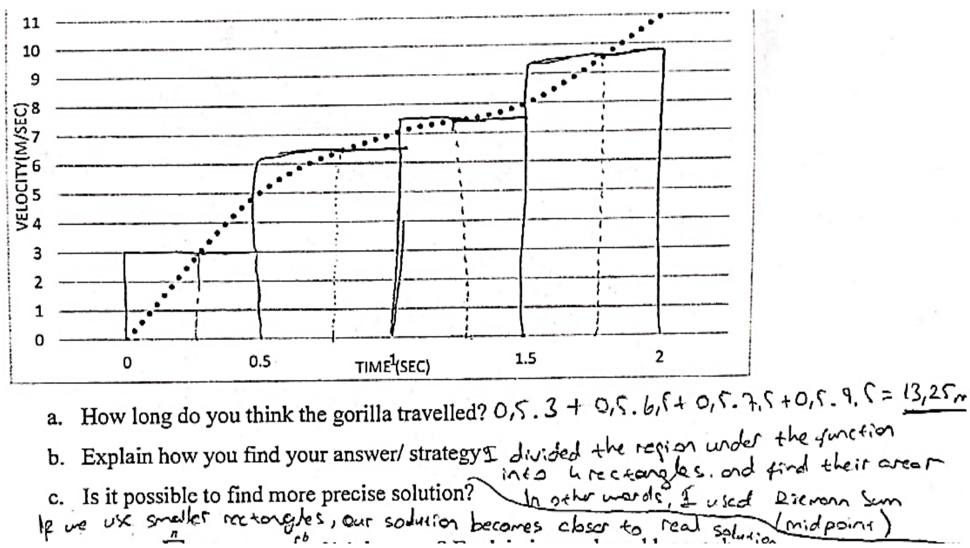


Figure 4.34 Betül Riemann Sum Solution

In the interview, she solved inheritance question related to Riemann sum, and she made a distinction and said that “*if we can find know or find the equation of the function we can find the area by using integral if we do not know by measuring we can calculate the area with Riemann Sum*.” Moreover, she explained that to reduce the error, to get a more accurate result it is better to use the midpoint. In the initial interview, she could not place the coordinate system and said that “*I cannot solve* *this question because I do not know its lengths or function.*” However, in the last interview, she replaced the coordinate system and determined the variables quickly.

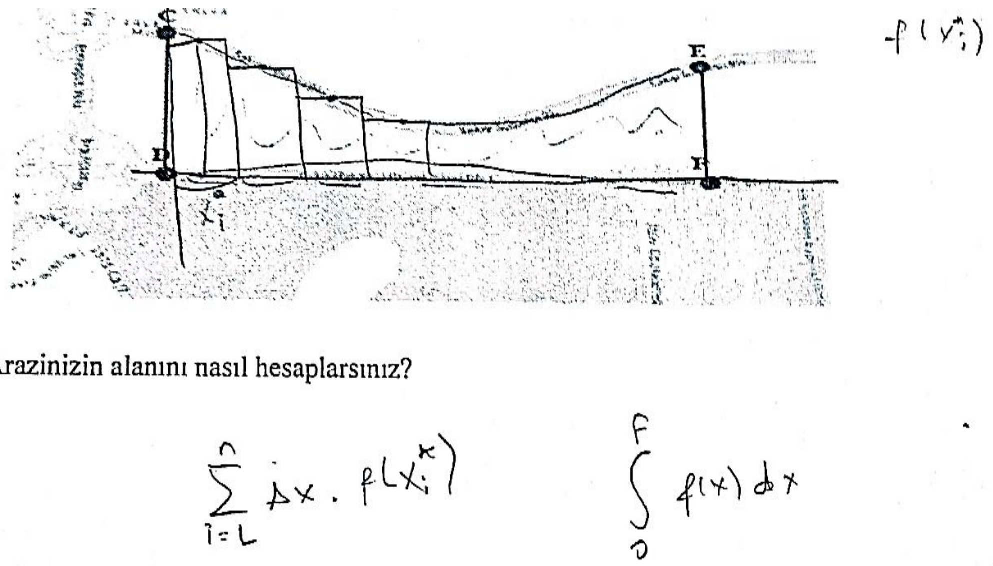


Figure 4.35 Betül’s Solution

3) *Betül explained the negative area with understanding the reasons lying under it.*

Previously, Betül could not explain the negative area: however, in the post-interview she answered as follows:

Betül	The reason area is negative when making a Reimann sum, we are looking at the height directly. The values here are negative *(referring to the y-axis)*, so our values are negative. Since the area cannot be negative, we have to multiply this by a minus.
Researcher	Why did you think like that?
Betül	y = 0, so that minus comes from here. We subtract y=0 and this curve.

4) *Betül transferred integral concepts into real-life situations.*

In the integral questionnaire, she was asked to find how much work is done when an elastic spring whose spring constant is k is extended 4 cm. Previously, she substituted the given directly into the formula; but this time, her reasoning was different. First, she drew a graph for $W = F.x$ Thinking F is constant, then she applied this knowledge to spring problem, then by using integral she solved the problem correctly.

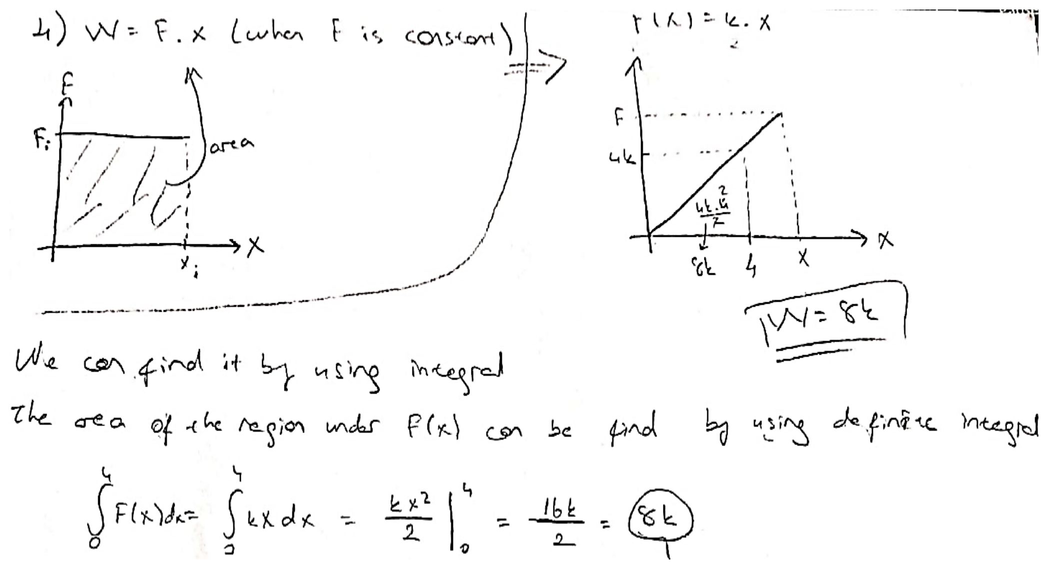


Figure 4.36 Work Problem Solution

There are two critical points; Betül interpreted the real-life problem in terms of integral and realized that force is dependent on x variable, and by changing x, F is affected, and she could define the variables correctly.

5) *She related and explained Riemann Sum and FTC inclusively.*

Through the interview and conceptual test, she explained FTC and Riemann Sum. Also, in the concept map, she related the accumulation of rate of change with FTC and integral. Also, she related integral with Riemann sum and infinitely small, which are the main and the sub-concepts of the integral. The question was about relating the Riemann Sum and FTC expressed algebraically. It can be seen from Figure 4.37 that she explained correctly and comprehensively.

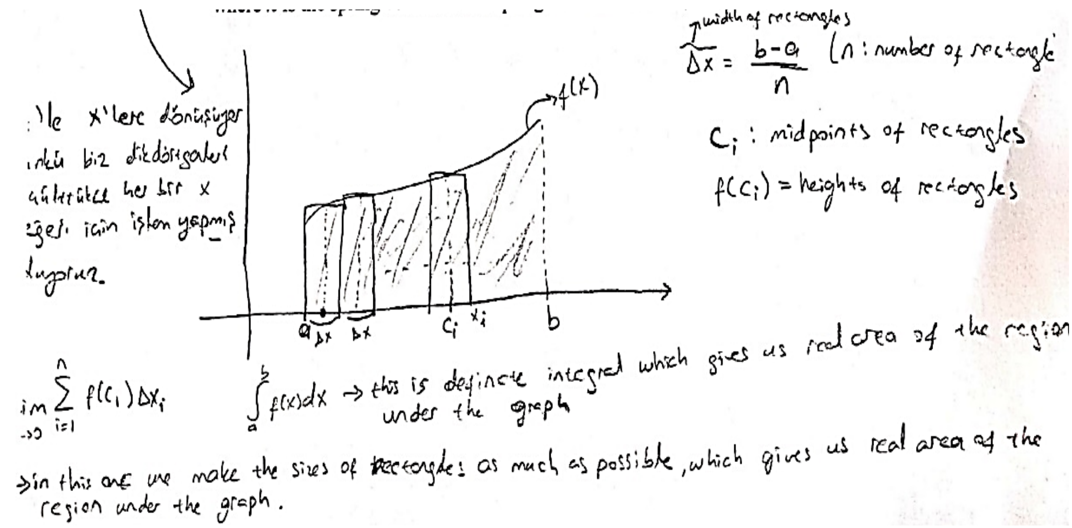


Figure 4.37 Betül Riemann Sum Solution

Taking into consideration developmentof Betül through the activities, concept map, conceptual questions, and the interview, she reconstructs her concept image about integral and reached an accurate concept image. In Figure 4.38 below, her final concept image can be seen.

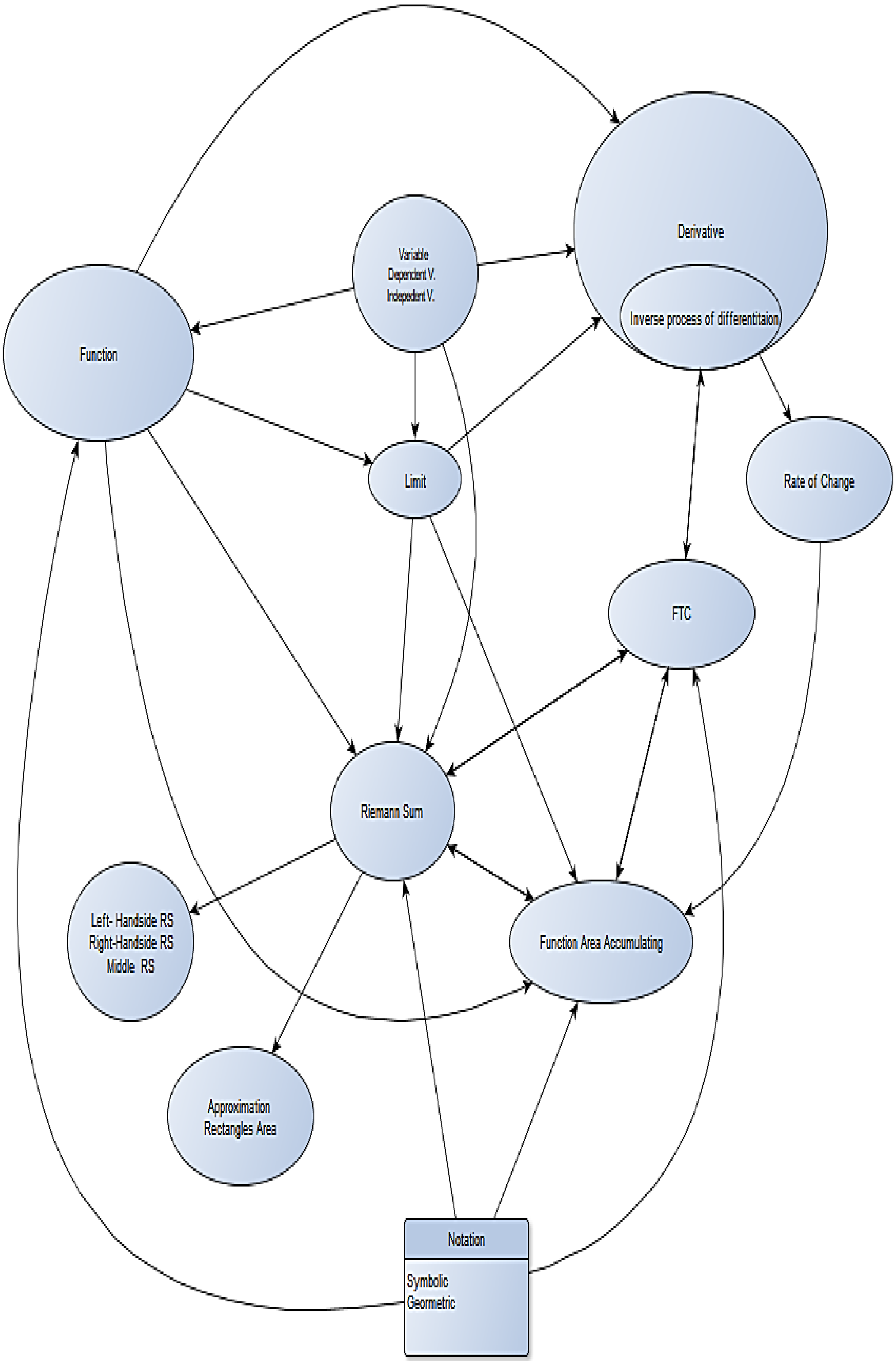


Figure 4.38 Final Concept Image of Betül

4.2 Developmental Phases of Funda

4.2.1 Funda’s Initial Concept Image

Funda has a discrete concept image (DCI). In this type of concept, images, Funda divided knowledge structures into sub-concepts and conceptualized these, either mathematically accurate or mathematically inaccurate. She structured all of her knowledge on integral as departmentalized, and neither mathematically accurate nor mathematically inaccurate knowledge is built on each other. Thus, she did not relate these structures between them. The properties of her image are;

1) *She explained integral through the geometric definition of “area under a curve.”*

Funda’s definition of integral was area-oriented, as Betül did. However, different from Betül, she gave a general description without mentioning the boundaries of integral and function.

Figure 4.39 Funda’s Integral Definition

She illustrated integral geometrically and symbolically in her concept map, and she did not define integral analytically.

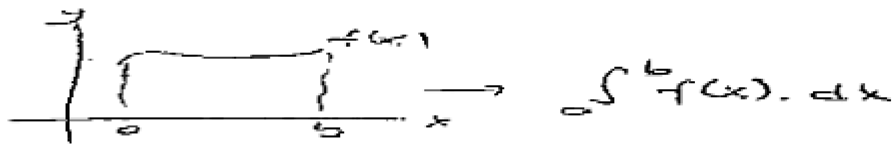


Figure 4.40 Funda’s Geometrical Representation of Integral

2) *She saw the whole concept as integral, and even she has an intuition about Riemann sum, she could not relate two of these two terms.*

In the **stuffed gorilla** problem, Funda stated that she calculated the function's integral, different from Betül; Funda divided the graph into parts then calculated the area.

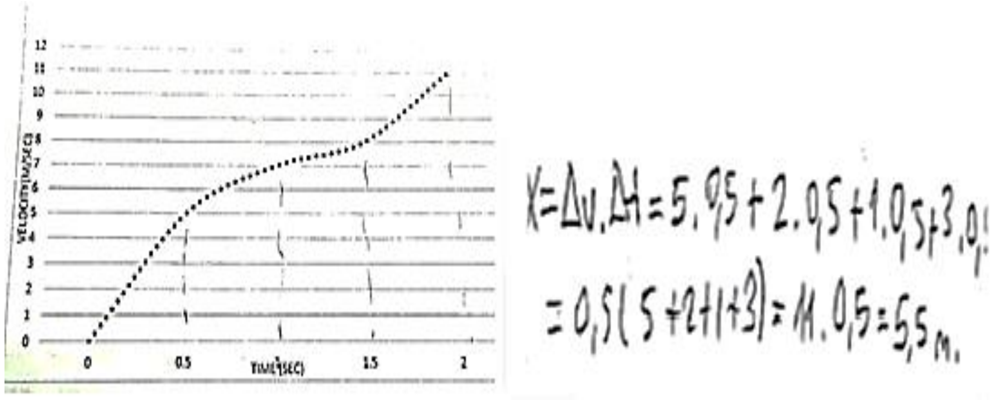


Figure 4.41 Funda's Solution in Question 1

In the interview, she stated that she could find the area under the curve by partitioning the graph. However, she did not specify or express the Riemann Sum term explicitly. Hence, Funda has an intuitive Riemann Sum as a partition. Since she mentioned before that she solved this kind of problems in the physics course. Betül was familiar with this kind of problem, and she defined as integral both area under a curve and antiderivative; she combined all of them intuitively not formally establish a connection between Riemann sum and the integral.

3) *Funda had a procedural approach to explaining antiderivative and area under the x-axis (negative area) concepts without making connections to the meanings underlying them*

In $F(\_\_) = \int_a^x f(t)dt$ problem: Funda also thought it would depend on the integrand's variable, and the boundaries should be constant. Moreover, when she was asked to explain dx, she made the same explanation with Betül, and she related it with derivative. Nevertheless, this relation was not so clear for her because she could not remember some details.

There is a relation btw. differentiation. When we took the integral, we actually used differentiation of some expression

$$\int 3x \, . \, dx = \frac{3x^2}{2} + c.$$

Figure 4.42 Funda's Explanation about Integral Variable

Researcher	So, is this dx related to the derivative or something?
Funda	Yes.
Researcher	How?
Funda	Derivative of x, well…when we take the derivative of x… it was returning. Sorry, or do we take the integral of the derivative? It was like…zero but not zero... Derivatives ...
Researcher	What happens if you take the integral of dx?
Funda	Do we take the integral of derivative it was happening … oh, I am confused. I do not remember all of them

Another point for Funda that was similar to Betül was that she stated dx shows according to what; the given function is integrated and related to the derivative. Nonetheless, when I asked her whether we have to write dx in the integral form:

Researcher	Well, if I write this question, integrate the following question according to x and never write dx . Is it okay?
Funda	No.
Researcher	Why?
Funda	dx is the opposite of the derivative because that is what you get this integral, and that cancels the dx. I guess it was necessary, so we cannot say that.
Researcher	Do you know why?
Funda	No, I do not know.

Another point that is worthy of mentioning is her interpretation of the negative area. She calculated an integral between the interval $(-1,1)$ without partition since her integral definition was "*the area under the curve*." When I asked her to take the integral of a function part by part, she found a negative area. She reached the same results with opposite signs for each integral. First, she thought that the negative area symbolized the placement through the region like under the $x \, axis$. Then she knows that area cannot be negative but, she cannot explain why.

Researcher	Why did you find that?
Funda	Well, that is because it was under x . Why is it like this? The minus is just showing that this is the bottom line, then…
Researcher	I get it. Can you calculate this area?
Funda	Yes. They are the same. If we look at that, why did you cancel each other? They cannot cancel each other since we find the total area.
Researcher	Ok...
Funda	Then, we will not only look at the negative area. There is a value of minus here.
Researcher	What value does it have?
Funda	How does the field have a negative value? I am confused now. Why is the field negative? So, we only do not think of it as an area?

4) *Funda used real-life situation examples to explain the integral meaningfully. She transferred her knowledge to another discipline to solve the integral problem in some cases.*

It is observed that mostly Funda went through the examples of the concepts, and she tried to link the given problem with that example and tried to interpret according to it. Her own constructed concept map and her statements during the interview can be evidence of this situation. She gave an example in her own constructed concept map as "*we can use it to predict something according to given values and variables. For instance, we know the frog population at a time, then we put the environment snakes, and we can try to find the formula for the frog population at any time. Also, if we have a pool and two taps, one works to fill the pool, and the other works to empty the pool. We can predict the volume of the water in that pool at any time* t". Moreover, even in finding the position of car A at $t = 1$, Funda answered the problem doubtfully with her concept definition and the procedural knowledge, which she remembered from physics courses.

Funda	I thought like this. Because it is a speed-time graph, it is going

	to give you this thing *(displacement)*. It will provide graphics with movement, which is probably the area under the graph.
Researcher	Well, you said it is the area under the curve. You are not sure why? Because the subject is integral, so you said it automatically.
Funda	Yes, I said that a little bit of that, but mostly, as we were doing like that continuously in the linear ones, the speeds - time graph. I mean from physics
Researcher	Do you know why did you do like that?
Funda	Hmm … Well. No.

Finally, in the **work problem** as Betül, she also substituted the spring extension to the force formula, then she applied the same to the work formula.

5) *Funda intuitively knew Riemann's sum. However, she did not relate it with integral, and she constructed these structures separately.*

During the interview, if there was a relationship between concepts, she was asked to show and explain the given concepts' relationships. She stated that she did not hear about the terms "accumulation of rate of change, infinity small." Moreover, as its name implies, she related FTC with derivative and explained that "*derivative is the fundamental concept in calculus*." She connected Riemann Sum with the area, function with a variable, lower and upper limit with integral.

The critical point was that she did not connect Riemann Sum with the limit concept and the integral. She could not relate the rate of change with any concept since she thought that it was Δx and may be related to infinity small. Finally, she associated work and motion with integral and explained that since she saw its examples. It can be seen that Funda had a conflicted understanding of integral in some problems she applied her area-oriented definition, and in others, she did not use it. Briefly, she has obvious problems in antiderivative, negative areas, interpreting the symbolic representation of integral, dx. She divided integral into sub-concepts, and they are not inter-related. Considering this point of view to provide a general view to the reader, Figure 4.43 represents Funda's initial concept image.

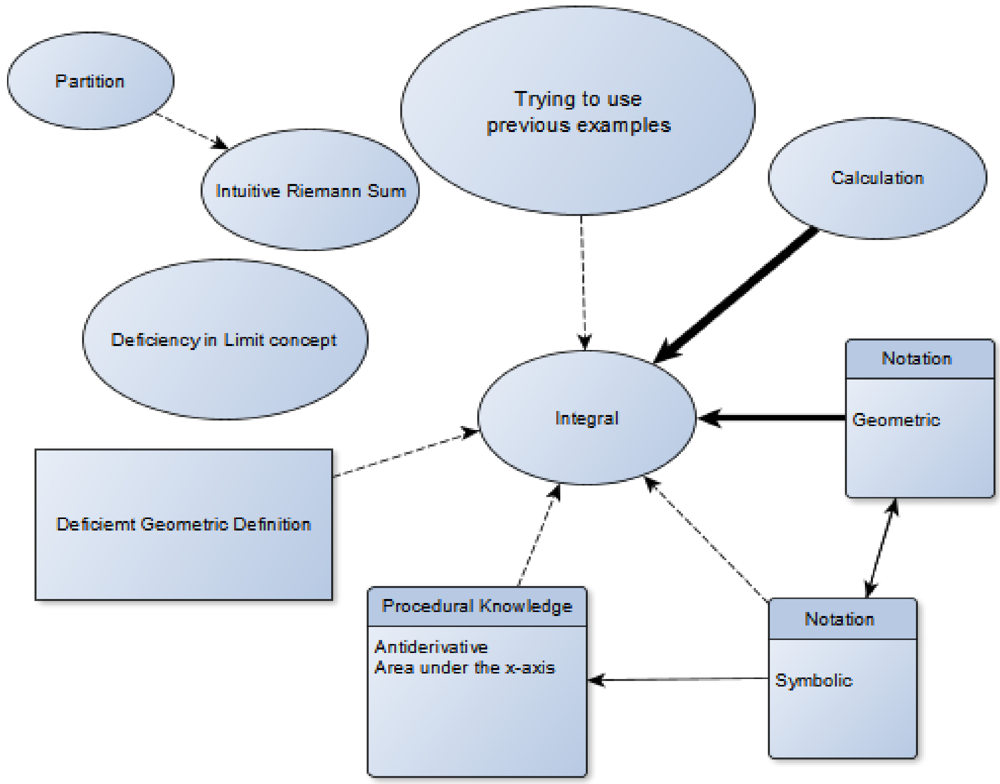


Figure 4.43 Initial Concept Image of Funda

As can be seen from Figure 4.43, her image is composed of separate structures such as calculation, antiderivative, the area under the x-axis, some examples, intuitive Riemann Sum. Compared with the mathematically accurate concept image, it can be more comfortable than Funda constructed her concept image on two main concepts: Riemann Sum and integral. Her Riemann Sum knowledge is only weakly fed by partition.

Moreover, the reason behind that is Funda has a limited understanding of the limit concept. Other factors that hinder her learning are procedural knowledge and deficient concept definition. Antiderivative and negative area concepts are affected by them. Although Funda has linked actively integral with geometric representation, her algebraic representation link was weak. Finally, as seen in Betül's case, Funda has powerful relationships between integral and procedural calculations.

4.2.2 Concept-Image development of Funda about integral through engineering Activities

Developments in prospective mathematics teachers' knowledge about integral were presented into two main sections: developments in prospective teachers' knowledge about (i) Riemann sum (ii) Fundamental Theorem of Calculus. These two main sections are also categorized according to Sfard's "Three-Phase Theory," namely Object-Process Perspective, and it has three main phases: interiorization, condensation, and reification. According to gained pre-determined objectives for each phase, students' development process is categorized. Whether the objective for the related big idea about integral is gained or not is determined from each participants' linkographs obtained from group discussion discourse.

4.2.2.1 Developments in Funda's knowledge about Riemann Sum in terms of Object -Process Perspective

According to the initial concept image of Funda, even she did not know what exactly the Riemann Sum is, she had some ideas about the concept, and the reason that prevents the development of the Riemann Sum is that the limit concept has not been learned before. Thus, she constructed integral and the concept separately. As aforementioned, her concept image around the integral concept and had no idea about FTC as Betül. Hence, she was not able to understand the between derivative and integral. Finally, she believes that the area under a curve is calculated by knowing the curve's function; otherwise, it could not be calculated. The most remarkable point in her concept image was she attempted to resemble the solutions that she experienced before, and this behavior was very dominant in all her moves. In the aforementioned previous section, due to the deficiency of essential concepts in Funda's concept image, it is determined that she was not at the interiorization stage.

During the application of engineering design activities, these difficulties that Funda confronted are considered, and appropriate feedback and scaffolding were given to her individually. Furthermore, her knowledge about Riemann Sum was developed through the engineering design activities and reached the reification process since she started to interpret and explain the Riemann sum as a whole and built a connection between Riemann Sum and integral. All the development processes of Funda in interiorization, condensation, and reification process are considered remarkable events, and under that headings, noteworthy points were given.

4.2.2.1.1 Interiorization Stage

In the first stage of the hierarchy, interiorization, students are in the lowest stages and become familiar with the concept and have limited skills about the concepts. In this regard, pre-interview data showed that Funda could not calculate unknown irregular areas by dividing a given interval into rectangles and calculating these rectangles areas, and adding them up. She was reasoned that by dividing the bounded area, she could find a more accurate solution. However, she was not aware that calculating the area with the limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten there will continue to be a "next piece," the accumulation is not complete. Furthermore, according to her, the approximated area cannot be calculated with rectangles without knowing the function. Hence, she did not know the Riemann Sum clearly and did not aware of the generalization process. However, she was very good at connecting mathematics with the real world. In this regard, she was not at the interiorization process.

Throughout the engineering design activities, Funda began to conceptualize calculating the unknown irregular area with rectangles. Moreover, she realized how approximation changes the area and how to identify the variables from real data in Activity 1 and Activity 2. In this regard, how and when Funda realized, conceptualized, and engaging through Activity 1 and Activity 2, how she met the interiorization phase's objectives was presented under the remarkable events heading.

4.2.2.1.1.1 Remarkable Event in Activity 1

In general, in this activity, Funda behaves as a moderator between İlke and Betül. Since her skill in design thinking and mathematical thinking was nearly the same, she was able to explain the mathematical part to İlke and also in the same way she also explained to Betül when she was not able to understand the design part. Moreover, during the discussions, she contributed when necessary, such as Betül and İlke did not understand each other. Hence, during the activity, when Betül had difficulty understanding the design part, she helped her understand, and the same for İlke. At first, she listened to her friends for most of the task and then declared her ideas or points that they have conflict. When she did not understand the design part, she asked İlke, and when not understand the mathematical session, she asked for help from Betül. In Figure 4.44, blue triangles are represented as small concepts that Funda went through it. It can also be seen that Funda articulated her ideas as chunked, then she linked with other thoughts, and she constructed the concept as below.

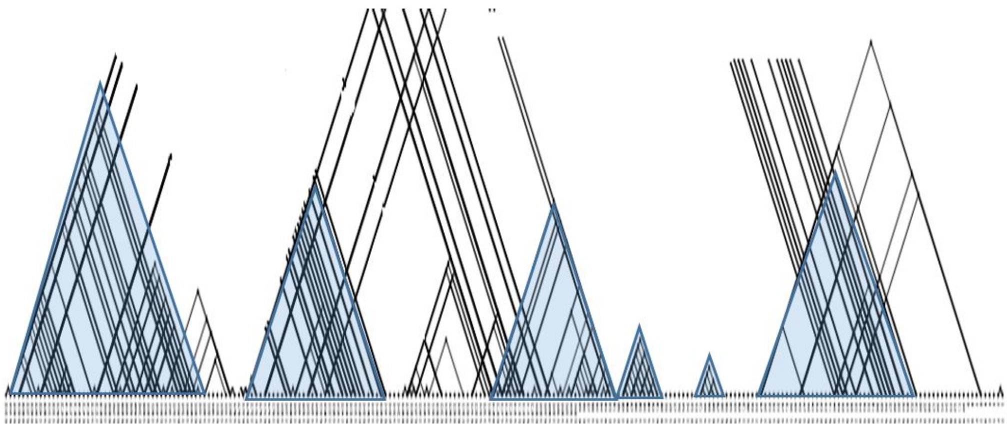


Figure 4.44 Funda's Learning Pattern

As shown in Figure 4.46, Funda's contribution to the discussion is formed in two sections; design and mathematical discussion. Through the design discussion, her codes were synchronized with the discussion codes: UP, shape, balance, buoyancy, max load, and material. Since Funda behaved as moderator and Betül contributed to the design discussion part very limited, she and İlke mostly discussed together.

In Figure 4.45, during the mathematical discussion, while part second layer represents the general flow of the discussion, the third layer shows the concepts that Funda concentrated, measurement of the canoe, construction of approximation, integral of a function, insist on finding function, realization process, Riemann Sum (RM), and physics connection. As mentioned earlier, the tendency to solve the solution of the known concepts that Funda knew, she was very dominant in this activity, and this situation affected Funda both positively and negatively in terms of producing different solutions. However, this also prevents her from seeing the critical point and linking parts that form a big picture. For instance, her group friends started to think about Riemann sum intuitively; she still insisted on calculating the area from known function, which caused a hindrance in seeing and accepting different solutions.

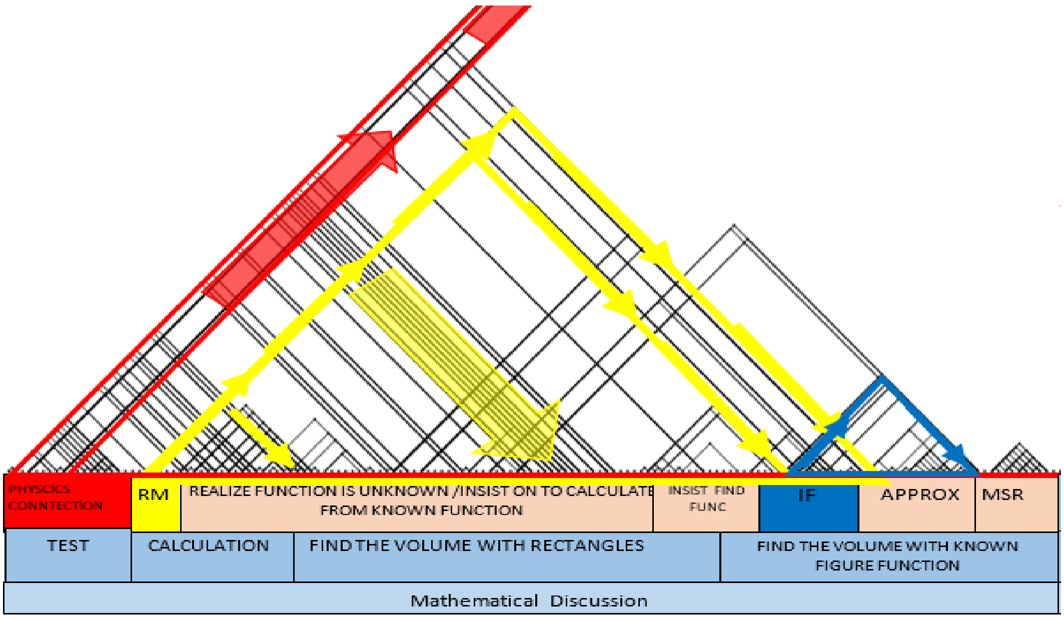


Figure 4.45 Mathematical Discussion Part of Funda

In the following conversation, at the beginning of the conversation, Funda tried to find the area approximately by imitating the shape of the canoe to a prism. In her initial concept image, starting from a known situation was dominant so, in this activity, she was so resistant to solve the problem by using her previous experience. In this way, she produced different solutions; this was advantageous for her and her friends in terms of thinking differently. On the other hand, it was a disadvantage that she could not reach the taught concept immediately. In the following conversation, this hindrance can be seen

Researcher	How will you calculate the volume?
Betül	Teacher, we have no idea
Funda	Can we measure it approximately?
Researcher	Yes, how can you measure it?
Funda	For example, I can assemble it in a known shape then I can measure it. For example, I can assume this as a prism according to its wide side.
Researcher	What else?
Funda	We can divide it into two parts, and we can take this bigger part to the bottom, where it is smaller than this part.
Betül	I did not understand
Researcher	OK, are you sure that you can find the answer?
Funda	What else can I compare? It is not like a cylinder
Researcher	What else?
İlke	How about using rectangles?
Researcher	OK, how can you use rectangles?
İlke	We will think of rectangles in such a way that we will lay rectangles by expanding or narrowing them.
Funda	If we draw a line and create the best curve according to the rectangles.
Researcher	Here, you do not know the function. Can you explain to me how you can find the equation of the best curve?
Funda	Hmm, we can assemble the curve into a circle; we can find its diameter.
Betül	I do not understand; what do we think of as a circle?
Funda	We can measure its circumference; then from the formula, we can find its r
Betül	Is this a circle?
Funda	A-line passing through 3 points is not a curve.
Researcher	OK, you said that you have a real measurement of the canoe, then how can you find the area of this part?
Betül	We can use unit square; then we can find the approximate area

Funda	I could not remember.
İlke	We can find with rectangles, it will be symmetrical already or say, for example, these rectangles will constantly be shrinking, for example, this will be the same there will be two, for example, the other one is going to be 0 here, so we will be like r.
Funda	How will these lines be?
İlke	The area below is a curve.
Betül	But there we know the function.
Funda	Then the circle's graphic is not?
Researcher	But is this circle.
Funda	It is a circle, isn't it?

.

Another essential factor in Funda's learning process is Funda's tendency to learn the answer step by step to such questions. She constantly asked the researcher to give a sheet that shows the steps that they will go through. Since she was not familiar with this kind of teaching, the instructor had to give her much scaffolding. After the scaffolding process, she constructed the Riemann Sum. As Betül, she also thought that those measurements were different. When she realized that her measurements are the same, she understood that she was finding the canoe's real volume. The moment she made all the connections about Riemann Sum, she realized that her measurement was the same. Figure 4.46 yellow lines show the connection process after Funda's realization and her construction of the process of finding the approximate area of a canoe with an unknown function.

Funda	I do not understand where the curve will come from.
İlke	Look, we will measure the canoe, then we will draw it on paper.
Funda	Then you will only measure here, but the base of the canoe is not like that.
İlke	Measure it ok; then we will draw on paper we measure from there, they will be the same.
Funda	Let us see if it is the same, then.
İlke	Look, we will take the real measurement, then we will calculate its area using rectangles. I will assume here as the origin and here is 10.5 cm and here is the same.
Funda	Yes, exactly, we measured 21 cm.
İlke	Okay. By drawing rectangles under here, we can measure them.
Funda	Ah, Since the rectangle's size will not change at all, we

	will draw something where it points to this so that you will draw the same rectangle here, then. We will divide by four. Let us take a part of it, rectangles like this, but you will look from here.
İlke	Exactly.
Funda	Hmm, then we will multiply it by four, then we will multiply by height. Got it.
Betül	I did not understand how we will find the height of the rectangles.
Funda	Look, since the rectangles' height are the same as the real height, we can measure it with the ruler. We divide them 1 cm, right? So their area will be the 1 multiply its height.

While Funda did not have any difficulty transforming images 3D to the 2D, she had a problem in imagining when transferring the curve 2D from 3D and did not understand how İlke replaces the rectangles the canoe in different planes.

