UNBOUNDED p-CONVERGENCE IN LATTICE-NORMED VECTOR LATTICES

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ABSTRACT. A net x_α in a lattice-normed vector lattice (X, p, E) is unbounded p-convergent to $x \in X$ if $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$ for every $u \in X_+$. This convergence has been investigated recently for $(X, p, E) = (X, |\cdot|, X)$ under the name of uo-convergence, for $(X, p, E) = (X, \|\cdot\|, \mathbb{R})$ under the name of un-convergence, and also for $(X, p, \mathbb{R}^{X^*})$, where $p(x)[f] := |f|(|x|)$, under the name uaw-convergence. In this paper we study general properties of the unbounded p-convergence.

1. Introduction and preliminaries

Lattice-valued norms on vector lattices provide natural and efficient tools in the theory of vector lattices. It is enough to mention the theory of lattice-normed vector lattices (see, for example, [19, 20, 10]). The main aim of the present paper is to illustrate usefulness of lattice-valued norms for investigation of different types of *unbounded convergences* in vector lattices, which attracted attention of several authors in series of recent papers [15, 12, 14, 13, 9, 7, 16, 3, 30, 4, 24, 5, 11, 25, 17, 8].

The *uo-convergence* was introduced in [23] under the name *individual convergence*, and the *un-convergence* was introduced in [26] under the name *d-convergence*. We refer the reader for an exposition on uo-convergence to [14, 15] and on un-convergence to [7] (see also recent paper [16]). For applications of uo-convergence, we refer to [9, 14, 15, 13, 22]. Throughout the paper, all vector lattices are assumed to be real and Archimedean.

Recall that a net $(x_\alpha)_{\alpha\in A}$ in a vector lattice X is order convergent (or o-convergent, for short) to $x \in X$, if there exists another net $(y_\beta)_{\beta\in B}$ satisfying $y_\beta \downarrow 0$, and for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. In this case we write $x_\alpha \xrightarrow{o} x$. In a vector lattice X, a net x_α is unbounded order convergent (or uo-convergent, for short) to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X_+$. In this case we write $x_\alpha \xrightarrow{uo} x$. The uo-convergence is an abstraction of a.e.-convergence in L_p-spaces for $1 \leq p < \infty$, [14, 15]. In a normed lattice $(X, \|\cdot\|)$, a net x_α is unbounded norm convergent to $x \in X$, written as $x_\alpha \xrightarrow{un} x$, if $\||x_\alpha - x| \wedge u\| \to 0$ for every $u \in X_+$. Clearly, if the norm is order

Date: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.

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2010 *Mathematics Subject Classification.* Primary 46A40; Secondary 46E30.

Key words and phrases. vector lattice, lattice-normed vector lattice, uo-convergence, un-convergence, mixed-normed space.

continuous, then *uo*-convergence implies *un*-convergence. For a finite measure μ, *un*-convergence of sequences in $L_p(\mu)$, $1 \leq p < \infty$, is equivalent to convergence in measure, see [7, 26]. Recently, Zabeti [30] introduced the following notion. A net x_α in a Banach lattice X is said to be *unbounded absolute weak convergent* (or *uaw-convergent*, for short) to $x \in X$ if, for each $u \in X_+$, $|x_\alpha - x| \wedge u \to 0$ weakly.

Let X be a vector space, E be a vector lattice, and $p : X \to E_+$ be a vector norm (i.e. $p(x) = 0 \Leftrightarrow x = 0$, $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, $x \in X$, and $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$), then the triple (X, p, E) is called a *lattice-normed space*, abbreviated as LNS. We say that elements x and y of an LNS X are p-disjoint if their lattice norms are disjoint, and abbreviate this by $x \perp_{\mathbf{p}} y$. The lattice norm p in an LNS (X, p, E) is said to be *decomposable* if, for all $x \in X$ and $e_1, e_2 \in E_+$, from $p(x) = e_1 + e_2$ it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$. We abbreviate the convergence $p(x_\alpha - x) \xrightarrow{o} 0$ as $x_\alpha \xrightarrow{p} x$ and say in this case that x_α p-converges to x. We refer the reader for more information on LNSs to [19, 20].

If, in addition, X is a vector lattice and the vector norm p is monotone (i.e. $|x| \leq |y| \Rightarrow p(x) \leq p(y)$), then the triple (X, p, E) is called *lattice-normed vector lattice*, abbreviated as LNVL. In an LNVL (X, p, E), p-disjointness implies disjointness. Indeed, let $x \perp_{\mathbf{p}} y$ and $0 \leq u \leq |x| \wedge |y|$. Then $p(u) \leq p(|x| \wedge |y|) \leq p(x) \wedge p(y) = 0$ and hence $u = 0$. Thus $x \perp y$. We shall make difference between two notions of bands in an LNVL $X = (X, p, E)$. More precisely, a subset B of X is called a *band* if it is a band in the usual sense of the vector lattice X. Following to [20, 2.1.2.], we say that a subset B of X is a *p-band* if

$$B = M^{\perp_{\mathbf{p}}} = \{x \in X : (\forall m \in M) \ x \perp_{\mathbf{p}} m\}$$

for some nonempty $M \subseteq X$. In general, there are many bands which are not p-bands. To see this, consider the normed lattice $(\mathbb{R}^2, \|\cdot\|, \mathbb{R})$. It has four bands, but only two of them are p-bands. It is easy to see that any p-band is an order ideal. The following example shows that a p-band may not be a band in general.

Example 1.1. Consider the LNVL (c, p, c) with

$$p(x) := |x| + (\lim_{n\to\infty} |x_n|) \cdot \mathbf{1} \qquad (x = (x_n)_n \in c),$$

where $\mathbf{1}$ denotes the sequence identically equal to 1. Take $M = \{e_1\}$. Then the p-band $M^{\perp_{\mathbf{p}}} = \{x \in c_0 : x_1 = 0\}$ is not a band.

In Proposition 2.18, we show that, under some mild conditions, every p-band is a band. Unless otherwise stated, we do not assume lattice norms to be decomposable. While dealing with LNVLs, we shall keep in mind also the following examples.

Example 1.2. Let X be a normed lattice with a norm $\|\cdot\|$. Then X is the LNVL $(X, \|\cdot\|, \mathbb{R})$.

Example 1.3. Let X be a vector lattice. Then X is the LNVL $(X, |\cdot|, X)$.

Example 1.4. Let $X = (X, \|\cdot\|)$ be a normed lattice. Consider the closed unit ball B_{X^*} of the dual Banach lattice X^*. Let $E = \ell^\infty(B_{X^*})$ be the vector lattice of all bounded real-valued functions on B_{X^*}. Define an E-valued norm p on X by

$$p(x)[f] := |f|(|x|) \quad (f \in B_{X^*})$$

for any $x \in X$. The Hahn-Banach theorem ensures that $p(x) = 0$ iff $x = 0$. All other properties of lattice norm are obvious for p. Thus (X, p, E) is an LNVL. Notice also that the lattice norm p takes values in the space $C(B_{X^*})$ of all continuous functions on the w^*-compact ball B_{X^*} of X^*. Hence, instead of $(X, p, \ell^\infty(B_{X^*}))$, one may also consider the LNVL $(X, p, C(B_{X^*}))$.

Example 1.5. Let X be a vector lattice, $X^\#$ be the algebraic dual of X, and Y be a sublattice of $X^\#$ such that $\langle X, Y\rangle$ is a dual system. Define $p : X \to \mathbb{R}^Y$ by $p(x)[f] := |f|(|x|)$. Then $(X, p, \mathbb{R}^Y)$ is an LNVL.

The LNVLs in Examples 1.1, 1.2, and 1.3 have decomposable norms. It can be shown easily that in Examples 1.4 and 1.5 the lattice norms are decomposable iff $\dim(X) = 1$.

We refer the reader for further examples of LNSs to [20]. It should be noticed that the theory of lattice-normed spaces is well developed in the case of decomposable lattice norms (cf. [19, 20]). In general, we do not assume lattice norms to be decomposable.

The structure of the paper is as follows. In Section 2, we study several notions related to LNVLs in parallel to the theory of Banach lattices. In particular, an LNVL (X, p, E) is said to be: *op-continuous* if $X \ni x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{p} 0$; a *p-KB-space* if, for any $0 \leq x_\alpha \uparrow$ with $p(x_\alpha) \leq e \in E$, there exists $x \in X$ satisfying $x_\alpha \xrightarrow{p} x$. We give a characterization of op-continuity in Theorem 2.9, and study some properties of p-KB-spaces, e.g. in Proposition 2.14 and in Proposition 2.15. A vector $e \in X$ is called a *p-unit* if, for any $x \in X_+$, $p(x - ne \wedge x) \xrightarrow{o} 0$. Any p-unit is a weak unit, whereas strong units are p-units. For a normed lattice $(X, \|\cdot\|)$, a vector in X is a p-unit in $(X, \|\cdot\|, \mathbb{R})$ iff it is a quasi-interior point of the normed lattice $(X, \|\cdot\|)$.

In Section 3, some basic theory of *unbounded p-convergence* in LNVLs is developed in parallel to uo- and un-convergences. For example, it is enough to check out the uo-convergence at a weak unit, while the un-convergence needs to be checked only at a quasi-interior point. Similarly, in LNVLs, *up-convergence* needs to be examined at a p-unit by Theorem 3.9.

In Section 4, we introduce and study *up-regular* sublattices. Majorizing sublattices and projection bands are examples of up-regular sublattices by Theorem 4.3. Also some further investigation of up-regular sublattices is carried out in certain LNVLs in this section.

In the last section, we study properties of mixed-normed LNVLs in Proposition 5.1, in Theorem 5.2, and in Theorem 5.5. We also prove that in a certain LNVL, the up-null nets are "p-almost disjoint" (see Theorem 5.6). Those results generalize correspondent results from [14, 7].

We refer the reader for unexplained notions and terminology to [2, 20, 21].

2. p-Notions in lattice-normed vector lattices

Most of notions and results of this preliminary section are direct analogies of well-known facts of the theory of normed lattices. We include them for convenience of the reader. They are also of certain proper interest and some of them will be used in further sections. In the present section, we define and study certain necessary notions such as: *op*-continuity of LNVLs, p-KB-spaces, p-Fatou spaces, p-units, etc. In particular, we characterize the *op*-continuity, prove some properties of p-KB-spaces, discuss p-dense subsets, and study p-units in LNVLs.

2.1. **p-Continuity of lattice operations in LNVLs.** The lattice operations in an LNVL X are p-continuous in the following sense.

