Optimal premium allocation under stop-loss insurance using exposure curves

Ozenc Murat Mert* ⬗, A. Sevtap Selcuk-Kestel ⬗

Institute of Applied Mathematics, Middle East Technical University, 06800, Ankara, Turkey

Abstract

Determining the retention level in the stop-loss insurance risk premium for both insurer and reinsurer is an important factor in pricing. This paper aims to set optimal reinsurance with respect to the joint behavior of the insurer and the reinsurer under stop-loss contracts. The dependence between the costs of insurer and reinsurer is expressed as a function of retention (d) and maximum-cap (m) levels. Based on the maximum degree of correlation, the optimal levels for d and m are derived under certain claim distributions (Pareto, Gamma and Inverse Gamma). Accordingly, the risk premium and exposure curves for both parties are based on the selected distributions. Quantification of the premium share over derived exposure curves based on the optimized retention and maximum levels and the maximum loss risk is obtained using VaR and CVaR as risk measures.

1. Introduction

A reinsurance contract shows different characteristics depending on the type of agreement. Among all other types of reinsurance forms, stop-loss is the most common type of agreement at which the partition rule of the risk is determined by a prescribed retention level. Additional to this constraint, depending on the type of the risk, the reinsurer may decide to set an upper bound (cap or maximum) on the severity of the risk (claim amount, cost). The partition of the risk with respect to the levels of these factors (d and m) between two parties creates a natural correlation that contributes to the set the optimal value maximizing the interest of insurer and reinsurer. The stop-loss reinsurance has an interesting property such that its optimal value is depicted when the variance of the cost, especially for the insurer, is minimized. Most of the optimal reinsurance studies in literature minimize the variance of cost for the insurer but rarely consider the reinsurer's aspect (see [5, 11–13]). The insurer's optimal strategy under the standard deviation premium principle with several constraints [11], optimal reinsurance arrangements under various mean-variance premium principles, the relation between the adjustment coefficient and maximum expected utility of wealth with respect to the retained risk [12, 13], the

optimal retentions by minimizing the value-at-risk (VaR) and the conditional tail expectation (CTE) of the insurer's total risk, and some other risk measures [2,8] can be counted among many remarkable studies in the reinsurance optimization. Some other literature, such as Markowitz type efficient frontier solution to determine the optimal retention and limiting levels under a joint survival probability [10], designing an optimal reinsurance contract maximizing the joint survival and profitable probabilities [6], two state-of-the-art evolutionary and swarm intelligence approaches [16], and optimal reinsurance contracts minimizing the convex combination of the VaR for both parties [7] concentrate on the joint behavior of the insurer and reinsurer in optimizing the retention offer different approaches. Additionally, the optimal values of the contract by minimizing the total risks of the insurer VaR, CVaR, and CTE can be found in [8,16,19]. The other risk measures, such as ES (expected shortfall) and RCVaR (robust conditional value-at-risk [14]), can also be considered in this framework. However, we stick with the traditional measures to be consistent with insurance literature.

On the other hand, for the partition of risk premium between insurer and reinsurer, exposure curves are commonly used in practice. Based on the level of retention, the premium share is mostly considered in non-proportional reinsurance contracts. The "rating by a layer of insurance" method evaluates the proportion of losses with respect to the size of loss from aggregate loss distributions [17]. In [15], it is studied on the liability insurance in which the claim sizes can not be assumed to be scaled by sums insured. They analyze Riebesell's system introduced in 1936 from the perspective of the collective risk model theory. The well-known exposure curves, which are a form of analytic function with two parameters by using Maxwell-Boltzman, Bose-Einstein, Fermi-Dirac (MBBEFD) distribution class for a single risk in [3] and extended in [1] for two dependent risks, is derived as a sole function of the retention. The dependence between the costs of insurer and reinsurer measured by the correlation coefficient as a function of retention and maximum levels under Normal, Gamma, and translated Gamma loss distribution assumptions are introduced by [9], at which the optimal stop-loss contract attains the maximum correlation. However, the literature lacks studies about the exposure curves for stop-loss contracts under optimal constraints and dependence framework.

Due to the other varieties in the loss distributions and the direct influence of the dependence created with respect to the upper and lower bounds (retention and cap) in reinsurance agreements, we aim to examine the influence of the correlation coefficient on the valuation of premiums for insurer and reinsurer, and then, evaluate the optimal stop-loss contract in the frame of Pareto, Gamma, and Inverse Gamma as aggregate claim distributions under two cases. **Case I** refers to the contracts only with retention (d); **Case II** quotes the ones with both retention and maximum (m) as defining constraints. The optimal value for d and m are determined in such a way that the maximum level of these bounds optimizes (minimizes) the cost of both insurer and reinsurer with respect to the selected loss distributions. To do so, we derive the expressions for the mean and variance of the aggregate claims for insurer and reinsurer and the correlation coefficient (ρ) between the costs of both parties. Keeping up with the value of the correlation between their respective costs, we determine the optimum premium and its partition between the insurer and the reinsurer in accordance with the selected distributions. Under this framework, the exposure curves as functions of retention and maximum levels for each distribution are analytically derived. Monte Carlo simulations are employed to illustrate the impact of parameters on determining the premium share. We employ a dynamic optimization scheme to determine the retention and maximum levels, which yields an equilibrium maximizing risk premiums for both parties. The convergence to optimal bounds in two cases is determined according to the values achieving the maximum correlation minimizing the expected cost. Under the proposed framework, we investigate the effectiveness of the optimized d and m levels compared to the risk measures VaR and CVaR. Our results show

that the optimal values for both cases also maximize the correlation between the costs. We expect the findings are utilized to determine the optimum level, its related retention value with respect to their cost distribution under derived exposure curves.

The remainder of the paper is structured as follows. In the next section, the derived expressions for the expected value, variance, and correlation coefficient for the two cases are presented. Moreover, the derivations of these statistics under Pareto, Gamma, and Inverse Gamma assumptions are introduced. Section 3 presents the premium share between the insurer and reinsurer by implementing the proposed exposure curves under three distributional assumptions. Section 4 sets up an optimization problem and corresponding algorithms to obtain the solutions. Section 5 is devoted to numerical illustrations at which the simulations based on the framework for the insurer and the reinsurer are presented as well as their behavior with respect to VaR and CVaR risk measures. Section 6 provides concluding remarks.

2. The costs of insurer and reinsurer

Once the claim occurs, the stop-loss contract is activated or dismissed with respect to conditions set on the retention or retention and cap levels. Based on the values of these parameters, the expected costs for both parties change as well as their premium share creating a natural dependence between the claims to be paid. Figures 1(a) and 1(b) show that the risk share follows a contrary pattern between insurer and reinsurer for each case.
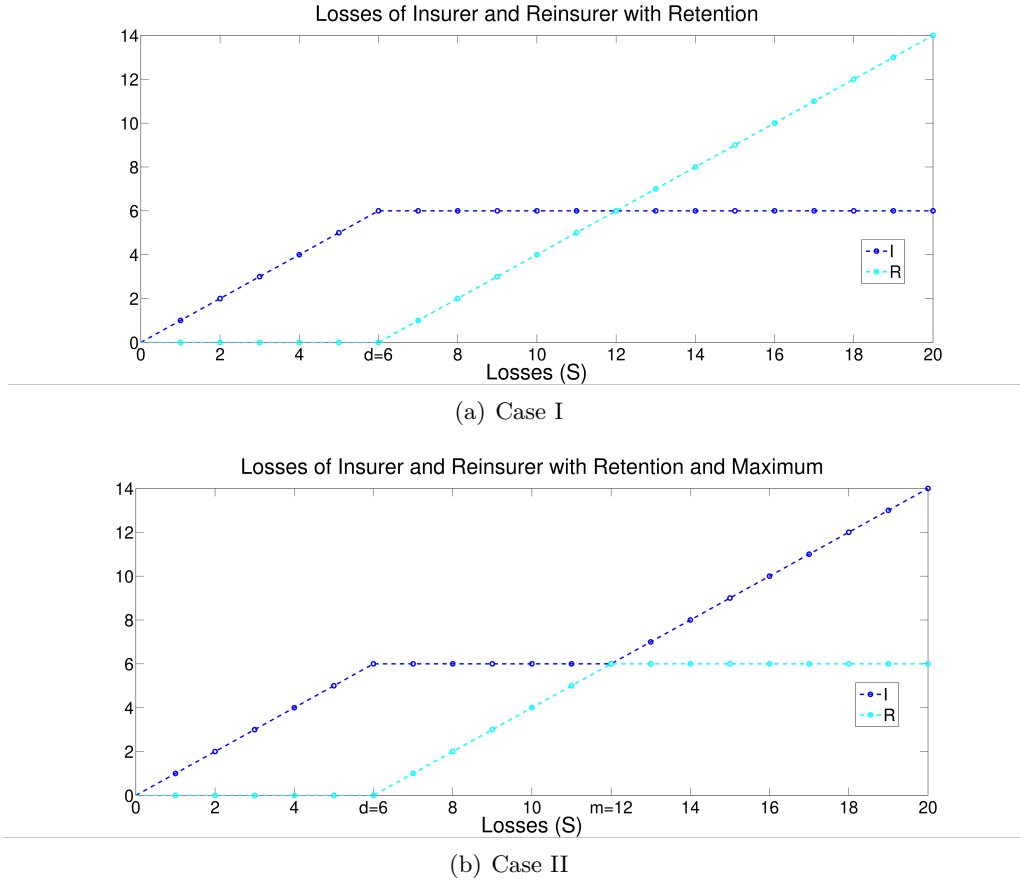
(a) Case I