At the end of the lesson, they calculated the buoyancy force and decided how many grams they would propose. Funda did not understand where to use g. After calculating and decided to determine how much their canoe would carry, she connected the design part and the mathematical discussion. She explained the process in her reflection paper as "*We took the surrounded wire and drew it to paper. It was an elliptical shape. We divided it into four equal parts. It was enough to find the area of only one of those parts. Now, we had a curve which we do not know its equation. We divided one of its axes into equal-sized parts and drew rectangles beneath the curve with equal-sized parts bases. We calculated the areas of those rectangles so that we can get a close value of this part's area. Then, we multiplied this part's area by four and found the area of the whole base. Moreover, again, we multiplied with the height and found our canoe's volume, which was 380 cm^3. The weight of our canoe was 27 g. So, we subtracted 27 from 380 and found approximately 350 cm^3. Thus, our model could carry until 350 g with a little margin of error.*"

It indicates she was making a connection to why they had to find the canoe's volume and why shape and volume were necessary to solve the problem. In brief,

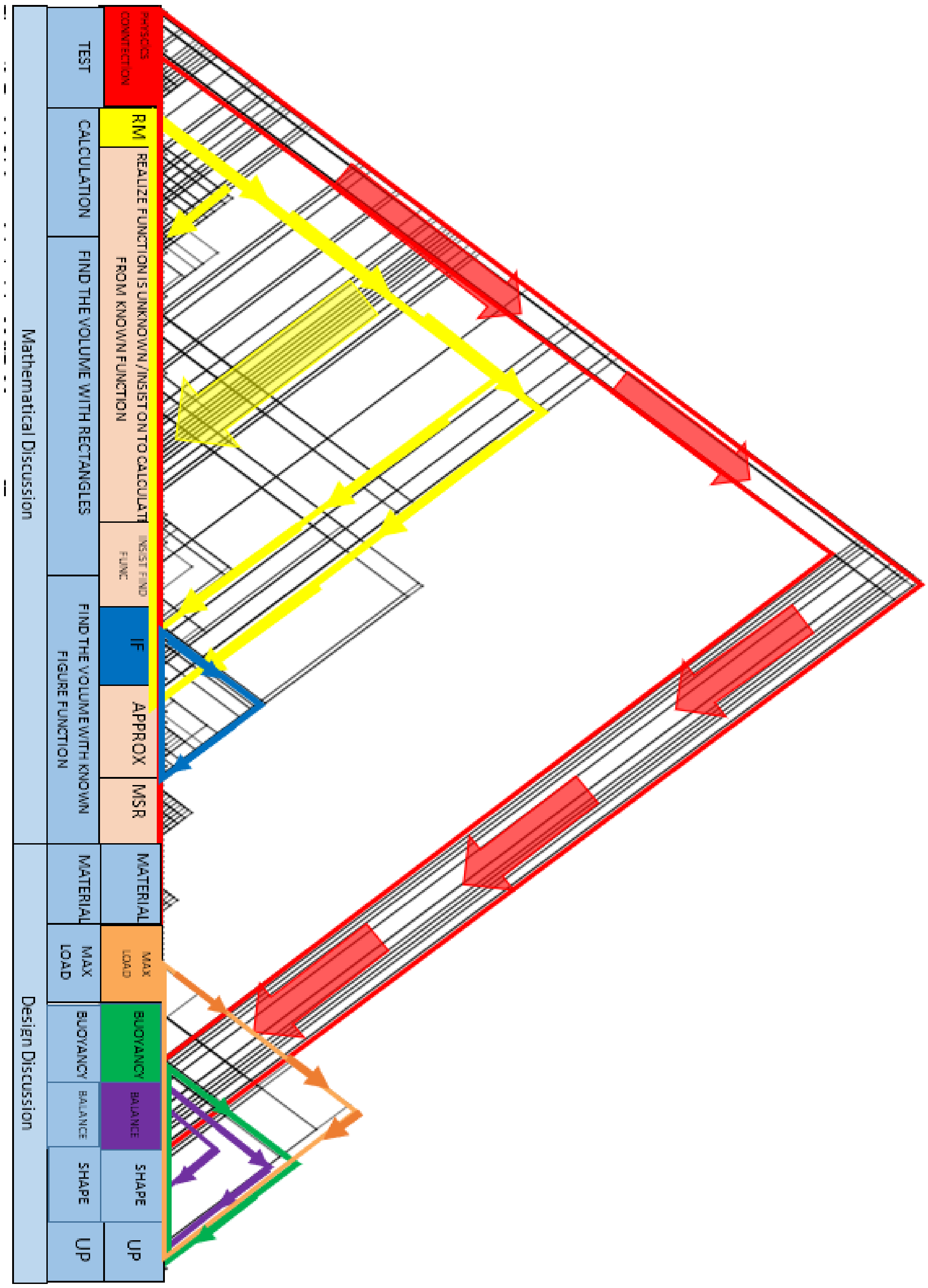


Figure 4.46 Funda’s Linkograph in Activity 1 MSR: Measurement IF: Integral of function

through the activity, Funda calculates unknown irregular areas by dividing a given interval into rectangles, calculating these rectangles areas, and adding them up without knowing the function; with the rectangles' help, the approximated area can be calculated.

Although she was able to conceptualize being aware of calculating the area with the limited number of rectangles will not give an exact solution, she could not be aware that by increasing the number of the rectangles, the more accurate calculation can be obtained. As a result, Funda was not able to reach the interiorization process. The reason Funda could not reach the interiorization process is that she has problems understanding the limit concept. Since she did not understand the term "$n \to 0$". Thus, in her mind, there are two types of thought; in the limit, only integers can be used, so it does not give the exact value when we use it in Riemann Sum and could not understand that when n goes to infinity, the length of intervals goes to 0.

For these, she was not able to comprehend "calculating the area with the limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten there will continue to be a "next piece," that is, the accumulation is not complete." As a result, she was not able to express this mathematical process algebraically hence generalization objective was not able to be achieved.

At the end of the lesson, what Funda achieved and what she did not achieve is given in the table below.

Table 4.5 Funda's Development Through Interiozation Process

Interiozation Stage	*Reached*
Calculate unknown irregular areas by dividing a given interval into rectangles and calculating these rectangles' areas, and add them up.	+
Without knowing the function, with the help of the rectangles approximated area can be calculated.	+

Table 4.5 (continued)

Start to be aware of calculating the area with a limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten there will continue to be a "next piece," that is, the accumulation is not complete.	-
Be aware of the generalization of the process.	-
Connect with the real world.	+

4.2.2.1.1.2 Remarkable Event in Activity 2

In this activity, students are asked to design a new two-story building to be used as an art museum that would be one of the city's signature buildings. Since the building will be mainly used for art education and exhibits, they are also asked to design each story's base. The towers must remain standing for the simulated earthquake. Through Activity 2, students must build earthquake-resistant structures. While building the tower, they should know how to deal with seismic forces coming from the ground.

Moreover, they should calculate the center of mass and the center of gravity to make their building resistant to earthquakes. Moreover, through the process, they learned how to transform their ideas into practical solutions by testing, evaluating, and modifying. The mathematical objectives were notating Riemann Sum algebraically, setting up a definite integral to explain the relation between Riemann Sum notation and the definite integral.

The group discussion linkograph in Figure 4.47 showed that they made three types of discussions; design discussion was about how to design the building, the mathematical discussion was about finding the Center of Gravity (CoG) of the building and its calculations, and the design part was about the how to design the building. The codes that indicate that the discussion's focus during the discussion part was UP, appearance, mezzanine floor construction, second-floor design, the whole appearance of the building, making the prototype of their building. During

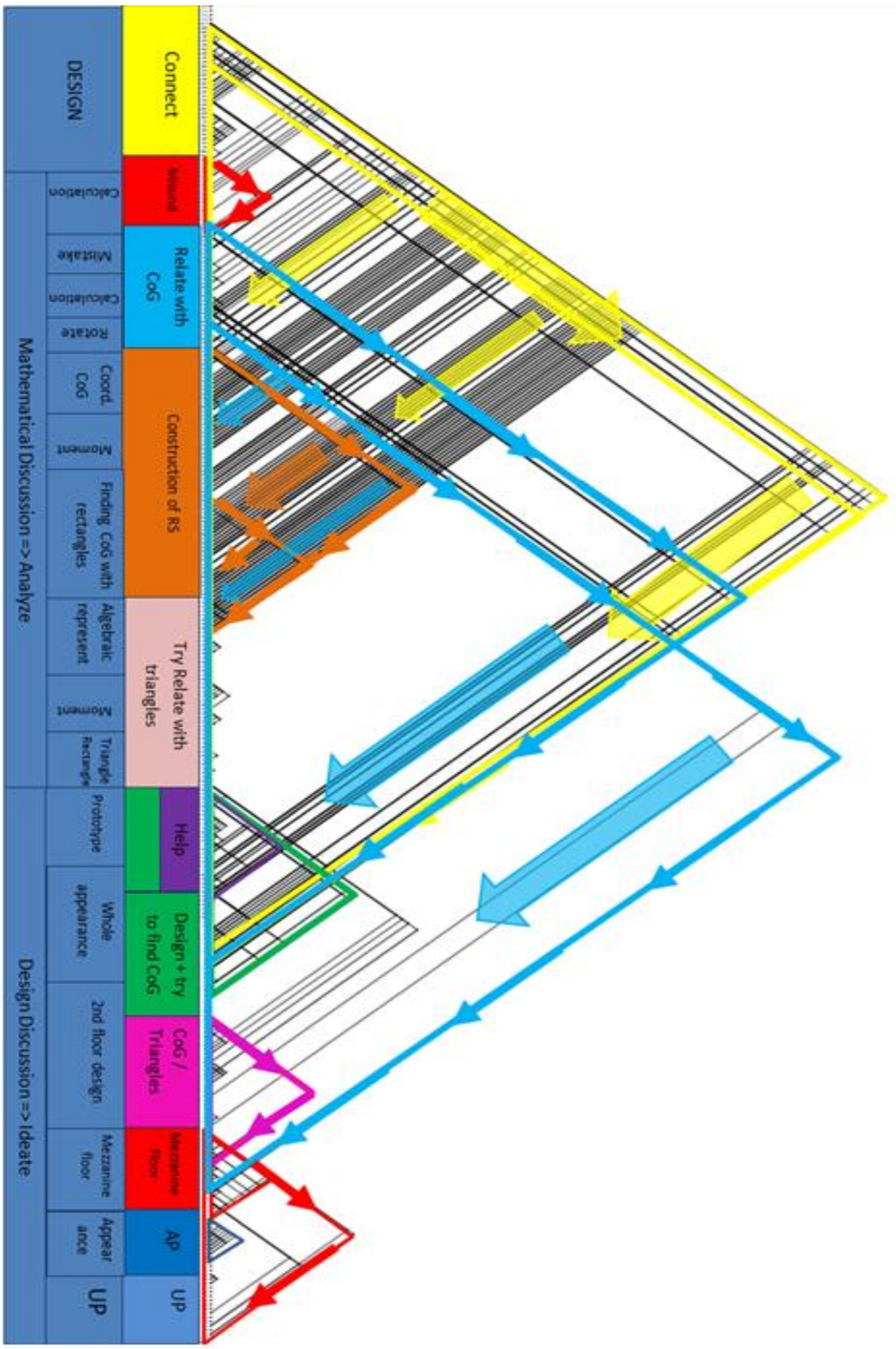


Figure 4.47 Funda’s Linkograph in Activity 2

the second Activity 2, Funda contributed the discussion codes as UP (understand the Problem), AP (Appearance), Mezzanine Floor, CoG and Triangles, Design try to find CoG, try to relate with the rectangles, Construction of RS, Relatewith CoG, Misund (Misunderstand), Connect.

As explained in Table 4.5, Funda was not able to achieve the interiorization objective. That is, she did not understand that a given area could be calculated with a limited number of rectangles and enhancing the number of the rectangles error decrease between the calculated area and the actual one since she did not conceptualize the limit concept completely, which was detected in her initial concept image.

To overcome the limit concept's difficulty during the construction of the Riemann Sum part, the researcher investigates with the group members about approximation. In Activity 1, Funda started to be aware of approximation, with the researcher's help, and the group members think about how to get more accurate results.

Moreover, in Activity 2, during group discussion, Betül explained that the more they took rectangles, the more precise they could find. Moreover, in the design part, when they designed and applied the formula into the prototype, she realized that increasing rectangles decreased the error and stated this as *"look, if we increase the number of rectangles, the error will be decreased, right? …* I mean the more rectangles means that, the more accurate results we can get". Additionally, during the construction part, the instructor leads them to think at first arithmetically or deal with numbers then tried to write that expression in general. Hence Funda achieved all the objectives of the interiorization phase.

Table 4.6 Funda's Development Through Interiozation Process

Interiozation Stage	*Reached*
Calculate unknown irregular areas by dividing a given interval into rectangles and calculating these rectangles' areas, and add them up.	+
Without knowing the function, with the help of the rectangles	+

Table 4.6 (continued)

approximated area can be calculated.	
Start to be aware of calculating the area with a limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten there will continue to be a "next piece," that is, the accumulation is not complete.	+
Be aware of the generalization of the process.	+
Connect with the real world.	+

4.2.2.1.2 Condensation Step

Sfard (1991) described this phase as "squeezing lengthy sequences of operations into more manageable units"(p.19). Moreover, she emphasized that the learner can be more skillful in thinking about the given concept in this stage. She resembled this process, a computer program that works with input-output relations instead of showing all detailed processes. That means learners do not go into details during the processes. In this step, a new concept is "officially" born when the learner confronts any difficulty that provides a new entity. In this step, learners may also connect the current process with another process and compare it within old and new knowledge and start to generalize it. Also, learners started to use different representations. In this regard, the study participants should be able to deal with various representations of the Riemann Sum, such as writing algebraic form of Riemann Sum by using their graphical representation or vice versa. They also should be able to handle the whole process within an awareness. For example, in this process, while the pre-service mathematics teachers are drawing rectangles on graphs within taking into consideration the variable of the functions, they should also explain the type of Riemann Sum accurately, differences between different types, and the margin of the error.

In this regard, the participants of the current study should be able to deal with various representations of the Riemann Sum, such as writing algebraic form of

Riemann Sum by using their graphical representation or vice versa. They also should be able to handle the whole process within an awareness. For example, in this process, while the pre-service mathematics teachers are drawing rectangles on graphs within taking into consideration the variable of the functions, they should also explain the type of Riemann Sum accurately, differences between different types, and the margin of the error.

Considering Condensation's objectives, at the end of the second activity, Funda almost reached the condensation stage. At this process, she understood that the results would be more accurate by adding more rectangles, and the Riemann Sum is an accumulation process. Moreover, Funda was able to draw and explain rectangles on the function's graphs by considering the variable of the functions. According to the left, middle, or right-hand side Riemann Sum was flexible at transferring graphical representation to algebraic representation or vice versa. However, throughout this activity, trying to resemble the solution, previous learned examples were still very active, preventing Funda from producing new ideas. The almost whole session, she attempted to combine the solution with the old one, which caused Funda to break out of the instruction and had difficulty tracing the course and connecting all pieces for the solution to the problem.

In the following section, how Funda met the condensation phase's objectives through Activity 2 was given, and how and when she contributed the discussion and conceptualized the objectives were presented.

4.2.2.1.2.1 Remarkable Event in Activity 2

As explained before in Activity 2, students were asked to design a new two-story building to be used as an art museum that would be one of the signature buildings of a city. In this activity, Funda was more active in the ideate part compared to the first activity. She started to construct her ideas into another one or combine them. In this part of the activity, she always acted like a bridge between İlke and Betül.

During the discussion about the design, she produced arguments to Betül or explained the model where she could not understand. For example, when thinking about constructing the second floor, Betül was always opposed to İlke about the unrestrained places and insisted that they prove their claims mathematically or intuitively. Especially in the red triangle in Figure 4.48, Funda tried to explain those unrestrained places would be supported with straws, and with the help of CoG of triangles, they strengthen their building. On the other hand, in the purple triangle in Figure 4.48, Funda helped and explained to İlke how to find CoG mathematically or with triangles.

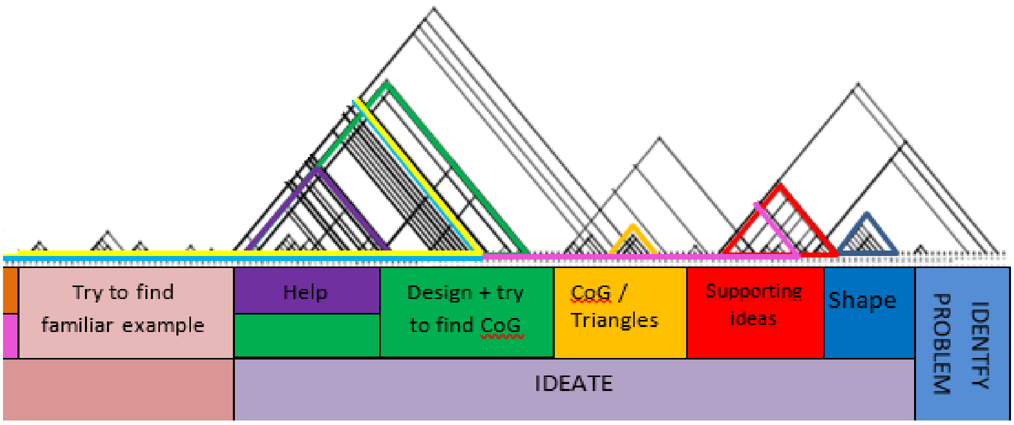


Figure 4.48 Funda's Linkograph Ideation Part

Throughout this activity, her initial concept image trying to resemble the solution to the previous learned examples was still very active, and that prevented Funda from producing new ideas. Moreover, she was still insisted on solving the problem step by step. Hence, Funda had difficulty in replacing triangles and comprehend calculating triangles CoG. It can be seen that most of the ideation process Funda focused on finding CoG of semi-circle by using triangles. However, it was asked for them to find CoG of the semi-circle within minimum error. At first, she tried to increase the number of triangles. However, she was not able to handle it. When her groupmates started to discuss finding CoG with rectangles and went through the analysis session, she still tried to find familiar examples. Since this problem was new for her, she could not find a pattern and was not able to produce any idea, and for a while, she had confusion.

With the researcher's guidance and discussion between İlke and Betül, she combined approximation, and increasing the number of rectangles decreases the error. After that, she realized that she could find triangles as in the Archimedes method. However, Funda thought that it needs cumbersome calculation then she gave up. It was essential to the point that Funda understood that accumulation is a process and adds new increments and continuously yields more accurate results. In the following dialog, it can be seen that Funda tried to find the CoG with known procedures, the researcher tried to refute her arguments and lead her to join the group discussion when she was not able to produce any idea she realized that all the pieces in her mind were separated and she was not able to connect them. Since that moment, Funda did not realize that finding the CoG and constructing the building based on CoG would prevent the building from earthquake waves. After all her arguments finished, she started to ask questions why she should know f or CoG. Then she started to construct the Riemann sum.

Funda	We can hang it from a corner. Can we go through folding like these diagonals?
İlke	But it has no diagonal (Cuts the shapes it draws).
Betül	You cannot fold any more if you have one symmetry axis.
Funda	Let us find the first center of gravity of this from triangles.
Researcher	How do you fold it on the real model (how do we find it on the building so).
Betül	We draw the edge centers first of the triangles.
Researcher	In the real building? It is a triangle. You may have balanced this axis with this axis. Well, where will you balance this axis with this?
Funda	Hmm.
Researcher	If you want to solve the problem from triangles, you can try it
Funda	OK, *(after a while)…*, it is cumbersome, I cannot find CoG, and there are f and m. I cannot relate to them. Why do we use them?

This process is also represented in Figure 4.47 and coded as trying to find the solution with triangles. It can be seen that there are some little links in that time interval, which means Funda produced ideas. However, she does not use them later. Besides, some of the orange triangles are interrelated with the s trying to find the solution with triangles codes, which coincides with the beginning of that intersection. Moreover, it can be seen that she started to construct the Riemann sum after finishing all the ideas about the triangles. Another exciting moment for Funda is that she tried to relate it with triangles unconsciously, and that again caused broke her out of the discussion and misunderstood essential points that are she interpret moment as weight, so her friends explain her and drawing rectangles on the semi-circle and finding their area. After explanations from her friends, she joined their construction process and understood what the formula means. The Riemann Sum transformation to relating it with the CoG happens after Funda listened and saw the prototype model of the building. Since in that meantime, she understood why they used f and m which she was not able to understand why they need those variables. Hence Funda, Determinate the accumulation of the changes according to the independent variable.

Funda:	We divide the model into four pieces then solve the problem with the rectangles.
İlke:	The model will be as we want, but we have to find the center of gravity; this is what asked us, right?
Funda:	Exactly.
Betül:	We find the center of the gravity from rectangles, not from the triangles.
Funda:	We should consider its maintenance. We can provide it with straws
Betül:	I think we should rotate the second layer around the center of the gravity so it will not be a problem, but we need to support there with straws.
Funda:	Yes, it will be original, so we rotate the layer around the center of gravity.
Betül:	Yes, let us rotate it from the center of gravity to gather in the middle of each floor.
Funda:	Exactly but we will do it at the intersection points, right?
Betül:	How?
Funda:	With the help of the straws, hmm, what did you calculate?
İlke & Betül:	Momentum.

Funda	Its weight.
İlke & Betül:	No, weight x force arm.

Her reflection paper is examined to check whether Funda understood the approximation concept. In her reflection paper, she explained that she divided the intervals, then Funda calculated and told why she found less area than the actual one. She stated her solution process as *"First, I divided the interval between 0 to 1 to 10 equal-sized parts and drew left-end point rectangles. The total area of those rectangles will be close to the area under the curve. Therefore, I found the heights of those rectangles. Then, I found the sum of the heights of those nine rectangles. And then multiplied the sum with Δx.*

$$(0{,}81+0{,}64+0{,}49+0{,}36+0{,}25+0{,}16+0{,}09+0{,}04).0{,}1=2{,}84.0{,}1=0{,}284$$

So, with my calculation I found the sum of the rectangles' areas as 0,284 which is close to $\frac{1}{3}$ *= 0,33. I found the total area less than 0,33 because I calculated the left Riemann sum."*

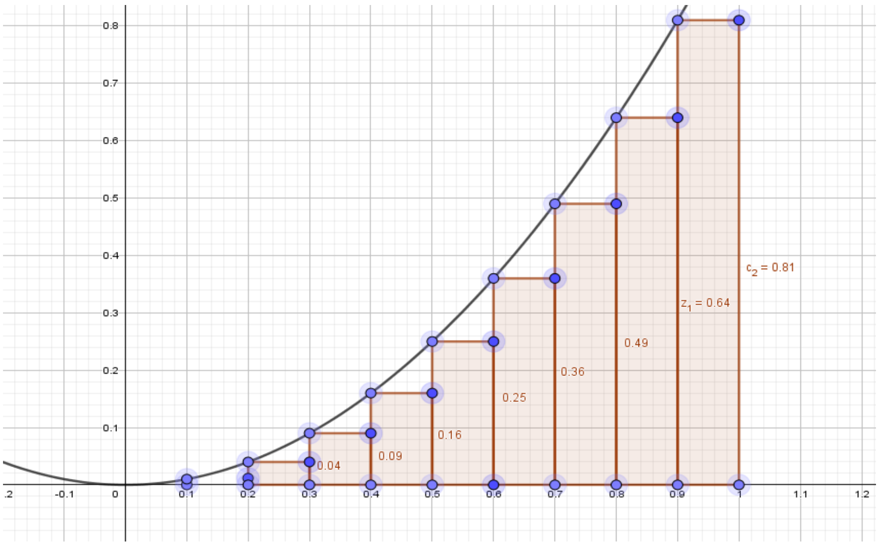


Figure 4.49 Funda's Solution in the Reflection Paper

Moreover, to verify her development, an interview had been conducted with her about her reflection paper and asked about the Riemann Sum. In her interview, she explained that because of the gaps between the rectangles and the function graph she obtained less area and told that in other situation taking a right Riemann sum she said that she would get more area than the actual one and by increasing the number of rectangles more accurate results can be obtained. That means she conceptualized that with the number of limited rectangles actual area.

Funda	I looked at this, the area of it will be calculated. This graphic is very similar to this linear line.
Researcher	Yes.
Funda	If we knew the function of the curve, we could solve it with integral. However, we cannot know so we can find it with the rectangles.
Researcher	So, how would you calculate it with rectangles?
Funda	Drawing like this (Drawing left-hand side Riemann sum)
Researcher	What will change if we take it from outside, and what will change if we take it from inside?
Funda	We find more value than the actual one. Because we have already taken all the gaps in the range, let us say it stays like an error over there.
Researcher	Okay, now you said if we draw a rectangle from the outside, we get a larger area; if we draw it from the inside, we get a smaller area. So, where is the real value of the area? What can you say about that?
Funda	Maybe it is about the between of them.
Researcher	So, how do you calculate it with rectangles?
Funda	For this problem, I can find its equation, then find its integral; for Riemann Sum, I will divide it into equal parts, then I can write it according to y.
Researcher	OK .
Funda	Hmm. If those lengths change to y,
Researcher	Yes.
Funda	x times all of them, if x is. I guess you would have xy_i
Researcher	This is your area, am I right? You wrote Δx. What does it mean? What does y_i mean?
Funda	Because I divide this interval into equal parts. Δx is one of these equal parts y_i . Is the rectangle's height which is drawn according to function.?
Researcher	What can you say about the area?
Funda	I can get an approximate area, and I can get more accurate results when I add more rectangles because the error will decrease and converge to a real area.

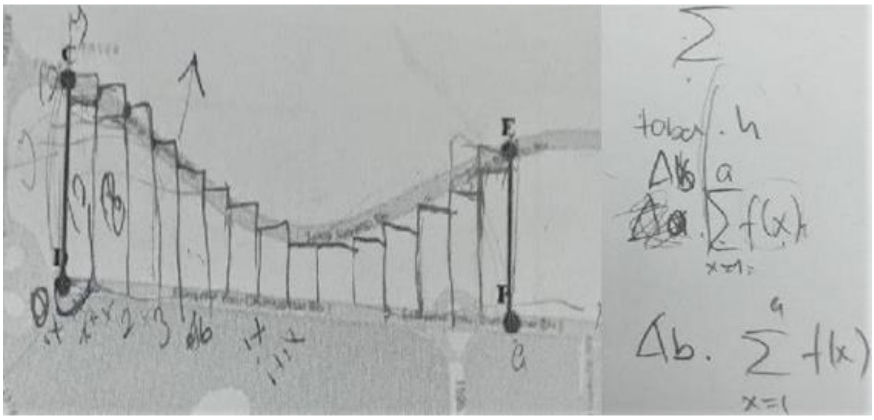


Figure 4.50 Funda’s Solution in the Interview

From Figure 4.50 and the dialog can be concluded that Funda knew the Riemann Sum and its types. She was also aware of their area and the error relations. In this regard, it can be concluded that Funda started to calculate the area with the limited number of rectangles will not give an exact solution. She was aware of by increasing rectangles, the more accurate calculation can be obtained, and this process can be expressed by the general formula. Accumulation is not complete and generalization of the process. During the interview and the lesson, it is detected that she is flexible in interpreting and writing $f(x_i)\Delta x_i$.

Moreover, she explained each variable in the expression by relating those with the figure. It seems that Funda had a conceptual idea about the Riemann sum and the error raised from left, right Riemann Sum. In brief, at the end of Activity 2, Funda’s development is presented.

Table 4.7 Funda’s Development Through Condensation Process

Condensation Stage	*Reached*
Write and interpret the product of $f(x_i)\Delta x_i$	+
Determinate the accumulation of the changes according to the independent variable.	+
Grasp that the quantity which is accumulating has a multiplicative nature.	+
Flexibly use various representations.	+

Table 4.7 (continued)

Differentiate left-hand side and right-hand side Riemann sum	+
Comprehend that accumulation is a process of adding new increments.	+
Smaller and smaller changes in the independent variable yield more precise estimates of the change in accumulation.	+

In the next chapter, it was presented that how Funda achieved the objectives of the reification process objectives.

4.2.2.1.3 Reification Process

Sfard (1991) defines the reification process as a shift. Students interpret familiar context with a new perspective. That means “reification is an instantaneous quantum leap”(Sfard, 1991,p.20) In this process, students transform the concept into a more concrete and static object. Students utilized different representations of the taught concept meaningfully. In the end, students detach newly learned concepts from the step. This detached concept becomes a member of a class and becomes a basis for another new entity. Moreover, in this process, students use different representations and try to generalize the concept.

Pre-serve mathematics teachers reach the reification stage when they can operate the Riemann sum as an object. Thus, treat Riemann sum as a whole and translate it into graphical representations uses algebraic methods to find the solution.

4.2.2.1.3.1 Remarkable Event in Activity 3

Activity 3 was about designing an arch dam. Students will build a dam with given conditions such as using the minimum amount of material and optimum surface area in this activity. By considering these conditions, the participants of the study should satisfy the persistency of their dam. Thus, throughout the design process, they have to think critically, do the proper calculation, then design the model. At the end of the activity, the study participants should be able to explain hydrostatic

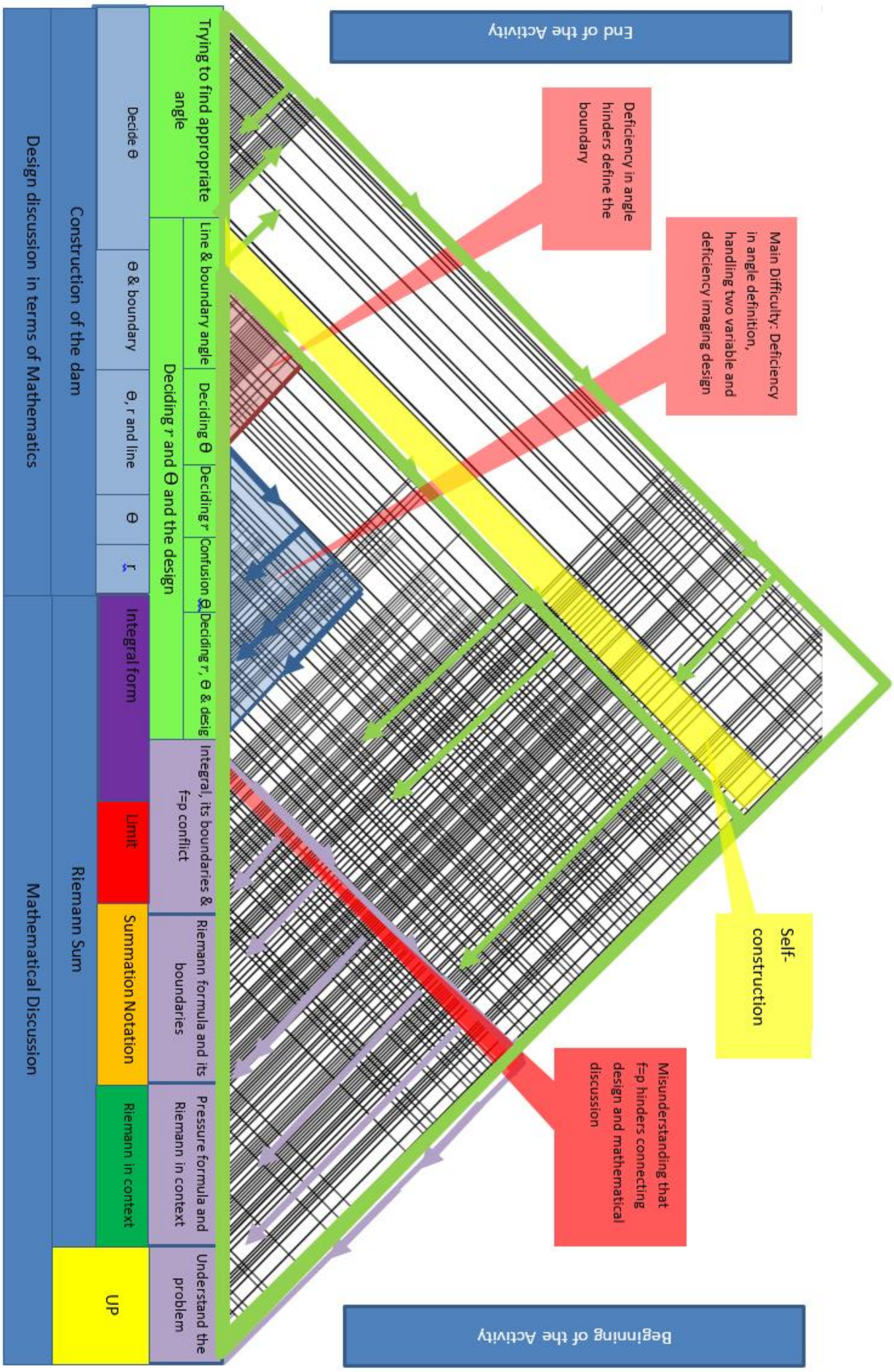


Figure 4.51 Funda’s Linkograph in Activity 3

pressure, and students will be able to explain hydrostatic forces. Besides, students will be able to explain the relationship between Riemann sum notation and integral form. Students will be able to set up integral, students will be able to determine the boundary of the integral, and finally, students will be able to explain Riemann sum in polar coordinates.

Activity 3, designed as the participants, should have determined the variables, formulate the collapsing force, and apply that formula in their design. In this activity, the theoretical aspect was also dominant as in the second one. Moreover, there were several different variables that students have to think about together and decide which variables' effect is more on the durability of the dam. In this way, it was aimed that students were made to conceptualize the variable concept, and also, they were able to see how and which variables affect their design. Moreover, they learned how to interpret the formula to express variables in theory.

Unlike previous activity, theory and practice were intertwined entirely in this one. The critical point of the activity was determining the boundary of the integral in polar coordinates. Because both situated abstraction of the defining boundary was prevented, and the participants could see how the theoretical calculation affected their design directly.

In Figure 4.51, the amount and nature of Funda's contribution to the discussion are represented by the number of lines throughout the activity. As shown in Figure 4.51, the number of links increased, and there is an almost balanced distribution between the discussion parts. On the contrary to her previous linkograph, her links were overlaid in this activity, which means she constructed all her ideas on each other. Moreover, the number of her contribution increased at the end of the mathematical discussion part. However, she participated when it is necessary at the beginning of the discussion. The reason for that context was known for her, so she could not produce so many ideas, and Betül and İlke understood each other.

The remarkable points for Funda were represented in the linkograph in Figure 4.51, and her contributions were coded in as Understand the problem, pressure formula and Riemann in context, Riemann formula, and its boundaries, integral, its boundaries and $f = p$, deciding r, θ and design confusion θ, deciding r, deciding θ, line, and boundary angle, trying to find an appropriate angle. Unlike the other linkographs, in this linkograph, Funda started to construct the concept by herself then contributed to the discussion.

This construction process is coded as self-construction in the linkograph. Also, three main parts in the linkograph represent the main difficulties or hindrance process, preventing understanding the concept. It can be seen from Figure 4.51 that after overcoming those difficulties, Funda fully achieved the objective of the activity.

During the UP code, she investigated how dam functions and its shape, and she also tried to substitute that knowledge into their problem. After explaining that liquid pressure is exerted equally in all directions, pressure force which is force causes pressure, and it is always vertical to the surface, the researcher asked where the extension of the pressure force would pass and after this moment group began to try to find the solution of the problem.

In the Riemann in context, with the researcher's guidance, she and her group friends discussed the forces on the dam and its components. End of the discussion, they decided that the Fy component of the force will collapse the dam. Contrary to the previous activity, Funda did not solve the problem by assimilating it into known shapes. She focused on how a dam works and how to write the Riemann sum. Hence, she did not go through the area concept; she thought she should use the arc length and should like very small units like points to shade the dam's arc.

It showed that she understood the accumulating process and why the limit was used in the Riemann Sum. In Figure 4.52, her formalization of the Riemann sum can be seen. Conceptually, she understood how to write the boundaries, deficiency in her angle definition, and not write the boundaries accurately. By discussing with Betül and İlke, she realized her mistake and wrote the boundaries accurately. The following excerpts show that Funda writes and understands the Riemann sum.

Moreover, she also conceptualized that every expression in the formula. Namely, during the formalization process, she wrote every expression by understanding and connecting them with the integral formula and context.

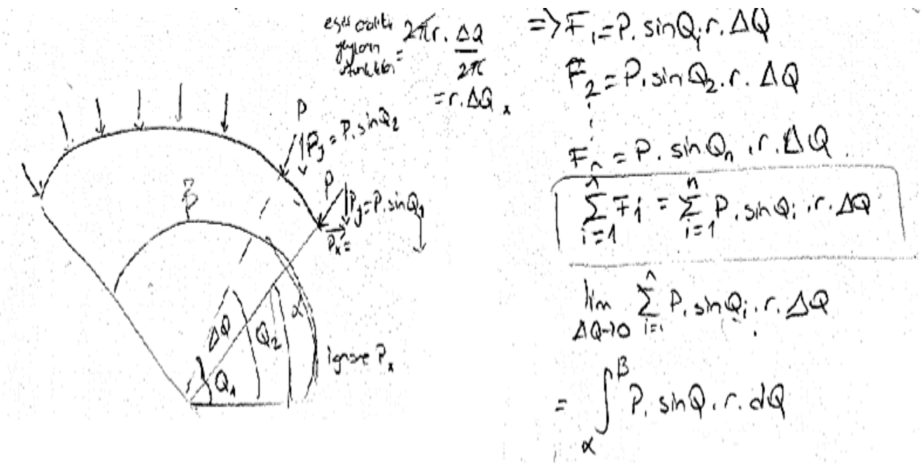


Figure 4.52 Funda’s Formalization of Riemann Sum

Researcher	Ok, you are going right, think a little more, and how we express it depends on y We see the $\Delta\theta$, what about each a?
İlke	a yes, it is the sheer force on the dam.
Funda	Is the average of P here is the distribution of pressure ın the arc in that circle?
Betül	Uh, we should found the area of the circle slice right now, but we found the length of the arch, think of the arch part, then if we take something like $2\pi rh$ instead of πr^2. Teacher, it is not an area. It is a length. *(The breaking point for Betül)*
Researcher	Why? Can you explain it?
Funda	I got it at a certain level, and then we have to write the pressures differently. If we are going to write a lot of different pressures, it said a single surface. Think of it as a surface
Betül	Exactly, I understood. If I write πr^2 it will be a circle slice, but there is no water there. However, we should consider the length.
Researcher	Ok else?
Funda	We are going to write $\Delta\theta$
Researcher	So, how do you write forces?
İlke	Do we think like this r multiply and this length?