Lemma 2.1. *Let $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two nets in an LNVL (X,p,E). If $x_\alpha \xrightarrow{p} x$ and $y_\beta \xrightarrow{p} y$, then $(x_\alpha \vee y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{p} x\vee y$. In particular, $x_\alpha \xrightarrow{p} x$ implies that $x_\alpha^- \xrightarrow{p} x^-$.*

However this result seems to be well-known, we did not find appropriate references for it and therefore, its elementary proof is included for the reader's convenience.

Proof. There exist two nets $(z_\gamma)_{\gamma\in\Gamma}$ and $(w_\lambda)_{\lambda\in\Lambda}$ in E satisfying $z_\gamma \downarrow 0$ and $w_\lambda \downarrow 0$, and for all $(\gamma,\lambda)\in\Gamma\times\Lambda$ there are $\alpha_\gamma\in A$ and $\beta_\lambda\in B$ such that $p(x_\alpha - x)\le z_\gamma$ and $p(y_\beta - y)\le w_\lambda$ for all $(\alpha,\beta)\ge(\alpha_\gamma,\beta_\lambda)$. It follows from the inequality $|a\vee b - a\vee c|\le|b-c|$ that

$$\begin{aligned} p(x_\alpha\vee y_\beta - x\vee y) &= p(|x_\alpha\vee y_\beta - x_\alpha\vee y + x_\alpha\vee y - x\vee y|)\\ &\le p(|x_\alpha\vee y_\beta - x_\alpha\vee y|) + p(|x_\alpha\vee y - x\vee y|)\\ &\le p(|y_\beta - y|) + p(|x_\alpha - x|) \le w_\lambda + z_\gamma \end{aligned}$$

for all $\alpha\ge\alpha_\gamma$ and $\beta\ge\beta_\lambda$. Since $(w_\lambda + z_\gamma)\downarrow 0$, then $p(x_\alpha\vee y_\beta - x\vee y)\xrightarrow{o} 0$. □

Definition 2.2. Let (X,p,E) be an LNVL and $Y\subseteq X$. Then Y is called *p-closed* in X if, for any net y_α in Y that is p-convergent to $x\in X$, it holds that $x\in Y$.

Remark 2.3.

(1) Every band is p-closed. Indeed, given a band B in an LNVL (X,p,E). If $B\ni x_\alpha\xrightarrow{p} x$, then, by Lemma 2.1, $|x_\alpha|\wedge|y|\xrightarrow{p}|x|\wedge|y|$ for any $y\in B^\perp$. Since $|x_\alpha|\wedge|y| = 0$ for all α, then $|x|\wedge|y|=0$, and so $x\in B^{\perp\perp}=B$.

(2) Every p-band is p-closed. Indeed, let $B = M^{\perp_{\mathrm{p}}}$ for some nonempty $M\subseteq X$ and $B\ni x_\alpha\xrightarrow{p} x_0\in X$. Take any $m\in M$. It follows from

$$p(x_0)\wedge p(m)\le (p(x_0 - x_\alpha) + p(x_\alpha))\wedge p(m)\le$$

$$p(x_0 - x_\alpha)\wedge p(m) + p(x_\alpha)\wedge p(m) = p(x_0 - x_\alpha)\wedge p(m)\xrightarrow{o} 0,$$

that $p(x_0)\wedge p(m) = 0$. Since $m\in M$ is arbitrary, then $x_0\in B$.

The following well-known property is a direct consequence of Lemma 2.1.

Proposition 2.4. *The positive cone X_+ in any LNVL X is p-closed.*

Proposition 2.4 implies the following well-known fact.

Proposition 2.5. *Any monotone p-convergent net in an LNVL o-converges to its p-limit.*

Proof. It is enough to show that if $(X,p,E) \ni x_\alpha \uparrow$ and $x_\alpha \xrightarrow{p} x$, then $x_\alpha \uparrow x$.

Fix arbitrary α. Then $x_\beta - x_\alpha \in X_+$ for $\beta \geq \alpha$. By Proposition 2.4, $x_\beta - x_\alpha \xrightarrow{p} x - x_\alpha \in X_+$. Therefore, $x \geq x_\alpha$ for any α. Since α is arbitrary, then x is an upper bound of x_α.

If $y \geq x_\alpha$ for all α, then, again by Proposition 2.4, $y - x_\alpha \xrightarrow{p} y - x \in X_+$, or $y \geq x$. Thus $x_\alpha \uparrow x$. □

2.2. **Several basic p-notions in LNVLs.** We continue with several basic notions in LNVLs, which are motivated by their analogies from vector lattice theory.

Definition 2.6. Let $X = (X,p,E)$ be an LNVL. Then

(i) a net $(x_\alpha)_{\alpha\in A}$ in X is said to be *p-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ p-converges to 0;

(ii) X is called *p-complete* if every p-Cauchy net in X is p-convergent;

(iii) a subset $Y \subseteq X$ is called *p-bounded* if there exists $e \in E$ such that $p(y) \leq e$ for all $y \in Y$;

(iv) X is called *op-continuous* if $x_\alpha \xrightarrow{o} 0$ implies that $p(x_\alpha) \xrightarrow{o} 0$;

(v) X is called a *p-KB-space* if every p-bounded increasing net in X_+ is p-convergent;

(vi) p is said to be *additive on* X_+ if $p(x+y) = p(x) + p(y)$ for all $x, y \in X_+$.

Remark 2.7.

1. p-convergence, a p-Cauchy net, p-completeness, and p-boundedness in LNVLs are also known as *bo-convergence*, *a bo-fundamental net*, *bo-completeness*, and *norm-boundedness* respectively (see, e.g. [20, p.48]).
2. Clearly, any LNVL $(X,|\cdot|,X)$ is op-continuous.
3. In Definition 2.6(v) we do not require p-completeness of X.
4. It is easy to see that a p-KB-space $(X,\|\cdot\|,\mathbb{R})$ is always p-complete (see, e.g. [29, Ex.95.4]). Therefore, the notion of p-KB-space coincides with the notion of KB-space.
5. Clearly, an LNVL $X = (X,|\cdot|,X)$ is a p-KB-space iff X is order complete.
6. Notice that, for a p-KB-space $X = (X,p,E)$ the vector lattice $p(X)^{\perp\perp}$ need not to be order complete. To see this, take a KB-space $(X,\|\cdot\|)$ and $E = C[0,1]$. Then the LNVL $X = (X,p,E)$ with $p(x) := \|x\| \cdot \mathbf{1}_{[0,1]}$ is clearly a p-KB-space, yet $p(X)^{\perp\perp} = E$ is not order complete.

Lemma 2.8. *For an LNVL (X,p,E), the following statements are equivalent.*

(i) *X is* op-continuous*;*

(ii) *$x_\alpha \downarrow 0$ in X implies $p(x_\alpha) \downarrow 0$.*

Proof. The implication $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (i)$: Let $x_\alpha \xrightarrow{o} 0$, then there exists a net $z_\beta \downarrow 0$ in X such that, for any β there exists α_β so that $|x_\alpha| \leq z_\beta$ for all $\alpha \geq \alpha_\beta$. Hence $p(x_\alpha) \leq p(z_\beta)$ for all $\alpha \geq \alpha_\beta$. By (ii), $p(z_\beta) \downarrow 0$. Therefore, $p(x_\alpha) \xrightarrow{o} 0$ or $x_\alpha \xrightarrow{p} 0$. □

From this proposition, it follows that the *op*-continuity in LNVLs is equivalent to the order continuity in the sense of [20, 2.1.4, p.48]. In the case of a p-complete LNVL, we have further conditions for *op*-continuity.

Theorem 2.9. *For a p-complete LNVL (X,p,E), the following statements are equivalent:*
(*i*) X *is op*-continuous*;*
(*ii*) *if* $0 \leq x_\alpha \uparrow \leq x$ *holds in X, then* x_α *is a p-Cauchy net;*
(*iii*) $x_\alpha \downarrow 0$ *in X implies* $p(x_\alpha) \downarrow 0$*.*

Proof. $(i) \Rightarrow (ii)$: Let $0 \leq x_\alpha \uparrow \leq x$ in X. By [2, Lm.12.8], there exists a net y_β in X such that $(y_\beta - x_\alpha)_{\alpha,\beta} \downarrow 0$. So $p(y_\beta - x_\alpha) \xrightarrow{o} 0$, and hence the net x_α is p-Cauchy.

$(ii) \Rightarrow (iii)$: Assume that $x_\alpha \downarrow 0$ in X. Fix arbitrary α_0, then, for $\alpha \geq \alpha_0$, $x_\alpha \leq x_{\alpha_0}$, and so $0 \leq (x_{\alpha_0} - x_\alpha)_{\alpha \geq \alpha_0} \uparrow \leq x_{\alpha_0}$. By (ii), the net $(x_{\alpha_0} - x_\alpha)_{\alpha \geq \alpha_0}$ is p-Cauchy, i.e. $p(x_{\alpha'} - x_\alpha) \xrightarrow{o} 0$ as $\alpha_0 \leq \alpha, \alpha' \to \infty$. Since X is p-complete, then there exists $x \in X$ satisfying $p(x_\alpha - x) \xrightarrow{o} 0$ as $\alpha_0 \leq \alpha \to \infty$. By Proposition 2.5, $x_\alpha \downarrow x$ and hence $x = 0$. As a result, $x_\alpha \xrightarrow{p} 0$ and the monotonicity of p implies $p(x_\alpha) \downarrow 0$.

$(iii) \Rightarrow (i)$: It is just the implication $(ii) \Rightarrow (i)$ of Lemma 2.8. □

Corollary 2.10. *Let (X,p,E) be an op-continuous and p-complete LNVL, then X is order complete.*

Proof. Assume $0 \leq x_\alpha \uparrow \leq u$, then by Theorem 2.9 (ii), x_α is a p-Cauchy net and since X is p-complete, then there is x such that $x_\alpha \xrightarrow{p} x$. It follows from Proposition 2.5 that $x_\alpha \uparrow x$, and so X is order complete. □

Corollary 2.11. *Any p-KB-space is op-continuous.*

Proof. Let $x_\alpha \downarrow 0$. Take any α_0 and let $y_\alpha := x_{\alpha_0} - x_\alpha$ for $\alpha \geq \alpha_0$. Clearly, $y_\alpha \uparrow \leq x_{\alpha_0}$. Hence $p(y_\alpha) \uparrow \leq p(x_{\alpha_0})$ for $\alpha \geq \alpha_0$. Since X is a p-KB-space, there exists $y \in X$ such that $p(y_\alpha - y) \xrightarrow{o} 0$. Since $y_\alpha \uparrow$ and $y_\alpha \xrightarrow{p} y$, Proposition 2.5 ensures that

$$y = \sup_{\alpha \geq \alpha_0} y_\alpha = \sup_{\alpha \geq \alpha_0} (x_{\alpha_0} - x_\alpha) = x_{\alpha_0},$$

and hence $y_\alpha = x_{\alpha_0} - x_\alpha \xrightarrow{p} x_{\alpha_0}$ or $x_\alpha \xrightarrow{p} 0$. Again by Proposition 2.5 we get $p(x_\alpha) \downarrow 0$. So by Lemma 2.8, X is *op*-continuous. □

Proposition 2.12. *Any p-KB-space is order complete.*

Proof. Let X be a p-KB-space and $0 \leq x_\alpha \uparrow \leq z \in X$. Then $p(x_\alpha) \leq p(z)$. Hence the net x_α is p-bounded and therefore, $x_\alpha \xrightarrow{p} x$ for some $x \in X$. By Proposition 2.5, $x_\alpha \uparrow x$. □

The following question arises naturally.