(b) Case II

Figure 1. The risk share between insurer and reinsurer, (a) Case I, (b) Case II.

Assume that the random variable S represents the total cost of claims in one period, which is shared between insurer, I, and reinsurer, R, as

$$S = I + R. \tag{2.1}$$

Given a known claim distribution on S, we can write the probability distributions for insurer and reinsurer, which lead us to derive the analytical expressions for the expected value and the variance for each case when they exist. Even though these derivations are straightforward in actuarial literature, the specifics, especially custom-tailored analytical derivations based on certain distributions are uniquely listed in this paper.

Given the aggregate loss distribution function, $F_S(s)$, and the distributions functions of insurer and reinsurer, $F_I(s)$ and $F_R(s)$, the expected value, $\mathbb{E}[.]$, the variance, $\mathbb{V}[.]$ and the correlation coefficient, $\rho(.,.)$ are given by [9]. For the contracts only with retention (Case I), the losses and their distributions are defined as

$$I = min(S, d), \text{ with } \qquad F_I(s) = \begin{cases} F_S(s), & s < d \\ 1, & s \geq d \end{cases} \tag{2.2a}$$

$$R = max(S - d, 0), \text{ with } \qquad F_R(s) = F_S(s + d). \tag{2.2b}$$

Proposition 2.1. *The expectation, the variance of the costs of insurer and reinsurer and the correlation coefficient are given as, [9],*

$$\mathbb{E}[I(d)] = \mathbb{E}[S] - \mathbb{E}[R(d)], \tag{2.3}$$

$$\mathbb{E}[R(d)] = \int_d^\infty [1 - F_S(s)]ds, \tag{2.4}$$

$$\mathbb{V}[I(d)] = \mathbb{V}[S] - \mathbb{V}[R(d)] - 2Cov[I(d), R(d)], \tag{2.5}$$

$$\mathbb{V}[R(d)] = 2\int_d^\infty s[1 - F_S(s)]ds + \mathbb{E}[R(d)](-2d - \mathbb{E}[R(d)]), \tag{2.6}$$

$$\rho(I(d), R(d)) = \frac{\mathbb{E}[R(d)](d - \mathbb{E}[S] + \mathbb{E}[R(d)])}{\sqrt{\mathbb{V}[R(d)]\left(\mathbb{V}[S] + \mathbb{E}[R(d)](d + \mathbb{E}[S]) - 2\int_d^\infty s[1 - F_S(s)]ds\right)}}. \tag{2.7}$$

It should be noted that the term in Equation (2.7) is a pure function of d. Thus, when the correlation between the costs of the insurer and the reinsurer is maximum, the retention level satisfying this yields the best choice for both parties.

A stop-loss contract having both retention, d, and maximum, m, yields the partition between insurer and reinsurer,

$$I(d, m) = min(S, d) + max(S - m, 0), \tag{2.8}$$
$$R(d, m) = min(m - d, max(S - d, 0)), \tag{2.9}$$

with the distribution functions

$$F_I(s) = \begin{cases} F_S(s), & s < d \\ F_S(s + m - d), & s \geq d, \end{cases} \tag{2.10a}$$

$$F_R(s) = \begin{cases} F_S(s + d), & s < m - d \\ 1, & s \geq m - d, \end{cases} \tag{2.10b}$$

respectively.

Similar to Case I, the expected values, variances and correlation coefficient are expressed as follows.

Proposition 2.2. *The expectation, variance of cost of insurer and reinsurer with respect to retention level d and maximum m and the correlation coefficient are, [9],*

$$\mathbb{E}[I(d, m)] = \mathbb{E}[S] - \mathbb{E}[R(d, m)], \tag{2.11}$$

$$\mathbb{E}[R(d,m)] = \mathbb{E}[R(d)] - \mathbb{E}[R(m)], \tag{2.12}$$

$$\mathbb{V}[I(d,m)] = \mathbb{V}[S] - \mathbb{V}[R(d,m)] - 2Cov[I(d,m), R(d,m)], \tag{2.13}$$

$$\mathbb{V}[R(d,m)] = \mathbb{V}[R(d)] - \mathbb{V}[R(m)] + 2\mathbb{E}[R(m)](\mathbb{E}[R(d)] - \mathbb{E}[R(m)] + d - m) \tag{2.14}$$

$$\rho(I(d,m), R(d,m)) = \frac{Cov[I(d), R(d)] - (2d - m)\mathbb{E}[R(m)]}{\sqrt{\mathbb{V}[R(d,m)](\mathbb{V}[S] - \mathbb{V}[R(d,m)] - 2Cov[I(d,m), R(d,m)])}}. \tag{2.15}$$

where $\mathbb{E}[R(.)]$ and $\mathbb{V}[R(.)]$ are given previously. It is also noted that $\rho(I(d,m), R(d,m))$ is the function of the bounds set on the claim amounts. The detailed derivations under the assumption of Normal, Gamma, and translated Gamma distributions for the loss can be found in [9]. On the other hand, the most commonly used distributions for claim amounts in literature, [4] and [18], are Pareto, Gamma, and Inverse Gamma due to their properties such as: (i) the Pareto distribution has extremely thick tail quoting to capturing the largest losses, (ii) the Gamma distribution is closed under convolution, therefore, it is infinitely divisible. Furthermore, it is right-skewed, which makes it suitable for expressing extreme losses; (iii) the Inverse Gamma distribution is a heavy-tailed so that larger claims can be analyzed explicitly, whereas, Gamma has a relatively thin tail than the Pareto and Inverse Gamma. Moreover, Gamma and Inverse Gamma have nice statistical properties, such as if a random variable $Y \sim Ga(a,b)$ then $\frac{1}{Y} \sim IG(a, \frac{1}{b})$. We derive the analytical formulations of the expected value, variance and the correlation coefficient for these distributions. The proofs in the propositions for the proposed statistics are not given due to space limits.

2.1. Aggregate claims with Pareto distribution

Assume $S \sim Pareto(a,b)$, $a, b \in \mathbb{R}^+$. The expressions derived for the mean, the variance, the covariance under Case I and Case II are shown below.

Proposition 2.3. *If $S \sim P(a,b)$, the expected values, variances of insurer and reinsurer costs and their covariance under Case I are*

$$\mathbb{E}[I(d)] = \frac{b(a - (b/d)^{(a-1)})}{a - 1}, \tag{2.16}$$

$$\mathbb{E}[R(d)] = \frac{b^a d^{1-a}}{a - 1}, \tag{2.17}$$

$$\mathbb{V}[I(d)] = \frac{ab^2 - 2a(a-2)b^a d^{1-a}[d - b] + (a-2)b^{a+1}d^{2-2a}[b^{a-1} - 2]}{(a-1)^2(a-2)}, \tag{2.18}$$

$$\mathbb{V}[R(d)] = \frac{b^a d^{(2-a)}\Big[2 - (b/d)^a(a-2)\Big]}{(a-1)^2(a-2)}, \tag{2.19}$$

$$Cov[I(d), R(d)] = \frac{ab^a d^{1-a}(d - b) + b^a d^{(2-a)}\Big[(b/d)^a - 1\Big]}{(a-1)^2}, \tag{2.20}$$

respectively.