Funda	We think of the length of the arc, not the area.
Betül	Think like this, the extension of the pressure force passes through the center, so if we say πr^2, we find the area there, but we do not take thickness in it as if we think it acts as a single point; it just acts here.
Funda	Since the height everywhere is the same in all.
İlke	Let us think of a very thin circle slice of r$\Delta\theta$ delta we think as much as we do not think r$\Delta\theta$
Betül	No, think as if you find the length of here.
Funda	It is not related to the area of the circle slice. It is related to the length.
İlke	Okay, but why do we say $2\pi\Delta\theta$?
Funda	look that is coming from this formula. Showing the $2\pi\alpha$ / 360. We find it from here.
İlke	Hmm.
Researcher	What do we do next?
Funda	We take the limit thinking as $\Delta\theta$ going to zero
Researcher	Why?
Funda	We divided the angles very tiny angles because we think it is smaller, smaller, smaller.
Researcher	How am I going to write the boundaries?
Funda	We will write from 0 to 360. I mean, it will be the length of the arc, right?
Betül	I am not sure (in the meantime, she tried to write the boundaries).
Funda	It depends on our porotype, right if we take like this it will be 0 to 180.
Betül	I think it will be from q_1 to q_n.
Funda	Yes, that is what I mean.
İlke	I did not understand why we try to write the summation form, then we write it as an integral form.

Even though she did not have any problem writing the formula of the Riemann sum except the boundaries, her misunderstanding of the $f = p$, prevent her from interpreting the formula correctly. Moreover, she had confused about which variable she should handle. She was not able to decide r or θ will affect the design more. This process is shown in her linkograph Figure 4.51 with blues links. In that discussion process, it can be seen that Funda tried to understand previous discussions' bases. However, her main focus on determining the primary variable. Hence, she was not able to write the boundaries and had difficulty on where to start. This confusion also prevented her from imagining the arc dam. Thus, she could not imagine the dam with the coordinate system, so she confused the initial boundary of the angle. She thought that there was no angle, so the initial value should be zero. This kind of thinking was deficiency imagining the dam without a coordinate system and deficiency in defining the angle. Thus, she was not able to write the boundaries of the integral and the Riemann sum. Until the line and boundary angle code Funda tried to find which variable is more effective on the formula. After having more information about the dam's shape and examining the dam's materials, Funda started to imagine the dam more concretely.

Then by discussing with Betül about replacing the coordinate system on the dam, she realized that she was taking the lower boundary of the integral wrong. During the self-construction time interval, she examined her notes and tried to conceptualize the boundary value and the primary variable. At the end of that interval, she wrote the angle in the Riemann sum accurately. From Figure 4.51, it can be seen that during the deciding r and θ code, links condensed around the integral, its boundaries & $f = p$ conflict, and Riemann formula codes. After she understood that the angles are the primary variable, she joined group discussion and built a connection between the Riemann Sum formula and the previous codes. Moreover, she started to make a link, which means she conceptualized the previous process and produced the ideas based on that process to find the appropriate angle. In the following conversation evolution of Funda's ideas can be seen.

Funda	I think from 0 because there is no angle, then from 0 to n
Betül	But there is the angle.
Funda	We ignore that. Accept it as an x-axis. I think it starts from 0.
Betül	Now I will say something when we take the integral of this and get this one, and in it, we will write θ_1 instead of θ now hmmm no.
Funda	We need to determine that angle.

Betül	Hmm so let us do something like that (thinks) when we write θ_n - θ_1 , we have to choose r as big, hmm no we have to say r smaller.
Funda	No, if this grows, p grows. We need to choose r small because we try to decrease p, so let us say f here because r is increasing pressure decreases because of f right?
Betül	No, this is the total f for us, and we should reduce the force so that our dam become sustainable.
Funda	Do we have to reduce p, right?
İlke	I understood and came.
Funda	We are trying to understand how we need to choose our values.
Betül	I think we cannot reduce p; we should manipulate other variables.
Funda	p and f are inversely proportional, but p and f are the same things, aren't they?
İlke	They are not the same.
Betül	They are not the same.
İlke	Think of it as a total force, and this is a which force acting per unit area.
Betül	Yes $p = f / a$
İlke	It is a very small unit.
Funda	How do we solve the problem?
Betül	We will manipulate the r and θ.
İlke	How do we decide on r?
Betül	We cannot decide p at the moment; it depends on water, right?
İlke	Yes.
Betül	So, we have to play with r and θ? Wait, r is also constant because we should determine the boundaries according to angle.
Funda	Yes.
Betül	We should decrease r but how about θ?
Funda	But if r decreases, f decreases.
Betül	So, we have to play with r and θ? Wait, r is also constant because we should determine the boundaries according to angle.
Funda	Which one do we should this?
Betül	If we select $\theta = 90$, can we construct our dam like this?
Funda	Should we choose $\theta = 60$ and let us r be big, or

	should we make it 80 and decrease r?
Betül	θ =90, the center of it will be 90. It is 90 from here.
İlke	If we put forward this acetate as a dam, consider these as mountains, the dam will be connected to the mountain at one point, but if we lean on this corner, I think the force will change.
Funda	In the second model, the dam will be connected to the mountain with 2 points.
İlke	Yes, but the angle will change in two models; it will be connected at an angle of 90 in one and more than 90 in the other.
Funda	Even if the angles change, two sides will be connected with the mountain.
İlke	I did not mean that. I said this first, and I create it by putting the acetate paper in the box in the corner. The second thing I said is that I create it by putting it in the middle of the box like this, so according to these shapes, the angle is changing.
Funda	The length of the arc will change so that r will change
Betül	θ_n Is not here. I mean, when we say that angle is 90, it is not mean that the angle of the center of the dam will start from here; it also includes the complementary angle.
Funda	How do we find it from that starting point?
Betül	It has become messy now.
Funda	Aa thinks like this is like the $x - axis$, and our angle starts here to select the angle by starting from here.
Betül	Exactly.

As seen from the conversation, Funda combined the line and the shape of the dam. Hence, she was able to write the boundaries soon. Previously, Funda solved the problems by assembling them to another problem that she knew; however, this problem was new for her, so she could not assemble to another problem. Hence, she had difficulty in handling the variables and the problem. Moreover, she was not able to see the big picture and interpret the formula comprehensively. At the end of the lesson, Funda wrote and interpreted the $\lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)\Delta x)$ and easily transfer between representations. However, it is not clearly determined that she was able to understand that estimating the accumulating functions' change via its rate of change and the Riemann functions equals to area of a graph in an arbitrary interval.

4.2.2.1.3.2 Remarkable Event in Activity 4

Activity 4 was a reverse engineering activity. The context of the activity was finding the volume of a cooling tower. In this activity, a cooling tower in a power plant was collapsed. Hence that tower should be repaired. In this context, the student's task is to calculate by using only the photos and the tower's height. However, they are supposed the find the exact volume of the tower in a limited time. Thus, the size of the new tower should be the same as the previous one. They do not have enough information even to estimate the cooling towers' volumes, so throughout the design process, they have to think critically, do the proper calculation, then design the model. At the end of the activity, the participants will be able to solve the problems related to solids of revolution and explain the Bernoulli principle. Besides, students will be able to explain the relationship between Riemann sum notation and integral form. Students will be able to apply FTC to problems.

Activity 4 was designed as the participants should have determined the variables, write the volume formula, and then have to calculate the tower's volume. In this sense, Funda realized that the tower's radius was changing, and they should write the integral according to it. She stated this in her reflection paper as "*Since the volume is filling inside a solid with the base as time as the height if we find the base area(s) we can find the volume. However, in our problem, the area of the base is not constant; it varies. So, we can find all those varied areas with the integral concept. Moreover, with the radii's changing, the number of the disc was increasing we got more accurate volume.*

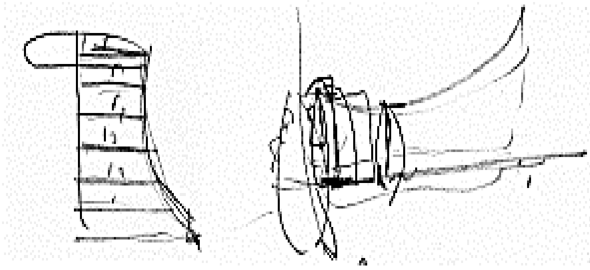


Figure 4.53 Funda's Drawing in Activity 4

Funda	Here is what we need—basement area times height.
Betül	All basement areas are circle.
Funda	Radius is constantly changing in all circles, but the height should be defined first to set up the integral.
Betül	Yes.
Funda	Think of it this way; we can find the volume by the basement area x the height. The height is constant, I divide it into fixed intervals, and if r varies according to the function, then is not this $\pi r^2 dx$ for one?
Betül	Yes.
Funda	we can find the volume of the interval between these two points that we have determined. If we slip them into the piece, add them up, and take the limit of the sum that I found.
Betül	OK, let us do it that way

Here, Funda explained the disc method without referring to it. She reached this method by using the Riemann Sum. This means she conceptualized the Riemann Sum and could easily transfer this knowledge to the integral representation.

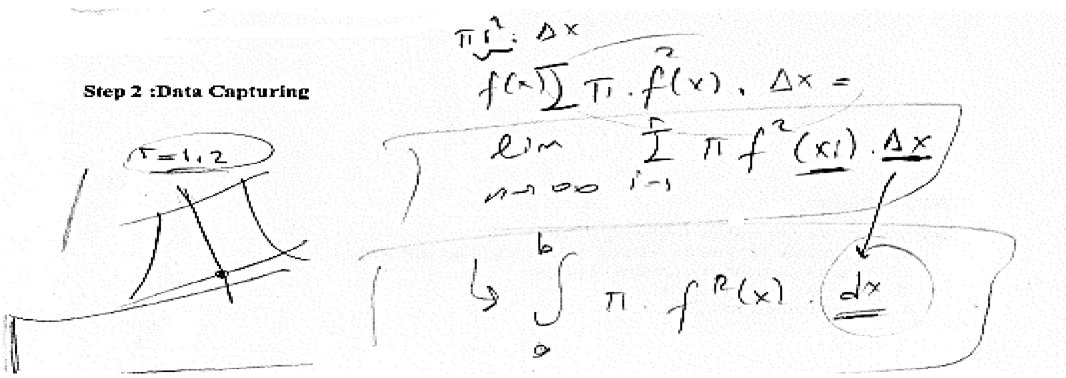


Figure 4.54 Funda's Obtaining the Formula

Considering these developments of Betül is showed in the table below. In the next chapter, it was presented that how Betül achieved object- process phase.

Table 4.8 Betül's Development through Reification Process

Objectives	***Reached***
Write and interpret the $\lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)\Delta x)$	+
Can easily transfer between representation	+
Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.	+
The accumulation of any function can be generalized by means of the graph surrounded by change I, independent variable, and the value of the function.	+

4.2.2.2 Developments in Funda's knowledge about Fundamental Theorem of Calculus in terms of Object -Process Perspective

Until this section, how Funda constructed the Riemann sum in terms of the Object-Process perspective was presented. According to Sfard's Theory, after learned knowledge was detached, it was categorized, and this new concrete object became a base for the new entity. In other words, new knowledge was constructed on this object, and that was performed as an input. Namely, it became a new process for a new knowledge series. In this regard, Riemann sum was an object which is the ultimate phase of the theory concurrently it was a first phase process for the Fundamental Theorem of Calculus. As being in the Riemann Sum Object-Process, Funda went through the interiorization, condensation, and reification process.

4.2.2.2.1 Interiorization Phase

Sfard (1991) described this phase as the students started to be familiar with the new concepts and had limited skills about the concept. However, through the process, the students became skillful. In this regard, in the interiorization process for the

Fundamental Theorem of Calculus, students get acquainted with the integral representation of the Riemann sum.

As seen from Figure 4.54, Funda could write the Riemann Sum and could easily use flexible representations. Moreover, she explained the Riemann function and transformed it into an integral representation.

In the next chapter, Funda's construction of the Fundamental Theorem of Calculus is presented. Through the activities, the important points that Funda achieved pre-determined objectives for the phases were presented. Thus, she met all the objectives of the interiorization phase.

4.2.2.2.2 Condensation Phase

Sfard (1991) described this phase as "squeezing lengthy sequences of operations into more manageable units" (p.29). Moreover, she emphasized that the learner can be more skillful in thinking about the given concept in this stage. She resembled this process, a computer program that works with input-output relations instead of showing all detailed processes. That means learners do not go into details during the processes. In this step, a new concept is "officially" born when the learner confronts any difficulty that provides a new entity. Namely, that difficulty may be a trigger for learning. In this step, learners may also connect the current process with another process and compare it within old and new knowledge and start to generalize it. Also, learners started to use different representations. In this regard, the participants of the current study should be able to set up the integral and be aware of the relation between the accumulation and the integral. That means while $\Delta x \to 0$, area accumulates under the f curve in $[a, b]$And it will equal to the total change in primitive function F.

As mentioned in the previous paragraph, the difficulty is a primary trigger for a new mathematical idea. This point was the students' starting point since this difficulty was presented to the students as dealing with two variables that vary simultaneously. In this way, the Fundamental Theorem of Calculus had been presented to them, and their construction process started officially. Moreover, they started to understand the relation between the antiderivative and the integral. In a very general point of view, during the condensation process, preservice mathematics teachers should be able to

- Comprehend and explain that two quantities differ smoothly and simultaneously.
- The increments can be small enough, therefore no matter how much the accumulations change, they covary in relation to each other with increases at a fundamentally constant rate of change.
- The rate of change of accumulations related to each other is the rate of change in their increase.
- Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.

▪ At a point piling up of infinitely smaller changes in the independent variable with limit results with an instantaneous change in accumulation which equals to the rate-of-change function

▪ Explain $\int_a^b f(t)dt = F(b) - F(a)$ as the accumulated area under the curve of f from a to b is equal to the total change in F, the accumulation function, from a to b.

Taking into consideration these objectives, Funda achieved all of them at the end of Activity 5. In the following section, how Funda met the condensation phase's objectives through Activity 5 was given and how and when she contributed the discussion and conceptualized the objectives were presented.

4.2.2.2.2.1 Remarkable Events in Activity 5 for Condensation Phase

The remarkable points for Funda were coded in Figure 4.58 as physics design and material, design, math, and design, T and x MU (Misunderstanding), T and

x covariation, variable confusion delta x, and summation, determine variables, derivative and integral boundary and formulate. During the whole discussion, she built forward and backward links and produced ideas to solve the problem. As seen from the Figure 4.58 first three and last two codes are the same as the group discussion. That means she participated in the discussion during that time interval, and in the mathematical discussion process, she followed a different path from her friends. The reason for this difference misunderstood the T and x relation. As in Betül, Funda theoretically knows the concepts of temperature, but in practice, she misused these two concepts and treated them as they were the same.

Moreover, although it was stated in the course that the heat moved from high temperature to low temperature, she considered the heat as if it were the same and thought that the heat moved from low temperature to high temperature. Finally, she thought that by increasing the thickness, the loss of temperature would below. Thus she noted in her worksheet as if T and x were the direct proportion.

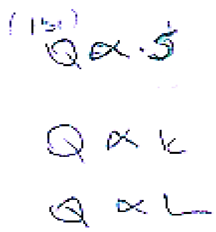


Figure 4.55 Misunderstanding of T and x

Moreover, she stated this situation as "When the thickness increased, the temperature also increased. Wasn't this the example we wore thick in winter?" This misunderstanding hindered Funda from solving the problem easily. After asking her to explain how she can change the temperature by wearing more clothes, she understood her mistake and realize that T and x were changing at the same time and stated as "*I guess we will add it like this. For example, if it loses T as at* $1x$*, he will lose* $2T$ *at* $2x$*. Yes, I guess it goes directly proportional. I mean at the same rate*" That means she realized explained that T and x were the quantities vary simultaneously. This process in Figure 4.58 was represented with the purple and yellow arrows. The change from purple to yellow in the arrows shows that Funda corrects his misunderstanding. While the purple arrows extend to the beginning of the lesson, no further connection has been established after turning yellow. Although she realized that covariation, she was not able to understand that this covariation affects the heat. Moreover, since still she was treating the temperature and the heat were the same, she had variable confusion. This process was represented with the red and orange arrows. As in the same purple arrows, the red arrows reached till the beginning of the lesson; after changing to orange, they were not built anymore. Although she was able to write and explained the "Δx" by drawing the figure (Figure 4.58**Hata! Başvuru kaynağı bulunamadı.**), she was ot able to substitute the variables into the formula and had confusion.

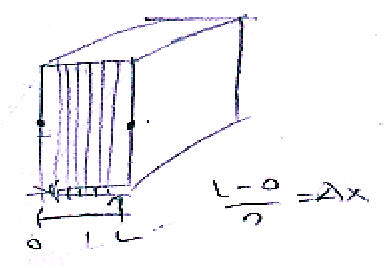


Figure 4.56 Funda's Drawing About the Division of x

This confusion is that Funda treats heat and temperature as the same concepts as in Betül. In the previous explanations, she explained how t and x are covariates; however, she linked the x with the head in writing the formula process. This construction process of Funda coincided with Betül's trying to establish the connection between T and x. With the explanation of İlke and the warning of Betül, Funda revised her equation. At this stage, the researcher also asked Funda which variables she wanted to find. After this conversation, Funda started to establish the formula. However, she still could not combine the rate of x, and the Q was related.

İlke: So, when these variable decreases and increases, we try to calculate the thing with the formula. It should give that change in the formula we will write. So, we must write things like that; it really should give me a bigger value of q when I increase them.

Betül: Yes, why did we write this?

Funda: We will claim these as Q_1 and Q_2And we will find the energy in each x. Am I not sure to do we find the heat? These correspond $to\ x$, so we can say that $x_{1,}, x_{2,}, x_3$?

Betül: There is a relationship between ΔT $t$hickness. ΔT decreases with increasing thickness, but I do not know if we can find anything here.

Funda: I guess the two are increasing, right? It is going to be something like Qds1 q in the formula… It is as if we are going to change these x's in it. We need to find the changes for q with x. We think the temperature will increase…. The more I divide my x; the smaller my delta t will become….

As it is understood from the conversation, Funda behaves as if Q represents T. Verbally, she states that T and x were related; she wrote Q depended on x during the formula writing process. Although Funda knew the variables that she should use while writing the formula, she could not establish the connections between the variables. She tried to find his variables by reviewing what she did. However, Funda was not able to overcome the problem, so researchers helped Funda to determine variables. This process was presented in Figure 4.58 with the red and orange arrows.

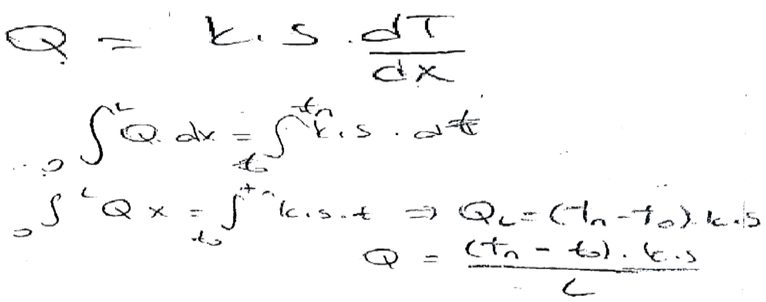


Figure 4.57: Construction of the Formula

Funda The more I divide my x, the smaller my ΔT….

Researcher What does it mean that your equation has both dx and dT

Funda: The derivative of T with respect to x. I mean these two changes at the same rate.

Researcher Can you explain to me the relationship between the variables?

Funda Now let us call the thickness of the wall x; if we call the inside temperature T_1 and the outside T_2 , the energy will flow to T_2 and the wall will lose bit T for each x it moves. If we write this for very small intervals, we can say dt and dx. When we first created the formula, we found the correlations between them or writing them instead of the formula from there. It shows us instantaneous changes; we want to see the change in a certain time frame. For this reason, we write the boundaries of T to T_1 T_2, while dx is about the thickness, we will write the thickness of the wall there as the boundary. This value we found will give us heat loss.

Taking into consideration developments of Funda, following table can be given. In the next section, it was presented how Funda achieved the reification process objectives' objectives.

Table 4.9 Developments of Funda in the Condensation Phase

Condensation Phase	*Reached*
Grasp and express that two quantities vary at the same time	+
The increments can be small enough, therefore no matter how much the accumulations change, they covary in relation to each other with increases at a fundamentally constant rate of change.	+
The rate of change of accumulations related to each other is the rate of change in their increase.	+
Grasp that a constant rate-of-change can predict the shift of an accumulating function in the interval multiplied by the shift in the independent variable.	+
At a point piling up of infinitely smaller changes in the independent variable with limit results with an instantaneous change in accumulation which equals to the rate-of-change function	+
Explain $\int_a^b f(t)dt = F(b) - F(a)$ as the accumulated area under the curve of f from a to b is equal to the total change in F, the accumulation function, from a to b.	+

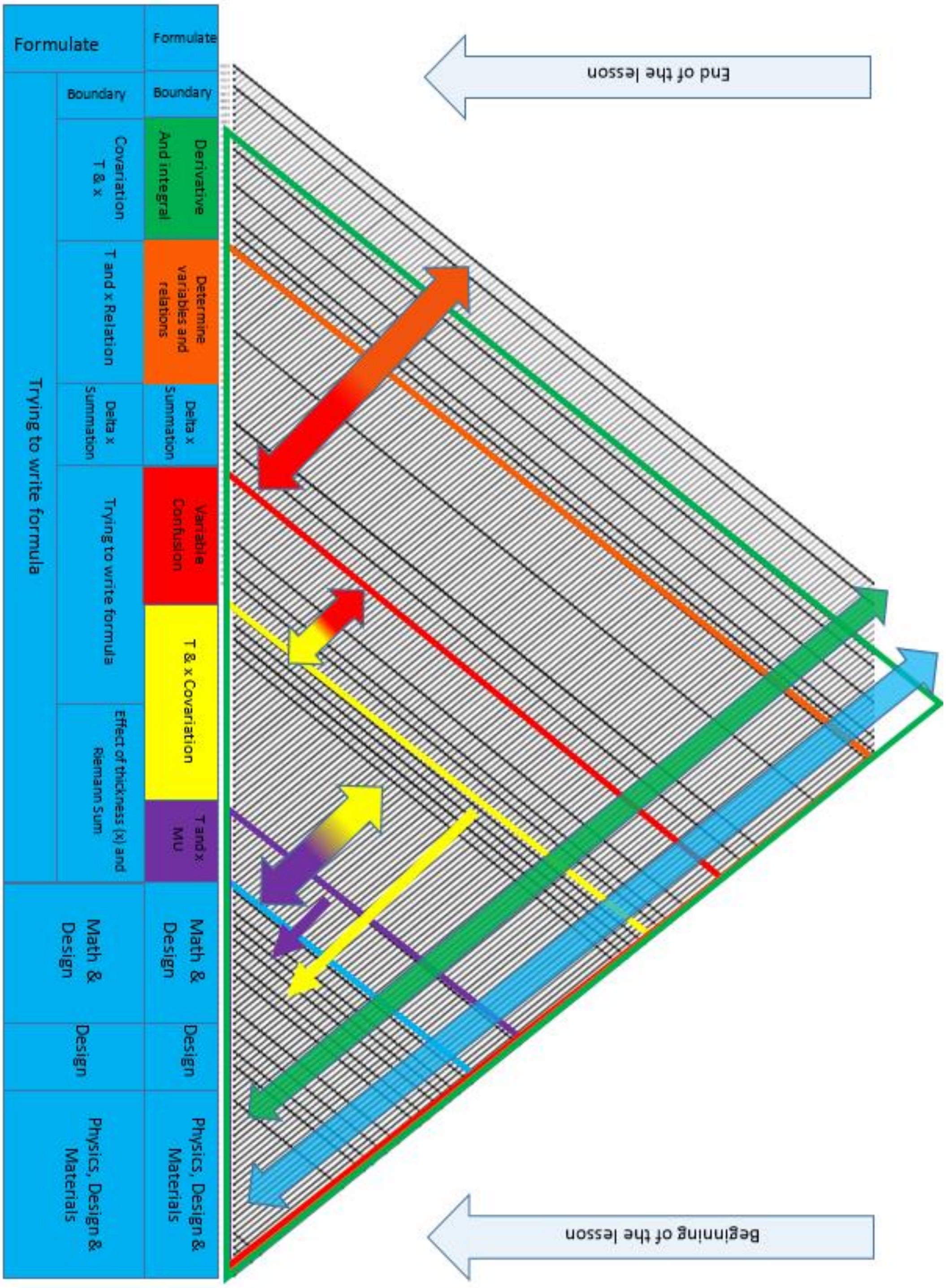


Figure 4.58 : Funda’s Linkograph in Activity 6

4.2.2.2.3 Reification Phase

Sfard (1991) defines the reification process as a shift. Students interpret familiar context with a new perspective. That means "reification is an instantaneous quantum leap" (Sfard,1991, p.20). In this process, students transform the concept into a more concrete and static object. Students utilized different representations of the taught concept meaningfully. In the end, students detach newly learned concepts from the step. This detached concept becomes a member of a class and becomes a basis for another new entity. Moreover, in this process, students use different representations and try to generalize the concept.

4.2.2.2.3.1 Remarkable Events in Activity 6 for Reification Phase

Pre-serve mathematics teachers reach the reification stage when they can operate the Fundamental Theorem of Calculus as an object. Thus, they had to treat the Fundamental Theorem of Calculus as a whole and translate it into two types of theorem. In this regard, it was expected from Funda;

- Write and express the relationship between accumulation and rate of change by writing the formula.
- Explain $F(x) = \int_a^x f(t)dt$ As "the value of $F(x)$ represents the accumulated area under the curve from a to x and F is a function that can be found from the accumulation of the rate of change of F, with independent variable t and x represents the independent variable of a second "pass" through the function f with a view for coordinating the rate of change $(f(x))$ with the change of the independent variable to determine the multiplicative structure of the accumulation, F.
- Express the $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ as "the instantaneous rate of change of the accumulation function at x is equal to the values of the function at x" (Radmehr & Drake, 2017, p.1057).

- Express the $F(x) = \int f(x)$ as "the antiderivative of f is F, F is an accumulation function and f is the function that describes the rate of change of F"

The focus of her discussion was coded in Figure 4.62 as understand the problem, properties of the bridge, forces on the bridge, equation of the slope, tension, load and angle, derivative & integral, Riemann dx, finding the equation and boundary, writing the equation, integral and derivative relation. During the whole discussion process, Funda produced both backward and forward links, and mostly, her discussion focus was synchronized with the group discussion. In the previous lesson, Funda constructed the Fundamental Theorem of calculus, and she realized the covariation between T and x. Hence, in this lesson, she also constructed the second part of the Fundamental Theorem of Calculus quickly compared with her friends. Thus, most of the time, she guided her friends through the lesson. As can be seen from her linkograph, Funda has shown the same pattern as in its previous ones. That means mostly she listened to her friends, and when it was needed, she participated in the discussion. After learning the physics part from the researcher, she drew the forces on the bridge and thought about how to find the slope of T, and expressed that "this comes from the derivative. Then she showed the variables in her drawing.

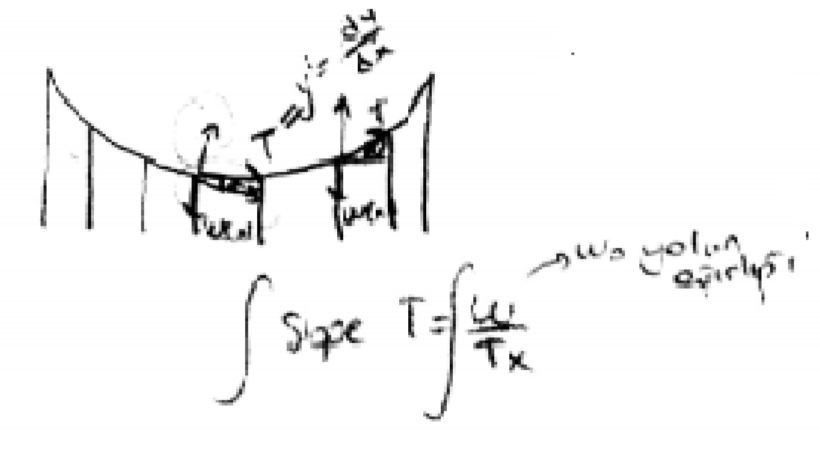


Figure 4.59 Funda's Drawing of the Forces on the Bridge

Writing this equation was one of the important parts of this activity. It means Funda knew how to find the whole tension through the rope. After writing, Funda and her friends needed some help because of the representation of the W.

Explain $F(x) = \int_a^x f(t)dt$ As "the value of $F(x)$ represents the accumulated area under the curve from a to x and F is a function that can be found from the accumulation of the rate of change of F, with independent variable t and x represents the independent variable of a second "pass" through the function f with a view for coordinating the rate of change ($f(x)$) with the change of the independent variable to determine the multiplicative structure of the accumulation, F.

There is no variable depends on the division of the road. There seems Tx can be variable. However, it is a constant. Thus, at that moment of the discussion, the road load can be represented as "Ux," U is a unit load of the road, and the x is the length of every unit.

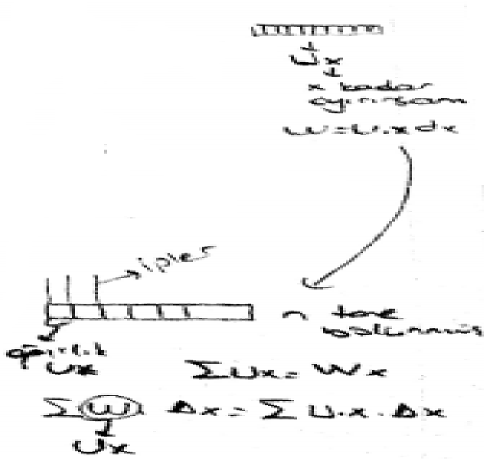


Figure 4.60 Funda's Writing of U

As seen from Figure 4.61 in the W's equation and substitute it with the main equation and integrate it.

Fundamental Thm of Calculus

$$\int \text{Slope } T = \int \underbrace{\frac{U.x.dx}{Tx}}_{\text{sabit}} = \frac{U}{2Tx} x^2 + c$$

Figure 4.61 Calculation of Integral

After this calculation, she realized that they were dealing with the parabola equation.

Considering these developments of Funda is that she was able to achieve all the Reification process objectives. Moreover, she was able to explain the various representation, and she was flexible during the transformations. Although she knew the Fundamental Theorem of Calculus and the Riemann Sum, she did not deal with much detail and treated the integral as an object. Moreover, this activity was a simple initial value differential problem, and she solved it. That means the Fundamental Theorem of Calculus as an object for her ad also process for the differential equations. In the next chapter, the final situation of Betül was discussed, and the findings of their final interview are presented. Moreover, the structure of her final concept image is given

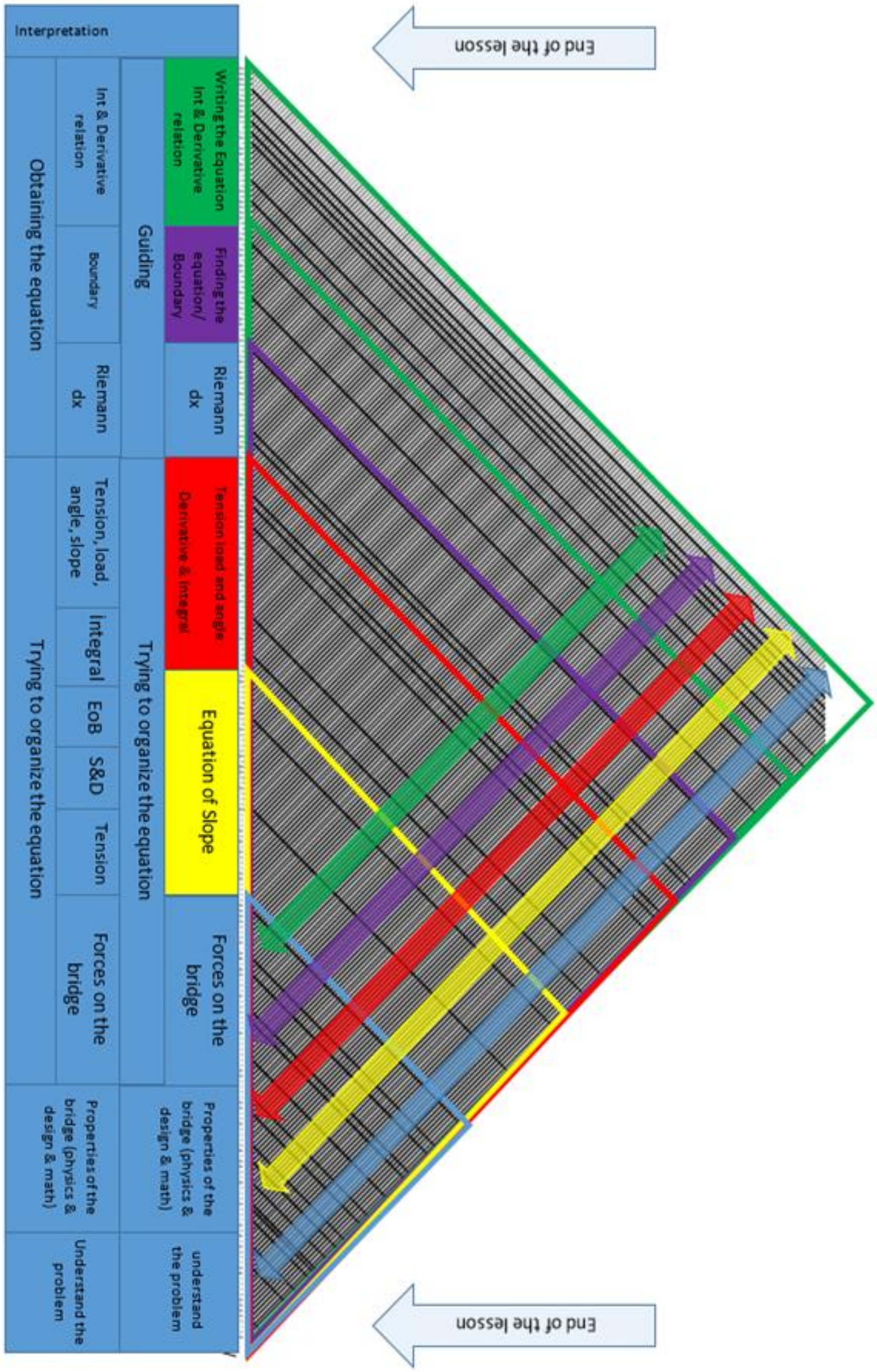


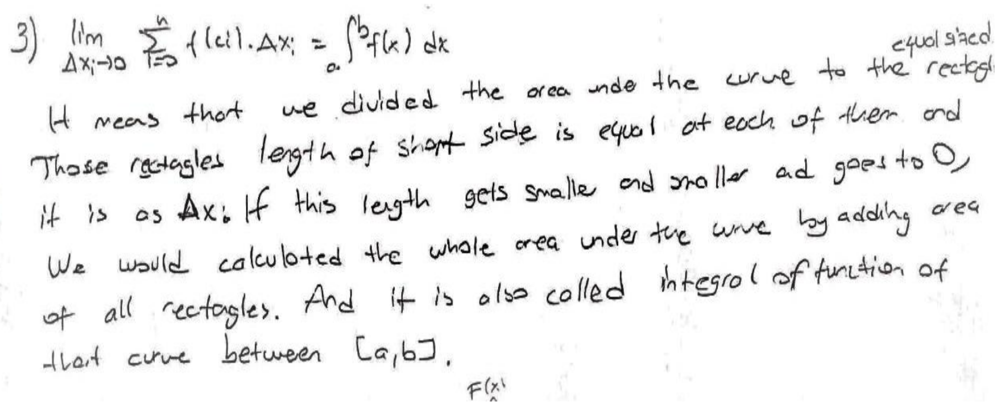
Figure 4.62 Funda’s Linkograph in Activity 6

4.2.2.3 Funda’s Final Concept Image

In the beginning, Funda has a discrete concept image (DCI). Her departmentalized knowledge was connected during activities and scaffolding her in scientific learning was rebuilt again. Thus, she had a relatively coherent concept image. The properties of her image are:

1) *She expanded her definition of integral.*

At the beginning of the terms, Funda made her definition of integral was area-oriented. It was seen that she altered her definition and mentioned the boundaries and the function. Both Funda related it with analytic definition, and also, she could explain its symbolic illustration accurately.