Problem 2.13. Let (X,p,E) be a p-KB-space. Is (X,p,E) p-complete?

We do not know the answer to Problem 2.13 even under the assumption that E is order complete.

Proposition 2.14. *Let (X, p, E) be a p-KB-space, and $Y \subseteq X$ be an order closed sublattice. Then (Y, p, E) is also a p-KB-space.*

Proof. Let $Y_+ \ni y_\alpha \uparrow$ and $p(y_\alpha) \leq e \in E_+$ for all α. Since X is a p-KB-space, there exists $x \in X_+$ such that $y_\alpha \xrightarrow{p} x$. By Proposition 2.5, we have $y_\alpha \uparrow x$, and so $x \in Y$, because Y is order closed. Thus (Y, p, E) is a p-KB-space. □

It is clear from the proof of Proposition 2.14, that every p-closed sublattice Y of a p-KB-space X is also a p-KB-space.

Proposition 2.15. *Let (X, p, E) be a p-complete LNVL, E be atomic, and p be additive on X_+. Then X is a p-KB-space.*

Proof. Let a net x_α in X_+ be increasing and p-bounded by $e \in E_+$. If the net x_α is not p-Cauchy, then there exists an atom $a \in E$ such that $f_a(p(x_\alpha - x_{\alpha'})) \not\to 0$, where f_a is the biorthogonal functional of a. Then there exist $\epsilon > 0$ and a strictly increasing sequence (α_n) of indices such that

$$f_a(p(x_{\alpha_n} - x_{\alpha_{n-1}})) \geq \epsilon > 0 \quad (\forall n \in \mathbb{N}).$$

Thus

$$\begin{aligned} n\epsilon &\leq \sum_{k=2}^{n+1} f_a(p(x_{\alpha_k} - x_{\alpha_{k-1}})) \\ &= f_a\Big(\sum_{k=2}^{n+1} p(x_{\alpha_k} - x_{\alpha_{k-1}}\Big) = f_a\Big(p\Big(\sum_{k=2}^{n+1} x_{\alpha_k} - x_{\alpha_{k-1}}\Big)\Big) \\ &= f_a(p(x_{\alpha_{n+1}} - x_{\alpha_1})) \leq 2f_a(e). \end{aligned}$$

Thus $n\epsilon \leq 2f_a(e)$ for all $n \in \mathbb{N}$, and hence $\epsilon \leq 0$; a contradiction. □

Remark that the LNVL $(c_0, |\cdot|, \ell_\infty)$ is not p-complete, yet the norming lattice ℓ_∞ is atomic and its lattice norm is additive on $(c_0)_+$. Consider the sequence $x_n = \sum_{i=1}^n e_i$, where e_n's are the standard unit vectors of c_0. Then $0 \leq x_n \uparrow$ and x_n's are p-bounded by $\mathbf{1} = (1, 1, \cdots) \in \ell_\infty$. Clearly, it is not p-convergent, so the LNVL $(c_0, |\cdot|, \ell_\infty)$ is not p-KB-space. Notice also that $(c_0, |\cdot|, \ell_\infty)$ is *op*-continuous.

2.3. **More details on Example 1.4.** Let us discuss Example 1.4 in more details.

(i) If X is an order continuous Banach lattice, then $(X, p, \ell_\infty(B_{X^*}))$ is *op*-continuous.

Proof. Assume $x_\alpha \downarrow 0$, we show $p(x_\alpha) \downarrow 0$. Our claim is the following: $p(x_\alpha) \downarrow 0$ iff $p(x_\alpha)[f] \downarrow 0$ for all $f \in B_{X^*}$.

For the necessity, let $p(x_\alpha) \downarrow 0$ and $f \in B_{X^*}$. Trivially, $|f|(x_\alpha) \downarrow$. If there exists $z_f \in \mathbb{R}$ such that $0 \leq z_f \leq |f|(x_\alpha)$ for all α, then

$$0 \leq z_f \leq |f|(x_\alpha) \leq \|f\| \, \|x_\alpha\| \downarrow 0.$$

Hence $z_f = 0$ and $p(x_\alpha)[f] = |f|(x_\alpha) \downarrow 0$.

For the sufficiency, let $p(x_\alpha)[f] \downarrow 0$ for every $f \in B_{X^*}$. Since p is monotone and $x_\alpha \downarrow$, then $p(x_\alpha) \downarrow$. If $0 \leq \varphi \leq p(x_\alpha)$ for all α, then

$$0 \leq \varphi(f) \leq p(x_\alpha)[f] = |f|(x_\alpha) \quad (\forall f \in B_{X^*}).$$

So by the assumption, we get $\varphi(f)=0$ for all $f\in B_{X^*}$, and hence $\varphi=0$. Therefore, $p(x_\alpha)\downarrow 0$. □ □

(ii) If $(X,\|\cdot\|)$ is a KB-space, then (X,p,E) is a p-KB-space.

Proof. Suppose that $0\le x_\alpha\uparrow$ and $p(x_\alpha)\le\varphi\in\ell_\infty(B_{X^*})$. As

$$\begin{aligned}\|x_\alpha\| &= \sup_{f\in B_{X^*}}|f(x_\alpha)|\le \sup_{f\in B_{X^*}}|f|(x_\alpha)\\ &= \sup_{f\in B_{X^*}}p(x_\alpha)[f]\le\varphi[f]\le\|\varphi\|_\infty\le\infty \quad (\forall\alpha),\end{aligned}$$

and since X is a KB-space, we get $\|x_\alpha-x\|\to 0$ for some $x\in X_+$. So, for any $f\in B_{X^*}$, we have $|f|(|x_\alpha-x|)\to 0$ or $p(x_\alpha-x)[f]\to 0$. Thus $p(x_\alpha-x)\xrightarrow{o}0$ in $\ell_\infty(B_{X^*})$ and hence $x_\alpha\xrightarrow{p}x$. □

Recall that a vector lattice X is called *perfect* if the natural embedding from X into $(X_n^\sim)_n^\sim$ is one-to-one and onto, where $X_n^\sim$ denotes the order continuous dual of X [2, pp.58-59]. If X is a perfect vector lattice, then $X_n^\sim$ separates the points of X [2, Thm.5.6(1)].

Proposition 2.16. *Let X be a perfect vector lattice, $Y=X_n^\sim$ and $p:X\to\mathbb{R}^Y$ be defined as $p(x)[f]:=|f|(|x|)$, where $f\in Y$. Then the LNVL $(X,p,\mathbb{R}^Y)$ is a p-KB-space.*

Proof. Assume $0\le x_\alpha\uparrow$ in X and $p(x_\alpha)\le\varphi\in\mathbb{R}^Y$. Then, for all $f\in Y$, we have $p(x_\alpha)[f]\le\varphi(f)$ or $|f|(x_\alpha)\le\varphi(f)$. So for all $f\in Y$, $\sup\limits_\alpha|f|(x_\alpha)<\infty$, and hence, by [2, Thm.5.6(2)], there is $x\in X$ with $x_\alpha\uparrow x$. An argument similar to (i) above shows that X is op-continuous. Therefore, $x_\alpha\xrightarrow{p}x$. □

2.4. p-Fatou space.
In this subsection, we introduce and discuss p-Fatou spaces.

Definition 2.17. An LNVL (X,p,E) is called *p-Fatou space* if $0\le x_\alpha\uparrow x$ in X implies $p(x_\alpha)\uparrow p(x)$.

Note that (X,p,E) is a p-Fatou space iff p is order semicontinuous [20, 2.1.4, p.48]. Clearly any op-continuous LNVL (X,p,E) is a p-Fatou space. It is easy to see that the LNVL (c,p,c) in Example 1.1 is not a p-Fatou space. Moreover the p-Fatou property ensures that p-bands are bands.

Proposition 2.18. *Let B be a p-band in a p-Fatou space (X,p,E). Then B is a band in X.*

Proof. Let $B=M^{\perp_p}=\{x\in X:(\forall m\in M)\ p(x)\perp p(m)\}$ for some nonempty $M\subseteq X$. Since B is an ideal in X to show that B is a band it is enough to prove that if $B_+\ni b_\alpha\uparrow x\in X$, then $x\in B$. That is easy, since $p(b_\alpha)\uparrow p(x)$ as X is a p-Fatou space. By o-continuity of lattice operations in E, we obtain that

$$0=p(b_\alpha)\wedge p(m)\xrightarrow{o}p(x)\wedge p(m)\qquad(\forall m\in M).$$

Therefore, $p(x)\wedge p(m)=0$ for all $m\in M$, and hence $x\in B$. □

In connection with Proposition 2.18 and Example 1.1, the following open problem arises.

Problem 2.19. Let (X,p,E) be a decomposable LNVL p-Fatou space in which every p-band is a band. Is X a p-Fatou space?

2.5. **p-Density and p-units.** In the present paper, we use the following definition of a p-dense subset in an LNS, which is motivated by the notion of a dense subset of a normed space.

Definition 2.20. Given an LNS (X,p,E) and $A \subseteq X$. A subset $B \subseteq A$ is said to be *p-dense in A* if, for any $a \in A$ and for any $0 \neq u \in p(X)$ there is $b \in B$ such that $p(a-b) \leq u$.

Remark 2.21.

(1) Consider the LNVL (X,p,E) with $p = |\cdot|$, $E = X$, and let Y be a sublattice X. If Y is p-dense in X, then Y is order dense. Indeed, let $0 \neq x \in X_+$, then there is $y \in Y$ such that $|y - \frac{1}{2}x| \leq \frac{1}{3}x$ which implies $0 < \frac{1}{6}x \leq y \leq \frac{5}{6}x$, and so $0 < y \leq x$.
(2) c is order dense, yet is not p-dense in both of the following LNVLs: $(\ell_\infty, \|\cdot\|_\infty, \mathbb{R})$ and $(\ell_\infty, |\cdot|, \ell_\infty)$.
(3) If $X = (X, \|\cdot\|)$ is a normed lattice, $p = \|\cdot\|$ and $E = \mathbb{R}$, then clearly a subset Y of X is p-dense iff Y is norm dense.