Proposition 2.4. *If $S \sim Pareto(a,b)$, the expected values, variances of insurer and reinsurer costs and their covariance under Case II are*

$$\mathbb{E}[I(d,m)] = \frac{a^2 b - ab - b^a[d^{1-a} - m^{1-a}]}{a - 1}, \tag{2.21}$$

$$\mathbb{E}[R(d,m)] = \frac{b^a}{a - 1}\Big[d^{1-a} - m^{1-a}\Big], \tag{2.22}$$

$$\mathbb{V}[I(d,m)] = -\frac{2b^a dm^{1-a}}{a-1} + \frac{b^{2a}m^{2(1-a)}\left[2d^{2(1-a)} - 1\right]}{(a-1)^2} \tag{2.23}$$

$$+ \frac{b^a d^{1-a}\left[2(1-a)d + 2b^a - d(b/d)^a\right] + 2b^{a-+1}d^{1-a}\left[a - b^{a-1}d^{1-a}\right]}{(a-1)^2}$$

$$+ \frac{ab^2 - 2b^a\left[d^{2-a} - m^{2-a}\right]}{(a-1)^2(a-2)},$$

$$\mathbb{V}[R(d,m)] = \frac{b^{2a}}{(a-1)^2}\left[(m^{1-a} - d^{1-a})^2 - 2(m^{2(1-a)} + d^{1-a})\right] \tag{2.24}$$

$$+ \frac{2b^a}{(a-1)^2(a-2)}\left[d^{2-a} - m^{2-a} + (a-1)(a-2)m^{1-a}(d-m)\right],$$

$$Cov[I(d,m),R(d,m)] = \frac{(2d-m)b^a m^{1-a}}{a-1} + \frac{b^a d^{1-a}}{(a-1)^2}\left[a(d-b) + d^{1-a}(b^a - d^a)\right], \tag{2.25}$$

respectively.

2.2. Aggregate claims with Gamma distribution

In case when the aggregate claim, S, distribution follows $Ga(a,b)$ with $a > 0$, $b > 0$, the derivations of mean, variance and correlation coefficient for each case are found analogously.

Proposition 2.5. *The expected values, the variances of insurer and reinsurer and their covariance for Case I under $Ga(a,b)$, assumption are found as*

$$\mathbb{E}[I(d)] = \frac{(ab+d)\Gamma(a,bd) - b\Gamma(a+1,bd) + ab\gamma(a,bd)}{\Gamma(a)}, \tag{2.26}$$

$$\mathbb{E}[R(d)] = \frac{1}{\Gamma(a)}\left[b\Gamma(a+1,bd) - d\Gamma(a,bd)\right], \tag{2.27}$$

$$\mathbb{V}[I(d)] = ab^2 + \mathbb{E}[R(d)]\left(2d + \mathbb{E}[R(d)]\right) \tag{2.28}$$

$$+ \frac{\Gamma(a,bd)}{\Gamma(a)}\left[2abd(b^{a+1} - 1) - d^2(2b^{a+1} + 1)\right]$$

$$- \frac{b^2\Gamma(a+1,bd)}{\Gamma(a)} - 2\left(\frac{1}{\Gamma(a)}\left[b\Gamma(a+1,bd) - d\Gamma(a,bd)\right]\right)^2, \tag{2.29}$$

$$\mathbb{V}[R(d)] = \mathbb{E}[R(d)]\left(-2d - \mathbb{E}[R(d)]\right) \tag{2.30}$$

$$+ \frac{1}{\Gamma(a)}\left[b^2\Gamma(a+1,bd) - d^2\Gamma(a,bd)\right],$$

$$Cov[I(d),R(d)] = \left(\frac{1}{\Gamma(a)}\left[b\Gamma(a+1,bd) - d\Gamma(a,bd)\right]\right)^2 \tag{2.31}$$

$$- \frac{d\Gamma(a,bd)}{\Gamma(a)}\left[(d-ab)(b^{a+1}d^{a-1} - 1)\right],$$

respectively.

These solutions require the upper and lower incomplete gamma functions ($\Gamma(.,x)$, $\gamma(.,x)$, $x > 0$), and gamma ($\Gamma(.)$) functions.

Proposition 2.6. *If $S \sim Ga(a, b)$, the expected values, the variances of insurer and reinsurer and their covariance for Case II are*

$$\mathbb{E}[I(d, m)] = \frac{\left[ab - 2(b^2 - 1)d\right]\Gamma(a, bd) + \left[ab - 2(b^2 - 1)m\right]\Gamma(a, bm)}{2\Gamma(a)} \qquad (2.32)$$

$$+ \frac{ab\left(\gamma(a, bd) + \gamma(a, bm)\right) + (bd)^a e^{-bd} + (bm)^a e^{-bm}}{2\Gamma(a)},$$

$$\mathbb{E}[R(d, m)] = \frac{b^2 - 1}{\Gamma(a)}\left[d\Gamma(a, bd) - m\Gamma(a, bm)\right] - \frac{b^a}{\Gamma(a)}\left[d^a e^{-bd} - m^a e^{-bm}\right], \qquad (2.33)$$

$$\mathbb{V}[I(d, m)] = ab^2 + \mathbb{E}[R(d, m)]\left[\mathbb{E}[R(d, m)] + 2ab\right] \qquad (2.34)$$

$$- \frac{1}{\Gamma(a)}\left[b^2(\Gamma(a + 2, bd) - \Gamma(a + 2, bm)) + 2b(m - d)\Gamma(a + 1, bd)\right]$$

$$+ \frac{1}{\Gamma(a)}\left[(m^2 + 2dm)\Gamma(a, bd)\right]$$

$$- 2\left(\mathbb{E}[R(d, m)] - \frac{1}{\Gamma(a)}\left[b\Gamma(a + 1, bm) - d\Gamma(a, bm)\right]\right)^2,$$

$$\mathbb{V}[R(d, m)] = \frac{1}{\Gamma(a)}\left[b^2(\Gamma(a + 2, bd) - \Gamma(a + 2, bm)) - d^2\Gamma(a, bd) - m^2\Gamma(a, bm)\right] \qquad (2.35)$$

$$+ \mathbb{E}[R(d, m)]\left[-\mathbb{E}[R(d, m)] - 2d\right],$$

$$Cov[I(d, m), R(d, m)] = \left(\mathbb{E}[R(d, m)] - \frac{1}{\Gamma(a)}\left[b\Gamma(a + 1, bm) - d\Gamma(a, bm)\right]\right)^2 \qquad (2.36)$$

$$+ \frac{m - d}{\Gamma(a)}\left[b\Gamma(a + 1, bd) - d\Gamma(a, bd)\right] + \mathbb{E}[R(d, m)](d - ab),$$

respectively.

2.3. Aggregate claims with Inverse Gamma distribution

When inverse Gamma with $a, b > 0$, $S \sim IG(a, b)$, is taken as the aggregate claim distribution, the average costs, variances and the correlation for insurer and reinsurer under two cases can be derived.

Proposition 2.7. *If $S \sim IG(a, b)$, the expected values, the variances of insurer and reinsurer and their covariance for Case I are expressed as*

$$\mathbb{E}[I(d)] = \frac{b\left[\Gamma(a, b/d) - (a - 1)\gamma(a - 1, b/d)\right] + \gamma(a, b/d)\left[1 + (a - 1)d\right]}{(a - 1)\Gamma(a)} \qquad (2.37)$$

$$\mathbb{E}[R(d)] = \frac{1}{\Gamma(a)}\left[b\gamma(a - 1, b/d) - d\gamma(a, b/d)\right], \qquad (2.38)$$

$$\mathbb{V}[I(d)] = \frac{b^2}{(a - 1)^2(a - 2)} + \left(\mathbb{E}[R(d)]\right)^2 \qquad (2.39)$$

$$+ \frac{b^2\gamma(a - 2, b/d) + 2(bd - 1)\gamma(a - 1, b/d) + (2 - d^2)\gamma(a, b/d)}{\Gamma(a)}$$

$$- 2\left[\frac{1}{\Gamma(a)}[b\gamma(a - 1, b/d) - d\gamma(a, b/d)]\right]^2,$$

$$\mathbb{V}[R(d)] = \frac{1}{\Gamma(a)}\Big[b^2\gamma(a-2,b/d) - 2db\gamma(a-1,b/d) + d^2\gamma(a,b/d)\Big] \tag{2.40}$$
$$- \Big(\mathbb{E}[R(d)]\Big)^2,$$

$$Cov[I(d),R(d)] = \left[\frac{1}{\Gamma(a)}\Big[b\gamma(a-1,b/d) - d\gamma(a,b/d)\Big]\right]^2 \tag{2.41}$$
$$+ \frac{1}{\Gamma(a)}\left[\Gamma(a,b/d) - \Gamma(a-1,b/d) + (b-d)\left(d - \frac{b}{a-1}\right)\right],$$

respectively.