3) $\lim_{\Delta x_i \to 0} \sum_{i=0}^{n} f(c_i).\Delta x_i = \int_a^b f(x)\,dx$

It means that we divided the area unde the curve to the equal sized rectangl
Those rectagles length of short side is equal at each of them and
it is as Δx_i If this length gets smaller and smaller and goes to 0,
We would calculated the whole area under the curve by adding area
of all rectagles. And it is also called integral of function of
that curve between [a,b].
F(x)

Figure 4.63 Funda’s Explanation of Expressions

2) *She learned Riemann Sum comprehensively.*

At the beginning of the semester, Funda could not relate Riemann sum with the limit concept, and she believes that with Riemann Sum, we cannot find the exact area. The final conceptual test showed that she overcame her understanding of the limit difficulty and explained that the exact area could be found. Moreover, she wrote that the middle Rieman sum approach is better than the left-hand and right-hand side Riemann sum. At the beginning of the course, she could not differentiate them, and she also could not explain why they are different. Especially curves like Picture 9, she started to draw left-hand side Riemann sum correctly by starting left side; however, when she came to the right side and depicted as right-hand side

Riemann sum, and Funda did not know which point, or she took as reference. In the final interview, she explained the types of Riemann Sum and showed them accurately.

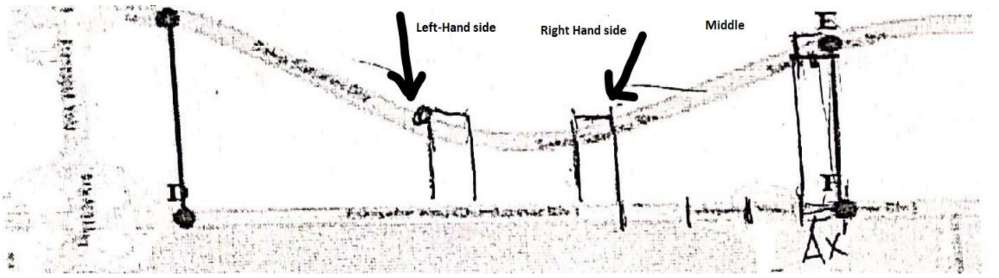


Figure 4.64 Funda's Solution about Riemann Sum

In the following excerpts, the explanation of Funda can be seen.

Funda	$\lvert CD\rvert$, $\lvert EF\rvert$.We would divide this into areas because it is hard to find the functions of this thing. So, I would divide the equally spaced lengths here into equal-length intervals. Then I would do something here. I would open the rectangle. I would take the violence.
Researcher	Why would you take the violence?
Funda	Middle more accurate results. Because if we do the left. Is it right here? Here comes the right.
Researcher	what is the right-left?
Funda	The point at which the end of the rectangle is doing something. For example, when it is over here.
Researcher	Ok
Funda	Yes, that is right. I say left. Functional, since it touches the left side.

In her conceptual test, she emphasized that it is better to take a middle the Riemann Sum; during the interview, she asked her to explain it. She said that "*the error rate would be less than the other approximations. Because the gap is on top of the rectangle. However, if we take the middle of it, it is like the two sides are equal. So, in this way, we draw a rectangle and take the average values to calculate all the* *rectangles.* From her statement, it can be understood that Funda understood the Riemann Sum approximation, and she learned that overestimation and underestimation. The critical point is that Funda knows that while the number of rectangles is increasing, the width of them is decreasing, and finally, every rectangle shades the given area. After discussing the approximation in Riemann Sum, Funda asked to write it. When explaining the accurately Riemann Sum, she could not write the appropriate notation.

$$\lim \sum_{i=0}^{n} f(x) . \Delta x$$

Figure 4.65 Funda's Limit Notation

While the summation part of the notation is correct, Funda did not describe either i or n. In her statements, she said that she wanted to get an infinite number of rectangles. However, she did not write n approaches the infinity. In her concept map, she wrote the notation as

$$\lim_{\Delta x_i} \sum_{i=0}^{n} f(x) . \Delta x_i$$

Figure 4.66 Funda's Limit Notation

The same notation is the same again. She wrote as if Δx was changing and $f(x)$ is not changing; however, it can be inferred that she is aware of every $\Delta x \to 0$ as she explained that "In this question, it is divided the functions into ranges. As these ranges go to 0, the intervals become so small it becomes as only the function's value remains. At the same time, we find the area under this function in that range. This means the Reimann Sum part, which is taken from the rectangles. The closer these intervals here are to 0, the smaller it is. All the rectangles make up the lines and cross the whole area, which equates to the integral, the Fundamental Theorem of Calculus. Then it is said to Funda that notation is for functions whose equation is known. Then she said that "we can use this for the function whose equation is known. In this problem, we can find its equation with the help of the GeoGebra, or we can measure, and by dividing rectangles, we can find the area then we can compare it with the scale on google maps." Until here, it can be inferred that Funda understood big ideas about Riemann Sum. Even though she explains that several rectangles go to infinity, she could not notate. It can be either; Funda had a detailed explanation before or still, she has some problems about limit notation because she insistently did not show n goes to infinity and wrote as if it was a limited number. Overall, Funda was able to apply approximation accurately, and she described how to calculate an area with the Riemann Sum with a high degree of accuracy.

3) She made the relation between dx sum and Δx

Moreover, when she is asked to explain dx, she made the same explanation with Betül, and she related it with dx. She stated that "It's the range here that has been transformed, but here. Δx is every single change out there. Every change in x and also, she added that dx shows not just according to what; the given function is to be integrated and but related with the derivative. "

4) Funda transferred integral knowledge into real-life situations.

In the integral questionnaire, it is asked to find how much **work is done** when an elastic spring whose spring constant is k is extended 4 cm. In the previous test, she substituted the given directly into the formula when she made different reasoning. In this time, she realized that $F(x)$ is a function and dependent on x so she could set up the integral formula and calculated it correctly.

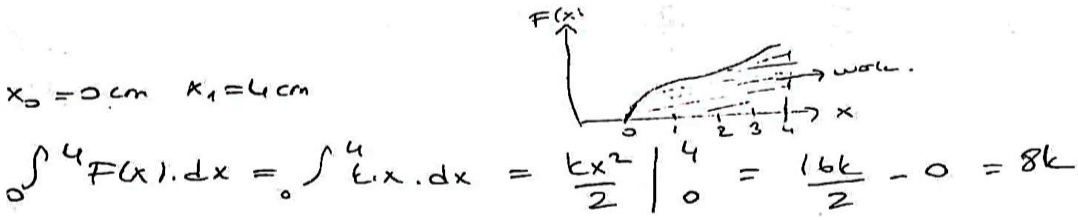


Figure 4.67 Funda's Solution in Real-Life Application

From this solution it can three main results can be found:

- Funda saw function as an object and interpret the whole picture.
- She could determine the dependent and independent variables.
- She could set the integral up and calculate accurately.
- She was able to use her mathematical knowledge in a different field problem.

5) *She explained the area under the x-axis concepts.*

Funda has successfully solved the question and explain the reasons that one of the negative coordinates yields the "negative" area. It does not mean that the area is negative. It is related to the coordinates. Moreover, even if it was shown in the graph, the area under the curve is bounded by the line $y = 0$.

Researcher	We will find this area. How do you write this with integral?
Funda	I think we did something like this in a parabola.
Researcher	What did you do there?
Funda	We subtracted the lower curve from the upper curve
Researcher	Good. What are we going to do here?
Funda	Since our line $y = 0$ line, we have never written it, but our area is limited by that line. By subtracting the $y = 0$ line from the equation of the curve given above, we find the remaining is which we want to find out.
Researcher	What did you mean?
Funda	$\int_{-1}^{0} x^2\,dx + \int_{0}^{1} 0 - (x^2 + 2x)dx$. It should be like this. This is the expression of the shaded area in the form of integral.
Researcher	Ok. You said that this is the reason for obtaining a negative result, is it true?
Funda	Yes. Because when we define definite integral just under a curve, we cannot see the y=0 line. Actually, we never thought about the subtraction process there.
Researcher	OK. Where does this minus come from?
Funda	It could be because it is below the x-axis.
Researcher	Okay, because it's under the x-axis. What do you mean by that? Can you elaborate what is the relationship between the minus and the x-axis?
Funda	Can I explain withdrawing rectangles?
Researcher	Sure.

Funda	Like we did in Reimann Sum. I Just draw one rectangle and explain on it .
Researcher	OK.
Funda	Hmm. -2 is the area of this place because the value of this place is -2 in the $y - axis$.
Researcher	Then what happens?
Funda	It is because it is negative; in fact, that is the negative area there. So, the field is not minus. The coordinate is a minus due to its sign.

It can be concluded from the excerpt that Funda was aware of that area is a positive quantity, but due to the coordinate of the point which in coordinate system determine the sign. Moreover, she is aware of why the minus is under the horizontal axis. Because most of the time among the students due to deficiency definition of the area under a curve, students have difficulty in understanding bounded regions below and under the axis.

6) Funda has a conflict about the Fundamental Theorem of Calculus

At the initial concept image, Funda expressed that integration is the inverse operation of differentiation. Even though this problem is proved in the course and emphasized with activity, she had a problem with applying this knowledge in this problem. This problem reveals that, without seeing this part of the problem in a physical problem, she cannot relate or understand it conceptually. She could not interpret FTC and stated that "I suppose the variable to be written in parentheses in the function F is $F(t)$. Here a, x in limit has been just variables. It changed according to t. I think it would be $F(t)$. However, I am not sure" In $F(\_\_) = \int_a^x f(t)dt$ problem: Funda also thought it would depend on the integrand's variable, and the boundaries should be constant. Moreover, Funda was able to interpret the Fundamental Theorem of Calculus.

Researcher	So, what is over there?
Funda	f function. $f(t)$ derivative. It is integral.
Researcher	Integral of what? Which is your original function

	there?
Funda	My original function is $f(t)$. This turns out after you take the multiple integral. This is an integral function.
Researcher	Well, would you look at this? The first part of the Fundamental. What did he write there?
Funda	Then $f(x), g(x)$ would be the derivative version of it. However, here is $f(x)$ if I integrate $g(x)$. The opposite happens then what I say. I said if I integrate $f(x)$ then $g(x)$ comes, but if I integrate gx here, $f(x)$ comes. So g is the integral of f. It is antiderivative. So was that right? However, here comes $f(x)$ if I also integrate $g(x)$
Researcher	So what does this part tell you?
Funda	The two of them are antiderivative to each other.
Researcher	So?
Funda	Here it said that $g(x)$ function is the antiderivative of f. That is why he said that the derivative of g is the function $f(x)$. So when we integrate $f(x)$ it turns out $g(x)$
Researcher	So why did it say x here?
Funda	I thought of it as a boundary, but I was not sure. But here x is a variable change between (a, b). However, this function variable is t.
Researcher	What does the Fundamental Theorem of Calculus tell us?
Funda	The Fundamental Theorem tells us: The derivative of a function, the derivative of a function within a certain boundary with respect to any x value, the derivative of this function, excuse the integral, gives it. It gives the area under this area. The one below says the opposite of each other, but ...
Researcher	So, with the derivative.
Funda	exactly. It says that the derivative and the integral are the opposite of each other. That is, it says we can always translate to each other in this way. It establishes the connection between derivative and integral.
Researcher	So, what does this provide for us, what kind of convenience does it provide?

Funda	For example, derivative meant slope. From the slope, we are looking at the area under that thing, under the line.
Researcher	If we did not know the basement. Think of the lesson during a lesson. What have we done? What have we always done until we learned about the fundamentality theorem? How did we calculate the integral?
Funda	We calculated it with Reimann Sum.
Researcher	However, what did we do after we switched to Fundamental?
Funda	We established a direct relationship between integral and derivative. We have found the equivalent of that point, not the area under it individually, for all points in the function. Since we know the derivative of that function, we wrote it by saying that it is mathematically inverse. So, we have never been in the Reimann Sum total. So, it provided us such a convenience.

This situation showed that Funda knew they were the reverse of each other, but she could not succeed when she tried to coordinate this with differentiation. Since her belief about boundaries should be the constant and definite integral of a function should be number, it cannot function. Moreover, their understanding of the derivative of a function is separated, and it is based on just applying derivative rules. Even though she has got accurate knowledge about derivative and integration that when a function is differentiated or integrated, a new function is obtained, she has lacked knowledge about which characteristics of the function are changed or not. In other words, this inverse process and the function concept are not structured within a high accuracy. Thus, she was not able to ignore neither the variable of the function nor the boundaries. She holds the incorrect belief that the name of the independent variable is an essential feature of the function and one which may be affected when a process such as differentiation is applied to the function and due to the definite integral boundary should be constant.

This faulty construction remained undetected until it was exposed by the attempt at coordinating these two parts of their mathematical knowledge.

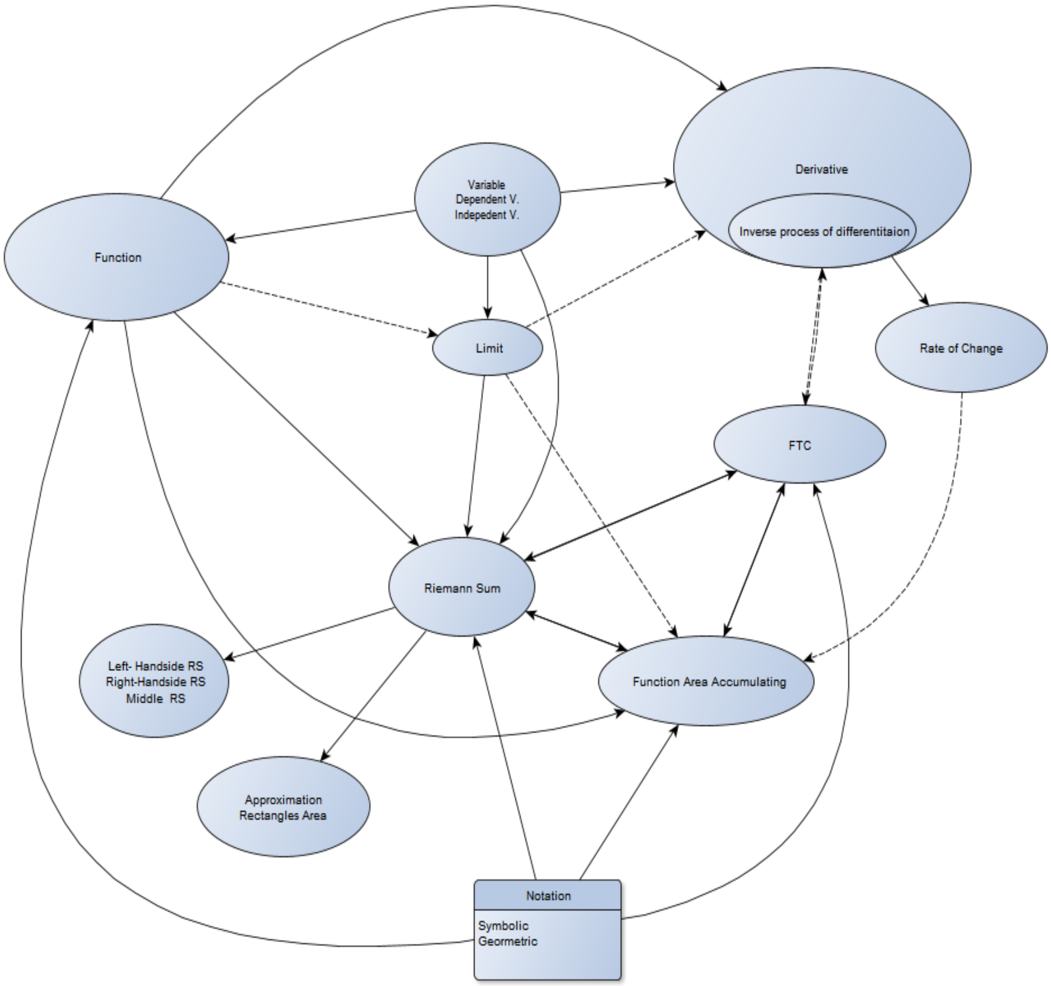


Figure 4.68 Final Concept Image of Funda

Under the light of the whole process, Funda constructed the definite integral mathematically accurate. However, deficiencies that were brought from the previous learning and not directly related with integral hinder her from constructing strong links between some concepts. For instance, in the limit concept, she has some deficiencies; even if those are compensated during the whole process, strong links were not produced. Because the limit is a complex concept and composed of different sub-concepts, due to its nature, time problem, and also the focus of the study is not limit; the concept was not taught efficiently. Nevertheless, Funda connects accurate links. A similar situation is valid for the rate of change. Again, because this concept is a subject that needs to be given about derivatives and Funda could not learn this concept sufficiently, Funda could not fully learn the Fundamental Theorem of Calculus.

4.3 Developmental Phases of İlke

4.3.1 İlke’s Initial Concept Image

İlke had a hybrid concept image (HCI). She constructed mathematically inaccurate knowledge on scientific knowledge or vice versa. This kind of image is fragmented, and learners create irrational knowledge of scientific knowledge or combine them intuitively according to context.

İlke internalized her knowledge on integral procedurally, not conceptually. By experiencing integral many times in different contexts, she conceptualized the term as a situated abstraction. Namely, she could solve relatively difficult tasks; however, she could not explain the reasons behind her solutions.

Thinking in terms of integral concept, she constructed her definitions around “area under a curve and antiderivative” and most commonly on deficit procedural knowledge. In general:

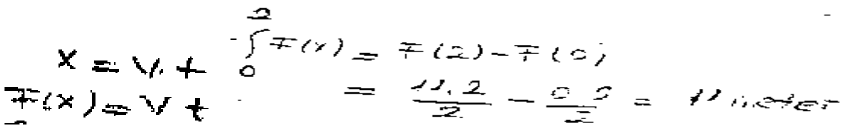
1) *İlke explained the integral through geometric definition (area of a region under a curve) or formal procedural arguments, but she could not relate them.*

Like Betül and Funda, İlke provided an area-oriented definition of integral by giving a general description of “the area under the function’s curve instead.” Besides, she defined integral as a procedure of calculation.

$$\int_a^b F(x)dx = F(b) - F(a)$$

Figure 4.69 İlke’s Procedural Calculation Definition

It can be seen that İlke constructed integrand as the result of the function instead of $F'(x)$ Moreover, she used this definition regularly in her works. For example, in the **stuffed gorilla** problem, İlke also stated that *“we can focus on velocity for the times which are the* ***extreme values*** *of a function.”*



During the interview, İlke used an area-oriented definition of integral while working on finding the position of car A at $t = 1$. Moreover, in $F(\_\_) = \int_a^x f(t)dt$ problem İlke used the same definition.

İlke:	I guess it T
Researcher	Why?
İlke:	Because here is the standard representation $F(x) - F(a)$ so we can write like this.
Researcher:	So?
İlke:	Is $F(T)$ equals to $F(t)$? *(Thinks a little bit)* ...Well... I am not sure but, yes, it is so we can write.

Here, there are two points that the first point is İlke tried to remember the Fundamental theorem of Calculus; however, since she just memorized it, she got troubled to apply it. The second point is that at first, she has problems determining the variable why she said, “it will be “T” instead of “t.” Moreover, as seen in Betül and Funda, İlke also thought that the boundaries must be constant.

Researcher	Well, is that x variable or constant?
İlke	Here, it is constant.
Researcher	Why?
İlke	This is something that shows where the integral starts and ends. Why should I write as a variable?
Researcher	So, we cannot write in here as a variable.
İlke	Here, yes
Researcher	So, we can write in general?
İlke	when determining the limits?
Researcher	Do you think that is that right? I mean, when determining boundaries?
İlke	None.

Researcher	So, if it is written in this way, x is constant
İlke	absolutely.

Different from them when is asked to İlke to integrate the $\int_0^1 xdt$ She said that she could not solve that since there is another variable as x. To understand what she meant, I asked her to take the derivative of xt with respect to t she answered it as x. Then it is asked her what the reason behind them is; she said I think they are the same, but I do not know.

6) *She could set up an integral function for prototypical examples correctly, and she calculated integral quickly.*

Another point that is worthy of mentioning is her interpretation of the negative area. She calculated an integral between the intervals$(-1,1)$. She found the area as a negative value; she explained that the area could not be negative; it is always positive. She explained the reason for the negative area correctly; however, she misinterpreted the integral. Both negative area interpretation and her definition about integral is "*the area under the curve*" she said that she finds the area under the curve. However, it is negative. She said that she took the area over the curve by showing the shaded area.

Figure 4.70 İlke's Solution

Researcher:	Let us look at this question
İlke:	I thought that it would be from -1 to 0.
Researcher:	Why did you think like that?
İlke	Because it seems as discontinuous
Researcher:	Do you think this is a discontinuous function?
İlke	No, it is not… I just say that. ... well, we define integral as the area under the curve... I mean on the positive $y-axis$ or upper part of the $x-axis,$ the area under the curve gives me here *(shows shaded area)*, but when I go negative part, the area under the curve will be here (shows the shaded area)
Researcher	What is the reason for the negative area? That minus, due to what
İlke:	Just because it is under the x-axis. Because every time I enter an input, it will give me $f\ (x)$, but here for every x I have already found a positive value.
Researcher	I get it. So, you mean that it is only here that the minus is on the $y-axis$ and that you took the shaded area?
İlke	Yes.

Finally, as İlke tried to write an integral from a work problem, she did not write the accurate representation, and she just tried to write an integral without knowing the reasons.

In her first concept map, she just defined integral and representations and gave some application examples. In the interview part, she also stated that she did not hear about the terms "accumulation of rate of change, infinity small, FTC and Riemann Sum." She related infinitely small, lower, and upper limit with a limit, function with variable, and definite integral with an area. The interesting point is that she refers to force and motions with derivative. However, she said that "*they are not related to integral. Because, as far as I remember, we find acceleration with taking derivative of velocity, so I do not think they are related*". Another point that her mathematical interpretation was so authoritative, so even if she does not know the accumulation rate of change, she explained it almost accurately and related it with integral.

It can be seen that İlke has hybrid understandings. She can build unscientific knowledge structures on scientific knowledge as in negative area question, or she applies her personal procedure of calculation definition and interprets the antiderivative and boundaries inaccurately. Briefly, she has apparent problems in antiderivative, negative areas, interpreting the symbolic representation of integral, dx . Finally, according to İlke, the definite integral and integral concept refers to the same concept. She did not think about other types of integral.

To provide the reader with a general view, Figure 4.71 represents the general form of the İlke's initial concept image.

As shown in Figure 4.71, her image comprises sub-concepts such as; calculation, İlke's conflicted deficient area oriented definition, some examples, intuitive Riemann Sum and intuitive accumulated change, geometric and symbolic representation. It can be seen that symbolic representation and procedural have substantial effects on her image and her interpretation skills, and the combination of scientific and unscientific combination of knowledge plays a crucial role in her concept image.

1) Definite integral concepts are constructed in pieces.
2) Scientific and unscientific integral sub-concepts are combined.
3) The student does not relate to the Riemann Sum and FTC.

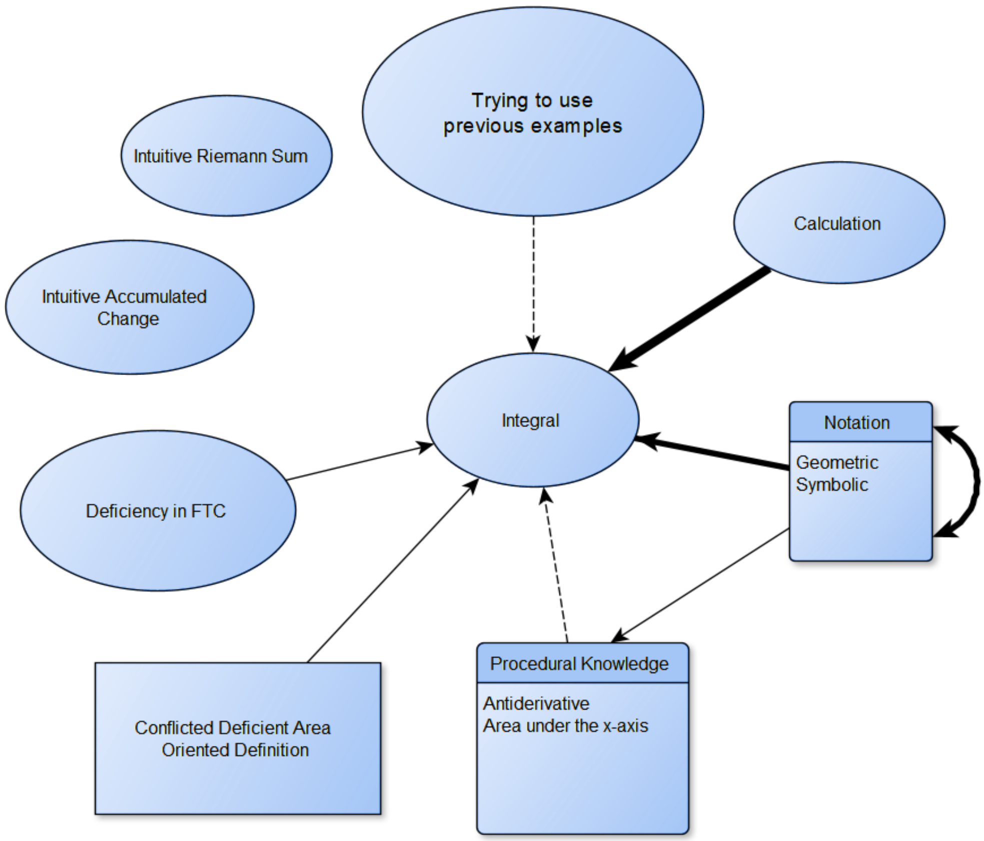


Figure 4.71 İlke's Initial Concept Image

4.3.2 Concept-Image development of İlke about integral through engineering Activities

According to initial concept images, Funda did not know what Riemann Sum is; however, she had some ideas about the concept due to her interpretation skill. She knew approximation, and she was able to adapt her knowledge to different examples. However, in her integral knowledge, misconstruction was preventing her from understanding the integral conceptually, which also caused misunderstandings in the application of the Fundamental Theorem of Calculus and had conflicts in her interpretations. Thus, she constructed a personal negative concept and the Fundamental Theorem of Calculus based on two different concept definitions. Finally, as her friends, she also believed that the area under a curve is calculated by knowing the curve's function; otherwise, it was not calculated. During the activation process, İlke was very active in the designing part, and even it consisted of some deficient information, she was able to support her claims with physics. However, she had some difficulty connecting with mathematics. During the activity, the dominance of designing skills helped her to join the mathematical discussion part with design and also helped her to understand easily. For most of the activity, she joined her ideas in the mathematical discussion part with the design part.

4.3.2.1 Developments in İlke's knowledge about Riemann Sum in terms of Object -Process Perspective

In this section, how İlke went through the object from the process was presented.

4.3.2.1.1 Interiorization Stage

In the aforementioned previous section, due to the deficiency of essential concepts in İlke's concept image and did not meet the criteria in chapter 2, it is determined that she was not at the interiorization stage.

4.3.2.1.1.1 Remarkable Event in Activity

As seen in Figure 4.73, İlke was more active, according to her group mates. She produced an idea about the canoe's shape, whether it should be sharp or flat. According to the activity's demand, she also added buoyancy and carry max load in her claims. After having designed the canoe, she could not realize what to do, and she did not understand Funda and Betül conservation. She listened to all of the group's conversations, and then she asked about the find the area with integral and the rectangles. At first, she also tried to find the area with a known function as her group mates. Then with the help of the instructor, she attempted to solve them with rectangles. Because of conflict in her integral definition and its dominance in her concept image, even though she had started accurately with rectangles, then she expressed the process with integral and tried to calculate with her definition, which is $\int^{b} f(x)dx = F(b) - F(a)$. Moreover, she drew rectangles to solve the question, then articulate it as if she answered the problem using integral. After realizing they were working with unknown functions, she was confused for a while.

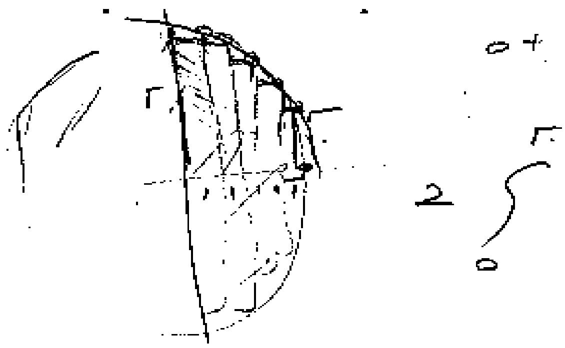


Figure 4.72 İlke’s Solution

İlke	We can find with rectangles
Researcher	Excellent, how?
İlke	It will be symmetrical already; for example, these rectangles will be continuous; for example, this will be the same.
Researcher	Ok?
İlke	There will be two, for example, the other one will accept 0, so here we will like the value of r from 0 to r, for example, then the total of them.
Funda	How will you take those lines?
İlke	The curve or the area below it
Betül	But we do not know the function, where is r
İlke	The length of the last thing.
Researcher	But your y value will change continuously.
İlke	y value will be at most this function.
Researcher	Ok, how can you calculate?
İlke	Let us say here is one and first will be the first 0. Let us say the other is one, and there will be r, and this will change then we can say 0 to r and subtract them.

İlke did not have any difficulty transforming images from 3D to 2D or vice versa and quickly replaced the rectangles into the canoe in different planes. In this way, she was able to connect design with mathematical discussions.

After calculating and decided to determine how much their canoe would carry, she connected the design part and the mathematical discussion at the end of the lesson.

At the end of the lesson, İlke was able to;

- Calculate unknown irregular areas by dividing a given interval into rectangles, calculating these rectangles' areas, and adding them up.
- Without knowing the function, with the help of the rectangles approximated area can be calculated.
- Do not be aware of calculating the area with a limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten, there will continue to be a "next piece," that is, the accumulation is not complete.
- Do not aware of the generalization of the process.

Table 4.10 İlke Improvement in Interiorization Stage

Interiozation Stage	***Reached***
Calculate unknown irregular areas by dividing a given interval into rectangles, calculating these rectangles' areas, and adding them up.	+
Without knowing the function, with the help of the rectangles approximated area can be calculated.	+
Start to be aware of calculating the area with a limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten there will continue to be a "next piece," that is, the accumulation is not complete.	-
Be aware of the generalization of the process.	-
Connect with the real world.	+

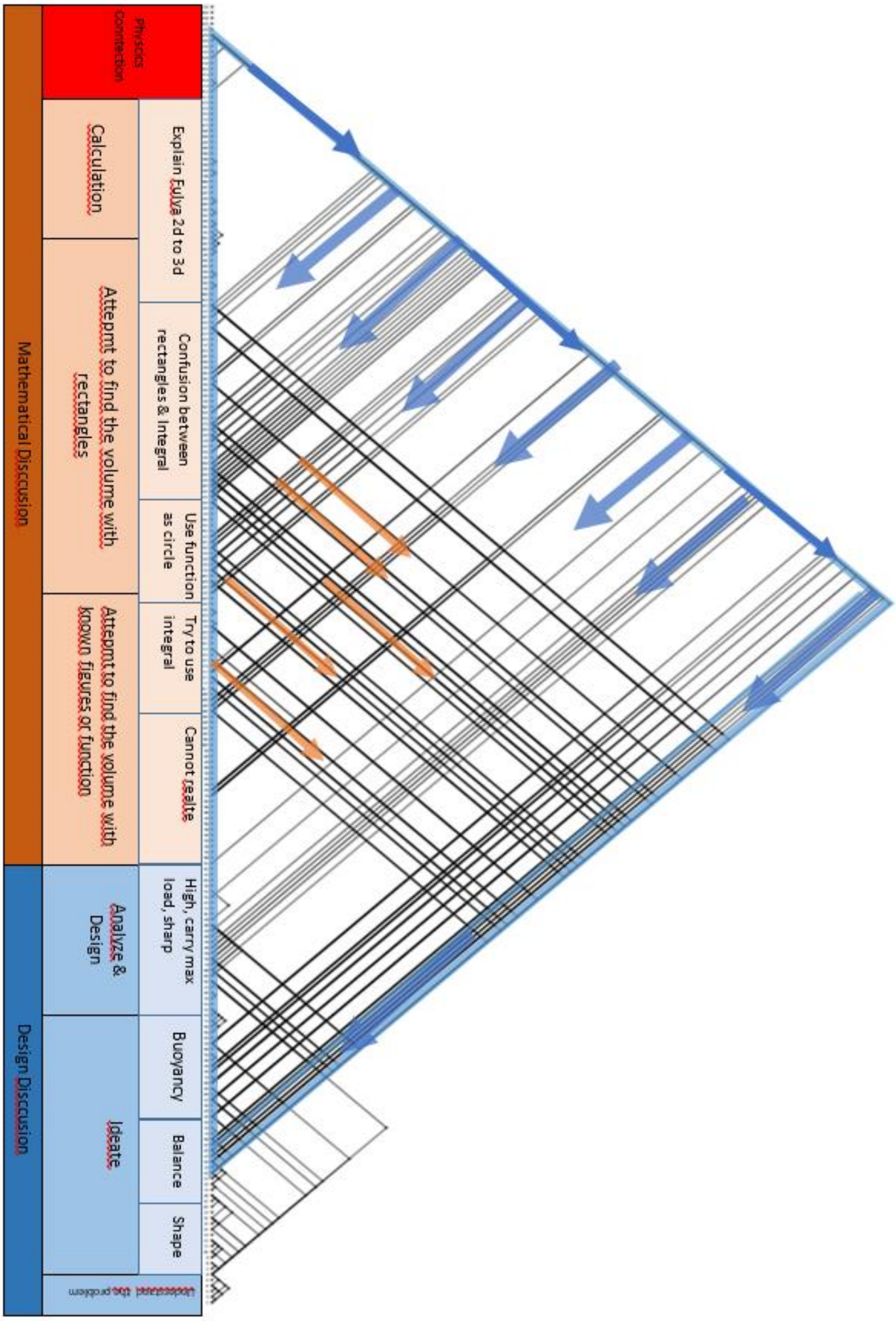


Figure 4.73 İlke’s Linkograpgh in Activity 1

4.3.2.1.1.2 Remarkable Event in Activity 2

Since the first activity researcher could not detect whether İlke had any idea about approximation or error, she investigated approximation with the group members. In this way, the researcher will consolidate the approximation construction of Funda and detect İlke's comprehension level. In this way, the researcher makes them think about the approximation process. At that point, İlke asked that "*why do we draw rectangles? Gaps also remain in this"* That means İlke did not have a whole structure about approximation in her mind. So, the researcher scaffolded her about how the number of rectangles might affect the results. During their discussion, Betül explained that the more they took rectangles, the more accurate the result they could find. By trying some small examples, she understood approximation.

Moreover, when they design and apply the formula into the prototype in the design part, she realized that increasing rectangles decreases the error. Hence İlke, started to be aware of calculating the area with the limited number of rectangles will not give an exact solution, but by increasing it, the more accurate calculation can be gotten there will continue to be a "next piece," that is, the accumulation is not complete and generalization of the process. When all of these are considered together, it is seen that İlke is making progress, and she reached all the objectives in the interiorization stage and passed through the condensation process.

4.4 Condensation Phase

At this process, she understood Riemann sum is an accumulation process. However, throughout due to the task's nature, which mostly emphasizes mathematics and design together, she had difficulties thinking mathematics since design thinking is dominant in her concept image. Moreover, she was not accustomed to such activities; İlke had trouble following the task and combine design pieces and mathematical calculation.

4.4.1.1.1.1 Remarkable Event in Activity 2

Compared with the first activity, for İlke, design and the mathematical processes were separated. The reason for that is İlke focused on its appearance. Then she thought about distributing the eight equally through the bases, so she tried to find CoG of triangles.

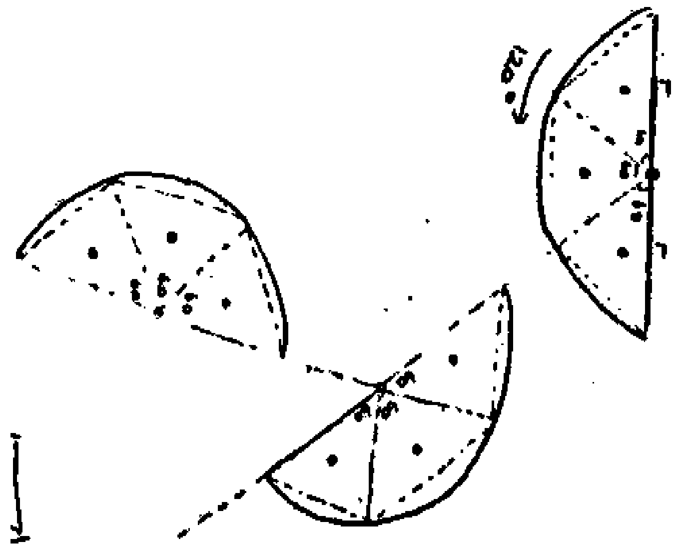


Figure 4.74 İlke's First Idea

With the improvement in İlke's thinking as her friends, she started to base her opinions on the previous one. it can be seen that blue, yellow, and red triangles are constructed on each other (See Figure 4.75.) After that, İlke affected by Betül's ideas and tried to connect rectangles with the moment. Since design thinking is active, then mathematical thinking, İlke could not combine them and could not transfer knowledge into design. Besides these, the activity was not familiar for her and composed different sub-sections, and to reach the solution, it was necessary to combine and think all of the pieces together. Therefore, she had confused and could not focus on the construction of the Riemann Sum and always asked about "how this will affect our design?". In her mind, pieces are constructed separately.