The following notion is motivated by the notion of a weak order unit in a vector lattice $X = (X, |\cdot|, X)$ and by the notion of a quasi-interior point in a normed lattice $X = (X, \|\cdot\|, \mathbb{R})$

Definition 2.22. Let (X,p,E) be an LNVL. A vector $e \in X$ is called a *p-unit* if, for any $x \in X_+$ we have $p(x - x \wedge ne) \xrightarrow{o} 0$.

Remark 2.23. Let (X,p,E) be an LNVL.

(1) If $X \neq \{0\}$ then, for any p-unit e in X it holds that $e > 0$. Indeed, let e be a p-unit in $X \neq \{0\}$. Trivially, $e \neq 0$. Suppose $e^- > 0$. Then, for $x := e^-$, we obtain that
$$p(x - x \wedge ne) = p(e^- - (e^- \wedge n(e^+ - e^-))) =$$
$$p(e^- - (e^- \wedge n(-e^-))) = p(e^- - (-ne^-)) = p((n+1)e^-) =$$
$$(n+1)p(e^-) \not\xrightarrow{o} 0$$
as $n \to \infty$. This is impossible, because e is a p-unit. Therefore, $e^- = 0$ and $e > 0$.
(2) Let $e \in X$ be a p-unit. Given $0 < \alpha \in \mathbb{R}_+$ and $z \in X_+$. Observe that, for $x \in X_+$, $p(x - n\alpha e \wedge x) = \alpha p(\frac{x}{\alpha} - ne \wedge \frac{x}{\alpha})$ and $p(x - n(e+z) \wedge x) \leq p(x - x \wedge ne)$, from which it follows easily that αe and $e + z$ are both p-units.
(3) If $e \in X$ is a strong unit, then e is a p-unit. Indeed, let $x \in X_+$, then there is $k \in \mathbb{N}$ such that $x \leq ke$, so $x - x \wedge ne = 0$ for any $n \geq k$.
(4) If $e \in X$ is a p-unit, then e is a weak unit. Assume $x \wedge e = 0$, then $x \wedge ne = 0$ for any $n \in \mathbb{N}$. Since e is a p-unit, then $p(x) = 0$ and hence $x = 0$.
(5) If X is op-continuous, then clearly every weak unit of X is a p-unit.

(6) In $X=(X,|\cdot|,X)$, the lattice norm $p(x)=|x|$ is always order continuous. Therefore, the notions of p-unit and of weak unit coincide in X.
(7) If $X=(X,\|\cdot\|)$ is a normed lattice, $p=\|\cdot\|$, $E=\mathbb{R}$, and $e\in X$, then e is a p-unit iff e is a quasi-interior point of X.

In the proof of the following proposition, we use the same technique as in the proof of [1, Lm.4.15].

Proposition 2.24. *Let (X,p,E) be an LNVL, $e\in X_+$, and I_e be the order ideal generated by e in X. If I_e is p-dense in X, then e is a p-unit.*

Proof. Let $0\neq u\in p(X)$. Let $x\in X_+$, then there exists $y\in I_e$ such that $p(x-y)\leq u$. Since $|y^+\wedge x-x|\leq|y^+-x|=|y^+-x^+|\leq|y-x|$, then, by replacing y by $y^+\wedge x$, we may assume without loss of generality that there is $y\in I_e$ such that $0\leq y\leq x$ and $p(x-y)\leq u$. Thus, for any $m\in\mathbb{N}$, there is $y_m\in I_e$ such that $0\leq y_m\leq x$ and

$$p(x-y_m)\leq\frac{1}{m}u.$$

Since $y_m\in I_e$, then there exists $k=k(m)\in\mathbb{N}$ such that $0\leq y_m\leq ke$, and so $0\leq y_m\leq ke\wedge x$.

For $n\geq k$, $x-x\wedge ne\leq x-x\wedge ke\leq x-y_m$, and so $p(x-x\wedge ne)\leq p(x-y_m)\leq\frac{1}{m}u$. Hence $p(x-x\wedge ne)\xrightarrow{o}0$. Thus e is a p-unit. ☐

3. Unbounded p-convergence

The *up-convergence* in LNVLs generalizes the *uo*-convergence in vector lattices [14, 12, 15], the *un*-convergence [7] and the *uaw*-convergence [30] in Banach lattices. We study basic properties of the *up*-convergence and characterize the *up*-convergence in certain LNVLs.

3.1. **Main definition and its motivation.** Let (X,p,E) be an LNVL. The following definition is motivated by its special case when it is reduced to the *un*-convergence for a normed lattice $(X,p,E)=(X,\|\cdot\|,\mathbb{R})=(X,\|\cdot\|)$.

Definition 3.1. A net $x_\alpha\subseteq X$ is said to be unbounded p-convergent to $x\in X$ (shortly, x_α *up*-converges to x or $x_\alpha\xrightarrow{up}x$), if

$$p(|x_\alpha-x|\wedge u)\xrightarrow{o}0\quad(\forall u\in X_+).$$

It is immediate to see that *up*-convergence coincides with *un*-convergence in the case when p is the norm in a normed lattice, and with *uo*-convergence in the case when $X=E$ and $p(x)=|x|$. It is clear that $x_\alpha\xrightarrow{p}x$ implies $x_\alpha\xrightarrow{up}x$, and for order bounded nets *up*-convergence and p-convergence agree. It should be also clear that, if an LNVL X is *op*-continuous, then *uo*-convergence in X implies *up*-convergence. The *uaw*-convergence is also a particular case of *up*-convergence as it follows from the next proposition.

Proposition 3.2. *In the notation of Example 1.5, $x_\alpha\xrightarrow{up}0$ in X iff for every $u\in X_+$, $|x_\alpha|\wedge u\xrightarrow{|\sigma|(X,Y)}0$.*

Proof. $x_\alpha \xrightarrow{up} 0$ in X iff for all $u \in X_+$, $p(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E iff for every $u \in X_+$, $p(|x_\alpha| \wedge u)[y] \to 0$ for all $y \in Y$ iff for every $u \in X_+$, $|y|(|x_\alpha| \wedge u) \to 0$ for all $y \in Y$ iff for every $u \in X_+$, $|x_\alpha| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$. □

In particular, if X is a Banach lattice, $Y = X^*$, the topological dual of X, $E = \mathbb{R}^Y$ and $p : X \to E$ as defined above, then $x_\alpha \xrightarrow{up} 0$ in X iff $x_\alpha \xrightarrow{uaw} 0$.

3.2. **Basic results on up-convergence.** We begin with the next list of properties of up-convergence which follows directly from Lemma 2.1.

Lemma 3.3. *Let $x_\alpha \xrightarrow{up} x$ and $y_\alpha \xrightarrow{up} y$ in an LNVL (X, p, E), then:*

(i) $ax_\alpha + by_\alpha \xrightarrow{up} ax + by$ for any $a, b \in \mathbb{R}$, in particular, if $x_\alpha = y_\alpha$, then $x = y$;

(ii) $x_{\alpha_\beta} \xrightarrow{up} x$ for any subnet x_{α_β} of x_α;

(iii) $|x_\alpha| \xrightarrow{up} |x|$;

(iv) if $x_\alpha \geq y_\alpha$ for all α, then $x \geq y$.

Lemma 3.4. *Let x_α be a monotone net in an LNVL (X, p, E) such that $x_\alpha \xrightarrow{up} x$, then $x_\alpha \xrightarrow{o} x$.*

Proof. The proof of Proposition 2.5 is applicable here as well. □

The following result is a p-generalization of [16, Lm.1.2 (ii)].

Theorem 3.5. *Let x_α be a monotone net in an LNVL (X, p, E) which up-converges to x. Then $x_\alpha \xrightarrow{p} x$.*

Proof. Without loss of generality we may assume that $0 \leq x_\alpha \uparrow$. From Lemma 3.4 it follows that $0 \leq x_\alpha \uparrow x$ for some $x \in X$. So $0 \leq x - x_\alpha \leq x$ for all α. Since, for each $u \in X_+$, we know that

$$p((x - x_\alpha) \wedge u) \xrightarrow{o} 0.$$

In particular, for $u = x$, we obtain that

$$p(x - x_\alpha) = p((x_\alpha - x) \wedge x) \xrightarrow{o} 0.$$

□

Similar to [12, Lm.1.2.(1)] we have that if $x_\alpha \xrightarrow{up} 0$ in an LNVL (X, p, E), then $\inf_\beta |y_\beta| = 0$ for any subnet y_β of the net x_α. Indeed, let y_β be a subnet of x_α. Clearly, $y_\beta \xrightarrow{up} 0$. If $0 \leq z \leq |y_\beta|$ for all β, then $p(z) = p(z \wedge |y_\beta|) \xrightarrow{o} 0$, and so $z = 0$. Hence $\inf_\beta |y_\beta| = 0$.

The following two results, which are analogies of Lemma 2.8 in [7] and of Lemma 3.6 in [15], we have respectively.

Lemma 3.6. *Let (X, p, E) be an LNVL. Assume that E is order complete and $x_\alpha \xrightarrow{up} x$, then $p(|x| - |x| \wedge |x_\alpha|) \xrightarrow{o} 0$ and $p(x) = \liminf_\alpha p(|x| \wedge |x_\alpha|)$. Moreover, if x_α is p-bounded, then $p(x) \leq \liminf_\alpha p(x_\alpha)$.*

Proof. Note that

$$|x|-|x|\wedge|x_\alpha| = \big|\,|x_\alpha|\wedge|x|-|x|\wedge|x|\,\big| \leq \big|\,|x_\alpha|-|x|\,\big|\wedge|x| \leq |x_\alpha-x|\wedge|x|.$$

Since $x_\alpha \xrightarrow{up} x$, we get $p(|x|-|x|\wedge|x_\alpha|) \xrightarrow{o} 0$. Thus

$$p(x)=p(|x|)\leq p(|x|-|x|\wedge|x_\alpha|)+p(|x|\wedge|x_\alpha|).$$

So $p(x)\leq \liminf_\alpha p(|x|\wedge|x_\alpha|)$. Hence $p(x)=\liminf_\alpha p(|x|\wedge|x_\alpha|)$. □

Lemma 3.7. *Let (X,p,E) be an op-continuous LNVL. Assume that E is order complete and $x_\alpha \xrightarrow{uo} x$, then $p(|x|-|x|\wedge|x_\alpha|)\xrightarrow{o} 0$ and $p(x)=\liminf_\alpha p(|x|\wedge|x_\alpha|)$. Moreover, if x_α is p-bounded, then $p(x)\leq\liminf_\alpha p(x_\alpha)$.*

We finish this subsection with the following technical lemma.