Proposition 2.8. *If $S \sim IG(a,b)$, the expected values, the variances of insurer and reinsurer and their covariance for Case II are expressed as*

$$\mathbb{E}[I(d,m)] = \frac{b}{a-1} - \frac{b^a}{\Gamma(a)}\Big[d^{1-a}e^{-b/d} - m^{1-a}e^{-m/d}\Big] \tag{2.42}$$
$$- \frac{1}{\Gamma(a)}\Big[(b-d)\gamma(a-1,b/d) - (b-m)\gamma(a-1,b/m)\Big],$$

$$\mathbb{E}[R(d,m)] = \frac{1}{\Gamma(a)}\Big[(b-d)\gamma(a-1,b/d) - (b-m)\gamma(a-1,b/m)\Big] \tag{2.43}$$
$$+ \frac{b^a}{\Gamma(a)}\Big[d^{1-a}e^{-b/d} - m^{1-a}e^{-m/d}\Big],$$

$$\mathbb{V}[I(d,m)] = \frac{b^2}{(a-1)^2(a-2)} - m^2 + d^2 + \mathbb{E}[R(d,m)]\Big[\mathbb{E}[R(d,m)] - \frac{2b}{a-1}\Big] \tag{2.44}$$
$$- \frac{2(m-d)}{\Gamma(a)}\left[\frac{1}{\Gamma(a)}[b\Gamma(a+1,bm) - m\Gamma(a,bm)]\right]$$
$$+ \sum_{k=0}^{\infty}\frac{(-1)^k(m^{2-a-k} - d^{2-a-k})}{k!(a+k)(2-a-k)},$$

$$\mathbb{V}[R(d,m)] = m^2 - d^2 - \sum_{k=0}^{\infty}\frac{(-1)^k(m^{2-a-k} - d^{2-a-k})}{k!(a+k)(2-a-k)} \tag{2.45}$$
$$+ \mathbb{E}[R(d,m)]\Big[-\mathbb{E}[R(d,m)] - 2d\Big],$$

$$Cov[I(d,m),R(d,m)] = \left(\mathbb{E}[R(d,m)] - \frac{1}{\Gamma(a)}\Big[b\Gamma(a+1,bm) - m\Gamma(a,bm)\Big]\right)^2 \tag{2.46}$$
$$= \frac{m-d}{\Gamma(a)}\left[\frac{1}{\Gamma(a)}[b\Gamma(a+1,bm) - m\Gamma(a,bm)]\right]$$
$$+ \mathbb{E}[R(d,m)]\left(d - \frac{b}{a-1}\right),$$

respectively.

These analytical derivations are used to find the optimal retention and cap values combined with the exposure curves to quantify the optimal premium.

A summary of these derivations is presented in Appendix A (Tables A1 and A2), as well as the sketches of some propositions in Appendix B.

3. Premium shares using exposure curves

Exposure rating is related to risk profiles for the case of per risk based on historical experience. Similar sized risks with the same risk category are summarized within a risk band. The risks belonging to one specific band are considered homogeneous for the rating;

thus, they can be modeled with the help of one single loss distribution function. The exposure rating, which basically aims to achieve an agreeable division of the premium of certain risk between the insurer and reinsurer using the risk band, is performed in two steps. Firstly, the risk premiums are estimated as a proportion of the gross premiums by applying a suitable loss ratio. Secondly, the risk premiums are considered as it is determined for the retention and transferred to a reinsurer which requires an underlying loss distribution function. Moreover, the accurate loss distribution function is derived from large portfolios of similar risks. The functions of these distributions are called exposure curves $(G(.))$ which give a ratio of a risk premium as the function of a prescribed deductible. Based on the selected statistical distribution assumptions on aggregate loss, we derive analytically the forms of exposure curves under Case I and II.

Even though the aggregate claims range over in $\mathbb{R}^+$, it is more realistic to adjust the aggregate claim on the maximum possible loss, M, defining a new random variable X, as $X = \frac{S}{M}$, whose distribution function is denoted as $F_X(.)$.

The proportion of expected loss of insurer to the aggregate expected loss, $G(k)$ at $k = \frac{d}{M}$, is given as, [3],

$$G(k) = \frac{\int_0^k [1 - F(x)]dx}{\int_0^\infty [1 - F(x)]dx},\tag{3.1}$$

which is re-expressed for Case I as

$$G(k) = 1 - \frac{\int_k^\infty [1 - F(x)]dx}{\mathbb{E}[X]}.\tag{3.2}$$

We also derive and introduce the exposure curves for Case II which is also novel in this study, given as below.

Proposition 3.1. *The exposure curve having the constraints on limits, $k = \frac{d}{M}$ and $l = \frac{m}{M}$, $G(k, l)$, is found as*

$$G(k, l) = 1 - \frac{\int_k^\infty [1 - F(x)]dx - \int_l^\infty [1 - F(x)]dx}{\int_0^\infty [1 - F(x)]dx}.\tag{3.3}$$

Henceforth, by means of exposure curves, the risk premium $(P(.))$ partition between insurer $(PI(.))$ and reinsurer $(PR(.))$ for two cases can be determined as

$$PI(.) = G(.)P, \quad PR(.) = (1 - G(.))P,\tag{3.4}$$

where $G(.)$ corresponds to $G(k)$ and $G(k, l)$ for Case I and Case II, respectively. Moreover, the loss ratios under this scheme become

$$PIr(.) = \frac{\mathbb{E}[I(.)]}{PI(.)},\tag{3.5a}$$

$$PRr(.) = \frac{\mathbb{E}[R(.)]}{PR(.)},\tag{3.5b}$$

respectively.

The modification of upper limit on aggregate loss in accordance with maximum probable loss, the distribution of $X = \frac{S}{M}, M > 0$, under the assumption of Pareto distribution is $X \sim Pareto(a, b/M)$, whereas if $S \sim Ga(a, b)$ having $F_S(s) = \frac{\gamma(a, bs)}{\Gamma(a)}$, then with $F_X(x) = \frac{\gamma(a, \frac{b}{M}x)}{\Gamma(a)}$ yielding $X \sim Ga(a, b/M)$. Similarly, $S \sim IG(a, b)$ having $F_S(s) = \frac{\Gamma(a, b/s)}{\Gamma(a)}$, and $F_X(x) = \frac{\Gamma(a, \frac{b/M}{x})}{\Gamma(a)}$ follows $X \sim IG(a, b/M)$.

Proposition 3.2. *Assume X follows Pareto(a, b/M), Ga(a, b/M) and IG(a, b/M) distributions with their corresponding parameters and M. In Case I, the exposure curves for these distributions ($G_P(k)$, $G_G(k)$, $G_{IG}(k)$, respectively) are found as*

$$G_P(k) = \frac{ak^{a-1} - (bM)^{a-1}}{ak^{a-1}}, \tag{3.6}$$

$$G_G(k) = Ga(k; a+1, \frac{b}{M}) - \frac{kM}{ab}(1 - Ga(k; a, \frac{b}{M})), \tag{3.7}$$

$$G_{IG}(k) = \frac{(a-1)\Gamma(a-1, \frac{b}{Mk})}{\Gamma(a)} - \frac{kM(a-1)}{b}\frac{[\Gamma(a, \frac{b}{Mk}) - \Gamma(a)]}{\Gamma(a)}. \tag{3.8}$$

Proposition 3.3. *X having Pareto(a, b/M), Ga(a, b/M) and IG(a, b/M) distributions, under Case II the exposure curves are expressed as*

$$G_P(k, l) = \frac{aM^{a-1} - b^{a-1}(k^{1-a} - l^{1-a})}{aM^{a-1}}, \tag{3.9}$$

$$G_G(k, l) = 1 + Ga(k; a+1, \frac{b}{M}) - Ga(l; a+1, \frac{b}{M}) + \frac{(k-l)M}{ab} \tag{3.10}$$
$$- kGa(k; a, \frac{b}{M}) + lGa(l; a, \frac{b}{M}),$$

$$G_{IG}(k, l) = 1 - \frac{(a-1)\left[\Gamma(a-1, \frac{b}{Ml}) - \Gamma(a-1, \frac{b}{Mk})\right]}{\Gamma(a)} \tag{3.11}$$
$$+ \frac{M(a-1)}{b}\frac{[k\Gamma(a, \frac{b}{Mk}) - l\Gamma(a, \frac{b}{Ml})]}{\Gamma(a)} + \frac{(k-l)M(a-1)}{b}.$$

4. The optimization of d and m

As noticed, the derivations are the functions of d and m. Therefore, we need to find the values of the retention d and the maximum m, which maximize the dependence between the insurer and the reinsurer with the minimum cost to assure the plausible business. On the other hand, loss ratios of both parties are also affected by the values of d and m. For this reason, our approach to finding the optimal values maximizing the correlation between the expected costs attaining the loss ratios of both to be as close as to each other. In other words, the optimal d and m find the reasonable risk relation between parties for a reasonable debate on the loss and the premium share.