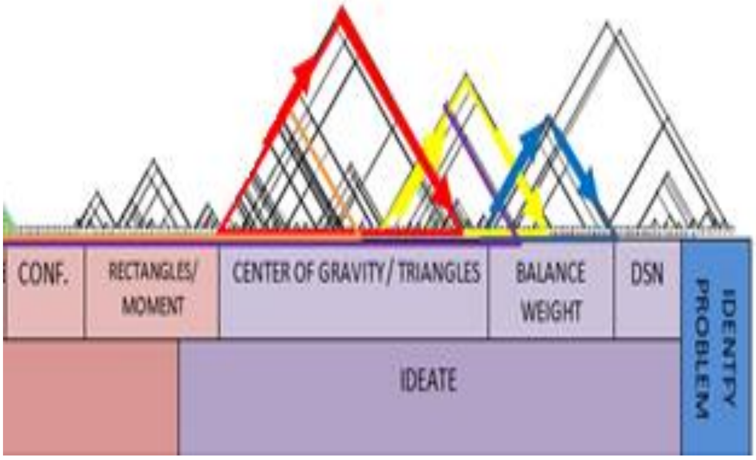


Figure 4.75 Some Part of İlke’s Linkograph

To help İlke connect the pieces, the researcher started to scaffold the first CoG and make students think about how Riemann Sum solve their problem. For İlke, this was important to help her to broke her difficulty in connection with mathematics. In this way, she was able to relate the concept with mathematics and reason about its applications. To help İlke generalize arithmetical calculations, the researcher first moved on real measurements and then expressed them algebraically. Another aspect that supports the development of İlke is the structure of activity, which is design, mathematics, and the use of mathematics with design. Namely, after establishing a connection with the design and Riemann sum, Betül’s application of the formula to prototype has enabled İlke to see how the method works and match each variable’s meaning in real-life. Thus, the discussions made during the activity were meaningful for İlke and enabled her to the reason for the problem’s solution. She also mentioned this as “I do not have any previous knowledge about the topic, and I thought my group mates also do not have. Our group discussions affected our decisions, mostly.” Moreover, Betül’s misinterpretation of the formula, provide İlke to reinterpret and join the formula with the design and CoG (see the orange triangle in Figure 4.78) .

Figure 4.76 İlke and Betül's Calculation

Her reflection paper is examined to see her development. The activity is designed as a transition from geometrical to algebraic representation; however, in the reflection paper, both representations are asked. In her reflection paper, it is observed that İlke used both representations correctly.

To find the area under the curve, we approximate the area by using rectangles and then use the limits to find the area.

I divided the interval (0, 1) into 3 subintervals of equal lengths, $\Delta x = \frac{1-0}{3} = \frac{1}{3}$ and this divides the interval into 3 subintervals.

(0, 1/3), (1/3, 2/3), (2/3, 1) with length $\Delta x = 1/3$.

Endpoints of these subintervals are $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$.

I drew a rectangle with height equal to height of function at the middle point of subinterval.

x_i	$X_0 = 0$	$X_1 = 1/6$	$X_2 = 3/6$	$X_3 = 5/6$
$f(x_i) = x_i^2$	0	1/36	9/36	25/36

We use the sum of the areas of the approximating rectangles to the approximate the area under curve.

$A = \sum_{i=0}^{1} f(xi)\Delta x = f(x_0).1/3 + f(x_1).1/3 + f(x_2).1/3 + f(x_3).1/3$

If we divide the area to n subintervals, we can find the summation as

$A = \sum_{i=0}^{n} f(xi)\Delta x \quad \int_0^1 f(x)dx = \int_0^1 x^{\wedge}2dx$

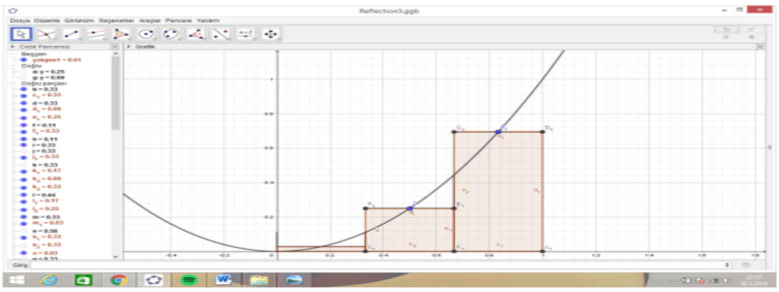


Figure 4.77 İlke Solution in a Reflection Paper

Considering these İlke processes is showed in the table below.

Table 4.11 İlke's Improvement in Condesantion Process

Condensation Stage	***Reached***
Write and interpret the product of $f(x_i)\Delta x_i$	+
Determinate the accumulation of the changes according to the independent variable.	+
Grasp that the quantity which is accumulating has a multiplicative nature.	+
Flexibly use various representations.	+
Differentiate left-hand side and right-hand side Riemann sum	+
Comprehend that accumulation is a process of adding new increments.	+
Smaller and smaller changes in the independent variable yield more precise estimates of the change in accumulation.	+

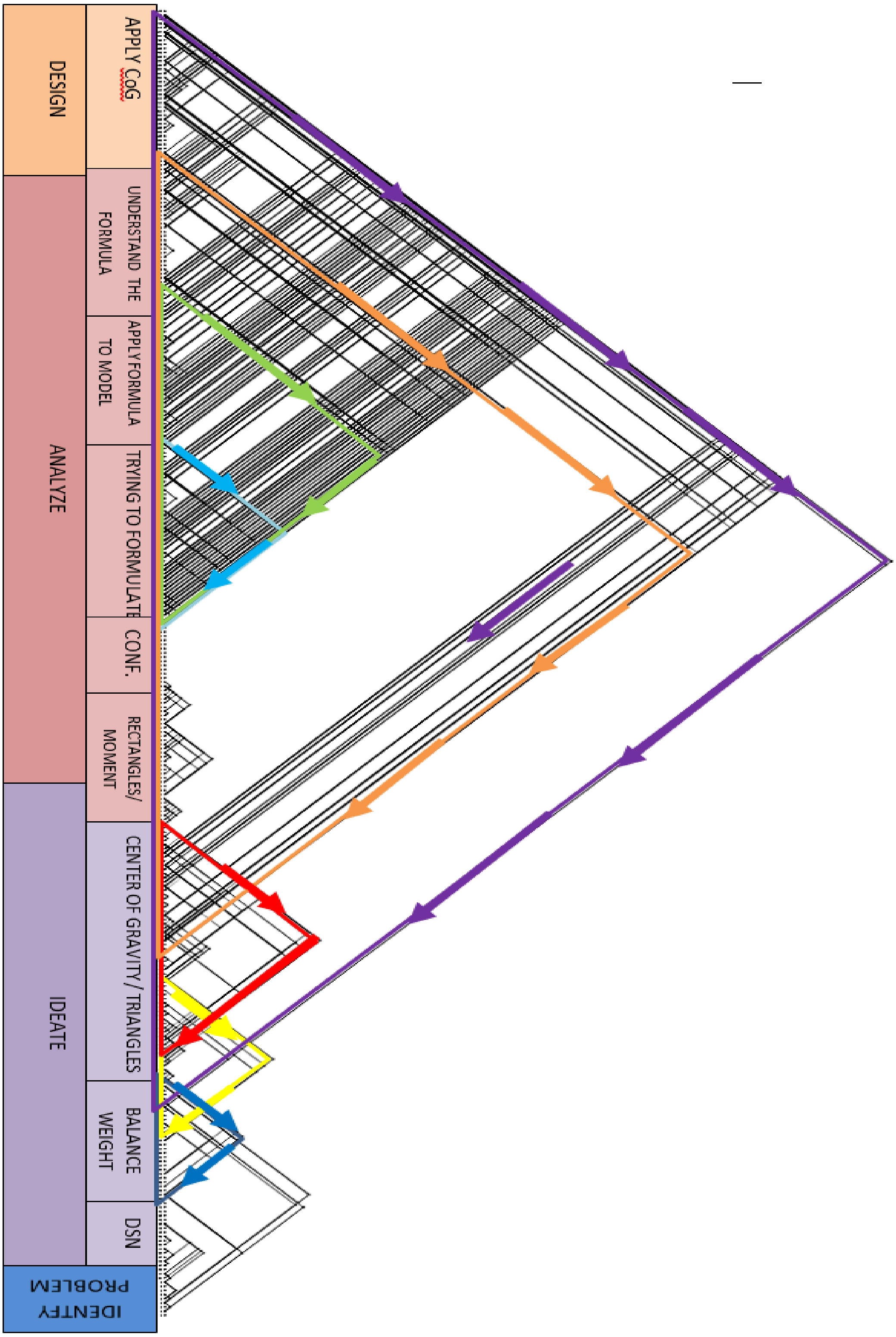


Figure 4.78 İlke’s linkograph in activity 2

4.4.1.1.1.2 Remarkable Events in Activity 3 for Transition to Reification Phase

Although İlke is at a higher level in this activity than the condensation phase, she could not clearly show the properties of the reification phase. Thus, this activity was a transition phase to reification like a preparation phase, and the foundations of the reification phase were laid.

The remarkable points for İlke were coded in Figure 4.79 as Understand the problem, pressure force components, finding pressure formula, try to relate the pressure formula with integral, deciding r and θ and the design, conflict in boundary angle, line and boundary angle, and finally trying to find the appropriate angle. Unlike the previous linkographs, İlke produced overlaid linkograph. That means she built connections by constructing the ideas on each other, and the linkograph indicated that she started to combine both mathematical parts and the design during the problem-solving process. Moreover, in some of the time intervals, she constructed the concept by herself then contributed the discussion. This construction process is coded as self-construction in the linkograph. That means İlke needs to combine both mathematical parts and design, construct, and understand the mathematical concept. However, as understood from the group discussion, codes are not the same, and in İlke's codes, the design codes. It is noteworthy that the design codes were predominant and almost no mathematical code related to the Riemann sum. That means İlke was starting to think mathematically and still need assistance to construct the Riemann sum concept. However, she started to construct the concept by herself. Moreover, her linkograph showed that both her initial concept image and misunderstanding the physical terms caused hindrance to understanding the concept easily. Her insufficient initial concept image, which is the integral area under a curve, prevented her from understanding and writing the Riemann Sum and the integral form. This process can be seen within the pırple triangles. In that area, İlke constructed the concept almost regularly and built overlaid links; however, in the "Summation Notation"

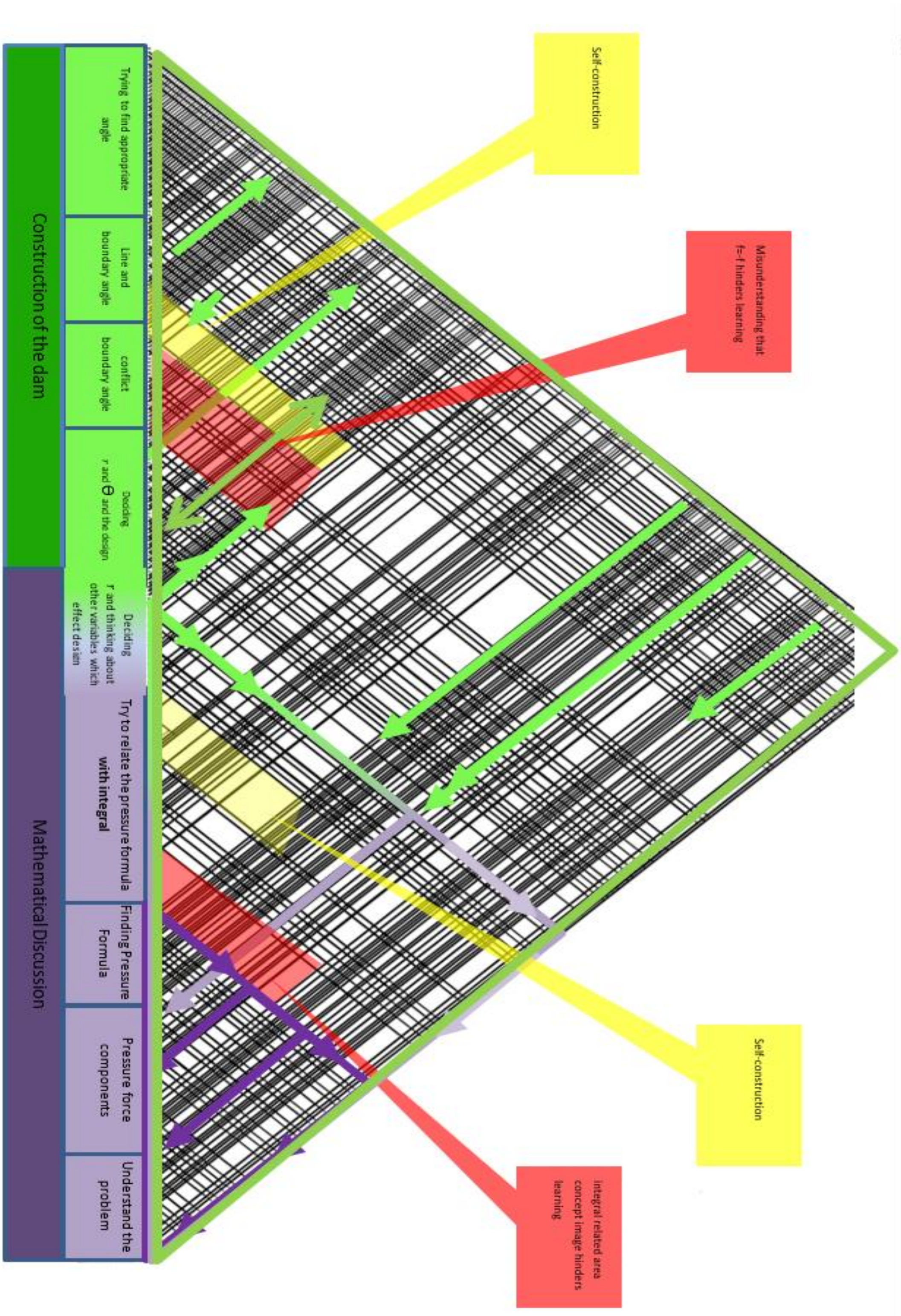


Figure 4.79 İlke’s Linkograph in Activity 3

codes, she suddenly stopped producing the links because of the mentioned hindrance.

After a while, she constructed the concept with the help of her friends. During this process, while İlke was reviewing her scribble, the following conversation occurred

İlke	I do not understand here. Will we use the area? So how will its integral be?
Funda	We consider the area as much as the length of the arc.
İlke	So is the area like this, and that is as much as r.
Betül	No, we will use length. Let us imagine that this is the arc. We bring r to the center from here, or so if we say πr^2. Here, we find the area of here, but our force only affects here; we do not take thickness in it; we think as if it is affecting a single point; it just seems to take the length of that.
Funda	Because the height is the same in all.
İlke	Let us think of a very thin circle like this, we think like r times this, do not we think as much as $r\Delta\theta$ (actually thinking right).
Betül	No, think as if you can find the length of here, not a circle segment, but if there is a circle segment, you will take the inside of it; there is no water in it; it applies pressure here
Funda	It is related to length, not the area of the circle.
İlke	Ok, why we call it $2\pi\Delta\theta$.
Funda	It comes from here. Look at this formula.
Betül	We are doing $2\pi r\alpha/360$ or look here is too small, or the height is the same then we will take the length of this small length.
İlke	Hmm, the length we take it from it does not go in, and they are all equal height ok, I get it.

After İlke overcame the area hindrance, she connected the physics part and the mathematical part and started to interpret. Since the formula involves different variables, she tried to decide which variable is the most important for the dam. This part was illustrated in Figure 4.79 as a gradient transition from purple lines to green lines. It can be seen that İlke built a backward link until the “deciding r and thinking about other variables which affect design” code; after self-construction, she began to produce both forward and backward links. That means she connected the whole process and started to interpret the formula and its effect on the design. From the following excerpts, İlke combined the process and started to produce ideas towards constructing the dam by making mathematical interpretations.

İlke:	When we integrate this formula but depends on r and the angle, how do we decide the length of r?
Betül:	We cannot decide p right now, is it depends on water, right?
İlke:	Yes.
Betül:	So we have to play with r and θ?
İlke:	yes
Betül:	We should decrease r but how about θ?
Funda:	but if r decreases, f decreases.
Betül:	I think our θ as much as big.
İlke:	The shape will be like this if we keep the angle large and r small (showing that the circle segment grows by opening the hands to the side).
Betül:	Yes, we need to keep the angle large because cos theta gets smaller as it approaches 90 or the sinus grows, or the cosine gets smaller.
İlke:	We assume that angle is bigger than 90.

In this activity, her initial concept image was so dominant that it hindered İlke to pass a new phase, reification. In this activity, it was not observed that neither İlke dealt with the integral form nor the Riemann sum easily and the other objectives belong to the reification process.

4.4.1.2 Remarkable Events in Activity 4 Transition to Reification Phase

Although İlke is at a higher level in this activity than the condensation phase, she could not clearly show the properties of the reification phase. Thus, this activity was a transition phase to reification like a preparation phase, and the foundations of the reification phase were laid. During the activity, she understood in a given interval Riemann function accumulates in a constant rate-of-change, and it can be expressed as a general formula. However, she was not able to see the Riemann sum as a static object or a structure. Moreover, every time she wrote every step of the

Riemann Sum with its details. This situation proves that she could not reach the reification process for the Riemann Sum at the same time she was beyond the Condensation process. İlke's this situation is represented in the following. During the conversation, it can be seen that İlke determined the variables and construct the formula. Whereas he could write the formula directly as Riemann sum or in integral form, İlke wrote the formula by explaining each step. When the researcher asked if he could write the final formula directly, she said she could not and write it; she should pass through all the steps. It showed that İlke was not able to think the Riemann Sum without going into details.

Researcher	How did you find the volume?
Betül	We will find the formula by multiplying the base area with height.
Funda	Exactly.
İlke	Yes.
Researcher	Then what is this?
İlke	Height.
Betül	Height.
Funda	Height.
Researcher	Why did you say dx?
İlke	Because it represents very small heights, and we call that variable x .
Researcher	What is $f(x)$?
İlke	Since c is $f(x)$, the number corresponding to r here is f(x), and we get $f(x)$ because it is always variable, right?
Researcher	Yes, what happens when we get a cross-sectional area at a time.
İlke	Yes, correspond to $f(x)$
Researcher	Yes.
İlke	We take at all cross-sectional areas. Then it will be related to f. hmm, I understood. Then I will write this as a summation and take the limit of that. Then I can write in the form of integral. Because why is this delta x, so my heights are getting smaller and smaller. What did I do when it got smaller? I called it dx. So, our formula comes out from here.

4.4.1.2.1.1 Remarkable Events in Activity 5 Transition to Reification Phase

Although İlke is at a higher level in this activity than the condensation phase, she could not clearly show the properties of the reification phase. Thus, this activity was a transition phase to reification like a preparation phase, and by recursive steps, which is a process composed of construction of Riemann sum and the integral, she began to approach the reification phase. Doing the recursive process, İlke strengthened the link between the concepts. On the other hand, through activity five, she also progressed in terms of interiorization and condensation phases for the Fundamental Theorem of Calculus. More information and her detailed process were given under the Fundamental Theorem of Calculus heading.

4.4.1.2.2 Reification Process

Sfard (1991) defines the reification process as a shift. Students interpret familiar context with a new perspective. That means “reification is an instantaneous quantum leap”(Sfard,1991, p.20) In this process, students transform the concept into a more concrete and static object. Students utilized different representations of the taught concept meaningfully. In the end, students detach newly learned concepts from the step. This detached concept becomes a member of a class and becomes a basis for another new entity. Moreover, in this process, students use different representations and try to generalize the concept.

4.4.1.2.2.1 Remarkable Events in Activity 6 Reification Phase

In this activity, it is observed that İlke met the objectives of the reification process objectives. Until this activity, İlker made great progress in determining the variables in the given problems, understanding the concept of accumulation she could represent the Riemann Sum. In this activity, İlke was able to write the general equation of the bridge and solve it using the integral, being aware of the fact that both variables change together, as in Activity 5. İlker fulfilled the objectives of this step, but it was observed that she could not develop the ability to "use the concept in problem-solving without going into details while problem-solving" expected from the reification step. İlker expressed this situation as follows while discussing with her group friends.

İlke	Why do we directly write the limit form?
Betül	Yes, we take it in infinitesimal form.
İlke	I do not say it, I know it, you wrote the sum, you wrote the direct limit, you missed some steps.
Betül	Yes.
İlke	We divided this place; why did we write the x s as sum, how did we write directly from here?
Betül	No, we know directly that we can write that it is equal to the integral this way.
İlke	Can we write like that? Can we write like this without writing intermediate digits?
Betül	Yes, we can write because we know.
İlke	Ok, I understood.

As understood from the excerpt, she met all the objectives of the reification of Riemann Sum. Moreover, in this time, İlke constructs the Fundamental Theorem of Calculus.

4.4.1.3 Developments in İlke's knowledge about Fundamental Theorem of Calculus in terms of Object -Process Perspective

4.4.1.3.1 Interiorization Stage and the Condensation Phase

Sfard (1991) described this phase as the students started to be familiar with the new concepts and had limited skills about the concept. However, through the process, the students became skillful. In this regard, in the interiorization process for the

Fundamental Theorem of Calculus, students get acquainted with the Riemann sum's integral representation. In the next title, how İlke met the interiorization phase's objectives in activity five was presented.

4.4.1.4 Remarkable Events in Activity 5

The remarkable points for İlke were coded in Figure 4.81 as Physics, design & materials, design, math& design, the effect of thickness, determining variables, covariation t and x (Riemann sum to integral), boundary, formulate. Unlike the previous linkographs, İlke produced a regular overlaid linkograph. That means she built connections by constructing the ideas on each other, and the linkograph indicated that she started to combine both mathematical parts and the design during the problem-solving process. The temperature and heat concept were relatively new for İlke, so she needed more assistance until the thickness code effect. However, İlke's construction time was quite short compared to construction time when writing formula code. As shown in Figure 4.81, İlke contributed the discussion synchronized with the group. However, in trying the formula part, she separated from the group and easily realized the covariation compared with the group. She produced forward and backward links through the discussion and constructed the whole concept on each other.

The main problem that İlke had difficulty in relating heat with the wall's thickness and interpreted the heat mistakenly. She thought they would find the heat needed to keep the room at a steady temperature with the formula. However, they need to find the heat that flows through the wall, and she explained this situation as *we said the size of the nest and you know the cat that will warm the inside with its breath. It will not do anything extra. That is why he will have to expend more energy when he is too big. For example, you know something like a maximum of 4 times the volume of the cat. I think there is such an effect of meant by size.* " During the conversation, she thought that thickness has no effect on heat, and it was impossible that having a very thick wall in terms of engineering respect. During the discussion in the following, Betül affected İlke’s thought, and she also thought like Betül that was “heat and the thickness have right proportion.”

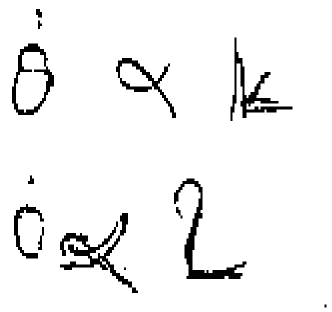


Figure 4.80 Heat and Thickness Relation

İlke	Thickness does not affect when we increase the thickness
Betül	Are you sure?
Funda:	When the thickness increased, the temperature increased, and was this not the example we wore thick in winter?
Betül	Does it decrease the speed of the conductivity as the thickness increases?
İlke	Not like that; for example, I used a thicker material of the same material.
Betül	Okay, is it the same thing that it only passes through this material and passes through the same material three times its thickness?
İlke	Instead of increasing the thickness instead of doing what you say, I put three different items on top of each other. If I put three of these spaces, my roof will increase in thickness
Betül	It could be better. But this does not mean that thickness does not affect when increases it.
Researcher	Look, think this way. This wall is our or our matter is okay. Here is the indoor environment, the external environment. Now our air temperature over there is 24 degrees. Our surface temperature. The warmth on every road from here to here slowly. what will happen to each molecule? There will be a flow of heat right here. So,

	there will be a temperature difference here. In other words, there will be a change in temperature according to the path it takes.
Betül	Then it is directly proportional to the thickness.
İlke	Yes. However, when it comes to the engineering part of this, how thick can you make the houses' walls? Since we cannot use Styrofoam side by side, we will use cardboard at work.
Betül	Well, you cannot use it right now, but you cannot ignore that reality after all
İlke	here I do not ignore the reality. I say because the material we have is limited.
Betül	Ok, not according to the material you have, but you have to think in general right now. We make a formula.
İlke	Yes, you are right. It is directly proportional. I have already figured it out.

This misunderstanding relation between the variables" prevented İlke to substitute the variables into the formula mistakenly. She explained that *"We are trying to calculate with the formula. When it decreases and increases, it needs to change the formula. So, we have to write things like that; it really should give me a bigger value of Q when I increase them."* By referring to the thickness of the wall and the inner volume of the house, she wrote the wrong formula. After Funda's explanation that she understood in every x wall loss, she understood which Q they had to find. Although İlke wrote the formula wrong, she determined the dependent variable and explained the change of the accumulation function value on very small intervals depends on the multiplication of the variables. It was one of the objectives belongs to the interiorization phase.

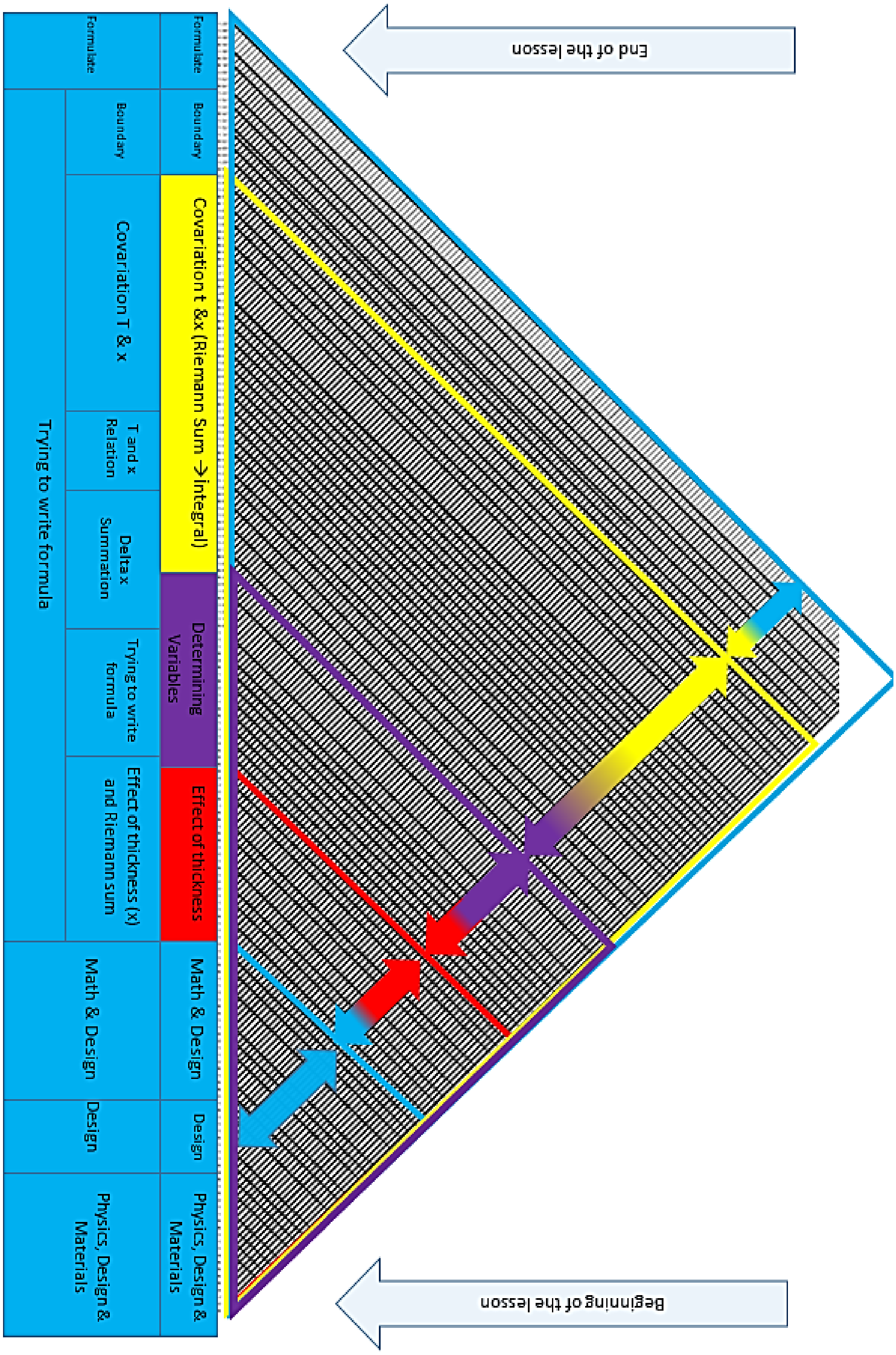


Figure 4.81 İlke's Linkograph in Activity 5

In her reflection paper, she explained what she learned from the course as follows;

> "By the help of relationships of surface area, type of material of house thickness of walls with heat, we discover the formula; Q= $k.s.\frac{\Delta T}{\Delta x}$. We used the ratio of change on T for each change on x since we should calculate the temperature difference between each layer. However, we recognized that we need to work with infinitely small thickness to approach real value. Then;
>
> Q= $k.s.\frac{\Delta T}{\Delta x} \quad \rightarrow \lim_{\Delta X\rightarrow 0} Q.\Delta x = \lim_{\Delta T\rightarrow 0} k.s.\Delta T \rightarrow \quad Q = k.s.\frac{dt}{dx} \rightarrow$
> $\int_{o}^{L} Q.dx = \int_{T0}^{Tn} k.s.dt$
> $\rightarrow Q = \frac{(Tn-To).k.S}{L}$ *where L is thickness, k is type of material and S* = $a^2\sqrt{3}$"

By stating the wall thickness as an infinitely small thickness and the change, she proved that she understood the Riemann function. Moreover, as she stated in her reflection paper and the lesson by stating "We used the ratio of change on T for each change on x since we should calculate the different temperatures between each layer," she understood that temperature and thickness were covarying. Thus, she explained that T and x vary accumulate simultaneously. Moreover, during the activity, she also used similar expressions throughout the discussion and stated that t and the x are covarying together, and the ratio between them will be constant as they change in the same amount. When it was taken into consideration both the statements in the following excerpt and also her reflection paper, it could be said that she understood that these variables with respect to each other cumulates at the same rate of change.

İlke	We say $b - a/n$. Let us take the length from zero to L, so $L - 0\ /\ n$ gives Δx
Researcher	Good.
Betül	Why didn't we say delta t? We said Δx
İlke	I mean, when I go to infinity, I reduce the Δx to the minimum. What?
Researcher	Well, when you reduce it to that minimum, how will t change? Did she ask this?
İlke	Change constantly with Δx If x decreases when I keep q constant. T should increase so that it remains constant.

Researcher	Is there a relationship between the changes in x and T?
İlke	Currently inverse ratio.

She determined the dependents and the independent variables correctly and constructed the covariation between t and t x correctly, as mentioned before. By linking this process with thinking the process in a very small interval and expressing it, it could be concluded that achieved all the objectives determined for interiorization and the condensation phases.

İlke:	What are you doing when you take the derivative of $a\ x\ b$ with respect to $dx\ atx$ with respect to dx? If you derive with respect to x, at will remain constant.
Funda:	Ok under dx.
İlke:	You know, when we calculate, it is atx^2 / 2. Because you are calculated by integrating with respect to x. Here so if we take this integral with respect to x it will be atx^2 / 2If we take it with respect to t, atx will be axt^2 / 2.
Betül:	We determine according to what it takes according to the one next to this d. There are both dt and dx here. I want to say what it will be?
İlke:	That is when I just said it. It is the rate of change, so I can get the inside-out product and dt here on the other side.

Table 4.12 İlke's development through Condensation process

Condensation Stage	*Reached*
Explain $\int_a^b \boldsymbol{f(t)dt = F(b) - F(a)}$ as the accumulated area under the curve of $\boldsymbol{f}$ from a to b is equal to the total change in $\boldsymbol{F}$, the accumulation function, from $\boldsymbol{a}$ to $\boldsymbol{b}$.	+

4.4.1.4.1 Reification Phase

Sfard (1991) defines the reification process as a shift. Students interpret familiar context with a new perspective. That means “reification is an instantaneous quantum leap” (Sfard,1991, p.20). In this process, students transform the concept into a more concrete and static object. Students utilized different representations of the taught concept meaningfully. In the end, students detach newly learned concepts from the step. This detached concept becomes a member of a class and becomes a basis for another new entity. Moreover, in this process, students use different representations and try to generalize the concept.

In this regard, İlke started to gain some of the reification phase objectives in Activity 5, and in Activity 6, she achieved all the objectives for the phase. However, her construction was so fragile and needed to be more practice to make it stronger. In the following titles, it was given that how İlke went through the processes.

4.4.1.4.1.1 Remarkable Events in Activity 5

After the lesson, İlke’s reflection paper was examined. It was seen that İlke started to conceptualize the Fundamental Theorem of Calculus naively, and she also started to interpret the Fundamental Theorem of Calculus. The reflection paper was asked to solve a pure mathematics problem related to using the theorem in a similar context as the lesson. In her reflection paper, she explained what they used in the lesson Fundamental Theorem of Calculus and explained it as in the following:

> The theorem has two versions;
>
> - $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
>
> If f is a continuous function, let us define a new function for the area under the curve y=f (t), then
> F(x) = $\int_a^x f(t)dt$
> According to this theorem, we can say that f is the derivative of F. In other words, F is an antiderivative of f. So, this theorem says that we can find the area under the function by

anti-differentiation to us.

- $\int_a^b f(x)dx = F(b) - F(a)$

This version says that we can find the area under the function
$y = f(x)$ $between$ $x = a$ and $x = b$.

From this explanation, it can be concluded that she started to construct the Fundamental Theorem of Calculus. However, the connections between the links were not strong. Because she explained the theorem, such as the sentences in the textbooks, and she could not explain the theorem in terms of the asked pure mathematics problem in the reflection paper.

4.4.1.4.1.2 Remarkable Events in Activity 6

During Activity 6, the focus of her discussion was coded in Figure 4.83 as understand the problem, properties of the bridge, forces on the bridge, S&D (Slope and Derivative), EoB (Equation of Bridge), integral, Tension load & design, Tension load, angle, slope, Riemann dx, Boundary, integral and derivative relation and finally interpretation. During the whole discussion process, İlke produced both backward and forward links, and mostly her discussion focus was synchronized with the group discussion. In the previous lesson, İlke constructed the Fundamental Theorem of calculus, and she realized the covariation between T and x. Moreover, she approached the solve the problem by thinking design part and the mathematical part together. Thus, she constructed the theorem based on variables in the problem within the design perspective. Hence in this lesson, she also constructed the second part of the Fundamental Theorem of the Calculus quickly compared with her friends. As can be seen from her linkograph, İlke has shown the same pattern as in its previous ones. That means she constructs each concept on each other, and mostly she directed the discussion.

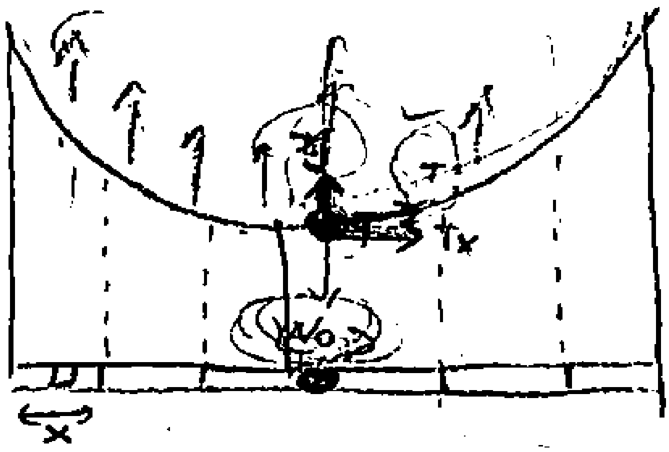


Figure 4.82 Free Body Diagram İlke

Since the beginning of the first activity, İlke's design thinking was dominant. Hence she did not have difficulty writing the free body diagram and interpreted it. By interacting with Funda, she also realized the connection between the rope and the derivative's tension. However, realizing the connection between derivative and the integral was not so clear and easy as Funda.