Lemma 3.8. *Given an LNVL (X,p,E). If $x_\alpha \xrightarrow{p} x$ and x_α is an o-Cauchy net, then $x_\alpha \xrightarrow{o} x$. Moreover, if $x_\alpha \xrightarrow{p} x$ and x_α is uo-Cauchy, then $x_\alpha \xrightarrow{uo} x$.*

Proof. Since x_α is order Cauchy, then $x_\alpha - x_\beta \xrightarrow{o} 0$ as $\alpha,\beta\to\infty$. So there exists $z_\gamma\downarrow 0$ such that, for every γ, there exists α_γ satisfying

$$|x_\alpha - x_\beta|\leq z_\gamma,\quad \forall\alpha,\beta\geq\alpha_\gamma. \tag{3.1}$$

By taking limit over β in (3.1) and applying Lemma 2.1, we get $|x_\alpha - x|\leq z_\gamma$ for all $\alpha\geq\alpha_\gamma$. Thus $x_\alpha\xrightarrow{o} x$.

For the uo-convergence, the similar argument is used, so the proof is omitted. □

3.3. **up-Convergence and p-units.** The following result is a generalization of [7, Lm.2.11] and of [14, Cor.3.5] in LNVLs.

First of all, we recall useful characterizations of order convergence. For any order bounded net x_α in an order complete vector lattice E, $x_\alpha\xrightarrow{o} x$ iff $\limsup\limits_\alpha |x_\alpha - x| = \inf_\alpha\sup_{\beta\geq\alpha}|x_\beta - x| = 0$. Moreover, for any net x_α in a vector lattice E, $x_\alpha\xrightarrow{o}0$ in E iff $x_\alpha\xrightarrow{o}0$ in E^δ (the order completion of E); see, e.g., [14, Cor.2.9].

Theorem 3.9. *Let (X,p,E) be an LNVL and $e\in X_+$ be a p-unit. Then $x_\alpha\xrightarrow{up}0$ iff $p(|x_\alpha|\wedge e)\xrightarrow{o}0$ in E.*

Proof. The "only if" part is trivial. For the "if" part, let $u\in X_+$, then

$$|x_\alpha|\wedge u\leq |x_\alpha|\wedge(u-u\wedge ne)+|x_\alpha|\wedge(u\wedge ne)\leq (u-u\wedge ne)+n(|x_\alpha|\wedge e),$$

and so

$$p(|x_\alpha|\wedge u)\leq p(u-u\wedge ne)+np(|x_\alpha|\wedge e)$$

holds in E^δ for any α and any $n\in\mathbb{N}$. Hence

$$\limsup_\alpha p(|x_\alpha|\wedge u)\leq p(u-u\wedge ne)+n\limsup_\alpha p(|x_\alpha|\wedge e)$$

holds in E^δ for all $n\in\mathbb{N}$. Since $p(|x_\alpha|\wedge e)\xrightarrow{o}0$ in E, then $p(|x_\alpha|\wedge e)\xrightarrow{o}0$ in E^δ, and so $\limsup\limits_\alpha p(|x_\alpha|\wedge e)=0$ in E^δ. Thus

$$\limsup_\alpha p(|x_\alpha|\wedge u)\leq p(u-u\wedge ne)$$

holds in E^δ for all $n \in \mathbb{N}$. Since e is a p-unit, we have that $\limsup_\alpha p(|x_\alpha| \wedge u) = 0$ in E^δ or $p(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E^δ. It follows that $p(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E and hence $x_\alpha \xrightarrow{up} 0$. □

3.4. ***up*-Convergence and sublattices.** Given an LNVL (X, p, E), a sublattice Y of X, and a net $(y_\alpha)_\alpha \subseteq Y$. Then $y_\alpha \xrightarrow{up} 0$ in Y has the meaning that

$$p(|y_\alpha| \wedge u) \xrightarrow{o} 0 \qquad (\forall u \in Y_+).$$

The following lemma is a p-version of [15, Lm.3.3].

Lemma 3.10. *Let (X, p, E) be the LNVL, B be a projection band of X, and P_B be the corresponding band projection. If $x_\alpha \xrightarrow{up} x$ in X, then $P_B(x_\alpha) \xrightarrow{up} P_B(x)$ in both X and B.*

Proof. It is known that P_B is a lattice homomorphism and $0 \leq P_B \leq I$. Since $|P_B(x_\alpha) - P_B(x)| = P_B|x_\alpha - x| \leq |x_\alpha - x|$, then it follows easily that $P_B(x_\alpha) \xrightarrow{up} P_B(x)$ in both X and B. □

Let (X, p, E) be an LNVL and Y be a subset of X. Then Y is called *up-closed* in X if, for any net y_α in Y that is up-convergent to $x \in X$, we have $x \in Y$. Clearly, every band is up-closed.

We present a p-version of [14, Prop.3.15] with a similar proof.

Proposition 3.11. *Let X be an LNVL and Y be a sublattice of X. Suppose that either X is op-continuous or Y is a p-KB-space in its own right. Then Y is up-closed in X iff it is p-closed in X.*

Proof. Only the sufficiency requires a proof. Let Y be p-closed in X and y_α be a net in Y with $y_\alpha \xrightarrow{up} x \in X$. WLOG, we assume $(y_\alpha)_\alpha \subseteq Y_+$ because the lattice operations in X are p-continuous. Note that, for every $z \in X_+$, $|y_\alpha \wedge z - x \wedge z| \leq |y_\alpha - x| \wedge z$ (cf. the inequality (1) in the proof of [14, Prop.3.15]). So $p(y_\alpha \wedge z - x \wedge z) \leq p(|y_\alpha - x| \wedge z) \xrightarrow{o} 0$. In particular, $Y \ni y_\alpha \wedge y \xrightarrow{p} x \wedge y$ in X for any $y \in Y_+$. Since Y is p-closed, $x \wedge y \in Y$ for any $y \in Y_+$. Since, for any $0 \leq z \in Y^\perp$ and for any α we have $y_\alpha \wedge z = 0$, then

$$|x \wedge z| = |y_\alpha \wedge z - x \wedge z| \leq |y_\alpha - x| \wedge z \xrightarrow{p} 0.$$

Therefore, $x \wedge z = 0$, and hence $x \in Y^{\perp\perp}$. Since $Y^{\perp\perp}$ is the band generated by Y in X, there is a net z_β in the ideal I_Y generated by Y such that $0 \leq z_\beta \uparrow x$ in X. Take for every β an element $w_\beta \in Y$ with $z_\beta \leq w_\beta$. Then $x \geq w_\beta \wedge x \geq z_\beta \wedge x = z_\beta \uparrow x$ in X, and so $w_\beta \wedge x \xrightarrow{o} x$ in X.

Case 1: If X is op-continuous, then $w_\beta \wedge x \xrightarrow{p} x$. Since $w_\beta \wedge x \in Y$ and Y is p-closed, we get $x \in Y$.

Case 2: Suppose Y is a p-KB-space in its own right. Let Δ be the collection of all finite subsets of the index set B. For each $\delta = \{\beta_1, \ldots, \beta_n\} \in \Delta$ let $y_\delta := (w_{\beta_1} \vee \ldots \vee w_{\beta_n}) \wedge x$. Clearly, $y_\delta \in Y$, $0 \leq y_\delta \uparrow$, and the net (y_δ) is p-bounded in Y. Since Y is a p-KB-space, then there is $y_0 \in Y$ such that $y_\delta \xrightarrow{p} y_0$

in Y and trivially in X. Since (y_δ) is monotone then it follows from Proposition 2.5 that $y_\delta \uparrow y_0$ in X. Also, we have $y_\delta \xrightarrow{o} x$ in X. Thus, $x = y_0 \in Y$. □

3.5. **p-Almost order bounded sets.** Recall that a subset A in a normed lattice $(X, \|\cdot\|)$ is said to be almost order bounded if, for any $\epsilon > 0$, there is $u_\epsilon \in X_+$ such that $\|(|x| - u_\epsilon)^+\| = \||x| - u_\epsilon \wedge |x|\| \leq \epsilon$ for any $x \in A$. Similarly we have:

Definition 3.12. Given an LNVL (X, p, E). A subset A of X is called a *p-almost order bounded* if, for any $w \in E_+$, there is $x_w \in X_+$ such that $p((|x| - x_w)^+) = p(|x| - x_w \wedge |x|) \leq w$ for any $x \in A$.

It is clear that p-almost order boundedness notion in LNVLs is a generalization of almost order boundedness in normed lattices. In the LNVL $(X, |\cdot|, X)$, a subset of X is p-almost order bounded, iff it is order bounded in X.

The following result is a p-version of [7, Lm.2.9], and it is also similar to [15, Prop.3.7].

Proposition 3.13. *If (X, p, E) is an LNVL, x_α is p-almost order bounded, and $x_\alpha \xrightarrow{up} x$, then $x_\alpha \xrightarrow{p} x$.*

Proof. Since x_α is p-almost order bounded, then it is easy to see that the net $(|x_\alpha - x|)_\alpha$ is also p-almost order bounded. So given $w \in E_+$. Then there exists $x_w \in X_+$ with

$$p(|x_\alpha - x| - |x_\alpha - x| \wedge x_w) \leq w$$

But $x_\alpha \xrightarrow{up} x$, so $\limsup\limits_\alpha p(|x_\alpha - x| \wedge x_w) = 0$ in E^δ. Thus, for any α,

$$p(x_\alpha - x) = p(|x_\alpha - x|) \leq p(|x_\alpha - x| - |x_\alpha - x| \wedge x_w) + p(|x_\alpha - x| \wedge x_w) \leq w + p(|x_\alpha - x| \wedge x_w)$$

Hence

$$\limsup_\alpha p(x_\alpha - x) \leq w + \limsup_\alpha p(|x_\alpha - x| \wedge x_w) \leq w$$

holds in E^δ. But $w \in E_+$ is arbitrary, so $\limsup\limits_\alpha p(x_\alpha - x) = 0$ in E^δ. Thus $p(x_\alpha - x) \xrightarrow{o} 0$ in E^δ, and so in E. □

The following proposition is a p-version of [15, Prop.4.2].