The optimization of determining values under Case I and Case II are expressed as

$$\text{Case I:} \quad \max_d \quad \rho(I(d), R(d))$$
$$\text{s.t.} \quad PI_r(d) - PR_r(d) = 0 \tag{4.1}$$
$$d > 0,$$

$$\text{Case II:} \quad \max_{d,m} \quad \rho(I(d, m), R(d, m))$$
$$\text{s.t.} \quad PI_r(d, m) - PR_r(d, m) = 0 \tag{4.2}$$
$$d > 0, \quad m > 0, \quad m > d.$$

Moreover, plugging into the expressions of

$$PI_r(d) = \frac{\mathbb{E}[I(d)]}{PG(k)}$$

and

$$PR_r(d) = \frac{\mathbb{E}[R(d)]}{P(1 - G(k))},$$

the constraint in Equation (4.1) becomes

$$\mathbb{E}[R(d)]G(k) = \frac{\mathbb{E}[S]}{2}.$$

Similarly, the constraint in Equation (4.2), $PI_r(d, m) - PR_r(d, m) = 0$, simplifies to

$$\mathbb{E}[R(d, m)]G(k, l) = \frac{\mathbb{E}[S]}{2},$$

where $k = \frac{d}{M}$, $l = \frac{m}{M}$ and M is the maximum possible loss.

Based on these representations, the optimization problems are re-written as

$$\text{Case I:} \quad \max_d \quad \rho(I(d), R(d))$$

$$\text{s.t.} \quad \mathbb{E}[R(d)]G(k) - \frac{\mathbb{E}[S]}{2} = 0 \tag{4.3}$$

$$d > 0,$$

$$\text{Case II:} \quad \max_{d,m} \quad \rho(I(d, m), R(d, m))$$

$$\text{s.t.} \quad \mathbb{E}[R(d, m)]G(k, l) - \frac{\mathbb{E}[S]}{2} = 0 \tag{4.4}$$

$$d > 0, \quad m > 0, \quad m > d.$$

The analytical solutions to the optimization problems are not easily tractable and do not have closed forms; therefore, they require numerical methods. By searching the behavior of the correlation and the premium share of the parties, we propose two algorithms, Algorithm (1) and Algorithm (2), to obtain optimal d under Case I and optimal d and m under Case II, respectively.

The values mM and M in these algorithms are considered as the interval of the possible retention and maximum levels. The loss amounts are generated by the proposed loss distribution. The optimal values are found by using the dynamic search of a maximum correlation of the parties satisfied by possible retentions and maximums in the interval $[mM, M]$. Moreover, the constraints given in Equations (4.3) and (4.4) are also checked by the possible retention and maximum levels. The small Δ choice increases the robustness of the optimization problem.

Algorithm 1 Optimal $d^\star$ under Case I

Consider the partition of the interval $[m \times M, M]$ as
$0 = d_0 < d_1 < \ldots < d_i < d_{i+1} < \ldots < d_n = T$ such that n many equidistant subintervals are obtained where $m \times M$ and M are minimium and maximum possible losses, respectively.
Let $\Delta = d_{i+1} - d_i$.
Let *tol* be a chosen small tolerance number.
for $i \leftarrow n$ **by** Δ **do**
    while $\mathbb{E}[R(d_i)]G(\frac{d_i}{M}) - \frac{\mathbb{E}[S]}{2} < tol$ **do**
        Calculate $\rho(d_i) = \rho(I(d_i), R(d_i))$
    end
end
Choose the maximum correlation $\rho(I(d_i), R(d_i))$ which is satisfied by $d_i = d^\star$.

Algorithm 2 Optimal $d^\star$ and $m^\star$ under Case II

Consider the partitions of the interval $[m \times M, M]$ as
$0 = d_0 < d_1 < \ldots < di < d_{i+1} < \ldots < d_n = T$ and
$0 = m_0 < m_1 < \ldots < m_j < m_{j+1} < \ldots < m_n = T$ such that n many equidistant
subintervals are obtained where $m \times M$ and M are minimium and maximum possible
losses, respectively.
Let $\Delta = d_{i+1} - d_i = m_{j+1} - m_j$.
Let *tol* be a chosen small tolerance number.
for $i \leftarrow n$ **by** Δ **do**
    for $j \leftarrow n$ **by** Δ **do**
        while $\mathbb{E}[R(d_i, m_j)]G(\frac{d_i}{M}, \frac{m_j}{M}) - \frac{\mathbb{E}[S]}{2} < tol$ & $m_j > d_i$ **do**
            Calculate $\rho(d_i, m_j) = \rho(I(d_i, m_j), R(d_i, m_j))$
        end
    end
end
Choose the maximum correlation $\rho(I(d_i, m_j), R(d_i, m_j))$ which is satisfied by $d_i = d^\star$ and
$m_j = m^\star$.

---

5. Numerical illustrations

To demonstrate the utilization of the proposed methodology, we construct exposure
curves based on the simulated aggregate losses from Pareto, Gamma and Inverse gamma
distributions. The parameters in the distributions are selected in such a way that the
expected value and variance yield the same values. The reason for this is to attain a
consistent base to make proper comparisons among loss distributions and their exposure
curves. Therefore, we select the parameters in such a way that each distribution yields
$\mathbb{E}[S] = 2$ and $\mathbb{V}[S] = 1$ resulting in $\mathrm{Pareto}(3.2361, 1.3820), \mathrm{Ga}(4, 0.5), \mathrm{IG}(6, 10)$.

The optimal retention and maximum levels are determined according to the condition
that their values give the highest correlation between the costs of insurer and reinsurer,
which influence the values of the mean, variance, premium partition, and the loss ratios
according to their expected losses simultaneously under the different distribution assumptions whose outcomes are summarized in Table 1 and Table 2 for Case I and Case II,
respectively.

Table 1. Optimal retention values in Case I.

	Pareto	**Gamma**	**InvGamma**
Max Correlation	0.3629	**0.4740**	0.4202
Optimal Retention ($d^\star$)	**2.2171**	**2.0996**	**2.1605**
Premium (P)	4.0416	**4.1859**	4.0626
$PI(d^\star)$	3.6076	3.4545	3.4641
$PR(d^\star)$	0.4340	0.7314	0.5985
$PI_r(d^\star)$	0.4948	**0.4748**	0.4923
$PR_r(d^\star)$	0.4948	**0.4748**	0.4923

For **Case I** simulations, it can be observed in Table 1 that the highest correlation
is obtained under Gamma distribution even though optimal retention levels are close
and vary around 2. The maximum premium on the insurer's side is achieved in Pareto,
whereas, for reinsurer in Gamma distribution. It is interesting to see also that even
the highest premiums are achieved under two different distributions; in terms of loss
ratios, the insurer and reinsurer attain the same values having their minimums in Gamma

distribution, concluding that the expected loss is smaller for both the insurer and the reinsurer. The tail behavior of each distribution is observable since our simulation includes the interval of minimum and maximum losses generated by proposed distributions. The Gamma distribution has a narrow interval compared to others (Figure 2(a)).

Table 2. Optimal retention and maximum values in Case II.

