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ON M-TH ROOTS OF NILPOTENT MATRICES*

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Dedicated to the memory of dear Professor Cem Tezer

Abstract. A new necessary and sufficient condition for the existence of an m-th root of a nilpotent matrix in terms of the multiplicities of Jordan blocks is obtained and expressed as a system of linear equations with nonnegative integer entries which is suitable for computer programming. Thus, computation of the Jordan form of the m-th power of a nilpotent matrix is reduced to a single matrix multiplication; conversely, the existence of an m-th root of a nilpotent matrix is reduced to the existence of a nonnegative integer solution to the corresponding system of linear equations. Further, an erroneous result in the literature on the total number of Jordan blocks of a nilpotent matrix having an m-th root is corrected and generalized. Moreover, for a singular matrix having an m-th root with a pair of nilpotent Jordan blocks of sizes s and l, a new m-th root is constructed by replacing that pair by another one of sizes $s+i$ and $l-i$, for special s, l, i. This method applies to solutions of a system of linear equations having a special matrix of coefficients. In addition, for a matrix A over an arbitrary field that is a sum of two commuting matrices, several results for the existence of m-th roots of A^k are obtained.

Key words. Jordan canonical form, Roots of nilpotent matrices, Rootless matrices, System of linear Diophantine equations, Nonnegative integer solutions.

AMS subject classifications. Primary 15A21; Secondary 15A24, 15A06, 15A16, 15B36, 15B05.

1. Introduction.

All matrices are assumed to be over a field $\mathbf{k}$ and of size $d \times d$ unless stated otherwise. We investigate m-th roots of matrices for an integer $m > 1$. Since all nonsingular complex matrices have m-th roots, the problem of existence of an m-th root for a singular complex matrix is reduced to considering a nilpotent one. Section 3 is devoted to our results on nilpotent matrices that have m-th roots or are rootless which constitutes the bulk of this article.

There has been continued interest on m-th roots of matrices even though sixth and seventh sections in Chapter VIII are devoted to the extraction of m-th roots in nonsingular and singular cases, respectively, in Gantmacher's book [2] dated 1959. Among the articles studying the roots of matrices or rootless matrices, we want to mention [5, 10, 11, 7, 8, 3, 9] in chronological order. More references can be found in [4], especially for square roots. For the use of matrix roots in medical imaging or more information on m-th roots, or Jordan forms, see [1]. Various characterizations of nilpotent matrices having an m-th root are found, one uses the sequence of the sizes of Jordan blocks of the matrix [5], another one uses the ascent sequence of the matrix [7], and another uses a_i's, where a_i is the multiplicity of the Jordan blocks of size i in the Jordan canonical form of the matrix [8]. Our initial inspiration for this article comes from Psarrakos's article on m-th roots of complex matrices [7] followed by Schwaiger's work on rootless nilpotent matrices [8]. Noticing that there is a mistake in Schwaiger's Theorem 1 (2) [8] that we could correct gave the final motivation to write this article. The erroneous statement and a counterexample is given in Remark 3.4. Our Corollary 3.3 (4) of Theorem 1.1 gives a more general result correcting the erroneous statement, and the other parts provide similar statements. Theorem 1.1 is obtained by a slight modification of Proposition 3.1 in [6]. For a nilpotent matrix A, we define *the Jordan type* of A as the vector $\underline{a} = (a_1, \ldots, a_d)$, or $\underline{a} = (a_1, \ldots, a_t)$ if the

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nilpotency t of A is explicit, where a_i is the multiplicity of the nilpotent Jordan block $[j_i]$ of size i in the Jordan canonical form of A, that is, $A \sim \operatorname{diag}([j_t]^{(a_t)}, \ldots, [j_1]^{(a_1)})$ where $\sim$ denotes the similarity relation between matrices and $\operatorname{diag}(X_1, \ldots, X_k)$ denotes a block diagonal matrix with the matrices X_i's as diagonal blocks.

THEOREM 1.1. *Suppose that A is a nonzero $d \times d$ nilpotent matrix over a field, with Jordan type $\underline{a} = (a_1, \ldots, a_d)$, and m is an integer with $1 < m < d$, then A has an m-th root if and only if there exist nonnegative integers $b_1, \ldots, b_d$ satisfying*

$$a_i = mb_{im} + \sum_{j=1}^{m-1} j\left[b_{(i-1)m+j} + b_{(i+1)m-j}\right] \quad \text{for } 1 \le i \le d, \tag{1.1}$$

where b_j is defined as 0 for $j > d$. In particular, if $\underline{b} = (b_1, \ldots, b_d)$ is the Jordan type for an m-th root B of A, and t, s are the nilpotencies of A, B, respectively, then

$$a_t = \sum_{j=1}^{m} jb_{(t-1)m+j}, \text{ where } b_j = 0 \text{ for } j > d \text{ if } tm > d, \tag{1.2}$$

$(t-1)m + 1 \le s \le tm$, and $t \le q + 1$ if $d = qm + r$, where $0 \le r < m$.