Proposition 3.14. *Given an op-continuous and p-complete LNVL (X, p, E). Then every p-almost order bounded uo-Cauchy net is uo- and p-convergent to the same limit.*

Proof. Let x_α be a p-almost order bounded *uo*-Cauchy net. Then the net $(x_\alpha - x_{\alpha'})$ is p-almost order bounded and is *uo*-converges to 0. Since X is *op*-continuous, then $x_\alpha - x_{\alpha'} \xrightarrow{up} 0$ and, by Proposition 3.13, we get $x_\alpha - x_{\alpha'} \xrightarrow{p} 0$. Thus x_α is p-Cauchy, and so is p-convergent. By Lemma 3.8, we get that x_α is also *uo*-convergent to its p-limit. □

3.6. rup**-Convergence.** In this subsection, we introduce the notions of rup-convergence and of an rp-unit. Recall that a net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E is *relatively uniform convergent* (or *ru-convergent*, for short) to $x \in E$ if there is $y \in E_+$, such that, for any $\varepsilon > 0$, there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \le \varepsilon y$ for any $\alpha \ge \alpha_0$, [21, Thm.16.2]. In this case we write $x_\alpha \xrightarrow{ru} x$.

Definition 3.15. Let (X, p, E) be an LNVL. A net $(x_\alpha)_\alpha \subseteq X$ is said to be *relatively unbounded p-convergent* (*rup-convergent*) to $x \in X$ if

$$p(|x_\alpha - x| \wedge u) \xrightarrow{ru} 0 \quad (\forall u \in X_+).$$

In this case we write $x_\alpha \xrightarrow{rup} x$.

Clearly, rup-convergence implies up-convergence, but the converse need not be true.

Definition 3.16. Given an LNVL (X, p, E). A vector $e \in X$ is called an rp-unit if, for any $x \in X_+$, we have $p(x - x \wedge ne) \xrightarrow{ru} 0$.

Obviously, every rp-unit is a p-unit. So, by Remark 2.23 (1) after Definition 2.22, if $e \in X \neq \{0\}$ is an rp-unit then $e > 0$. Not every p-unit is an rp-unit. To see this, take $X = (C_b(\mathbb{R}), |\cdot|, C_b(\mathbb{R}))$ and $e = e(t) = e^{-|t|}$. Then e is a p-unit. However, e is not an rp-unit since $p(1 - 1 \wedge ne)$ does not ru-converge to 0, where $1(t) \equiv 1$.

The following result generalizes [14, Cor.3.5] and [7, Lm.2.11].

Proposition 3.17. *Let (X, p, E) be an LNVL with an rp-unit e. Then $x_\alpha \xrightarrow{rup} 0$ iff $p(|x_\alpha| \wedge e) \xrightarrow{ru} 0$.*

Proof. The "only if" part is trivial. For the "if" part let $u \in X_+$, then

$$|x_\alpha| \wedge u \le |x_\alpha| \wedge (u - u \wedge ne) + |x_\alpha| \wedge (u \wedge ne) \le (u - u \wedge ne) + n(|x_\alpha| \wedge e),$$

and so

$$p(|x_\alpha| \wedge u) \le p(u - u \wedge ne) + np(|x_\alpha| \wedge e)$$

holds for any α and any $n \in \mathbb{N}$.

Given $\varepsilon > 0$. Since e is an rp-unit, then there is $y \in E_+$ and $n_0 \in \mathbb{N}$ such that

$$p(u - u \wedge n_0 e) \le \frac{\varepsilon}{2} y.$$

It follows from $p(|x_\alpha| \wedge e) \xrightarrow{ru} 0$ that there exists $z \in E_+$ and α_0 such that

$$p(|x_\alpha| \wedge e) \le \frac{\varepsilon}{2n_0} z$$

for any $\alpha \ge \alpha_0$. Take $w := y \vee z$, then

$$p(|x_\alpha| \wedge u) \le \varepsilon w$$

for any $\alpha \ge \alpha_0$. Therefore, $p(|x_\alpha| \wedge u) \xrightarrow{ru} 0$. □

4. *up*-Regular sublattices

The *up*-convergence passes obviously to any sublattice of X. As it was remarked in [7, p.3], in opposite to *uo*-convergence [14, Thm.3.2], the *un*-convergence does not pass even from regular sublattices. These two facts motivate the following notion in LNVLs.

Definition 4.1. Let (X, p, E) be an LNVL and Y be a sublattice of X. Then Y is called *up-regular* if, for any net y_α in Y, $y_\alpha \xrightarrow{up} 0$ in Y implies $y_\alpha \xrightarrow{up} 0$ in X. Equivalently, Y is *up*-regular in X when $y_\alpha \xrightarrow{up} 0$ in Y iff $y_\alpha \xrightarrow{up} 0$ in X for any net y_α in Y.

It is clear that if Y is a regular sublattice of a vector lattice X, then Y is *up*-regular in the LNVL $(X, |\cdot|)$; see [14, Thm.3.2]. The converse does not hold in general.

Example 4.2. Let $X = B([0, 1])$ be the space of all real-valued bounded functions on $[0, 1]$ and $Y = C[0, 1]$. First of all X under the pointwise ordering (i.e., $f \leq g$ in X iff $f(t) \leq g(t)$ for all $t \in [0, 1]$) is a vector lattice and if we equip X with the ∞-norm, then it becomes a Banach lattice.
We claim that the sublattice $Y = (Y, |\cdot|, Y)$ is a *up*-regular sublattice of $X = (X, |\cdot|, X)$. Let (f_α) be a net in Y such that $f_\alpha \xrightarrow{up} 0$ in Y. That is $|f_\alpha| \wedge g \xrightarrow{o} 0$ in X for any $g \in Y_+$. In particular, we have $|f_\alpha| \wedge \mathbf{1} \xrightarrow{o} 0$ in X, where $\mathbf{1}$ denotes the constant function one. Since $\mathbf{1}$ is a strong unit in X, then it is a p-unit for the LNVL $(X, |\cdot|, X)$. It follows from Theorem 3.9 in Subsection 5.1.3 that $f_\alpha \xrightarrow{up} 0$ in X. However, the sublattice Y is not regular in X. Indeed, for each $n \in \mathbb{N}$ let f_n be a continuous function on $[0, 1]$ defined as:

- f_n is zero on the intervals $[0, \frac{1}{2} - \frac{1}{n+2}]$ and $[\frac{1}{2} + \frac{1}{n+2}, 1]$,
- $f_n(\frac{1}{2}) = 1$,
- f_n is linear on the intervals $[\frac{1}{2} - \frac{1}{n+2}, \frac{1}{2}]$ and $[\frac{1}{2}, \frac{1}{2} + \frac{1}{n+2}]$.

Then $f_n \downarrow 0$ in $C[0, 1]$ but $f_n \downarrow \chi_{\frac{1}{2}}$ in $B([0, 1])$. So by Lemma 2.5 in [14], we have that Y is not regular in X.

Consider the sublattice c_0 of ℓ_∞. Then $(c_0, \|\cdot\|_\infty, \mathbb{R})$ is not *up*-regular in the LNVL $(\ell_\infty, \|\cdot\|_\infty, \mathbb{R})$. Indeed, (e_n) is *un*-convergent in c_0 but not in ℓ_∞. However, $(c_0, |\cdot|, \ell_\infty)$ is *up*-regular in the LNVL $(\ell_\infty, |\cdot|, \ell_\infty)$.

4.1. Several basic results.

We begin with the following result which is a p-version of [16, Thm.4.3]

Theorem 4.3. *Let Y be a sublattice of an LNVL $X = (X, p, E)$. Then Y is up-regular in each of the following cases:*
(i) Y is majorizing in X;
(ii) Y is p-dense in X;
(iii) Y is a projection band in X.

Proof. Let $(y_\alpha) \subseteq Y$ be such that $y_\alpha \xrightarrow{up} 0$ in Y. Let $0 \neq x \in X_+$.

(i) There exists $y \in Y$ such that $x \leq y$. It follows from

$$0 \leq |y_\alpha| \wedge x \leq |y_\alpha| \wedge y \xrightarrow{p} 0,$$

that $y_\alpha \xrightarrow{up} 0$ in X.

(ii) Choose an arbitrary $0 \neq u \in p(X)$. Then there exists $y \in Y$ with $p(x-y) \leq u$. Since

$$|y_\alpha| \wedge x \leq |y_\alpha| \wedge |x-y| + |y_\alpha| \wedge |y|,$$

then

$$p(|y_\alpha| \wedge x) \leq p(|y_\alpha| \wedge |x-y|) + p(|y_\alpha| \wedge |y|) \leq u + p(|y_\alpha| \wedge |y|).$$

Since $0 \neq u \in p(X)$ is arbitrary and $|y_\alpha| \wedge |y| \xrightarrow{p} 0$, then $|y_\alpha| \wedge x \xrightarrow{p} 0$. Hence $y_\alpha \xrightarrow{up} 0$ in X.

(iii) $Y = Y^{\perp\perp}$ implies that $X = Y \oplus Y^\perp$. Hence $x = x_1 + x_2$ with $x_1 \in Y$ and $x_1 \in Y^\perp$ Since $y_\alpha \wedge x_2 = 0$, we have

$$p(y_\alpha \wedge x) = p(y_\alpha \wedge (x_1 + x_2)) = p(y_\alpha \wedge x_1) \xrightarrow{o} 0.$$

Hence $y_\alpha \xrightarrow{up} 0$ in X. □

The following result deals with a particular case of Example 1.5.