	Pareto	**Gamma**	**InvGamma**
Max Correlation	**0.4412**	0.4346	0.4361
Optimal Retention $(d^\star)$	**3.5164**	**2.9214**	**3.1017**
Optimal Maximum $(m^\star)$	**9.1544**	**5.0814**	**6.0517**
Premium (P)	4.0810	**4.1149**	4.0916
$PI_r(d^\star, m^\star)$	0.4901	**0.4860**	0.4889
$PR_r(d^\star, m^\star)$	0.4901	**0.4860**	0.4889

Simulations in **Case II** creates a surface that is not tractable easily unless we fix the retention level to the optimal one and search for the maximal level to illustrate the impact of change in m according to the correlation coefficient and loss ratios. Table 2 shows that the maximum correlation is also achieved under the Pareto distribution assumption as in Case I. Optimal retention level, which is around 2.2, yields the maximum m level under Pareto distribution. On the other hand, the maximum premium and premium ratios for both parties are achieved under Gamma distributions. The behavior of the correlation with respect to m, (Figure 2(b)), is expected to become horizontal for maximum m values, as the charge for the reinsurer increases with respect to the increase in the maximum level. This means that the dependence between insurer and reinsurer is not affected after sufficiently high cap levels.
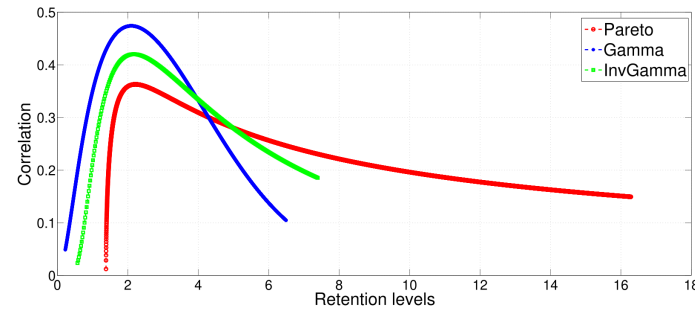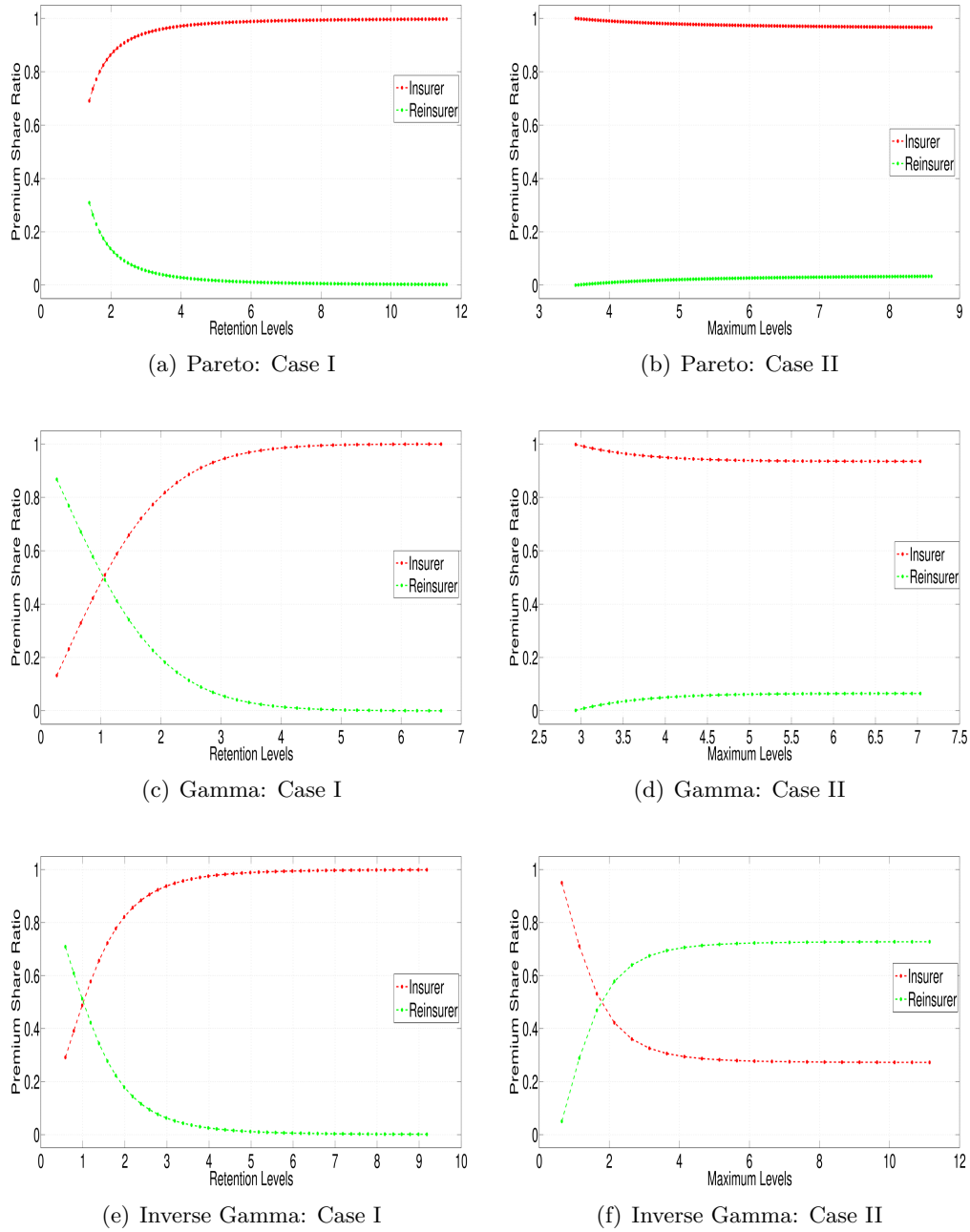

(a) Case I (only d)
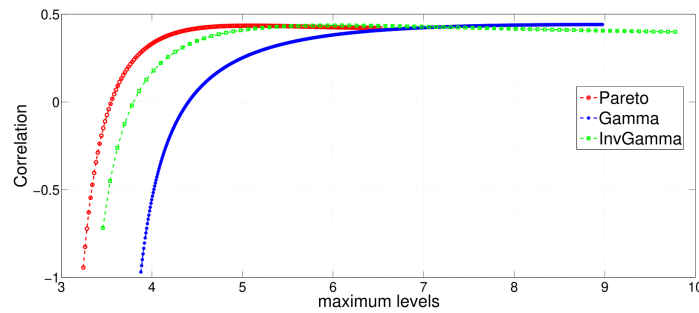
(b) Case II (d and m)

Figure 2. The correlation coefficient for the selected distributions.

The behavior of the premium share ratio between the insurer and the reinsurer using derived exposure curve functions for Case I is shown in Figures 3(a), 3(c), and 3(e). An increase in the retention level raises the burden on the insurer in favor of a higher premium share. However, it reduces the cost of the reinsurer and results in an unfair premium share for the reinsurer. Figures 3(b), 3(d), and 3(f) demonstrate the premium share ratio with respect to maximum levels for Case II using the exposure curves. While the maximum level is increasing, the costs of reinsurer become larger so that reinsurer's premium share ratio increases as well. On the other hand, this causes a decrease in the cost of the insurer so that the fair premium gained by the insurer diminishes.

(a) Pareto: Case I

(b) Pareto: Case II

(c) Gamma: Case I

(d) Gamma: Case II

(e) Inverse Gamma: Case I

(f) Inverse Gamma: Case II

Figure 3. Expsoure curves of the insurer and the reinsurer for Case I and Case II.

The capital and reserve requirements, which are mostly in relation to risk measures such as VaR and CVaR, are taken into account to quantify the maximum risk of each party. In Table 3, we demonstrate the VaR and CVaR values for certain α values. Based on the $d^\star$, we show that the difference between VaR and CVaR values of insurer and reinsurer decreases when α increases for Gamma distribution. Moreover, the highest correlation between parties in Case I is obtained under Gamma, which is another indicator that the optimal solution, $d^\star$, which maximizes the correlation between parties and finds the equilibrium between the loss ratios of each party, is confirmed when the risk measures VaR and CVaR for insurer and reinsurer are considered.

Table 3. VaR and CVaR for the insurer and the reinsurer under Case I for optimal d.