During the discussion, İlke constructed the integral concept via interacting with Betül. İlke used and improved Betül's ideas. For instance, in the conversation below, by connecting the derivative and the integral process, İlke listened to Betül. By interpreting her ideas on her crawling, she reached the connection between derivative and integral and explained the Fundamental Theorem of Calculus's implementation. During the discussion process, at first, İlke determines the variables and draws the free body diagram of forces on the rope, then by using the components of the tension, she formed the main equation (Figure 4.84).

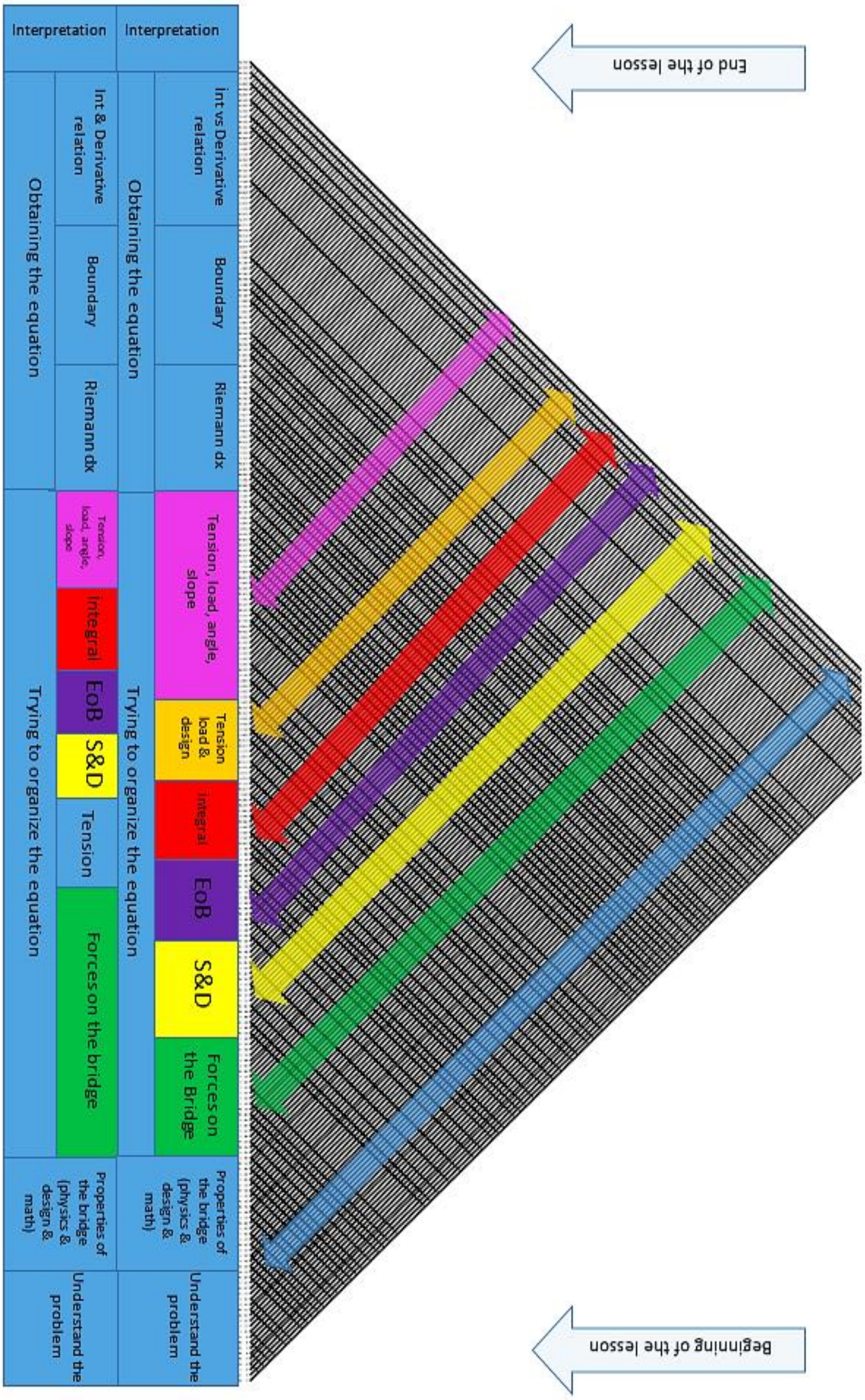


Figure 4.83 İlke’s Linkograph in Activity 6

İlke	Write the equation.
Researcher	OK, so I need the equation. How do I find the equation?
Betül	From two-point.
İlke	Here is that equality. Summary of all forces and the slope.
Researcher	How?
Betül	Derivative at that point.
İlke	Ty/Tx
Researcher	Equals what?
İlke	The derivative of the function at that point.
Researcher	What did you mean?
İlke	v/Fx
Researcher	If I know the equation of the slope of a line, where do I find that line equation?
Betül	$y = mx + n$
Researcher	What else?
İlke	$Y2 - Y1$.
Researcher	If you know your general equation, How?
İlke	teacher, I try two or three points.
Funda	By using derivative.
İlke	By using derivative.
Betül	Integral.
İlke	We integrate the right side.
Researcher	Can you explain?
İlke	T will remain constant. What if we hang a chain like that? You know, the T on it will be the same at every point. When we turn it into a bridge and the car passes over it, the tension will constantly change so, x and y will also change. You know, the component in xv will also change.
Funda	The x component will not change at all. y will always change.
İlke	Or so it changes, but it does not affect because it mutually cancels out.
Betül	Exactly.
İlke	That is, when T increases, it will also increase.
Funda	I think it should also increase. Doesn't it have to change because of the α changes?

İlke	Yes.
Funda	Because of α changes, sin alpha comes from there. Then does not it have to change from there.
İlke	The slope of f?
Betül	Yes.
İlke	Antiderivative of slope f is equal to the distributed charge divided by the integral of Tx.
Researcher	What does this mean equality?
İlke	ux unit weight, right? So, I divided it into x parts. The weight of each unit piece is u Is not it that we just found the cable's equation carrying the distributed load in a unit when we say ux, but we have to add it to the equation because the tension changes according to the angle; it comes as Tx. You divided it into x parts. In other words, not every single weight is thought of as follows. It says that I do this in x pieces. So, let me divide it into x n pieces to find the unit weight. As the sum of these distributed charges increases, it gives us the equation for the rope. The unit weight is the length of this length. Let this weight ux. What if the weight of this piece is ux ok then this will be x_3.
Researcher	So how would you explain this equation using derivative and integral?
İlke	Hmm, this is the application of the Fundamental Theorem of Calculus.
Betül:	Yes
Researcher:	Ok, how?
İlke	These uxes are accumulating as in their instantaneous velocity of this accumulation, as the tension across the threshold changes to me is equal to the rate at which its accumulation in change changes. Now x is the length of what I divide. We are using Δx because it is related to how small I split it. The thing coming from above is about how often I lay Δx … I mean. R: yes, so the more often I split, to more distribute the load. That means that the bridge should bear that much weight.
Researcher	So what is our formula? Our bridge's formula is what?
İlke	Parabola.

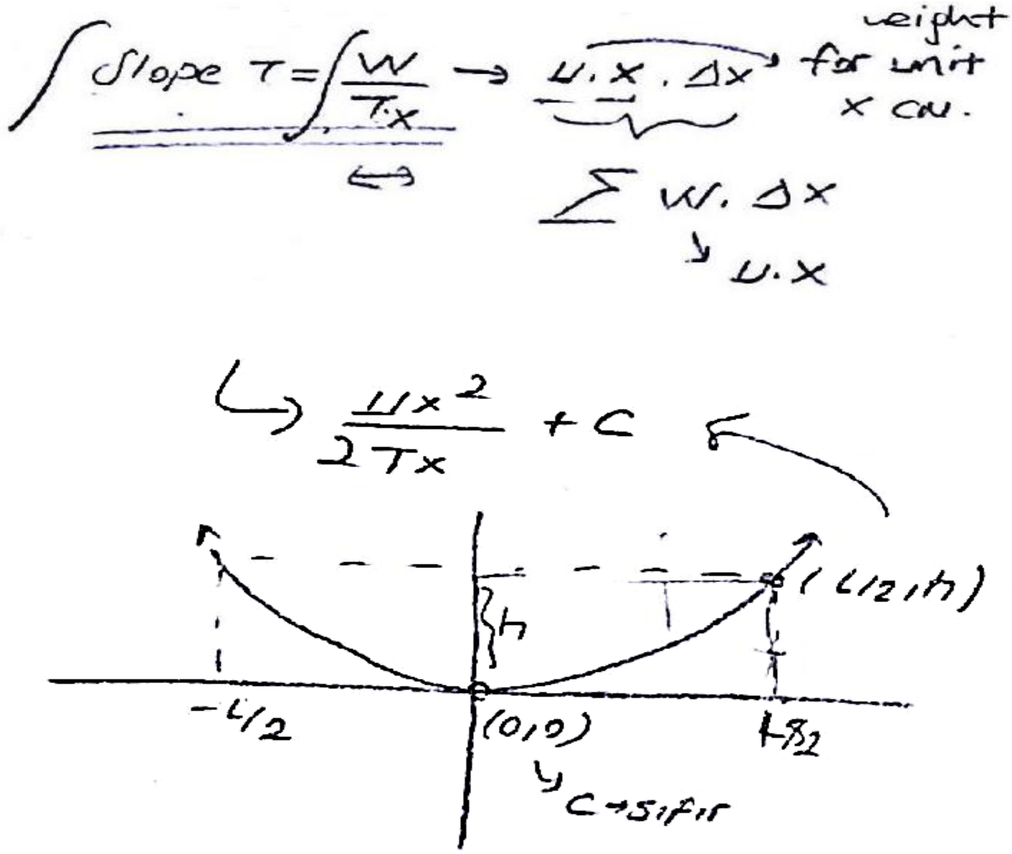


Figure 4.84 İlke's Construction of the Equation

As it can be seen from the conversation, İlke, by explaining the variables and connected them from the formula, stated the Fundamental Theorem of Calculus.

Considering these developments of İlke is that she was able to achieve all the Reification process objectives. Moreover, she explained the various representation, and she was flexible during the transformations. Although she knew the Fundamental Theorem of Calculus and the Riemann Sum, she dealt with so many details and could not treat the integral as an object. However, she understood that the Riemann sum and the Fundamental Theorem of Calculus are integral components. Therefore, although she reached the reification process, she could not see the integral as an object. However, by strengthening the links, she will see the integral as an object and can be used as a differential equations process. In the next chapter, the final situation of İlke was discussed, and the findings of their final interview are presented. Moreover, the structure of her final concept image was given.

4.4.2 İlke's Final Concept Image

İlke has a hybrid concept image (HCI). She constructed inaccurate mathematical knowledge on mathematically accurate knowledge or vice versa. This kind of image is fragmented, and learners create an irrational experience on mathematically accurate knowledge or combine them intuitively according to context.

1) *She enriched the personal definition of integral and gave its analytic description as well.*

In her conceptual test, İlke defined integral based on FTC and widened her definition. Since she wrote her paper, "one definition of" other definitions of the integral was asked. Moreover, she described the integral as the instantaneous rate-of-change of accumulation of function based on the FTC, and she also defined integral based on the Riemann sum, and she stated that by "*dividing very tiny wide rectangles of an interval under a curve, we could find the area*."

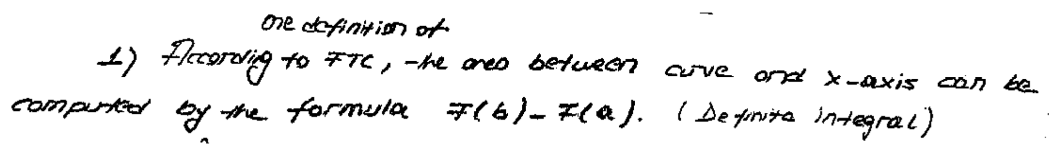

one definition of
1) According to FTC, the area between curve and x-axis can be computed by the formula F(b) - F(a). (Definite integral)

Figure 4.85 İlke's Definition On Integral

She also could write to them algebraically and explain them. That means she connected all the descriptions and the representation on her mind and go over them easily.

2) *She learned Riemann sum comprehensively.*

A contextual problem about a stuffed gorilla that was dropped down from the top of the building is asked to determine whether students know Riemann sum, and speed vs. time graph is given. İlke solved this question by using Riemann Sum, and she also mentioned the type of Riemann sum, which is the midpoint Rieman sum. Moreover, she also wrote that a more precise solution could be obtained (Figure 72).

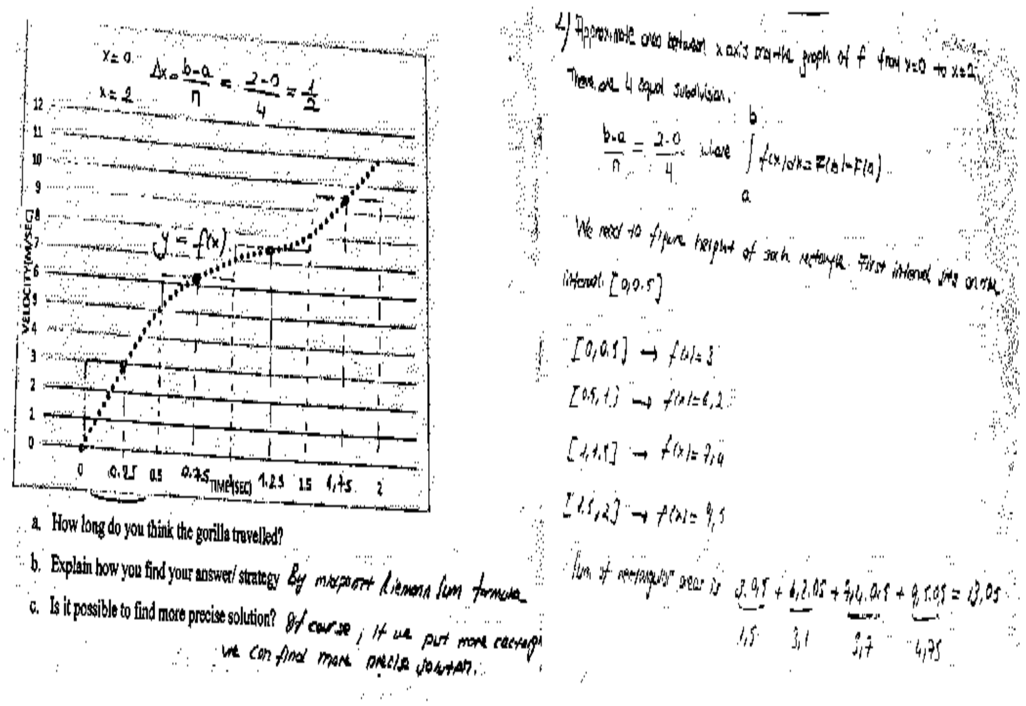


Figure 4.86 İlke's Solution of Question 2

In the interview, she solved inheritance questions related to Riemann sum, ad she said that "I can use rectangles to solve and as we did in the lesson by proportioning the image with the real one I can calculate it. Moreover, by using mathematical problems, I can fit the curve and find its equation, and by using integral, I can solve again". Moreover, she explained that reduce the error to get an accurate result, and by taking limit two of the results which she mentioned will be the same

3) İlke explained the area under the negative x-axis.

In the previous, İlke could not explain the negative area. Moreover, she had confusion on interpreting the "the area under a curve" phrase, and there was a conflict between her integral definition and its application. However, in the final interview, it was observed that she overcame this conflict and explained the area under the negative x-axis, and calculate it accurately.

$$\int_{-1}^{0}(x^2-2x)\,dx+\int_{0}^{1}(x^2-2x)dx$$

$$\frac{x^8}{3}-x^2\,x=-1\;and\;x=0\;(-1/3-1)$$

$$\frac{x^8}{3}-x^2\,x=0\;and\;x=1\;(-1/3+1)$$

Result: -2/3

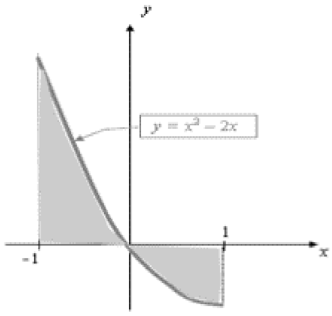


Figure 4.87 İlke’s Solution of Problem about Area

Researcher	Can you explain why you solved this problem in this way?
İlke	Actually, it is asked shaded area, and in origin, we have to find this area.
Researcher	Ok, you find negative results; what does it mean?
İlke	Yes, but it is not the area. That means this area is in the negative x region. It does not refer to the negative area. I mean, as it refers to the direction or region.

4) İlke transferred integral concepts into real-life situations.

In the integral questionnaire, it is asked to find how much work is done when an elastic spring whose spring constant is k is extended 4 cm. Previously İlke substituted the given directly into the formula. First, she drew a graph for = F.x , thinking F is constant, then she applied this knowledge to the spring problem, then by using integral she solved the problem correctly.

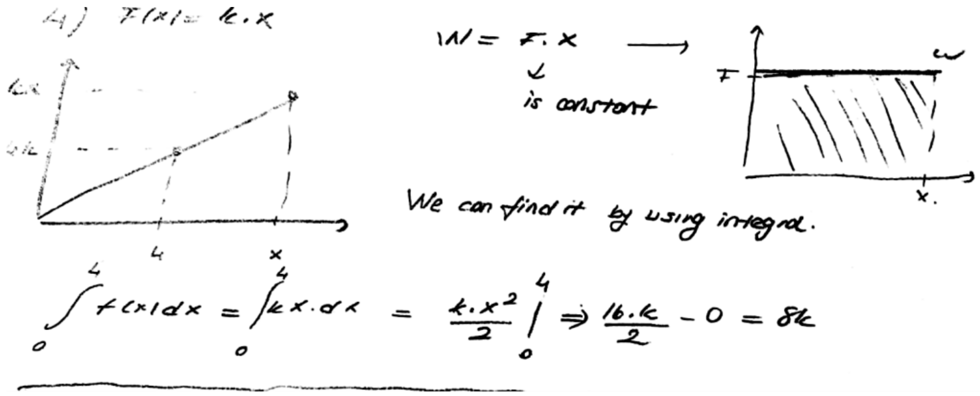


Figure 4.88 İlke's Work Problem Solution

5) *She related and explained Riemann Sum and FTC inclusively.*

Through the interview and conceptual test, she explained FTC and Riemann Sum. Also, in the concept map, she related the accumulation of rate of change with FTC and integral, and also, she related integral with Riemann sum and also infinitely small, which are the main and the sub-concepts of the integral. The question was about relating the Riemann Sum and FTC expressed algebraically. It can be seen from Figure 4.89 that she explained correctly and comprehensively. She related integral and explained every component of the equity.

3. $\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$ eşitliği size ne ifade ediyor?

Cevap: Sol taraf için f fonksiyonu için grafiğin altındaki alanı dikdörtgenler yardımıyla hesaplamak istersek f(ci) herhangi bir noktada o dikdörtgen için yüksekliği verir. Aynı şekilde delta xi de i'nci dikdörtgen için kısa kenarın uzunluğunu temsil eder. i 1 den n e kadar tüm dikdörtgenlerin alanları toplamını toplam sembolüyle gösterir, alanı da f(ci).delta xi şeklinde gösteririz. limit kısmında ise delta xi lerin sıfıra yaklaşması aslında hesaplamadaki hata payını azaltmak için dikdörtgenlerin eninin sıfıra çok çok yaklaştığı durumu temsil eder. Bu da çok sayıda ve eni çok küçük dikdörtgenlerle grafiğin altı ile ekseni arasında kalan bölgenin alanını bulmak için bizi mükemmel sonuca yaklaştırır.

Sağ tarafa baktığımızda da bunun integral tanımını yani f fonksiyonun altında x eksenine kadar olan bölgenin alanını bulma formülünü gösterir. Bu alanın sınırları x=a ve x=b ile sınırlıdır. Yani, biz x e göre integral alıyoruz ve burada dx x in türevini temsil eder.

Figure 4.89 İlke's Explanation about the Riemann Sum

6) *Although she knew the relationship between the Fundamental Theorem of Calculus and the Riemann Sum, she could not link them with the integral concept strongly.*

In her first concept map, she just defined integral and its representations and gave some application examples. In her last concept map, she organized the calculus concept map. However, she did not write the FTC under the integral part (Figure 4.90)

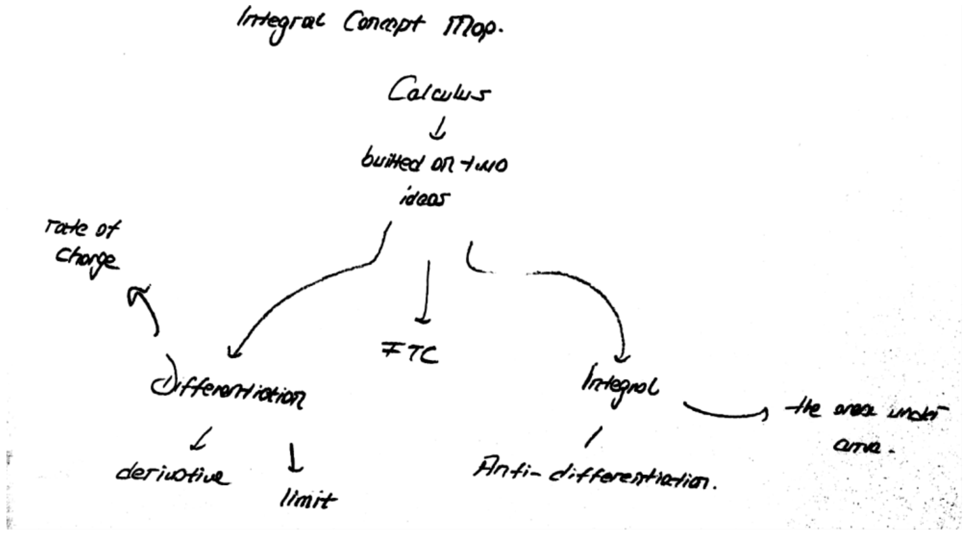


Figure 4.90 İlke’s Concept Map

By considering this during the interview, it is asked to clarify it. During the interview, she used Figure 4.91. Since she could not show other representations between concepts, she explained in the interview as “ I thought that area is related with integral and the Riemann Sum relation with integral. Hence area and the Riemann Sum are also related to each other.

Değişim oranı→ Türev

Sonsuz küçük → Türev İntegral

Birikimin Değişim oranı → analizin Temel teoremi → integral türev

Alan → İntegral

Rieman Toplamı→ İntegral

Değişken→ Fonksiyon→ Türev, İntegral

Alt limit, Üst limit → Limit→ Türev, İntegral

Figure 4.91 Relations Between the Concepts

In this sense, considering the whole process, İlke constructed a mathematically accurate concept image.

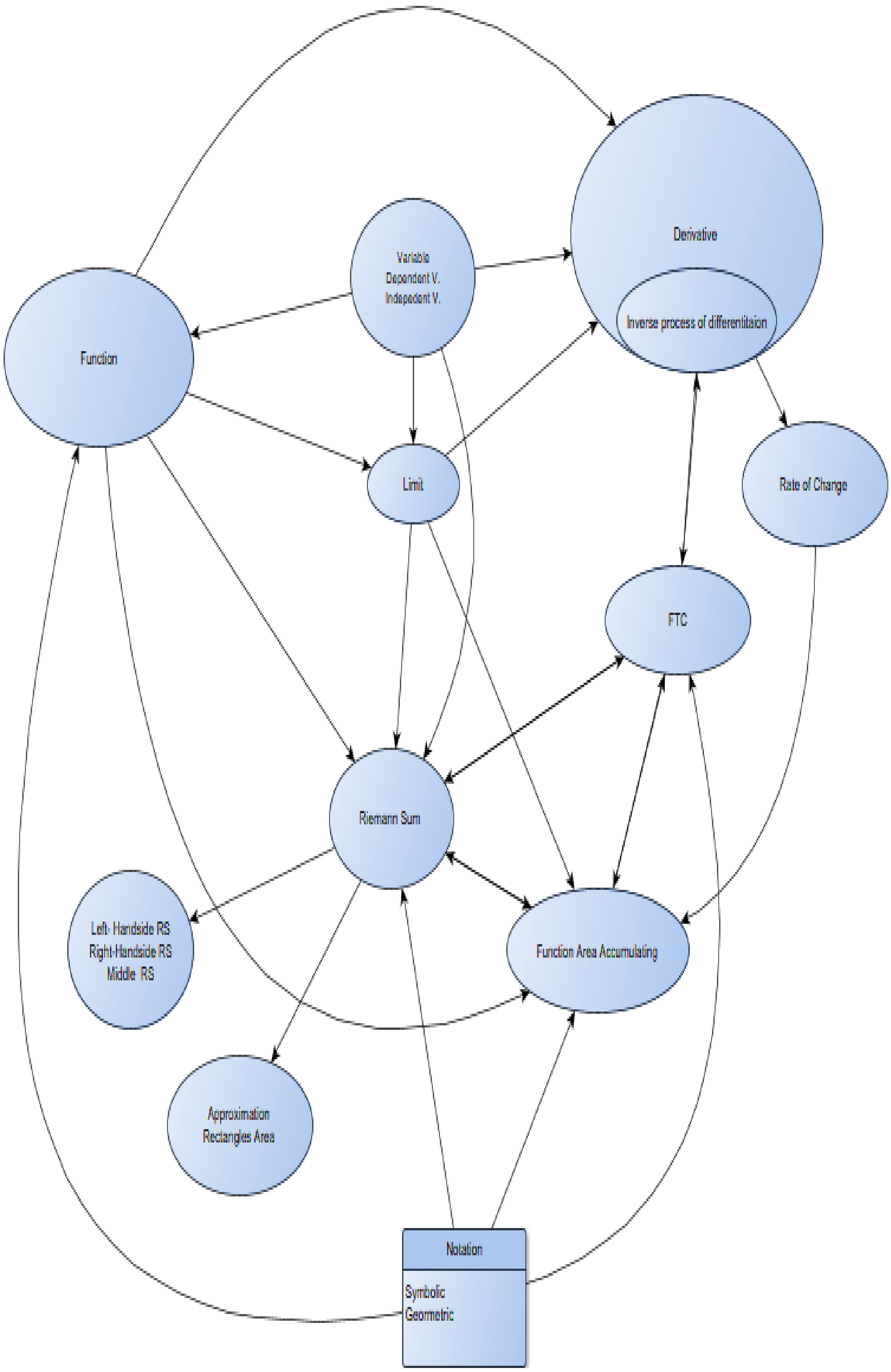


Figure 4.92 İlke's Final image

CHAPTER 5

CONCLUSIONS AND DISCUSSION

This study aims to investigate the nature and development of middle school mathematics teachers' concept images about integral throughout engineering design-based activities. In this chapter, the conclusions of the current study findings and the discussion of the critical results, including interpretation and justifications of the findings of the study, address the literature. In this regard, this chapter is divided into three parts. The first part is related to the prospective mathematics teachers' initial concept images common and individual characteristics at the beginning of the study. This part is essential because the concept images were classified and the general characteristics of their concept images' were determined. In the second part of this chapter, concept image developments of the prospective mathematics teachers throughout the engineering design-based activities to illustrate their construction details based on Sfard's three-phase theory were discussed. This subsection is also critical because the characteristics of the given instruction according to the prospective mathematics teachers' initial concept images and their dominant features in terms of engineering design process activities were discussed. Finally, the prospective mathematics teachers' final concept images were discussed in the last part of the section. Moreover, for integrity, implications of the study are given under the conclusions, if needed.

5.1 Common and Individual Characteristics of the Prospective Mathematics Teachers' Initial Concepts Images on Integral

In the current study, it is crucial to discuss similarities and differences in initial concept images of the prospective mathematics teachers. In this regard, first presented common characteristics of the pre-service mathematics teachers' initial concept images on integral were discussed and then discussed them individually

5.1.1 Nature and Common Characteristics of the Prospective Mathematics Teachers' Initial Concepts Images on Integral

I discussed prospective mathematics teachers' initial concept image on integral by focusing on their definitions, solutions, and connected links which they construct. The initial interviews and the concept maps revealed that pre-service mathematics teachers' concept images contained problems on fundamental concepts that composed integral concepts. In this regard;

Conclusion 1: Pre-service teachers' initial concept images were constructed on the "integral as the area under a curve" definition strongly related to geometrical representation.

The pre-interviews and the concept map results showed that three prospective mathematics teachers generally attempted to construct their concept image on a limited definition for integral and used it constantly. This result is compatible with the previously conducted studies (Fisher, Samuels, & Wangberg, 2016; Rasslan & Tall, 2002; Rösken & Rolka, 2007). Such result was expected because instructors emphasize the conception that "integral means the area under a curve" in the university entrance exam preperation and calculus courses. In addition to being incomplete, this definition also causes students to misinterpret of the integral when solving integral problems. Moreover, this also leads students to confuse finding the area between curves, especially the graphs of which some parts are on the second quadrant, and the other is on the fourth quadrant on the coordinate plane where y-axis takes negative values, which causes the student to perceive that there can be a negative area.

Suggestion: The implication of this conclusion is that teachers should be aware of students thinking. In some situations, teachers may not be aware of the student's thinking, and they may reinforce this type of misunderstanding. In this sense, they should emphasize the area under the x-axis and solve problems that prevent those kinds of misunderstandings. Moreover, to make students realize that integral is not simply equal to "area under a curve," they should solve different contextual problems requiring integral not only to calculate the area.

Conclusion 2: Pre-service mathematics teachers have intuitive knowledge or no knowledge about Riemann Sum.

During the interviews, conceptual tests, and concepts, none of the pre-service mathematics teachers talked about the Riemann Sum. Moreover, they stated that they did not know the Riemann Sum concept. However, İlke and Funda approached the problem by dividing the given interval into pieces and finding the area. There was an important point here; they were asked to find the area under the given curve, however they treated the curve as if it was a line. Although they divided the interval, they tried to find the exact area and thought that the given curve was a straight line and thought that it was the exact solution. Moreover, by dividing the interval into rectangles, they thought they would find an approximate solution and not find the exact one. However, they were not aware that they assumed that the curve were a line, they also find an approximate solution.

The reason for finding precise and exact results for the problems' solution can be the previous courses. This situation also leads students look for the exact solution instead of looking for hee other solutions. Another reason is that they realized that they had to find the area under the curve, and from this point of view, they tried to solve the problem by setting the integral, which led them to find an exact solution. However, the curve equation in the problem was not given, so they could not write and set up the integral. As a result, they assumed the curve was a line.

Conclusion 3: Pre-service mathematics teachers have procedural knowledge about the Fundamental Theorem of Calculus.

The pre-service teachers' initial concept images showed that they could do procedural procedures. However, they do not know why they do what they do. In the literature, the same conclusion was reported (Ely, 2012; Orton, 1983). In the conceptual questions test, pre-service mathematics teachers solved the set-up integration questions easily. At first glance, it seems that students understand the integral conceptually and can solve the question by determining variables. In fact, when asked the same question again in the same context by taking different variables or the inverse of the given function, it was seen that the students could not do it. When they had difficulty determining the function, they were asked to set up the integral for a general f function. However, they were not able to set up the integral and had difficulty determining the variables. It is referred to as "situated abstraction" (Hoyles & Noss, 1987) in the literature. This concept is based on the learners' constructions. This concept was introduced as "situated abstractions are invariants shaped by the specific situation in which the learner forges them. Although these invariants are situated, they simultaneously contain the general's seed that could be valid in other contexts" (Laborde, 2007, p.77). In seemed as in our concept, pre-service mathematics teachers knew how to solve the problem or set up the integral, but they could not apply the same solution when the conditions were changed in the same question. Hence that causes students to learn the general pattern of that problem; if they encounter that kind of problem, they apply the same solution. As a result, if the solutions work, they start to use the same solution for similar problems without knowing its reasons. Thus, it prevents pre-service mathematics teachers from understanding why integral and the derivative are inverses of each other.

Conclusion 4: Pre-service mathematics teachers have problems interpreting real-life situations with mathematics.

Interview results revealed that pre-service mathematics had serious problems representing real-life situations with mathematics. Real-life confusion and determining variables were the most significant difficulty that the pre-service mathematics teachers had. Furthermore, this confusion increases when applying the integral definition to physics contexts such as velocity, force, etc. Even though the context is familiar to students, when it was asked to change the function's variable then integrate it, they may not be able to apply the incomplete definition. That means they were not able to think mathematics and the given situation together. For example, when they were asked to find a cube's volume, they can compute it efficiently. However, when they were asked to compute the volume of a cube-shaped concrete object, they could not compute it. The reason for not computing the volume of the object, the pre-service mathematics teachers could not relate the real-life and mathematics. Hence, they were not able to think measure the cube's length and express it mathematically.

Moreover, they could not measure the cube faces and think of objects on the coordinate plane. It is due to students' abstract thinking constantly and trying to express the given object in an algebraic form. Hence, they were not able to utilize on real-life situations and also mathematics. Therefore, this hinders students learning and commitment cognitively to the activity. During the Activity 2, they could not overcome the difficulty which is building connections between the real world and mathematics until the part in which students measured the canoe with their own. Moreover due to this connection difficulty, they resist learning during the measurements of canoe and computation of the volume of the canoe. Especially for Betül, this difficulty was severe. Even though she was eager to learn and know integral concepts, she could not apply her knowledge to the design of the canoe and the mathematical part which is the computation of the canoe's volume.

Moreover, she was not able to commit cognitively. Hence, this difficulty hindered her learning of the Riemann Sum concept seriously. When they started to deal with the activities quantitatively, they started to adhere cognitively to the activity and took a lot of effort to learn.

Suggestion: Asking authentic problems or using physics contexts while teaching integral in Calculus courses will enable not only pre-service mathematics teachers but also provide learn integral conceptually from a wider group of students who enrolled in different majors (such as science, engineering etc). From this perspective the students who have difficulty in relating the real world to mathematicc can overcome that difficulty. Moreover, these students who have difficulty in interpreting real-life situation with the help of mathemtics or having difficulty in realting them should collect data in the real settings that means they should start with handling quantitative data. Furthermore, they gain confidence, be motivated, and become eager to learn. In this regard in the case of teaching the Riemann sum, design-based activities should be given. Hence, they can deal with the numbers. The student should collect with a limited number of rectangles quantitatively,, then increase it, and realize that the real value is approached as the disorder increases. Real data collections enable mathematics to gain meaning and become abstract, allowing them to imagine where each value in the formula corresponds to in practice. It makes it easier to interpret the formulas they see by increasing the application of mathematics and establishing real-life connections. This situation also caused why and how the derivative-anti-derivative relationship is established because the Fundamental Theorem of Calculus is not fully grasped.

5.1.2 Nature of Individual Characteristics of The Prospective Mathematics Teachers' Initial Concepts Images on Integral

The analysis of the initial interviews and the concept maps revealed that the pre-service mathematics teachers' concept image contains fundamental concepts such as the Riemann Sum, approximation, accumulation, covariation, and the Fundamental Theorem of Calculus which comprise definite integral. According to the results of the data, pre-service mathematics teachers have, their initial concept images were classified. In this regard;

Conclusion 5: Three of the prospective mathematics teachers displayed three different concept images about integral the partial primitive concept image (PPCI), discrete concept image (DCI), hybrid concept image (HCI).

Riemann sum and the Fundamental Theorem of Calculus are the essential phenomena of the definite integral. Therefore, the investigation of the prospective mathematics teachers' concept images showed how these three prospective teachers constructed and connected their knowledge about Riemann sum, the Fundamental Theorem of Calculus, and the basis of their related component for the definite integral. For example, if the pre-service mathematics teachers' had constructed the integral accurately, they would have a mathematically accurate concept image. However, their concept images were composed of mathematically accurate and inaccurate concepts (Vosniadou, 1994; Viholainen, 2008), mathematically accurate concepts with missing or no connections, and partially accurate concepts with missing connections (Greca & Moreira, 2002; Tall & Vinner, 1981). Hence, all three concept images were mathematically deficient because the fundamental concepts such as the Riemann sum were either not constructed or incorrectly constructed by the pre-service mathematics teachers.

These differences may arise from past experiences since they vary for every student (Tall & Vinner, 1981; Vinner, 1983; Vinner, 1991). Besides, every student's reasoning differs (Black, Freeman & Johnsan-Laird, 1986); hence, the concept's construction differs. However, in mathematics education, the aim is to enable all students to have an accurate conception of the integral, but they can vary due to the mentioned reasons.

Conclusion 6: One prospective mathematics teacher might hold mathematically accurate and inaccurate fragments in her concept image

İlke's explanations revealed that she had mathematically accurate and inaccurate fragments together in her mind. These constructions were composed of definite integral altogether; as mentioned before, she had two different concept definitions: deficient and mis constructed the Fundamental Theorem of Calculus. Moreover, by blending these constructions, she built definite integral in her mind. Therefore, during the interview, she used her concept image consistently and coherently. Hrepic et al. (2010) called this model a "hybrid model," which is explained composing this kind of concept image, it was necessary to use different domains such as mathematically accurate and inaccurate concepts. Also, Norman (1983) and Tall and Vinner (1981) emphasized that concept images may contain conflicting factors or erroneous concepts. This situation may arise from students' inclination to memorize the formula or the newly learned concept in the simplest way. Hence by omitting the complex part of the definition, they focused on the "area under a curve" part. The instructors may not enough emphasize relations between the Riemann sum and the Fundamental Theorem of Calculus, or there can be another reason.