Theorem 4.4. *Let X be a vector lattice and $Y = X_n^\sim$. Assume $X_n^\sim$ separates the points of X. Define $p : X \to \mathbb{R}^Y$ by $p(x)[y] = |y|(|x|)$. Then any ideal of X is up-regular in $(X, p, \mathbb{R}^Y)$.*

Proof. Let I be an ideal of X and x_α be a net in I such that $x_\alpha \xrightarrow{up} 0$ in I. We show $x_\alpha \xrightarrow{up} 0$ in X. By Example 3.2, this is equivalent to show $|x_\alpha| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$ for any $u \in X_+$. First note that if $v \in I^\perp$, then $|x_\alpha| \wedge |v| = 0$, and so, for any $w \in (I \oplus I^\perp)_+$, we have $|x_\alpha| \wedge w \xrightarrow{|\sigma|(X,Y)} 0$. Note also that $I \oplus I^\perp$ is order dense (see, e.g., [2, Thm.3.3.(2)]). Let $u \in X_+$ and $y \in Y$, then there is a net w_β in $(I \oplus I^\perp)_+$ such that $w_\beta \uparrow u$, and so $|y|(w_\beta \wedge u) \uparrow |y|(u)$. Given $\varepsilon > 0$. There is β_0 such that

$$|y|(u) - |y|(w_{\beta_0} \wedge u) < \frac{\varepsilon}{2}.$$

Also there is α_0 such that

$$|y|(|x_\alpha| \wedge w_{\beta_0}) < \frac{\varepsilon}{2}$$

for all $\alpha \geq \alpha_0$. Taking into account the inequality $|a \wedge c - b \wedge c| \leq |a - c|$ (cf. [2, Thm.1.6.(2)]) we have for any $\alpha \geq \alpha_0$,

$$\begin{aligned} |y|(|x_\alpha| \wedge u) &= |y|(|x_\alpha| \wedge u) - |y|(|x_\alpha| \wedge u \wedge w_{\beta_0}) + |y|(|x_\alpha| \wedge u \wedge w_{\beta_0}) \\ &\leq |y|(u) - |y|(w_{\beta_0} \wedge u) + |y|(|x_\alpha| \wedge w_{\beta_0}) < \varepsilon. \end{aligned}$$

Since $u \in X_+$ and $y \in Y$ are arbitrary, we get $|x_\alpha| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$ for any $u \in X_+$, and this completes the proof. □

The next Corollary might be compared with [16, Cor.4.6]

Corollary 4.5. *Let X be a vector lattice and $Y = X_n^\sim$ be the* order continuous dual. *Assume that $X_n^\sim$ separates the points of X. Define $p : X \to \mathbb{R}^Y$ by $p(x)[y] = |y|(|x|)$. Then any sublattice of X is up-regular in the LNVL $(X, p, \mathbb{R}^Y)$.*

Proof. Let X_0 be a sublattice of X and x_α be a net in X_0 such that $x_\alpha \xrightarrow{up} 0$ in X_0. Let I_{X_0} be the ideal generated by X_0 in X. Then X_0 is majorizing in I_{X_0} and, by Theorem 4.3(i), we get $x_\alpha \xrightarrow{up} 0$ in I_{X_0}. Now, Theorem 4.4 implies that $x_\alpha \xrightarrow{up} 0$ in X. □

4.2. **Order completion.** Denote by X^δ the order completion of a vector lattice X.

Lemma 4.6. *Let (X^δ, p, E) be an LNVL, where X^δ is the order completion of X. For any sublattice $Y \subseteq X$, if Y^δ is up-regular in X^δ, then Y is up-regular in $X = (X, p|_X, E)$.*

Proof. Take a net $(y_\alpha)_\alpha \subseteq Y$ such that $y_\alpha \xrightarrow{up} 0$ in Y. Then $p(|y_\alpha| \wedge u) \xrightarrow{o} 0$ for all $u \in Y_+$. Let $w \in Y^\delta$ and, since Y is majorizing in Y^δ, there exists $y \in Y$ such that $w \leq y$. Therefore, we obtain $y_\alpha \xrightarrow{up} 0$ in Y^δ. Since Y^δ is *up*-regular in X^δ, the net y_α is *up*-convergent to 0 in X^δ, and so in X. □

Lemma 4.7. *Let (X^δ, p, E) be an LNVL. For any sublattice $Y \subseteq X$, if Y is up-regular in X, then Y is up-regular in X^δ.*

Proof. Let $(y_\alpha)_\alpha \subseteq Y$ and $y_\alpha \xrightarrow{up} 0$ in Y. By assumption $y_\alpha \xrightarrow{up} 0$ in X. Let $u \in X_+^\delta$, then there exists $x \in X$ such that $u \leq x$. Therefore, we obtain $p(|y_\alpha| \wedge u) \leq p(|y_\alpha| \wedge x) \xrightarrow{o} 0$, i.e. $y_\alpha \xrightarrow{up} 0$ in X^δ. □

In connection with Lemma 4.7, the following question arises.

Problem 4.8. Is it true that I^δ is *up*-regular in X^δ, whenever I is a *up*-regular ideal in X?

Proposition 4.9. *Let (X, p, E) be an LNVL. Define $p_L^\delta : X^\delta \to E^\delta$ and $p_U^\delta : X^\delta \to E^\delta$ as follows: $p_L^\delta(z) = \sup\limits_{0 \leq x \leq |z|} p(x)$ and $p_U^\delta(z) = \inf\limits_{|z| \leq x} p(x)$ for all $z \in X^\delta$ (clearly, both p_U^δ and p_L^δ are extensions of p). Then:*

- *(i) p_L^δ is a monotone E^δ-valued norm;*
- *(ii) p_U^δ is a monotone E^δ-valued seminorm;*
- *(iii) if X is op-continuous, then p_U^δ is p-continuous (i.e. $z_\gamma \downarrow 0$ in X^δ implies $p_U^\delta(z_\gamma) \downarrow 0$ in E^δ);*
- *(iv) if X is op-continuous, then $p_U^\delta = p_L^\delta$.*

Proof. (*i*) Let $X^\delta \ni z \neq 0$. Since X is order dense in X^δ, there is $x \in X$ such that $0 < x \leq |z|$, and so $p_L^\delta(z) \geq p(x) > 0$.

Let $0 \neq \lambda \in \mathbb{R}$, then

$$p_L^\delta(\lambda z) = \sup_{0 \leq x \leq |\lambda z|} p(x) = \sup_{0 \leq \frac{1}{|\lambda|} x \leq |z|} p(x) = |\lambda| \sup_{0 \leq \frac{1}{|\lambda|} x \leq |z|} p(|\lambda|^{-1} x) = |\lambda| p_L^\delta(z).$$

Let $z, w \in X^\delta$, we show $p_L^\delta(z+w) \leq p_L^\delta(z) + p_L^\delta(w)$. Suppose $0 \leq x \leq |z+w|$, then $0 \leq x \leq |z| + |w|$. By the Riesz Decomposition Property, there exist $x_1, x_2 \in X$ such that $0 \leq x_1 \leq |z|$, $0 \leq x_2 \leq |w|$, and $x = x_1 + x_2$. So

$$p(x) = p(x_1 + x_2) \leq p(x_1) + p(x_2) \leq p_L^\delta(z) + p_L^\delta(w).$$

Thus $p_L^\delta(z+w) = \sup\limits_{0 \leq x \leq |z+w|} p(x) \leq p_L^\delta(z) + p_L^\delta(w)$.

Now, we prove the monotonicity of the lattice norm p_L^δ. If $|z| \leq |w|$ then, for any $x \in X$ with $0 \leq x \leq |z|$, we get $0 \leq x \leq |w|$. So $\sup\limits_{0 \leq x \leq |z|} p(x) \leq \sup\limits_{0 \leq x \leq |w|} p(x)$ or $p_L^\delta(z) \leq p_L^\delta(w)$.

(ii) We show firstly the triangle inequality. Let $z, w \in X^\delta$ and $x_1, x_2 \in X$ be such that $|z| \leq x_1$ and $|w| \leq x_2$, then $|z+w| \leq |z| + |w| \leq x_1 + x_2$. So

$$p_U^\delta(z+w) = \inf_{|z+w| \leq x} p(x) \leq p(x_1 + x_2) \leq p(x_1) + p(x_2).$$

Thus $p_U^\delta(z+w) - p(x_1) \leq p(x_2)$ for any $x_2 \in X$ with $|w| \leq x_2$. Hence $p_U^\delta(z+w) - p(x_1) \leq p_U^\delta(w)$ or $p_U^\delta(z+w) - p_U^\delta(w) \leq p(x_1)$, which holds for all $x_1 \in X$ with $|z| \leq x_1$. Therefore, $p_U^\delta(z+w) - p_U^\delta(w) \leq p_U^\delta(z)$ or $p_U^\delta(z+w) \leq p_U^\delta(w) + p_U^\delta(z)$.

Now, if $|z| \leq |w|$, then, for any $x \in X$ with $0 < |w| \leq x$, we have $|z| \leq x$. So $\inf\limits_{|w| \leq x} p(x) \geq \inf\limits_{|z| \leq x} p(x)$ or $p_U^\delta(z) \leq p_U^\delta(w)$.

(iii) Assume $z_\gamma \downarrow 0$ in X^δ. Let $A = \{a \in X : z_\gamma \leq a \text{ for some } \gamma\}$. Then $\inf A = 0$. Indeed, if $0 \leq x \leq a$ for all $a \in A$, then $0 \leq x \leq A_\gamma$ for all γ, where $A_\gamma = \{a \in X : z_\gamma \leq a\}$. So, by [14, Lm.2.7], we have $x \leq z_\alpha$. Thus $x = 0$.

Clearly, A is directed downward and dominates the net $(z_\alpha)_\alpha$. Since X is *op*-continuous, then $p(A) \downarrow 0$ and, by the definition of p_U^δ, we get that $p(A)$ dominates the net $(p_U^\delta z_\alpha)$. Therefore, $p_U^\delta z_\alpha \downarrow 0$.

(iv) Let $z \in X^\delta$, then $|z| = \sup\{x \in X : 0 \leq x \leq |z|\}$. By (iii), we have

$$\begin{aligned} p_U^\delta(z) = p_U^\delta(|z|) &= \sup\{p_U^\delta(x) : x \in X, 0 \leq x \leq |z|\} \\ &= \sup\{p(x) : x \in X, 0 \leq x \leq |z|\} = p_L^\delta(z). \end{aligned}$$

□

In connection with Proposition 4.9(iv), the following question arises.

Problem 4.10. Does the equality $p_U^\delta = p_L^\delta$ imply the *op*-continuity of X?

Proposition 4.11. *Let (X, p, E) be an LNVL. Then, for every net x_α in X,*

$$x_\alpha \xrightarrow{up} 0 \ \text{ in } (X, p, E) \quad \Leftrightarrow \quad x_\alpha \xrightarrow{up} 0 \ \text{ in } (X^\delta, p^\delta, E^\delta),$$

where $p^\delta = p_L^\delta$.

Proof. Assume $x_\alpha \xrightarrow{up} 0$ in (X, p, E). Then $p(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E for all $x \in X_+$, and so $p(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E^δ for all $x \in X_+$, by [14, Cor.2.9]. Hence

$$p^\delta(|x_\alpha| \wedge x) \xrightarrow{o} 0 \tag{4.1}$$

in E^δ for all $x \in X_+$. Let $u \in X_+^\delta$, then there exists $x_u \in X_+$ such that $u \leq x_u$, since X majorizes X^δ. From (4.1) it follows that $p^\delta(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E^δ. Since $u \in X_+^\delta$ is arbitrary, then $x_\alpha \xrightarrow{up} 0$ in $(X^\delta, p^\delta, E^\delta)$.

Conversely, assume $x_\alpha \xrightarrow{up} 0$ in $(X^\delta, p^\delta, E^\delta)$ then, for all $u \in X_+^\delta$, $p^\delta(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E^δ. In particular, for all $x \in X_+$, $p(|x_\alpha| \wedge x) = p^\delta(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E^δ. By [14, Cor.2.9], $p(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E for all $x \in X_+$. Hence $x_\alpha \xrightarrow{up} 0$ in (X, p, E). □

5. Mixed-normed spaces

In this section, we study LNVLs with mixed lattice norms.