α	Risk measure	**Pareto**	**Gamma**	**InvGamma**
0.90	VaR_I	2.2171	2.0996	2.1605
	CVaR_I	2.2171	2.0996	2.1605
	VaR_R	0.5990	1.2410	1.0130
	CVaR_R	5.8055	2.9455	6.5825
0.95	VaR_I	2.2171	2.0996	2.1605
	CVaR_I	2.2171	2.0996	2.1605
	VaR_R	1.2720	1.7780	1.6680
	CVaR_R	6.1420	3.2140	6.9100
0.99	VaR_I	2.2171	2.0996	2.1605
	CVaR_I	2.2171	2.0996	2.1605
	VaR_R	3.5350	2.9290	3.4530
	CVaR_R	7.2735	3.7895	7.8025

The VaR and CVaR values with selected α and distributions (Table 4) are calculated based on the simulated aggregate losses. In comparison to our findings in Case II, when we select the maximum level (m) as the VaR and CVaR values, for insurer and reinsurer, the mean values ($\mathbb{E}[I_v]$, $\mathbb{E}[I_{cv}]$, $\mathbb{E}[R_v]$, $\mathbb{E}[R_{cv}]$) and the change with respect to their variances ($\frac{\mathbb{V}[I]}{\mathbb{V}[I_v]}$, $\frac{\mathbb{V}[R]}{\mathbb{V}[R_v]}$, $\frac{\mathbb{V}[I]}{\mathbb{V}[I_{cv}]}$, $\frac{\mathbb{V}[R]}{\mathbb{V}[R_{cv}]}$) are calculated in Table 5. For instance, under Pareto and $\alpha = 0.975$, for insurer and reinsurer sequentially, the expected costs ($\mathbb{E}[I]$, $\mathbb{E}[R]$) are 1.9325 and 0.0675; whereas the same quantities in case of VaR ($\mathbb{E}[I_v]$,$\mathbb{E}[R_v]$) become 1.9718 and 0.0283 and for CVaR are ($\mathbb{E}[I_{cv}]$, $\mathbb{E}[R_{cv}]$) 1.9447 and 0.0555 (Table 5). These values are obtained when $m^\star$ is taken as its corresponding optimal value given in Table 2. We observe that the closest approximation is obtained in Gamma, which generates the lowest loss ratios obtained from our optimization algorithms and then is more suitable when VaR and CVaR risk measures are taken into account. Furthermore, the variance change is observed to be the lowest in Gamma distributions with each α level in Table 5.

Table 4. VaR and CVaR based on the selected loss ditributions.

α	Risk measure	**Pareto**	**Gamma**	**InvGamma**
0.975	VaR	4.3208	4.3860	4.5450
	CVaR	6.2540	5.0695	5.8206
0.99	VaR	5.7350	5.0290	5.6130
	CVaR	8.2972	5.6892	7.0823
0.995	VaR	7.1048	5.6879	6.5340
	CVaR	10.2891	6.1462	8.1793

Table 5. The expected costs and relative variances if $m^\star$ is taken as VaR and CVaR.

	Pareto			Gamma			InvGamma		
α	0.975	0.99	0.995	0.975	0.99	0.995	0.975	0.99	0.995
$\mathbb{E}[I]$	1.9325	1.9325	1.9325	1.8771	1.8771	1.8771	1.8990	1.8990	1.8990
$\mathbb{E}[I_v]$	1.9718	1.9489	1.9394	1.8881	1.8881	1.8772	1.8990	1.8990	1.8990
$\mathbb{E}[I_{cv}]$	1.9447	1.9347	1.9304	1.8771	1.8734	1.8720	1.9008	1.8940	1.8914
$\mathbb{E}[R]$	0.0675	0.0675	0.0675	0.1230	0.1230	0.1230	0.1010	0.1010	0.1010
$\mathbb{E}[R_v]$	0.0283	0.0509	0.0607	0.1120	0.1120	0.1229	0.0801	0.0974	0.1038
$\mathbb{E}[R_{cv}]$	0.0555	0.0654	0.0696	0.1228	0.1265	0.1277	0.0992	0.1060	0.1087
$\dfrac{\mathbb{V}[I]}{\mathbb{V}[I_v]}$	0.5948	0.7516	0.8748	1.0983	1.0983	1.0181	0.8319	0.9647	1.0301
$\dfrac{\mathbb{V}[I]}{\mathbb{V}[I_{cv}]}$	0.7672	0.9408	1.0372	0.9994	1.0224	1.0274	0.9819	1.0547	1.0893
$\dfrac{\mathbb{V}[R]}{\mathbb{V}[R_v]}$	10.1959	2.4040	1.4443	1.3478	1.3478	0.9166	2.0096	1.1293	0.9046
$\dfrac{\mathbb{V}[R]}{\mathbb{V}[R_{cv}]}$	1.8431	1.1537	0.8700	1.0033	0.8909	0.8549	1.0639	0.8297	0.7496

6. Concluding remarks

This paper aims to determine the optimal premium share between the insurer and the reinsurer with the use of exposure curves under certain aggregate loss distributions. The optimal premium is achieved in terms of the level of dependence- correlation coefficient between the costs of insurer and reinsurer through an optimization scheme. The analytical derivations of the exposure curves under Pareto, Gamma, and Inverse Gamma distributions under the standard deviation premium principle are counted as the main contributions to literature, which will enable researchers to understand the pricing behavior. The Monte Carlo simulations show that the proposed approach in this study also determines under which of the distributions the maximum correlation, the highest premium, and the smaller expected loss are observed with respect to the conditions on the reinsurance contract in terms of retention level and/or maximum level. Moreover, the optimized values based on the proposed approach are compared to the values that minimize parties' total risks with respect to the risk measures VaR and CVaR. The outcomes show that the optimized solution, which maximizes the correlation between parties and equates their loss ratios, is close enough to the values obtained by VaR and CVaR measures with high α levels for all selected distributions under Case I and Case II; however, Gamma distribution is more suitable when compared the others, which is the situation that we obtain by our proposed methodology.

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Appendix A. Summary table for derived statistics

Table A1. The derivations of the statistics under Case I.

	Pareto	Gamma	Inverse Gamma
$\mathbb{E}[R(d)]$	$\dfrac{b^a d^{1-a}}{a-1}$	$\dfrac{1}{\Gamma(a)}[b\Gamma(a+1,bd) - d\Gamma(a,bd)]$	$\dfrac{1}{\Gamma(a)}[b\gamma(a-1,b/d) - d\gamma(a,b/d)]$
$\mathbb{V}[R(d)]$	$\dfrac{b^a d^{(2-a)}[2 - (b/d)^a(a-2)]}{(a-2)(a-1)^2}$	$\mathbb{E}[R(d)](-2d - \mathbb{E}[R(d)])$ $+ \dfrac{1}{\Gamma(a)}[b^2\Gamma(a+1,bd) - d^2\Gamma(a,bd)]$	$\dfrac{1}{\Gamma(a)}[b^2\gamma(a-2,b/d) - 2db\gamma(a-1,b/d) + d^2\gamma(a,b/d)]$ $- (\mathbb{E}[R(d)])^2$
$Cov[I(d), R(d)]$	$\dfrac{ab^a d^{1-a}[d-b] + b^a d^{(2-a)}[(b/d)^a - 1]}{(a-1)^2}$	$\left(\dfrac{1}{\Gamma(a)}[b\Gamma(a+1,bd) - d\Gamma(a,bd)]\right)^2$ $- \dfrac{d\Gamma(a,bd)}{\Gamma(a)}[(d-ab)(b^{a+1}d^{a-1} - 1)]$	$\left[\dfrac{1}{\Gamma(a)}[b\gamma(a-1,b/d) - d\gamma(a,b/d)]\right]^2$ $+ \dfrac{1}{\Gamma(a)}\left[\Gamma(a,b/d) - \Gamma(a-1,b/d) + (b-d)\left(d - \dfrac{b}{a-1}\right)\right]$

Table A2. The derivations of the statistics under Case II.