Moreover, their knowledge about the Fundamental Theorem of Calculus is procedural, and they perceived that the Fundamental Theorem of Calculus as a computational aspect of integral. Hence, finding an area under a curve with the Fundamental Theorem of Calculus and the pre-service mathematics teachers combined each element through time. Moreover, as students were exposed to many problems of this type are given to students in textbooks or course recitations, so this situation is reinforced, and the combination of these two concept images is strengthened.

Conclusion 7: Tendency to use familiar contextS and experienceS was dominant in concept image.

The most dominant situation in Funda's interview was that she constantly tried to solve the problems by simulating the problems she had seen before. She tried to make connections between the problems then transfer the solution of the previous problem into the given one even if their objectives were not the same. When the context was familiar, she constantly tried to remember the previously worked out problems. This behavior has its advantageous and disadvantageous for Funda since she learned the concepts in this way. It is advantageous because if the concepts were constructed accurately, she could easily learn the new knowledge. However, her integral concept had some missing parts, which negatively affected Funda, hindered her, and made her focus just on the context. Hence, she was not able to solve the problems.

Moreover, at the beginning of the semester, this situation made her resistant to learning, and during the unfamiliar context, she could not produce any idea, and it was hard for her to change her habit of mind. Tall (2013) explained this "met before" to highlight the trace it leaves in mind from a previous experience that affects our current thinking. Also Tall and Vinner (1981) mentioned that concept images could be constructed through the experiences as in Funda’s example. Moreover, as in Funda’s case, having this kind of concept image might create resistance to learning (Gentner, 2002).

5.2 Common and Individual Developments in Prospective Mathematics Teachers’ Concept Images Through the Engineering Design Process Activities

The concept images that the students had at the beginning of the study have been discussed so far. Under this title, how students learn the definite integral, what stages they go through and how their concept images have changed while applying engineering design process activities have been discussed. In addition, these developments are given in two parts as common and individual developments.

5.2.1 Common Developments in Prospective Mathematics Teachers’ Concept Images Through the Engineering Design Process Activities

Common developments in prospective mathematics teachers’ concept images through the engineering design process activities were explained in this section.

Conclusion 8: Differences in their initial concept images caused prospective mathematics teachers to follow different learning paths.

Aforementioned, prospective mathematics teachers have three different concept images due to different configurations of the basic concepts related to integral. Since every prospective teacher has different deficiencies in the concepts, they construct different concepts regarding their deficiency in every activity. Hence, their focus on the discussion changed, and this caused them to follow different learning paths. However, although the activities are designed separately for each concept, all the prospective teachers learned all the objectives and the necessary concepts due to the activities' cumulative nature at the end of the semester. The only the difference was that the prospective teachers learned the same objectives or concepts in different activities due to their lack of prior knowledge. As a result, differences in their initial concept images caused the prospective teachers to pass the phases of Sfard in different activities.

Suggestion: One implication of this conclusion is that teachers should pay attention to the sequence and activity structure while designing the engineering design-based activity. The activity should serve both teaching the central concept or the objective and should consist of its sub-concepts. Moreover, the activity should be flexible and provide both students and the teachers to move between different sub-concepts. In this way, even if students follow different paths, eventually, they reached the same objective.

Conclusion 9: Experience played an essential role in establishing the connection between mathematics and real life. Moreover, it enabled students to understand the mathematical concepts conceptually.

There are two crucial parts of this conclusion: the experience and the real-life application. In this conclusion, the term experience is used in a wide range of behavior. It can be called re-discovering concepts by handling a real-world situation, developing or realizing the mathematical relationships between the mathematical ideas when designing a given task prototype, and finally managing with the real world. Experience is a process formed by relating real-world situations with mathematics and investigating and expressing the situation with it.

In this study, the participants went through both processes thanks to the engineering design process. As a result, they both experienced how to relate mathematics with real life, and while learning mathematics which is integral in our case, they experienced what each component in the integral corresponds with real life. This experience of both sides, namely applying mathematics to real life and seeing the connection of mathematics in real life, enabled students to use mathematics in different real-life contexts and interpret real-life problems in a mathematical thought-centered way. In this study, different from the literature, the problem situation was not given in the form of multiple representations; on the contrary, they used different representations in line with their understanding during the discussions. In this way, they all see different representations and also interpretations of one concept. Thus, they express themselves more clearly, and as a group, during the discussion, they can transfer the knowledge through the representations, so they connect more powerful and conceptual links between the concepts. Hence, they could connect the real-life variables via this transition with their representation in the mathematical expression.

Using real-life problems was a critical point that allowed students to understand the integral concept in a given context and facilitate them to express their experiential knowledge from informal to formal. Furthermore, with engineering design, by manipulating the variables, they conceptualize and experience the integral. Hence they both built strong links between mathematics and real life, the transition between them. In addition to these in line with the literature (Laurens, Batlolona, Batlolona, & Leasa, 2017), they also minimized their difficulties in understanding the concepts related to integral. In this sense, engineering design-based activities were a valuable tool for providing context and real-life experience.

Another vital point about engineering design activities was providing students the experience with the variable concept using the engineering design process. Because via the engineering design process, students were able to manipulate the variables, and they were able to observe the changes immediately on their prototype. Hence, they conceptualize which variable, determined by themselves, was influential on their design. Then they decided to use that variable while they were mathematizing their formula. This point of view is significant because pre-service mathematics teachers conceptualize the variable concept and learn how those variables affect their design and how to express it symbolically. Moreover, they also realize the relationships between variables such as dependent and independent variables. Through the semester, by experiencing variables, their conception about functional dependency and covariational reasoning developed. It has been known that students who cannot understand the relationship between input and output variables also lack a conceptual understanding of functions (Oehrtman, Carlson &Thompson, 2008). Moreover, according to related literature, to develop functional understanding, it is important to use supported activities using different representations and also enable students to interpret the given situation meaningfully (Antonini, Baccaglini, Lisarelli,2020; Falcade, Laborde & Mariotti, 2007; Oehrtman, Carlson &Thompson, 2008; Stalvey, Vidakovic, 2015; Thompson & Carlson, 2017). As it is known realizing the functional dependencies is fundamental for understanding the functions (Sekerák, Lukáč & Doboš, 2020) and also all these studies pointed out that to use dynamic tools are useful in understanding the relationships between the variables. However, none of these studies made students determine the variabled. On the contrary, pre-determined variables were given without using a context. Hence, even students could see the relationships between the variables, they could not conceptualize the concept deeply since they did not determine the variables and see why and how those variables are related to each other. McNair (2000) emphasized that students can understand and develop abstract ideas related to the given context if the familiar experience is used. Thompson (2019) pointed out that it is essential to define the learning goals quantitatively at the beginning to develop reasoning. In this sense, to remedy functional dependency and the covariational reasoning, different from the literature, for this study, functions were not the first step; determining and manipulating the variable was the first step. Moreover, they dealt with variables quantitatively then as students developed functional dependency throughout the activities, the activities became more abstract. As a result of this, their difficulties were overcome.

Suggestion: While applying the engineering design-based activity, instructors should pay attention to the characteristics of the concept, such as being a foundational concept or being known to the students, and a familiar real-life context should be chosen. When selecting this type of context, it should also be considered that the prototype of the context should be produced by the students easily, and the activity should be solved by using quantitative variables. The activity should enable students to interpret, manipulate and see the immediate changes on the prototype. In this way, students can discover the variables and decide which variable is necessary. Moreover, they also find out the relations between them, such as output/ input variable. They do the calculations quantitatively by using those variables. Through the activities with both quantitative calculations in the application process of real-life context and reinforcing the relations, teachers help students to realize the variation between those variables. In this way, students can make more meaningful interpretations about the variables' covariation, which is fundamental for functions.

Conclusion 10: After the pre-service mathematics teachers had experience with mathematics, discussing and letting them construct the concept to be learned by themselves provided meaningful learning. Moreover, this learning process provided them to recognize the connections between the variables and concepts. Hence they constructed the concepts accurately.

In STEM education, the inquiry is a shared pedagogical approach. In this approach, learners actively play a role and answer the question or pose ways to solve it. At the end of the entire process, they create an output and present them, and finally, the teacher gave feedback to students and summarized the subject taught (Aditomo, Goodyear, Bliuc, & Ellis, 2013; Chu, 2009; Chu, Reynolds, Tavares, Notari & Lee, 2017). Hence, with the aid of inquiry, which helps the students construct the given concept more meaningfully. It was the same process for this study. However, engaging with the prototype and experiencing mathematics led students to individual developments. In other words, engineering design-based process provides pre-service mathematics teachers more time to construct the concept individually. In group work, sometimes students with a high-level achievement can manage the whole process, and students with low achievement can generally repeat what pre-service mathematics teachers with high achievement levels do or say. Hence, teachers could not fully understand which pre-service mathematics teachers fully understood the concept and consider factors such as the limited duration of the course, pre-service mathematics teachers with low levels of achievement cannot fully understand the concept given. However, the engineering design process activities given in a specific order enable pre-service mathematics teachers who have different subject deficiencies to eliminate their deficiencies in the same activity and allow pre-service mathematics teachers to construct the concept given individually. The linkograph data indicated that, as pre-service mathematics teachers internalized and structured the concepts individually, they reinforce this information in a discussion environment. Because during the activity, pre-service mathematics teachers generate an idea and apply it to the activity to solve the problem, they see the result of their idea immediately. This situation is an critical evident to the researcher to determine whether the pre-service mathematics teachers have a deficiency or have difficulty in terms of providing pre-service mathematics teachers' feedback. Thus, the teacher will give feedback according to each pre-service mathematics teachers' lack of knowledge.

At the beginning of the semester, pre-service mathematics teachers were not able to construct the concepts themselves, and they tend to make short inferences and focused to the whole process on it. They assumed as if they knew that new knowledge was the same with the previous one. Moreover, they tended to memorize the given information instead of internalizing it. During the semester, to prevent memorizing, pre-service mathematics teachers were oriented to recognize the concepts taught. This orientation was achieved after they experienced mathematics by connecting it with real life. They realized that the information they memorized or tried to memorize did not work as they experienced mathematics.

Moreover, the real-life application of mathematics motivated and excited them. Hence, pre-service mathematics teachers started to think more about their calculations and tried to apply the information they had learned. After discussing, the researcher asked individual questions regarding their deficiency in the integral concepts and gave them time to think and reconsider the process. Therefore, they started to think about the whole process during the activity and then connect and construct the concepts. In other words, with the help of the researchers' questions, they produced the prototypes step by step with their own constructions and they reconstruct the whole process in their mind.

Moreover, they made the concepts more meaningful for themselves individually. This learning process provided them to recognize the connections between the variables and concepts. Hence, they constructed the concepts accurately. Most of the studies that focused on student-centered methods and group work emphasized the concept's summarization, which aimed to be learned at the end of the lesson.

Conclusion 11: According to construction of integral concepts of the pre-service mathematics teachers, different instructions and feedbacks should be given to them

There were two reasons for the differences in constructing the integral concept for the prospective mathematics teachers, one of them was their initial concept images, and the second one was differences in their tendency to solve the problem in tersm of mathematical thinking or design thinking, finally, their way of construction. The results of the study revealed that there were three main tendency which are that design thinking was dominant than mathematical thinking (D>M), mathematical thinking was dominant than design thinking (M>D), and the equivalence of them (M=D). In D>M, the prospective teacher prefers to design part more than the mathematical part and contributes to the design process during the discussion. Moreover, the prospective teacher was not able to inquiry about mathematical conceptions enough. At the beginning of the course, she could not understand and contribute to the mathematical part discussion. In the following processes, the student was able to connect with mathematics easily. In the condition of M>D, the prospective teacher prefer calculation or mathematical processes to design and was not able to contribute to the design discussion and contributed to the discussion by asking mathematical problems. During all processes, she thinks abstract and tried to express the process mathematically. At the beginning of the course, she could not understand and contribute to the design part discussion. Moreover, the prospective teacher had difficulty relating the real-world context with the mathematics. However, in the following processes, as the prospective mathematics teacher learned how to design, she was able to construct the integral concept as a whole more easily and progressed faster in contrast with the D>M condition. In the M=D condition, both of the aptitudes have the same weight. The prospective mathematics teachers in this condition were both contributed to the mathematical and design discussions. Moreover, she behaved as a bridge between D>M and M>D.

In the literature, most of the articles focused on the result or product of the activity (Crismond & Adams 2012; Diaz & King 2007; Moore et al. 2014) or, in general, mention the importance of establishing a collaborative group (English, King, Smeed, 2017). Furthermore, in the STEM or engineering design articles, it is not mentioned about the learners' tendency and how the instructor leads their discussion. However, determining this type of tendencies is essential for the literature because, in this way, the teacher can give feedback according to students' tendencies to take the most effective instruction style. Even it seems that in group discussion group made a consensus on an idea, it does not mean that all the group members construct the given concepts in the same way. Since a concept's construction is personal, it occurs in students' minds and is formed according to their characteristics (Tall, 2013; Tall & Vinner, 1981). Moreover, in the personal discussion with the instructor, they mostly use their tendency. In this sense, engineering design activity provides giving the instruction both to the group and also to the individuals. Furthermore, it also allows teachers to give students instruction according to their tendency and knowledge level. In this regard, all the group members construct the given concept in an intended way in the current study. In the literature, most of the articles emphasize the product of the STEM or design activity and assess the achievement with the activity's product, such as constructing the longest bridge or perfect catapult (Asghar, Ellington, Rice, Johnson & Prime, 2012; English & Hudson, 2013; English, King, Smeed, 2017; Petroski, 1998; Mehalik, Doppelt, & Schuun, 2008). However, not enough attention was given to the gained knowledge or the reasons for the failure or the success (Lewis, 2005). Those articles may only investigate the generic steps of the STEM activity or improve the students' skills such as problem-solving, creativity. Although this is the reason, content knowledge cannot be ignored in either case because knowledge and skill are two components that complement each other. Moreover, this kind of approach to the activity caused a misinterpretation of the students' developments in terms of learning and the group discussion's nature. Furthermore, some of the articles attribute this failure or the success only to the design process by assessing the result or the product of the activity (Asghar, Ellington, Rice, Johnson & Prime, 2012; English & Hudson, 2013; English, King, Smeed, 2017; Petroski, 1998, Mehalik, Doppelt, & Schuun, 2008). However, differences in students' aptitudes and disconnectivity in the discussion process can be the reason. Because of the groups formed with these students D>M or M>D, they would not reach the solution, and the activity may fail.

Suggestion: While applying the engineering design-based activity, it should be paid attention to the group dynamics and the student's aptitude. It can be done by listening to their discussion and also their representations. Moreover, rather than the product, the process of the design should be investigated. Students may be asked questions about the problems they encountered in the design process and how they overcame them. If the problem arises from the students' STEM aptitude, they can be formed as D=M, D>M, and M>D. In this way, disconnectivity between groups can be achieved. If the group is formed as D>M and M>D, the student who thinks mathematically cannot design thinking students and vice versa. Hence it causes a disconnectivity in the discussion process.

In groups with D> M, even if a product is obtained at the end of the activity, the information part will be incomplete, so questions can be asked about the concept to be taught during the design process and the presentation. Since the student is good at the design part, starting from the design process, instructors make the student step through the mathematical knowledge step by step or make the students build a connection between mathematical expression and design.

In groups with M>D, they may focus on the process and cannot reach the solution or produce the product. At first, the instructor makes students realize their problem, then leads them to discuss the design and prevent them from being lost in the mathematical part.

Moreover, contrary to D>M situation, the instructor should start from the mathematical part and move to the design, or by emphasizing the mathematical expressions, he/she may link it with the product and explain which mathematical expressions correspond in the product.

Conclusion 12: At the begging of the semester, students' discussions were disjointed during the activities on the linkographs, and they became synchronized toward the end of the semester.

In examining or extracting the discourse dynamics, it is recommended to examine students' thought processes in small intervals (Goldschmidt, 2014) to explore how students construct the given concept and its components. In the current study, examining students' thought processes in small intervals means investigating the pre-service mathematics teachers' construction process of the integral. Linkography is an efficient tool for understanding the nature of the students' construction process in detail. Two main factors affect the students' construction process: group work and the nature of the activity. Namely, during the group discussion, pre-service mathematics teachers learned some knowledge from each other and provided it to their group mates to realize each other's own mistakes.

Moreover, due to the activities' nature, they consisted of iterative cycles, and these cycles also affect the pre-service mathematics teachers' learning process. Therefore, it was necessary to examine the smaller intervals of the activities (Goldschmidt, 2014) to understand how the pre-service mathematics teachers construct integral. In this sense, the discussion between the pre-service mathematics teachers becomes more important. Because while analyzing the discussion links between the concepts their construction process of the definite integral can be determined. Hence, this allows the researcher to explore how the pre-service mathematics teachers connect the concepts during discussions.

Moreover, the researcher can determine which source, experience, discussion, or behavior allows pre-service mathematics teachers to connect or construct the integral concept. Furthermore, instructor can also determine the sources which hinder understanding or prevent pre-service mathematics teachers from connecting or constructing the definite integral concept. Thus, the researcher could make inferences about the sources and design the lesson according to those inferences. In addition, linkograph analysis allows the researcher to study the focus of the group as well as the individuals in the group.

In this regard, linkographs showed that pre-service mathematics teachers produced separated ideas at the beginning of the semester, and they were not connected. Moreover, they were not affected by each other's ideas; mostly, they were tied to their initial concept images. Also, since the participants' initial concept images were different, they focused on different points during the activity. Hence, the group discussions were not efficient enough. Another piece of information obtained from the linkographs was that the students mainly discussed mathematics and design separately and that they could not think of mathematics in a real-life context. Moreover, that means that the pre-service mathematics teachers were not able to develop the ideas or were not able to evaluate the ideas during discussions.

Because of amusement or not knowing how to apply mathematics or experience the mathematics in the real-life context. Hence, they were not able to progress through the activity.

Through the semester, the participants have learned to apply mathematics to real life, think, discuss, and construct their ideas on each other. Hence the links in the linkographs become overlapped more. Finally, linkographs allowed the researcher to identify the sources of information used in the transition between the ideas (Blom & Bogaers, 2020). Linkograph analysis helped the researcher to see where the participants used their previous thoughts and where new ideas were produced during the design (Blom & Bogaers, 2020). In this sense, the linkograph provided evidence to the researcher of how the pre-service mathematics teachers construct and built connections between the integral concepts.

At the beginning of the semester, all the pre-service mathematics teachers produced backlinks which means they could not think about how to design the given task or could not connect the context with the mathematics. Because their linkographs were separated, they could not think about how those ideas may be used in the design process. In this regard, mostly, they used their previous knowledge about the integral. According to their deficiency and content of their discussion the researcher oriented them through the activities. Moreover, with the help of the cognitive conflict that they experience and new context, their connectedness to their initial rote concept images decreased. In this way, they become more open to new knowledge. Besides, they experienced mathematics and learned how to evaluate the given situation using their mathematical knowledge, represented by the forward links in the linkographs.

Moreover, by evaluating their assertion, they are affected by each other's learning process. Thus, pre-service mathematics teachers construct the definite integral on each other's ideas. Another piece of information that linkograph allowed us that to see was the participants' way of learning. The linkograph illustrated the structure of the pre-service mathematics teachers' learning construction style and their reasoning during the activity. For example, at the beginning Betül's idea production was dispersed, however, through the semester she started to produce ideas building on each other. Linkographs' belong to Funda was always in the same pattern. Since Funda listens then thinks about the discussion and makes contribution to the group discussion.

Conclusion 13: Pre-service mathematics teachers' construction of the concepts was not linear, and they tended to solve the problem step by step with guidance.

This study showed that the pre-service mathematics teachers' construction processes were not linear, and it was different for each individual. Although each activity was designed as a cumulative structure, pre-service mathematics teachers showed different developments. Moreover, in each activity, pre-service mathematics teachers reused their knowledge for each concept and they constructed previous concepts on the new one. Even though they learned the same concept, they discovered different properties of the concept, such as writing the Riemann sum algebraically and relating it with geometrical illustrations. Thus, in each activity, they proceed the concepts in cycles.

Moreover, their progressions were not the same. For instance, for an activity, although they were all supposed to learn to set up integral, some of the pre-service mathematics teachers were not able to achieve that goal. Instead, they achieved the goal of the previous activity. These differences were due to the mathematical backgrounds of the participants and the differences between their collaboration styles. On the other hand, there were similarities between the pre-service mathematics teachers' constructing processes and their approach to the problem. The pre-service mathematics teachers tended to solve the problem step by step at the beginning of the semester. Hence they continuously asked for a guide that shows the solution steps since they were not able to interpret the problem and reasoning about it. Although some guidance was given to them, they were not sure about their solutions since they were not able to apply their theoretical knowledge to the problem. However, activities were planned as being more quantitative at the beginning. Hence, by beginning with the quantitative calculation, they become more confident and not afraid of making mistakes, and through the semester, their degree of dependence on the guidance decreased.

5.2.2 Individual Developments in Prospective Mathematics Teachers' Concept Images Through the Engineering Design Process Activities

In the current study, it is critical to discuss the individual developments of the pre-service mathematics teachers through the engineering design activities according to Sfard's three-phase theory by building connections with their concept images. In this regard, the pre-service mathematics teachers' individual developments by considering their initial concept images as a reference point was discussed This section will discuss how they moved on to the phases while dealing with the engineering design activities.

5.2.2.1 Developments in Betül's concept images through the engineering design process activities

How Betül's conception change during the activities and in which activity she improved her understanding and the points that affect her conceptions by focusing on Sfard's three-phase theory was discussed. Her developments through the phases were presented in Figure 5.1. In this chapter, the most prominent and dominant evidence which prevented Betül's learning was discussed.

The results of her pre-interviews showed that she had a partial-primitive concept image (PPCI). That means Betül had partially developed a knowledge scheme about definite integral at a basic knowledge level separately. Namely, she divided the whole definite integral concept into sub-concepts, and she either ignored some parts or learned them as basic level knowledge or basic sub-concepts about integral. Moreover, during the preparation activities, her mathematical thinking (qualitative reasoning) aspect was dominant, but this made it difficult for Betül to understand the integral concept, as she could not make mathematically meaningful real-life situations. As a result, she could not be confident enough in her own knowledge. Thus when she faced any difficulties, she did not make enough effort and put a barrier to her learning. In the beginning, to make more meaningful conceptions and construct her knowledge accurately, all calculations started quantitatively. Our assumption is also supported by Thompson’s study (2019), which also emphasizes that to develop students’ reasoning, learning activities should be started quantitatively. Through, progressing from quantitative to qualitative reasoning, we aimed to develop Betül’s qualitative reasoning and her design thinking.

Betül’s pre-interview data of also indicated that she had problems determining the independent and dependent variables from given data, and she had trouble interpreting the variables when the domain and the range of the given function are changed. Furthermore, she did not know the Riemann sum. Her initial concept image revealed that Betül was having difficulty connecting real life with mathematics, which was so dominant in her thinking process that it hindered Betül’s progression on connecting mathematics with real life. Moreover, linkograph analysis revealed that Betül’s contribution to classroom discussions was always mathematically oriented. The experience of measuring the canoe’s length and computing its volume was the breaking point for Betül, since she understood the concept and her produced ideas that helped her to solve the problem. Her difficulty about expressing the real life situation mathematically or connecting the real life and mathematics was robust, so she could not achieve all the objectives of the interiorization phase. However, in the second activity, since she started to overcome relating mathematics in the real situation, she conceptualized the Riemann Sum easily and met the requirements of interiorization and condensation phases.

In Activity 2, Betül started to determine the independent and dependent variables. The first key point for Betül was that she needed some scaffolding about how to connect Riemann sum with Center of Gravity. At that point, rather than teaching

Riemann Sum at first and combining it with the Center of Gravity, the instructor started to scaffold Center of Gravity first and started to ask why it is essential for the design and make the participants think about using Riemann Sum to solve their problem. Beginning in this fashion was essential to help Betül break her difficulty in connecting with mathematics and the real world. In this way, she could relate the Riemann Sum with the real world and started thinking about its applications.

Moreover, she built a connection between design of the earthquake resistant building, physics, and mathematics. The second turning point for Betül was the mistake they made when implementing the extracted formula in their own designs. It helped her understanding as to which variable belonged to the design corresponding to the mathematical representation. Thus, she was able to start to interpret the mathematical formula according to the design point of view. As a result, she was able to determine the variables in the activity.

In Activity 3, she was in the reification process. The most prominent evidence of this was Betül's concept image related to integral as "area under a curve." During the activity, she consistently tried to apply this definition until she started to think about all the variables in the problem together. At that point, she understood which variable is the compound of their design. As a result, by applying the formula, she realized that the definition she was trying to implement was not working and gave up. Hence, she reshaped her integral definition. Since there is no robust understanding about the Riemann Sum in her mind, she easily reached the object process.

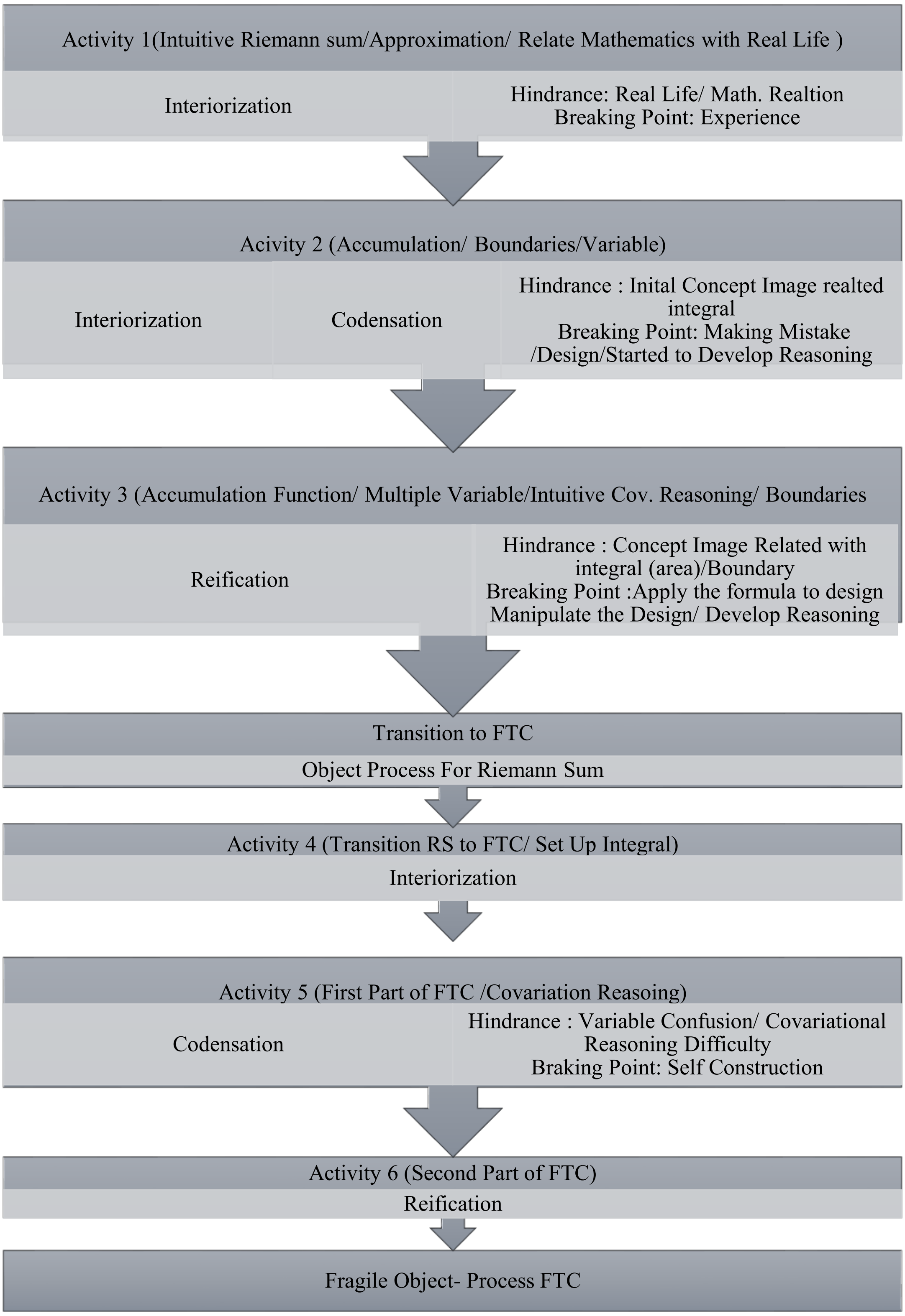


Figure 5.1 Whole development process belongs to Betül

Activity 4 was designed for the interiorization phase for the Fundamental Theorem of Calculus. Since Betül learned the integral before, she went through this phase quickly. However, since she learned integral in some respects as a situated abstraction (Hoyles, Noss & Kent, 2004), she had more difficulty in the forward stages. Namely, she was very competent in procedural calculations, and in a similar context, she solved the problem and set up integral correctly. At first glance, it seemed as if she learned the subject. However, she could not solve the problem in different contexts even if the problem was parallel to some of the prior problems. Activity 5 can be an example of this situation. In addition to her confusion in variables, covariational difficulty, situated abstraction was the implicit difficulty that supports those difficulties and prevents integral learning. At this stage, Betül needed help to clear up this confusion. After the researcher gave adequate assistance, Betül could build the process by herself, reconstructing the concepts. Hence, she reached the reification process. In the end, she met all the objectives of the reification process. However, she could not exactly reach the object-process stage, since she could not see separate integral forms of the whole concepts (Sfard, 1991). To reach this phase, she used the recently constructed integral concept many times. She needed to use integral and went through that process recursively.

5.2.2.2 Developments in Funda's concept images through the engineering design process activities

In this section, Funda's conception changes during the activities, the improvements in her understanding, and the points that affect her conceptions are discussed in relation to Sfard's three-phase theory. The common developments of her through the phases were presented in Figure 5.2. In this section, the most prominent and dominant evidence which prevented Funda's learning was also discussed.

Funda has a discrete concept image (DCI). In this type of concept, images, Funda divided knowledge structures into sub-concepts and conceptualized these, either mathematically accurately or mathematically inaccurately. She structured all of her own knowledge on integral as departmentalized, and neither mathematically accurate nor mathematically inaccurate knowledge was built on each other. Thus, she did not relate these structures between them.

In general, during the activity 1, Funda behaves as a moderator between İlke and Betül. Since her skill in design thinking and mathematical thinking was nearly the same, Funda was able to explain the mathematical part to İlke and also similaryly she could also explained to Betül when she could not understand the design part. Moreover, during the discussions, she contributed when necessary, especially when Betül and İlke did not understand each other. Hence, during the activity, when Betül had difficulty understanding the design part, Funda helped her understand, and she did the same for İlke. During most of the task, she listened to her friends and then declared her ideas or points of conflict. When she did not understand the design part, she asked İlke, and when not understand the mathematics, she asked for help from Betül. As mentioned earlier, the tendency to solve the problems of the known concepts that Funda knew, she was very dominant in this activity, and this situation affected Funda both positively and negatively in terms of producing different solutions. However, this also prevented her from seeing the critical point and linking parts that formed the big picture.

Another point that prevents Funda from learning is a tendency to imitate the given problem to the known problems. She was so resistant to solve the problem by using her previous experience. In this way, she produced different solutions; this was advantageous for her and her friends to think differently. On the other hand, it was a disadvantage for her because if she could not build a connection between their previous experiences and the given problem, she was stuck and could not focus on the problem and colud not produce any ideas.

Another tendency that hinders her is a tendency to find the answer step by step in such questions. She constantly asked the researcher to give an answer sheet that shows how they can go through the activity.

In Activity 2, she was between the interiorization and condensation processes. She started to build new links and constructed the Riemann sum. The most prominent difficulty was a deficiency in understanding the limit concept, especially when the limit value of the function diverged. Throughout this activity, her initial concept image of trying to resemble the solution of previous learned examples was still very active, preventing Funda from producing new ideas. Moreover, she was still insisted on solving the problem step by step.

To ease her difficulties, the researcher started to know situations or help her to experience the new situation with different examples. Moreover, she was not familiar with this kind of teaching; the instructor had to give her much scaffolding. After the scaffolding process, she constructed the integral. With the researcher's guidance and discussion between İlke and Betül, she related approximation, and increasing the number of rectangles decreases the error. After that, she realized that she could find triangles as in the Archimedes method.

In Activity 3, she was between the interiorization and condensation processes. This activity was a transition activity for Funda. She began to meet the condensation process's objectives while she was achieving the interiorization process's objectives. After she met all the interiorization process objectives and reached saturation, she started to see the big picture without going into the details. She began to detach from the phase.

The first hindrance for Funda was that she misunderstood the relation between f and p . It was because she was still trying to imitate the problem solution from a familiar prior problem. Her misunderstanding of the relation between f and p. Thus, this caused a more significant problem which is difficulty in relating mathematics and physics. Hence, she was not able to understand the mathematical part. Another hindrance was situated abstraction about defining the boundary of the integral. Funda used her initial concept image to determine the boundaries. However, in activity 3, angle was oriented and boundaries could not be determined by simply

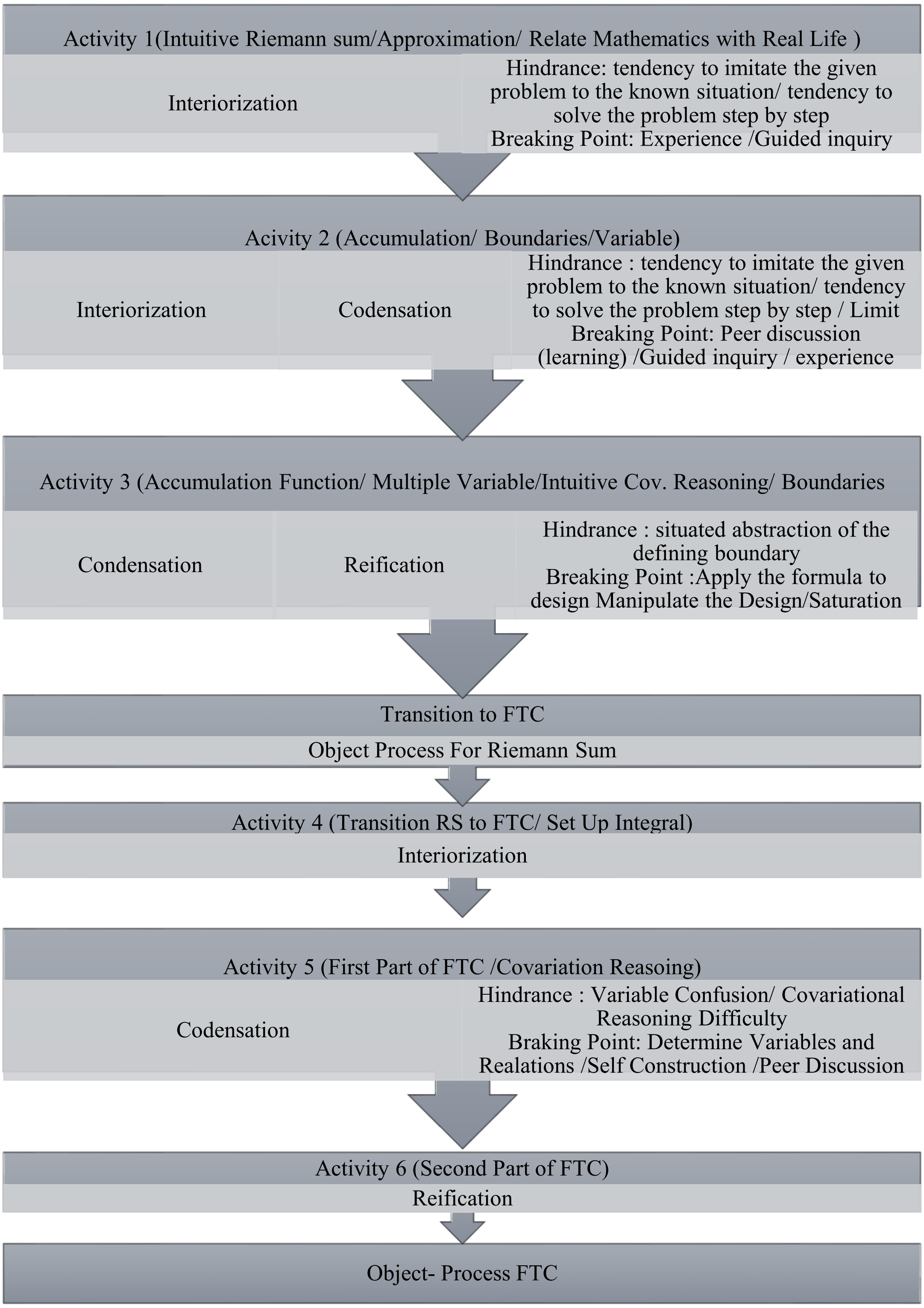


Figure 5.2 Whole development process belongs to Funda

writing the angles. The reason for behaving like this was that she had a cognitive conflict about integral in the previous lesson, and its effects and the construction of Riemann Sum was in process.