5.1. **Mixed norms.** Let (X, p, E) be an LNS and $(E, \|\cdot\|)$ be a normed lattice. The *mixed norm* on X is defined by

$$p\text{-}\|x\| = \|p(x)\| \quad (\forall x \in X).$$

In this case the normed space $(X, p\text{-}\|\cdot\|)$ is called a *mixed-normed space* (see, for example [20, 7.1.1, p.292])

The next proposition follows directly from the basic definitions and results, so its proof is omitted.

Proposition 5.1. *Let (X, p, E) be an LNVL, $(E, \|\cdot\|)$ be a Banach lattice, and $(X, p\text{-}\|\cdot\|)$ be a mixed-normed space. The following statements hold:*

(i) *if (X, p, E) is op-continuous and E is order continuous, then $(X, p\text{-}\|\cdot\|)$ is an order continuous normed lattice*;
(ii) *if a subset Y of X is p-bounded (respectively, p-dense) in (X, p, E), then Y is norm bounded (respectively, norm dense) in $(X, p\text{-}\|\cdot\|)$;*
(iii) *if $e \in X$ is a p-unit and E is order continuous, then e is a quasi-interior point of $(X, p\text{-}\|\cdot\|)$;*
(iv) *if (X, p, E) is a p-Fatou space and E is order continuous, then $p\text{-}\|\cdot\|$ is a Fatou norm,* [21, p.42];
(v) *if Y is a p-almost order bounded subset of X, then Y is almost order bounded set in $(X, p\text{-}\|\cdot\|)$.*

Theorem 5.2. *Let (X, p, E) and (E, m, F) be two p-KB-spaces. Then the LNVL $(X, m \circ p, F)$ is also a p-KB-space.*

Proof. Let $0 \le x_\alpha \uparrow$ and $m\big(p(x_\alpha)\big) \le g \in F$. Since $0 \le p(x_\alpha) \uparrow < \infty$ and since (E, m, F) is a p-KB-space, then there exists $y \in E$ such that $m\big(p(x_\alpha) - y)\big) \to 0$. Hence $p(x_\alpha) \uparrow y$. Thus the net x_α is increasing and p-bounded. Since X is p-KB-space, then there exists $x \in X$ such that $p(x_\alpha - x) \to 0$. As $(X, m \circ p, F)$ is clearly po-continuous, then $m\big(p(x_\alpha - x)\big) \xrightarrow{o} 0$ i.e. $m \circ p(x_\alpha - x) \xrightarrow{o} 0$. Thus $(X, m \circ p, F)$ is a p-KB-space. □

Corollary 5.3. *Let (X, p, E) be a p-KB-space and $(E, \|\cdot\|)$ be a KB-space. Then $(X, p\text{-}\|\cdot\|)$ is a KB-space.*

5.2. ***up*-Completeness.** The following well-known technical lemma is a particular case of Lemma 3.8.

Lemma 5.4. *Given a Banach lattice $(X, \|\cdot\|)$. If $x_\alpha \xrightarrow{\|\cdot\|} x$ and x_α is o-Cauchy, then $x_\alpha \xrightarrow{o} x$.*

Recall that a Banach lattice is called un-complete if every un-Cauchy net is un-convergent, [16].

Theorem 5.5. *Let (X,p,E) be an LNVL and $(E,\|\cdot\|)$ be an order continuous Banach lattice. If $(X,p\text{-}\|\cdot\|)$ is a un-complete Banach lattice, then X is up-complete.*

Proof. Let x_α be a up-Cauchy net in X. So, for every $u\in X_+$, $p(|x_\alpha - x_\beta|\wedge u)\xrightarrow{o}0$. Since E is order continuous, then, for every $u\in X_+$, $\|p(|x_\alpha - x_\beta|\wedge u)\|\to 0$ or for every $u\in X_+$, $p\text{-}\big\||x_\alpha - x_\beta|\wedge u\big\|\to 0$, i.e. x_α is un-Cauchy in $(X,p\text{-}\|\cdot\|)$. Since $(X,p\text{-}\|\cdot\|)$ is un-complete, then there exists $x\in X$ such that $x_\alpha\xrightarrow{un}x$ in $(X,p\text{-}\|\cdot\|)$. That is, for every $u\in X_+$, $\|p(|x_\alpha - x|\wedge u)\|\to 0$. Next we show the net $\big(p(|x_\alpha - x|\wedge u)\big)_\alpha$ is order Cauchy in E. Indeed,

$$\big|p(|x_\alpha - x|\wedge u) - p(|x_\beta - x|\wedge u)\big| \le p\big(\big||x_\alpha - x|\wedge u - |x_\beta - x|\wedge u\big|\big)\le p(|x_\alpha - x_\beta|\wedge u)\xrightarrow{o}0.$$

Now, Lemma 5.4 above, implies that $p(|x_\alpha - x|\wedge u)\xrightarrow{o}0$. □

5.3. **up-Null nets and up-null sequences in mixed-normed spaces.** The following theorem is a p-version of [7, Thm.3.2] and a generalization of [14, Lm.6.7], as we take $(X,p,E)=(X,\|\cdot\|,\mathbb{R})$.

Theorem 5.6. *Let (X,p,E) be an op-continuous and p-complete LNVL, E an order continuous Banach lattice, and $X\ni x_\alpha\xrightarrow{up}0$. Then there exist an increasing sequence α_k of indices and a disjoint sequence $d_k\in X$ such that $(x_{\alpha_k}-d_k)\xrightarrow{p}0$ as $k\to\infty$.*

Proof. Consider the mixed norm $p\text{-}\|x\|=\|p(x)\|$. Since $p(|x_\alpha|\wedge u)\xrightarrow{o}0$ for all $u\in X_+$, then $p\text{-}\||x_\alpha|\wedge u\|=\|p(|x_\alpha|\wedge u)\|\xrightarrow{o}0$ that means $x_\alpha\xrightarrow{un}0$ in $(X,p\text{-}\|\cdot\|)$ by o-continuity of $(E,\|\cdot\|)$.

By [7, Thm.3.2], there exists an increasing sequence α_n of indices and a disjoint sequence d_n in X such that $x_{\alpha_n}-d_n\xrightarrow{p\text{-}\|\cdot\|}0$. Now $(X,p\text{-}\|\cdot\|)$ is a Banach lattice by [20, 7.1.3 (1), p.294]. So, by [27, Thm.VII.2.1] there is a further subsequence (α_{n_k}) such that $|x_{\alpha_{n_k}}-d_{n_k}|\xrightarrow{o}0$ in X. By op-continuity of X, $p(x_{\alpha_{n_k}}-d_{n_k})\xrightarrow{o}0$. □

The next corollary is a p-version of [7, Cor.3.5].

Corollary 5.7. *Let (X,p,E) be an op-continuous LNVL, E be an order continuous Banach lattice, and $X\ni x_\alpha\xrightarrow{up}0$. Then there exist an increasing sequence α_k of indices such that $x_{\alpha_k}\xrightarrow{up}0$.*

Proof. Let α_k and d_k be as in Theorem 5.6. Since the sequence d_k is disjoint, then $d_k\xrightarrow{uo}0$ by [14, Cor.3.6.]. Since X is op-continuous, then $d_k\xrightarrow{up}0$. Since

$$p(|x_{\alpha_k}-d_k|\wedge u)\le p(x_{\alpha_k}-d_k)\xrightarrow{o}0 \quad (\forall u\in X_+),$$

then $x_{\alpha_k}-d_k\xrightarrow{up}0$. Since $d_k\xrightarrow{up}0$, then $x_{\alpha_k}\xrightarrow{up}0$. □

Next proposition extends [7, Prop.4.1] to LNVLs.

Proposition 5.8. *Let (X, p, E) be a p-complete LNVL, $(E, \|\cdot\|)$ be an order continuous Banach lattice, and $X \ni x_n \xrightarrow{up} 0$. Then there exist a subsequence x_{n_k} of x_n such that $x_{n_k} \xrightarrow{uo} 0$ as $k \to \infty$.*

Proof. Suppose $x_n \xrightarrow{up} 0$, then, for all $u \in X_+$ $p(|x_n| \wedge u) \xrightarrow{o} 0$, and so $\|p(|x_n| \wedge u)\| \to 0$ since E is order continuous. Thus $|x_n| \wedge u \xrightarrow{p\text{-}\|\cdot\|} 0$, i.e. $x_n \xrightarrow{un} 0$ in $(X, p\text{-}\|\cdot\|)$. It follows from [20, 7.1.2, p.293] that the mixed-normed space $(X, p\text{-}\|\cdot\|)$ is a Banach lattice, and so by [7, Prop.4.1] there is a subsequence x_{n_k} of x_n such that $x_{n_k} \xrightarrow{uo} 0$ as $k \to \infty$. □

Next result is a p-version of [7, Thm.4.4].

Proposition 5.9. *Let (X, p, E) be an op-continuous and p-complete LNVL such that $(E, \|\cdot\|)$ is an order continuous atomic Banach lattice. Then a sequence in X is up-null iff every subsequence has a further subsequence which uo-converges to zero.*

Proof. The forward implication follows from Proposition 5.8. Conversely, let x_n be a sequence in X and assume that $x_n \not\xrightarrow{up} 0$. Then there is an atom $a \in E_+$, $u \in X_+$, $\varepsilon_0 > 0$ and a subsequence x_{n_k} of x_n satisfying $f_a\big(p(|x_{n_k}| \wedge u)\big) \geq \varepsilon_0$ for all k. By the hypothesis there exist a further subsequence $x_{n_{k_j}}$ of x_{n_k} which uo-converges to zero. By the op-continuity of X we get $p(|x_{n_{k_j}}| \wedge u) \xrightarrow{o} 0$, and so $f_a\big(p(|x_{n_{k_j}}| \wedge u)\big) \to 0$, which is a contradiction. □

Our last result is a p-version of [7, Lm.5.1].

Proposition 5.10. *Let (X, p, E) be an op-continuous p-complete LNVL and $(E, \|\cdot\|)$ be an order continuous Banach lattice. If X is atomic and x_n is an order bounded sequence such that $x_n \xrightarrow{p} 0$ in X, then $x_n \xrightarrow{o} 0$.*

Proof. The mixed-normed space $(X, p\text{-}\|\cdot\|)$ is an atomic order continuous Banach lattice such that $x_n \xrightarrow{p\text{-}\|\cdot\|} 0$, and so $x_n \xrightarrow{o} 0$ by [7, Lm.5.1]. □

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