	Pareto	Gamma	Inverse Gamma
$\mathbb{E}[R(d,m)]$	$\dfrac{b^a}{a-1}[d^{1-a} - m^{1-a}]$	$\dfrac{b^2-1}{\Gamma(a)}[d\Gamma(a,bd) - m\Gamma(a,bm)]$ $- \dfrac{b^a}{\Gamma(a)}[d^a e^{-bd} - m^a e^{-bm}]$	$\dfrac{1}{\Gamma(a)}[(b-d)\gamma(a-1,b/d) - (b-m)\gamma(a-1,b/m)]$ $+ \dfrac{b^a}{\Gamma(a)}[d^{1-a}e^{-b/d} - m^{1-a}e^{-m/d}]$
$\mathbb{V}[R(d,m)]$	$\dfrac{b^{2a}}{(a-1)^2}[(m^{1-a} - d^{1-a})^2 - 2(m^{2(1-a)} + d^{1-a})]$ $+ \dfrac{2b^a}{(a-1)^2(a-2)}\left[d^{2-a} - m^{2-a} + (a-1)(a-2)m^{1-a}(d-m)\right]$	$\dfrac{1}{\Gamma(a)}[b^2(\Gamma(a+2,bd) - \Gamma(a+2,bm)) - d^2\Gamma(a,bd) - m^2\Gamma(a,bm)]$ $+ \mathbb{E}[R(d,m)][-\mathbb{E}[R(d,m)] - 2d]$	$m^2 - d^2 - \dfrac{1}{\Gamma(a)}\sum_{k=0}^{\infty} \dfrac{(-1)^k(m^{2-a-k} - d^{2-a-k})}{k!(a+k)(2-a-k)}$ $+ \mathbb{E}[R(d,m)][-\mathbb{E}[R(d,m)] - 2d]$
$Cov[I(d,m), R(d,m)]$	$\dfrac{(2d-m)b^a m^{1-a}}{a-1}$ $+ \dfrac{b^a d^{1-a}}{(a-1)^2}[a(d-b) + d^{1-a}(b^a - d^a)]$	$(\mathbb{E}[R(d,m)] - \dfrac{1}{\Gamma(a)}[b\Gamma(a+1,bm) - d\Gamma(a,bm)])^2$ $+ \dfrac{m-d}{\Gamma(a)}[b\Gamma(a+1,bd) - d\Gamma(a,bd)] + \mathbb{E}[R(d,m)](d-ab)$	$(\mathbb{E}[R(d,m)] - \dfrac{1}{\Gamma(a)}[b\Gamma(a+1,bm) - m\Gamma(a,bm)])^2$ $= \dfrac{m-d}{\Gamma(a)}\left[\dfrac{1}{\Gamma(a)}[b\Gamma(a+1,bm) - m\Gamma(a,bm)]\right]$ $+ \mathbb{E}[R(d,m)]\left(d - \dfrac{b}{a-1}\right)$

Appendix B. Proofs for some propositions

i) The proofs related to equations given in *Proposition (2.5)* and *Proposition (2.6)*:

$$
\begin{aligned}
\mathbb{E}[R(d)] &= \int_d^\infty (s-d)\frac{s^{a-1}e^{-\frac{s}{b}}}{b^a\Gamma(a)}ds \\
&= \int_d^\infty s\frac{s^{a-1}e^{-\frac{s}{b}}}{b^a\Gamma(a)}ds - d\int_d^\infty \frac{s^{a-1}e^{-\frac{s}{b}}}{b^a\Gamma(a)}ds \\
&= ab\int_d^\infty \frac{s^{a}e^{-\frac{s}{b}}}{b^{a+1}\Gamma(a+1)}ds - d\int_d^\infty \frac{s^{a-1}e^{-\frac{s}{b}}}{b^a\Gamma(a)}ds \\
&= ab(1-Ga(d;a+1,b)) - d(1-Ga(d;a,b)) \\
&= ab\frac{\Gamma(a+1,d)}{\Gamma(a+1)} - d\frac{\Gamma(a,d)}{\Gamma(a)}.
\end{aligned}
\tag{B.1}
$$

$\mathbb{E}[R(d,m)]$ is obtained by putting Equation B.1 into

$$
\mathbb{E}[R(d,m)] = \mathbb{E}[R(d)] - \mathbb{E}[R(m)].
$$

$$
\begin{aligned}
\mathbb{V}[R(d)] &= \int_d^\infty (s-d)^2\frac{s^{a-1}e^{-\frac{s}{b}}}{b^a\Gamma(a)}ds - (\mathbb{E}[R(d)])^2 \\
&= \mathbb{E}[R(d)](-2d - \mathbb{E}[R(d)]) + (a+1)ab^2(1-Ga(d;a+2,b)) \\
&\quad - d^2(1-Ga(d;a,b)) \\
&= \mathbb{E}[R(d)](-2d - \mathbb{E}[R(d)]) \\
&\quad + \frac{1}{\Gamma(a)}[b^2\Gamma(a+1,bd) - d^2\Gamma(a,bd)],
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\mathbb{V}[R(d,m)] &= \mathbb{V}[R(d)] - \mathbb{V}[R(m)] + 2\mathbb{E}[R(m)](\mathbb{E}[R(d)] - \mathbb{E}[R(m)] + d - m) \\
&= 2\int_d^m s[1-F_S(s)]ds + \mathbb{E}[R(d,m)][-\mathbb{E}[R(d,m)] - 2d] \\
&= \frac{1}{\Gamma(a)}[b^2(\Gamma(a+2,bd) - \Gamma(a+2,bm)) - d^2\Gamma(a,bd) - m^2\Gamma(a,bm)] \\
&\quad + \mathbb{E}[R(d,m)][-\mathbb{E}[R(d,m)] - 2d].
\end{aligned}
\tag{B.3}
$$

The proof is obtained with a revision on

$$
Cov[I(d,m),R(d,m)] = Cov[I(d),R(d)] - (2d-m)\mathbb{E}[R(m)]
$$

such that

$$
\begin{aligned}
Cov[I(d,m),R(d,m)] &= \mathbb{E}[R(d)](d - \mathbb{E}[S] + \mathbb{E}[R(d)]) - (2d-m)\mathbb{E}[R(m)] \\
&= (\mathbb{E}[R(d,m)] - \frac{1}{\Gamma(a)}[b\Gamma(a+1,bm) - d\Gamma(a,bm)])^2 \\
&\quad + \frac{m-d}{\Gamma(a)}[b\Gamma(a+1,bd) - d\Gamma(a,bd)] + \mathbb{E}[R(d,m)(d-ab).
\end{aligned}
\tag{B.4}
$$

ii) The proofs related to equations given in *Proposition (2.7)* and *Proposition (2.8)*:

$$\mathbb{E}[R(d)] = \int_d^\infty (s-d) \frac{b^a}{\Gamma(a)} s^{-a-1} e^{-b/s} ds$$

$$= \frac{b^a}{\Gamma(a)} \int_d^\infty s^{-a} e^{-b/s} ds - \frac{b^a d}{\Gamma(a)} \int_d^\infty s^{-a-1} e^{-b/s} ds \qquad \text{(B.5)}$$

$$= \frac{1}{\Gamma(a)} [b\gamma(a-1, b/d) - d\gamma(a, b/d)].$$

$\mathbb{E}[R(d,m)]$ is obtained by putting Equation B.5 into

$$\mathbb{E}[R(d,m)] = \mathbb{E}[R(d)] - \mathbb{E}[R(m)].$$

Finding the second moment of the cost of reinsurer with respect to d is enough to estimate the variance. Thus,

$$\mathbb{E}[R(d)^2] = \int_d^\infty (s-d)^2 \frac{b^a}{\gamma(a)} s^{-a-1} e^{-b/s} ds \qquad \text{(B.6)}$$

$$= \frac{b^a}{\Gamma(a)} \left[\int_d^\infty s^{1-a} e^{-b/s} ds - 2d \int_d^\infty s^{-a} e^{-b/s} ds + d^2 \int_d^\infty s^{-1-a} e^{-b/s} ds \right]$$

$$= \frac{1}{\Gamma(a)} [b^2 \gamma(a-2, b/d) - 2db\gamma(a-1, b/d) + d^2 \gamma(a, b/d)].$$

$$\mathbb{V}[R(d,m)] = \mathbb{V}[R(d)] - \mathbb{V}[R(m)] + 2\mathbb{E}[R(m)](\mathbb{E}[R(d)] - \mathbb{E}[R(m)] + d - m) \qquad \text{(B.7)}$$

$$= 2 \int_d^m s[1 - F_S(s)] ds + \mathbb{E}[R(d,m)][-\mathbb{E}[R(d,m)] - 2d]$$

$$= m^2 - d^2 - \sum_{k=0}^\infty \frac{(-1)^k (m^{2-a-k} - d^{2-a-k})}{k!(a+k)(2-a-k)}$$

$$+ \mathbb{E}[R(d,m)][-\mathbb{E}[R(d,m)] - 2d].$$