Moreover, even it has little effect on Funda, her previous difficulties in Activity 2 were active in her mind. Students rely on the concept image when solving problems (Rasslan & Tall, 2002). Because during to solve a problem, "temporal representations" employed and constructed (van der Veer et al., 1999; Vosniadou & Brewer, 1992, 1994; Vosniadou et al., 1999). Thus, she trusted her initial concept image and this resulted in a mistake that directly assumes the angle’s values as a boundary; hence, group mates’ had different calculations that are not related.

To solve her difficulties, the researcher started to know situations in Activity 3 and repeated the subjects they learned until that lesson. Since she was not familiar with this kind of teaching, the instructor gave her much scaffolding. With the researcher’s guidance and discussion between İlke and Betül, Funda started to think about the relations between Riemann sum and approximation. After that process, the researcher gave her time to construct the relation between the approximation and the Riemann sum on her own.

Activity 4 was designed for the interiorization phase for the Fundamental Theorem of Calculus. Since Funda was also familiar with integral before the study, she went through this phase quickly. Since her initial concept image changed and reconstructed, she had fewer problems than Betül.

In Activity 5, there were two hindrances which were confusion in variables and covariational difficulty. Betül had confusion about T and x. While she participated in the discussion during that time interval and in the mathematical discussion process, she followed a different path from her friends. The reason for this difference was misunderstanding T and x relation. As in Betül, Funda theoretically knows the concepts of temperature, but she misused the temperature and heat concepts and treated them as if they were the same in practice. Because still, her initial concept image was still resistant to change in terms of simulating to known situations. Norman (1983) explains this situation as uncertain about their existing knowledge and have "degree of certainty" elements in their model. In this sense, both kinds of questions were different for Funda, and her cognitive conflict experience may lead her to doubt her own knowledge. Since she trusted her knowledge of the known situation, she constantly tried to relate a given situation with her previous experiences, which she felt confident. At this stage, Funda needed help to clear up this confusion and she learned from her peer’s discussion. With İlke’s explanation and Betül’s warning, Funda revised her equation. After that, Funda was allowed to build the process by herself. By doing this, she reconstructs the concepts. In Activity 6, the effects of her initial concept image were not observed, and she reached equilibrium. In the end, she met all the reification process objectives, and she reached the object-process stage.

5.2.2.3 Developments in İlke’s concept images through the engineering design process activities

İlke had a hybrid concept image (HCI). She constructed mathematically inaccurate knowledge on mathematically accurate knowledge or vice versa. This kind of image is fragmented, and learners create irrational scientific knowledge or combine them intuitively according to context. Moreover, her design thinking aspect was dominant, and she always tends to design first. She had difficulty in the mathematical part and was not so successful in reasoning in the mathematical discussion. This feature was both advantageous and disadvantageous for her. As her group friends, she was not in the interiorization process. İlke internalized her knowledge of integral procedurally, not conceptually. By experiencing the integral several times in different contexts, she conceptualized the definite integral as a situated abstraction.

In Activity 1, thinking in terms of integral concept, she constructed her definitions around "area under a curve and antiderivative" and most commonly on deficit procedural knowledge. This was a big problem for her to produce ideas for solving the problem. Because in this activity, area definition was not applicable, and that was a conflict for İlke. Thus, conflict in her integral definition and its dominance in her concept image ended unsuccessfully, even though she had started accurately.

Another hindrance for İlke was her thinking style. Since she was able to produce ideas on design discussion part of the activity more easily than she could on the mathematical discussion; however, İlke could not combine design discussion with the mathematical discussion and could not transfer the idea obtained during the mathematical discussion into the design. Besides, activities were not familiar to her, and she composed different sub-sections, even though reach the solution, it was necessary to combine and think all of the pieces together. Therefore, she was confused and not able to focus on the construction of the Riemann Sum. Hence, she constructed the ideas as pieces in her mind. To prevent this type of disjoint the researcher gave instructions from an application to theory. By emphasizing the variables in the design, the researcher provides İlke suceed to related mathematics and design parts. Going from practice to theory, from theory to practice in the same activity, it made İlke realize the connections between them. While refering the points she mentioned in practice correspond to theory, she could refer practically where the parts of the theory correspond in practice, making it easy for İlke to establish a connection between theory and practice.

In Activity 3, İlke started to connect the whole process. However, her conflict was still continuing. In this activity, she balanced her mathematical thinking and design thinking. Nevertheless, she solved her difficulties related to the theoretical part in the design part. Her deficient integral concept image related to the area under a curve definition was very powerful and prevented her from improving in constructing the Riemann Sum. Besides her difficulty tracing all the processes in a line, her concept image was still active with losing its effect, which prevents seeing the whole picture and thinking about all the variables. To compensate for this situation researcher assisted İlke, and she reviewed the Riemann Sum by connecting it with their design.

Moreover, peer discussion helps İlke to check her ideas if they were correct. With these, the researcher provides İlke to reconsider the whole process. As a result, she self-constructed and connect each variable with the formula and thought about it on the prototype. This was breaking point for her to reconstruct the integral concept in her mind. Even though she started to construct the concept by herself, she still needs assistance to construct the Riemann sum concept more concretely. In this activity, her initial concept image was so dominant that it hindered İlke to pass a new phase, reification.

Activity 4 was a transition phase to reification. Namely, it was like a preparation phase, and the foundations of the reification phase were laid. However, she was not able to see the Riemann sum as a static object or a structure. Moreover, every time she wrote every step of the Riemann Sum with its details. This situation proves that she was not able to reach the reification process for the Riemann Sum; however, at the same time, she was beyond the Condensation process.

On the other hand, activity 4 was designed to prepare the participants for the Fundamental Theorem of Calculus, and İlke was familiar with integral before the activity. Thus, two different structures started to form in her mind. In other words, while Riemann sum continued to be constructed conceptually, an integral concept built on Riemann sum was formed step by step; on the other hand, a previously formed integral concept started to be rearranged and conceptually constructed. Interestingly, these two distinct structures merged in the following activities, and the integral construction is completed. As a result, all this process caused İlke to progress slowly. Therefore, in terms of Riemann Sum's construction, İlke's condensation process lasted until the 6th activity. This was due to the cataloging difficulties, which refers to the cognitive conflict in student's minds. Namely, this kind of difficulty occurs while students try to reason the concept based on their concept images, even if they know the accurate concept definition. Similar examples can be found in the literature (Bingolbali & Monaghan, 2008; Edwards & Ward, 2008; Vinner & Dreyfus, 1989). However, in our concept, İlke's situation is complicated. Because while İlke experience this difficulty, on the other hand, she was constructing the Fundamental Theorem of Calculus which was in the interiorization phase in Activity 4 and 5. She recently learned the Riemann Sum. Her concept image related to the Riemann Sum was active as well. Thus, progression in the Fundamental Theorem of Calculus was clearly observed in Activity 5, and the reification process is continued in Activity 5.

In Activity 5, İlke's design thinking style was still dominant. However, she was able to learn to combine it with mathematics. In this way, she can solve her difficulties. During her confusions, she solved the problem by skillfully shifting between design and mathematical knowledge. She was getting used to the activities' structure, and she traced all processes during the lesson. In this activity, İlke started to conceptualize the Fundamental Theorem of Calculus naively, and she also started to interpret the Fundamental Theorem of Calculus. As İlke's groupmates, İlke also had confusion on covariation between the variables. However, she was able to understand the situation by herself. İlke's design thinking style had a significant impact on clearing up the confusion because she could think the variables together, interpret the design, and prove to solve the problem. Activity 6 was an equilibrium and a connection activity for İlke. Because she completed the Riemann Sum construction and continued the process of building the Fundamental Theorem of Calculus and merged it with the Riemann Sum. Therefore, she constructed the Fundamental Theorem of Calculus based on the Riemann Sum. As a result, she understood that the Riemann sum and the Fundamental Theorem of Calculus are the components of the integral. Therefore, she reached the reification process, but she could not see the integral as an object. However, by strengthening the links, she will see the integral as an object and be used as a process for the differential equations.

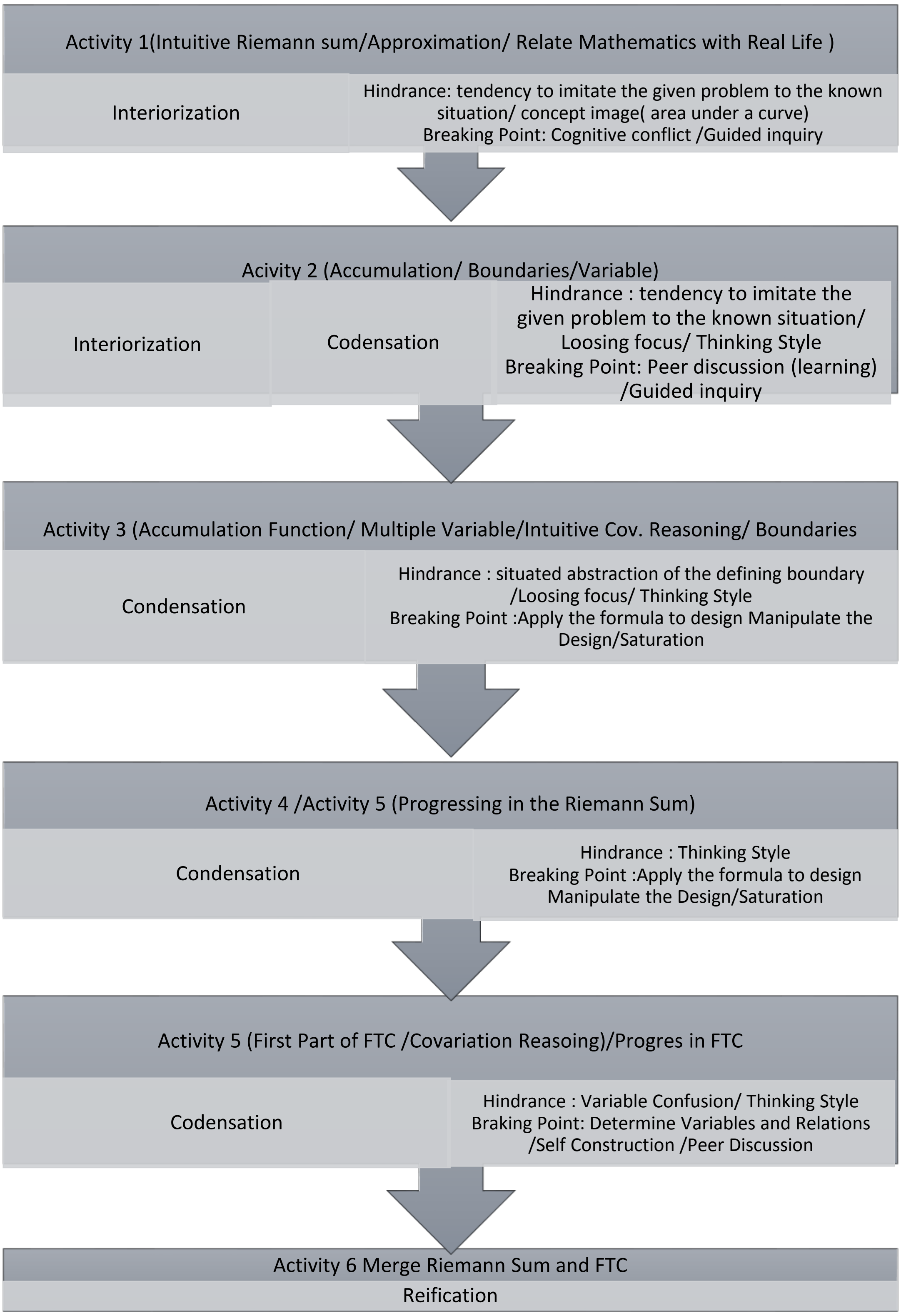


Figure 5.3 Whole development process belongs to İlke

5.2.2.4 Sfard's three-phase framework proposed by the current study

Sfard (1991) analysis the concept formation process and divided the process into three main parts. She stated that through analysis of the parts in concept formation, the knowledge is transferred from computational operations to abstract objects through the steps of interiorization, condensation, and reification. This process occurs in the hierarchy, which means that without reaching the former stage, the learner cannot go through the subsequent stages. Moreover, for the reification phase, Sfard stated that "it seems inherently so difficult that at certain levels it may remain practically out of reach for certain students" (Sfard,1991, p.1), and she defined the reification process as "a sudden ability to see something familiar in a new light" (Sfard,1991, p.19). Furthermore, according to the scheme, o enhance the "degrees of structuralization" (Sfard,1991, p.18), it should start with the familiar context or objects then students should be able to turn objects into "autonomous entity" (Sfard,1991, p.18), then combine all the new entity with the previous structure and see all the concept as a whole. Sfard (1991) defines the final part as reification. In the light of these explanations, there is a contradiction between the theory and its application. In theory, while a cyclic structure is mentioned in the first two steps, this structure is not mentioned in the reification process. Our data showed that that cycle goes on the reification process too. Because the acquisition of knowledge is in progress, the cyclic structure continues in reification. In our study, pre-service mathematics teachers used the structures they created in the process of acquiring knowledge repeatedly in different situations and put the concept into practice, developing and then putting the concept into a practice again. This process occurs in cycles, and learning occurs. Thus, by applying this concept that the student understands, s/he constructs the taught concept in her/ his mind. Hence, reification is also composed of cycles.

Another case that should be mentioned in theory is the reification step. According to the theory of Sfard (1991) to learn conceptually, the student must pass the three steps mentioned hierarchically and finally see the concept as a whole that step is the last step in which learning is conceptually learned and completed. The learned concept after this step constitutes the interiorization step for another higher-level concept. Also, reification is defined as "an ontological shift, a sudden ability to see something familiar in a totally new light." (Sfard, 1991), and not every student can reach this step (Sfard, 1991). Namely, the theory of Sfard (1991), which is generally based on the conceptual acquisition of the concept in mind, primarily creates a contradiction within itself, and there are two problems here; the first is if the student cannot reach this step, according to Sfard (1991), the student remained in the condensation step at best. When there is new information about the learned concept, this information will also merge with the existing ones, and because of the hierarchical structure of the Sfard's theory (1991), all the knowledge will be compiled in that step. As a result, students will not be able to learn the concept structurally and could not go further from the condensation step. In this example, assume that the learner may understand other parts of a phenomenon conceptually however did not learn the first step of the concept. Could we still say the learner is in the condensation phase? Or, according to what criteria could we determine the learners' phase? The theory does not give concrete answers to these questions.

Sfard (1991) has shown that this hierarchical structure is separate from each other. The study's data revealed that although these structures are hierarchical, they are not separate from each other, and there is a cloudy transition between them. Our data revealed that there is an intersection between these phases instead of a sharp line. Each step has certain levels in its loops, and the student is approaching the highest level closest to the upper step in the lower step and started to move from the lower step from the lowest level in the upper step gradually. In other words, when the state of knowledge acquired in a step is saturated, it is switched to the upper step over through the intersection region, which consists of lower and upper steps. It is observed in three of the pre-service mathematics teachers. While they achieved the interiorization phase's objectives and reached saturation, they started to go through the condensation phase.

Moreover, in order for structured learning to occur, students must correctly connect all the taught concept structures with each other (Sfard, 1991, Tall & Vinner, 1983). For this reason, these steps should not be separated when learning does not occur separately. There are some unclear explanations about the theory made by Sfard. For instance, it is not clear that the theory has a dynamic structure. As can be understood from the linkographs, all three students' knowledge structuring processes exhibit a dynamic structure in and between the phases. We can also support that this dynamic process with concept image theory. Namely, we know that the process that students obtain information and develop concept images is formed at the end of this process. Besides, we know that the concept image has a dynamic structure. (Johnson-Laird, 1983; Vosniadou et al., 1999; Viholainen, 2018) In this sense, when both situations are considered together, it is concluded that there is a dynamic process in structuring the knowledge. Adding a new situation to the structure formed in the process of learning the concept is proof that the process is dynamic.

Moreover, our data revealed that related concepts can be constructed and may feed each other simultaneously. For example, while İlke was in the condensation process for the Riemann Sum in Activity 5, she was also in the interiorization phase for the Fundamental Theorem of Calculus, and there was a gap between these two structures. However, through the activity, she reached the reification process and combined the Riemann Sum and the Fundamental Theorem of Calculus. On the other hand, while she was processing through the Riemann Sum, she also progressed in the Fundamental Theorem of calculus. It means that although the theory is hierarchical, this structure can sometimes be disconnected rather than continuous.

Moreover, different parts of the same concept can be structured synchronously with each other in different phased and by acquiring new knowledge to these parts in different phases may converge each other. Furthermore, when the acquiring new knowledge is complete this concepts merge at one point. As a result, this theory has a hierarchical, cyclic, and dynamic structure and the phases are not separated from each other. Finally, Sfard (1991) defines reification as "interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into an object, into a static structure" (p. 20). However, in our study, we observed that reification is a cyclic process. Since acquiring the information is in a cumulative cycle which composed of getting information, using and revising it then using the obtained information in different contexts, it can only be a basis for new knowledge at higher-order when it is saturated. Moreover, the concept is given to the student consciously and conceptually up to the finest detail. Hence, reification occurs with cumulative saturated knowledge cycles, not with sudden flashes. If it occurs only with flashes, it may mean ignoring all the student's awareness of the concept during the process and the importance of the previous steps. It can be asked that every sudden flash means that students learn the given concept or, inversely, could we say that if the students will not experience leap, does that mean they were not able to construct the concept structurally? In our study, there were sudden flashes. However, they were not related to the objective of the activity; they were related to irrelevant understanding flashes. As a result, the reification step also involves a cyclic process, and there may be a leap and reification definition belong Sfard (1991) did not work in the same way in our study.

Sfard states that the information learned here is wholly disconnected and argues that there is a transition to the reification step when this situation occurs. However, the data of our study followed a different path from this argument. There is a situation called object, but this is a step after reification, not in the reification step. Students may reach this stage quickly, but sometimes it may take time. Moreover, by using the taught concept in different contexts or high-level concepts, all students may reach the phase.

5.3 Suggestions for Future Studies

In this study, changes in pre-service mathematics teachers' concept images of the definite integral through the engineering design activities were investigated from a design-based research perspective. Engineering design-based activities were designed based on the STEM education perspective, definite integral literature. This study is based on the data obtained in the second cycle and focuses on the pre-service mathematics teachers' understanding and concept images. With the help of the linkograph analysis, both individual and group dynamics were observed. Moreover, each pre-service mathematics teacher's progress in the group is examined, and their construction of the concepts, points that hinder their learning, and the breakpoints of these hindrances are detected by analyzing the data with a linkograph. The construction of the definite integral concept is examined by the three-phase theory (Sfard,1991). In this regard, some suggestions were identified to be used in future studies.

First of all, I studied with three pre-service mathematics teachers in the "STEM Context in Mathematics Education" course. Similar studies can be conducted for all pre-service mathematics teachers in a classroom environment. The examined data was the second cycle of the study. Hence there is a need for validating and enhancing the effectiveness of the designed activities in different contexts. In this sense efficiency of the activities can be tested in the regular calculus courses, or the activities can be used in the high school context, and their development can be investigated. However, researchers may have problems in crowded classes due to the high number of groups. Because giving feedback to all groups and observing them may be difficult and take too much time. This kind of problem can be overcome in two ways. First, the problems in the groups may be written on the solution board with its hint. In this way, instead of answering the common questions repeatedly, the teacher may give feedback on critical problems in the groups for a long time. Another suggestion is to determine one leader from all groups and conduct the activities before the lesson. In this way, both leaders help their friends, and in critical points teacher gives feedback. Moreover, control of crowded groups becomes easier. The literature emphasizes that peer learning is an efficient way of learning (Boud, Cohen, & Sampson, 1999; Gosser & Roth, 1998).

In this study, I focused on the knowledge development of the pre-service mathematics teachers and developed activities. Instead of only using developed activities, after gave the theoretical aspects of the engineering design and conduct sample activities, it can be asked from the prospective mathematics teachers to design a STEM activity. Moreover, conduct those activities in the classroom environment during the given course and also in the school. They were also asked to determine the issues that they experienced. Thus, researchers may have the opportunity to observe pre-service mathematics teachers' thinking processes and their difficulties during the design process. This kind of activity enables us to understand how they interpret the critical point in the activities and give us clues about their thinking process during the designing activities. Moreover, in this way, researchers may focus on the professional development of the pre-service mathematics teachers in STEM activities.

Another essential issue is the used theories in the current study. To present the data as simple and understandable, classification of the concept images has been done on the basis of the big ideas of the mathematical concepts. Due to a limited number of students, those classifications are dependent on the data. Researchers may examine if this classification is valid for other students and whether it can be used in a generalized way. Because the classification is applicable to all students, it can be generalized to all mathematics concepts. In this way, a common language can be used. Since classification is based on big ideas, even if the concepts are changed, classification terminology can be used in the same meaning in classifying the concept images of the students. Thus, on a this of classification or using a general terminology, researchers may have an idea which big ideas are deficient or constructed accurately by the students.

Knowledge developments of the pre-service mathematics teachers are examined with three-phase theory (Sfard,1991), and it is observed that theory has some unclear points in the practice, so a revised framework is proposed. According to the proposed framework, there are four phases, interiorization, condensation, reification, and object. These steps are in a cyclic and dynamic structure. Moreover, phases are not separated, steps have different levels, and the boundary between the steps is not sharp and cloudy. This revised framework can be validated using the data that can be obtained in other contexts.

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APPENDICES

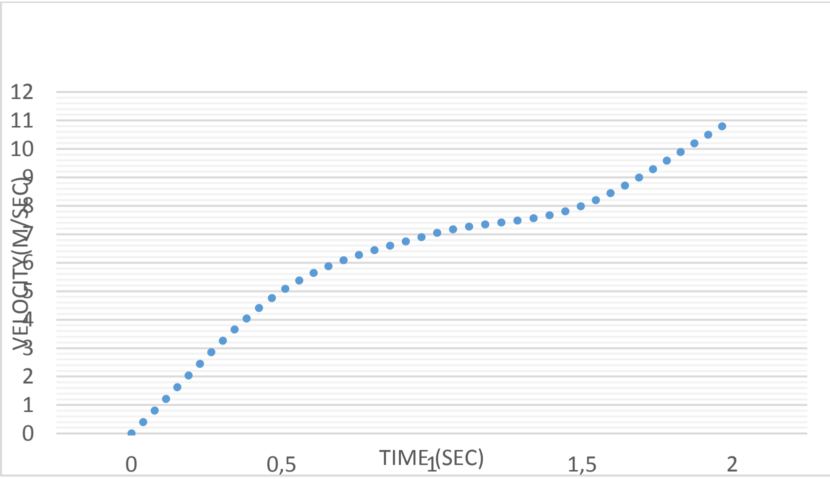
A. REFLECTION GUIDE

1) Explain problem, solution process and your ideas about the process from beginning to the end and different way of thinking emerged during this process? (**indicate even if they were wrong**) (How did you analyze the problem situation? What were the difficulties (if any) that you faced during the solution process? How did you overcome those difficulties? What factors influenced your decisions (group discussions, previous knowledge etc.) (20 points)
2) What did you learned after solving this question? (Please explain in detail)
   a. What were the mathematical ideas and concepts covered with this problem? (16 points)
   b. Did you learn a new idea (concept), or all the ideas were known for you?
   c. If the ideas (concepts) were already known, is there any change with these ideas? What kinds of change occurred?
3) If you look from a teacher perspective; (30 points)
   a. If you were given the same topic and asked to design a STEM activity for the same purpose what would you do differently?
   b. If you apply this problem in a classroom, which objectives do you expect students to learn (Write separately such as Math, Science etc.)?
   c. What can be students' possible answers?
   d. How do you modify this question for lower grades?
   e. In such a classroom application of this problem;
      i. Where and what kind of difficulties students may encounter?
      ii. What could you do to overcome the difficulties and mistakes that students have?

4) For $y = x^2$ function show that area of the approximating rectangles approaches in the range of (0,1). (34 points). Show your work by algebraically get a general formula.

B. Conceptual Questions

1) What does the integral of a function mean?
2) A stuffed gorilla (wearing a parachute) was dropped from the top of a building. His parachute opened the instant he was dropped. We were able to record the velocity of the gorilla with respect to time is shown below. Note that he touched the ground just after 2 seconds.



   a. How long do you think the gorilla travelled?
   b. Explain how you find your answer/ strategy
   c. Is it possible to find more precise solution?

3) What does $\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$ mean? Explain in words and by graph.
4) The force required to extend (or compress) an elastic spring to x units longer (shorter) is propotional to x. This is known as Hooke's law. If the force is denoted $F(x)$, then

$$F(x) = kx,$$

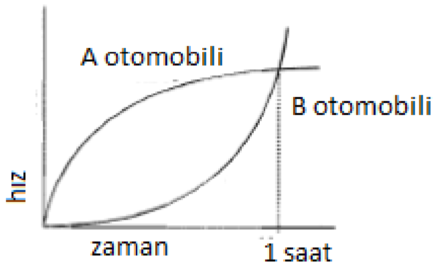
where k is the spring constant. If a spring is extended 4 cm, how much work is done?

C. Initial Interview Questions

Add Tarih Zaman:

Adı- Soyadı:

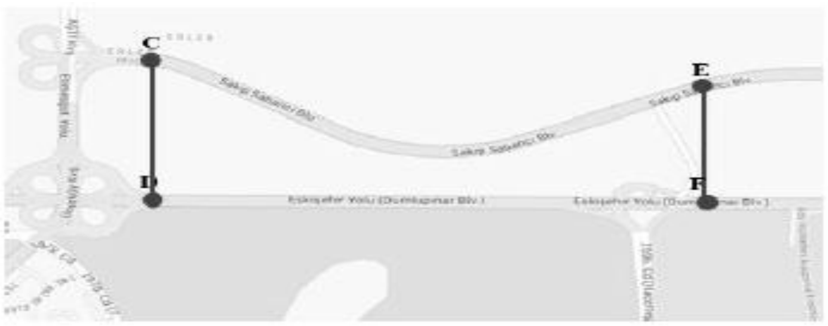
1. Aşağıdaki grafik iki otomobilin hız- zaman grafiğini göstermektedir (otomobillerin aynı yerden aynı yönde gittiklerini kabul ediniz).



Buna göre t=1 de A ve B otomobilerinin konumları arasındaki ilişki nedir?

a- Aynı yerderdirler.
b- Otomobil A, otomobil B nin önündedir.
c- Otomobil B, otomobil A yı geçmektedir.
d- A ve B otomobillleri çarpışmaktadırlar
e- a ve c.

2. Eskişehir Yolu ile Sakıp Sabancı Bulvarı arasında CD ve EF hatları arasında bir arazinin size miras kaldığını farzedin.



Arazinizin alanını nasıl hesaplarsınız? Hesapladığınız da bu alanı matematiksel olarak nasıl ifade edersiniz?

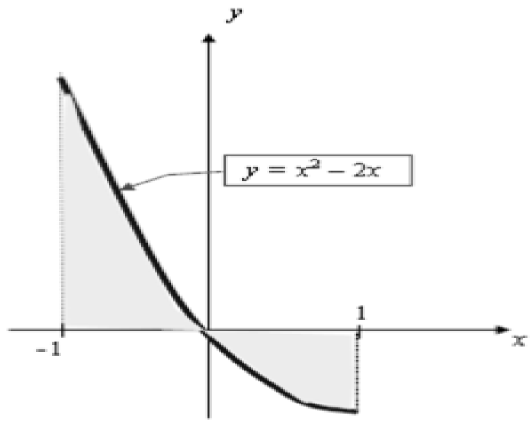
3. $\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$ eşitliğini açıklayınız.

4. F fonksiyonunda parantez içini doldurunuz nedenini açıkayınız
   $$F(\_\_) = \int_a^x f(t)dt$$

5. $\int_0^1 (x^2 + 2x + 1)\, dx$ Şeklinde bir ifadeyi çözdüğünüzde ne elde edersiniz? Belirsiz integral olsaydı aynı ifadeyi nasıl yorumlardın?

   C sabiti hakkında ne söyleyebilirsiniz?

6. Aşadğıdaki şekilde gösterilen tarali alanı bulunuz.



7. Aşağıdaki ifadeler arasındaki ilişkileri oklarla gösteriniz
   İş, türev, Fonksiyon, limit, Rieman toplamı, Alt limit, Üst limit, belirli integral,sonsuz küçük, alan, değişim oranı, birikimin değişim oranı, Değişken, Analizin temel teoremi, Hareket

D. Sample Consent Form

GÖNÜLLÜ KATILIM FORMU

Değerli katılımcı,

Bu ders, Prof. Dr. Erdinç Çakıroğlu tarafından yürütülen "İlköğretim Matematik Öğretmenliği Öğretmen Adaylarının STEM Aktiviteleri ve Mühendislik Tasarım Temelli Öğretim ile Belirli İntegrale dair Zihinsel Modellerinin Değişimi" projesi kapsamında içeriği oluşturulmuş olan matematik odaklı STEM aktiviteleri ile hizmet öncesi öğretmen eğitimini amaçlamaktadır. Araştırmamızın amacı matematik eğitiminde STEM etkinliklerini ve mühendislik tasarım yöntemini kullanarak öğrencilerin integral vb. konularda kavramsal olarak öğrenmelerine yardımcı olacak bir ders dizayn etmek, ders kapsamındaki etkinliklerin öğrencilerin ilgili kavramlara dair kavram görüntüleri üzerine etkisini araştırmaktır. Bu amaçlar için tasarlanan ders kapsamında 14 hafta sürmesi planlanan çalışma süresince (i) integral testi, (ii) kavram haritası, (iii) STEM etkinlikleri için grup çalışma raporları, (iv) bireysel çözüm kâğıtları, (v) ses kayıt ve video kayıt cihazlarıyla desteklenmiş gözlemler, (vi) görüşmeler, (vii) etkinlik sonrası düşünce raporları, (viii) gruplarca hazırlanan STEM soruları ve bu soruların uygulama planları (ix) öğretmen adaylarının sunumları (mikro-öğretim) temel veri kaynakları olacaktır. Bu kapsamda toplanacak veriler Araş. Gör. Ümmügülsüm Cansu Kurt'un doktora tez çalışmasında kullanılacaktır.

Çalışma süresince toplanacak veriler tamamıyla gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir. Elde edilecek bulgular tez çalışmasında ve bilimsel yayımlarda akademik etik kurallarına dikkat edilerek kullanılacaktır. Çalışmaya katılım tamamıyla gönüllülük temelindedir. Çalışma süresince katılımcılar için potansiyel bir risk öngörülmemektedir. Ancak, katılım sırasında farklı amaçlarla toplanan veya alınan dersin gerekleri olarak toplanacak verilerin bilimsel çalışma ve tez çalışması amaçları çerçevesinde kullanılmamasını isteyebilirsiniz. Bu durum ders performansınızın değerlendirilmesinde kesinlikle negatif bir durum oluşturmayacaktır.

Çalışma hakkında daha fazla bilgi almak için ODTÜ Eğitim Fakültesi Matematik ve Fen Bilimleri Eğitimi Bölümü öğretim üyesi Prof. Dr. Erdinç Çakıroğlu (erdinc@metu.edu.tr.) ve Araştırma Görevlisi Ümmügülsüm Cansu Kurt (e-posta: cansuummugulsum@gmail.com) ile iletişim kurabilirsiniz. Bu çalışmaya katıldığınız için şimdiden teşekkür ederiz

Bu çalışmaya tamamen gönüllü olarak katılıyorum ve istediğim zaman yarıda kesip çıkabileceğimi biliyorum. Verdiğim bilgilerin bilimsel amaçlı yayımlarda kullanılmasını kabul ediyorum. (Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz).

İsim Soyisim Tarih İmza

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E. Ethical Permission

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

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15 ARALIK 2017

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof.Dr. Ayhan Kürşat ERBAŞ ;

Danışmanlığını yaptığınız doktora öğrencisi Ümmügülsüm CANSU'nun **"Matematik Öğretmen Adaylarının STEM Etkinlikleri ve Mühendislik Tasarımı Temelli Öğretim i,le Belirli İntegral Kavramına Dair Zihinsel Modellerinin Değişimi"** başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülerek gerekli onay **2017-EGT-183** protokol numarası ile **05.11.2018-04.01.2019** tarihleri arasında geçerli olmak üzere verilmiştir.

Bilgilerinize saygılarımla sunarım.

Prof. Dr. Ş. Halil TURAN
Başkan V

Prof. Dr. Ayhan SOL
Üye

Prof. Dr. Ayhan Gürbüz DEMİR
Üye

Doç. Dr. Yaşar KONDAKÇI
Üye

Doç. Dr. Zana ÇITAK
Üye

Yrd. Doç. Dr. Pınar KAYGAN
Üye

Yrd. Doç. Dr. Emre SELÇUK
Üye

F. Sample Actvitiy

Shake It! Earthquake Tower Challenge

Problem: The world's third, which will be Turkey's second airport on the sea will be constructed in Rize. As there is inadequate territory in Rize, airport will be constructed on the see. Transportation to the airport will be provided by a suspension bridge which can carry the most weight cars. You have been hired as an engineer for an instruction company which is going to build the bridge. You are expected to design and build a model suspension bridge and your model bridge must meet the following requirements:

- The bridge must fit on the base.
- Your bridge must be at least 30 cm long
- Your bridge has at least 2 ropes that are at least 6 cm long.
- A construction drawing with measurements and analysis must be reported before testing.

ENGINEERING DESIGN PROCESS

Step 1. Identify Problem and Constraints

Step 2. Research & Step 3: Ideate

Step 4: Analyze & Step 5 Design

Step 6: Communication and Reflect

Storyboard should include following items.
Steps of finding general formula of cable.
Determined load for the bridge

Step 7: Test
This step will be conducted together.

G. HLT table of the integral

Sequence of Activity	Big Idea	Description of the Activity	Possible Discourse Topics
Preparation Activity	Determining Variable	Students will design and construct a parachute with polygons and a circumscribed circle. With the area of the parachute and length of segment of polygons, students will a construct a parachute and try to drop an egg from 3 meters height to the ground without breaking. During the activity students will learn area of the circle and polygons.	What are the variables?
Activity 1	Approximation	In this activity, students will learn construction of canoes from engineering perspective. Students will discover the principle of buoyancy. Throughout this activity, they will also learn the components of the engineering solutions which are shaped by the environmental conditions and availability of resources. Moreover, they will also learn to find approximate area of the irregular shapes.	Which variables do we need to solve the problem?
Activity 2	Accumulation/ Accumulation Function	In this project, students will construct eye-pleasing towers with using straws and semi-circle planes. The towers must	

		remain standing for the simulated earthquake. At the end of the activity, students will notate and interpret the Riemann Sum algebraically. They will also learn accumulation.	
Activity 3	Limit of Accumulation Function/Determining Boundary	In this activity students will build a dam with given conditions which is the material should be minimum, the surface area will be optimum, and dam should be the most persistent dam. Students will learn limit of accumulation function explain relationship between Riemann sum notation and integral form. Moreover, they will also learn determining the boundary of the integral in polar coordinates.	How do you calculate a given area without knowing the function?
Activity 4	Transition Riemann Accumulation Function from setting up integral	In this activity student will repair a cooling tower whose some part collapsed. During the activity they have to calculate by using only the photos of the tower. The student will learn relation between the Accumulation Function and integral representation. Moreover, they will also learn setting up the integral.	How can you measure canoes volume?
Activity 5	Fundamental Theorem of Calculus 2	In this activity students will construct an insulated house. Students will be informed about heat and temperature. Students will learn explain	How can measure the canoe's length?

		second part of the Fundamental Theorem of Calculus. They will also learn relation between antiderivative and definite integral	
Activity 6	Fundamental Theorem of Calculus 1	In this activity students will build a bridge. During the activity Students will learn tension and compression forces act on suspension bridge.	How do you provide your building to withstand the earthquake?

CURRICULUM VITAE

PERSONAL INFORMATION

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EDUCATION

Degree	Institution	Year of Graduation
MS	Selçuk University Mathematics	2010
BS	Cumhuriyet University Mathematics Education	2007
High School	Hazım Kulak Anadolu High School, Aksaray	2001

WORK EXPERIENCE

Year	Place	Enrollment
2019-Present	Bolu Abant İzzet Baysal University Dept. of Mathematics Education	Research Assistant
2010-2019	METU Dept. of Mathematics Education.	Research Assistant
2010- October	Bolu Abant İzzet Baysal University Dept. of Mathematics Education	Research Assistant
2008- 2010	Recep Tayyip Erdoğan University Dept. of Mathematics	Research Assistant

FOREIGN LANGUAGES

Advanced English, Elementary German

PUBLICATIONS

1. Garip, B., Cansu, Ü., Demirtaş, D., and Bülbül, M.Ş. “Görme engelliler için matematik öğretim materyali tasarımı: İğneli Sayfa.” İlköğretim Online, 11 (4), 1-9 (2012).
2. Kurt- Cansu, Ü. “Görme engelliler ve matematik eğitimi.” Sürdürülebilir ve Engelsiz Bilim Eğitimi Dergisi, 1 (1), 21-28 (2015)

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HOBBIES:

Writing Story, Painting, Cooking, Knitting, Embroidery, Modelling