Theorem 1.1 is proved in Section 3 and has many implications, namely Corollaries 3.3, 3.5, 3.6, 3.11, 3.12, 3.13, and 3.14. The equations (1.1) and (1.2) become very easy to use when expressed as a matrix equation. In Corollary 3.6, we write these equations in two equivalent matrix equations with nonnegative integer entries. One of the equations is of the form $M\underline{b}^T = \underline{a}^T$, where M is a special matrix with entries $0, 1, 2, \ldots, m$ and T in the exponent denotes the transpose. We can regard M as an additive function from the additive semigroup of d-tuples $\mathbf{N}^d$ to itself, where $\mathbf{N}$ is the set of all nonnegative integers. If a d-tuple $\underline{a}$ in $\mathbf{N}^d$ is fixed and A is any nilpotent matrix of Jordan type $\underline{a}$, then A has an m-th root if and only if $\underline{a}$ is in the image of this function and the set of all preimages of $\underline{a}$ is the set of all possible Jordan types of m-th roots of A. The equation $M\underline{b}^T = \underline{a}^T$ provides a new insight and a new easy algorithm to use in computer calculations of the Jordan form of the m-th power of a nilpotent matrix from that of the matrix by a single matrix multiplication. The matrix M carries the information on the splittings of the Jordan block sizes after taking the m-th power and avoids eigenvector computations. Conversely, when a nilpotent matrix A with the Jordan type $\underline{a}$ is given, if a solution $\underline{b}$ with all nonnegative integer entries exists, then $\underline{b}$ is the Jordan type of an m-th root of A. That is, the existence of an m-th root of A is reduced to finding a nonnegative integer solution $\underline{b}$ to the equation $M\underline{b}^T = \underline{a}^T$. The crucial point here is that the system $M\underline{b}^T = \underline{a}^T$ is always consistent as a system of linear equations and has free variables; however, there is no guarantee that there will be a nonnegative integer solution; for matrices having no third roots, see Examples 3.2 (3) and 3.10. Thus, whenever a method is developed to find all nonnegative integer solutions $\underline{b}$ of the system of linear Diophantine equations $M\underline{b}^T = \underline{a}^T$, we will have all possible Jordan types of m-th roots of a nilpotent matrix A with the Jordan type $\underline{a}$. Due to the lack of a general method for determining the existence of nonnegative integer solutions, and a lack of a method for finding all nonnegative integer solutions to a system of linear Diophantine equations when there is one, we give many examples in Corollary 3.11 and Corollary 3.14 of Jordan types of a nilpotent matrix and an m-th root of it. We give many examples of rootless matrices in Corollary 3.13.

We point out the relevance of several of the corollaries of Theorem 1.1 with the results in the articles mentioned above. For a prime number p, Corollary 3.5 (5) implies that the set of all p-th roots of

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$\mathrm{diag}([j_t], [j_{t-1}]^{(p-1)})$ is a singleton set; hence, it is contractible. This provides an example of an extreme case of Theorem 1.2 in [9] stating that the set of all m-th roots of a nilpotent complex matrix is path-connected. Corollary 3.5 (1) provides a stronger result than the second statement in Psarrakos's Theorem 3.2 [7] which is stated in Section 2 below. The parts (3) and (4) of Corollary 3.5 give many conditions implying rootlessness. In fact, there are rootless and not rootless matrices of any nilpotency t for $2 < t < d$ and there is no rootless nilpotent matrix of nilpotency 2 when $d > 2$ by Corollary 3.12 which is analogous to Theorem 2 in [8].

Section 4 contains proofs of the results for not necessarily nilpotent matrices stated below. In the first part, we use Proposition 3.1 to obtain Theorem 1.2. For a singular matrix A over a field containing all eigenvalues of A, the Jordan canonical form of A is $J_A = \mathrm{diag}(R, N)$ when A is not nilpotent, and $J_A = N$ when A is nilpotent, where R is a nonsingular matrix and N is a nilpotent matrix. We refer to N as *the nilpotent part of* A.

Theorem 1.2. *Let $m > 1$, $k \geq 0$, $u, s, l, i \geq 1$ be integers satisfying*

$$0 \leq mk \leq s < s + i \leq m(k+1) \quad \text{and} \quad 0 \leq mk \leq l - i < l \leq m(k+1),$$

E be any square matrix. Let $\underline{b}$ and $\underline{c}$ be the Jordan types of the nilpotent parts of B and C, respectively, where

$$B = \mathrm{diag}(E, [j_s]^{(u)}, [j_l]^{(u)}) \text{ and } C = \mathrm{diag}(E, [j_{s+i}]^{(u)}, [j_{l-i}]^{(u)}).$$

Then $B^m \sim C^m$ and $\underline{b}$ and $\underline{c}$ have the same entries except for $c_s = b_s - u$, $c_l = b_l - u$, $c_{s+i} = b_{s+i} + u$, and $c_{l-i} = b_{l-i} + u$.

Theorem 1.2 gives a method of obtaining new m-th roots from a given m-th root B of a singular matrix A, provided that there is a pair of special size nilpotent Jordan blocks, say s and l, in the Jordan canonical form of B. Namely, a new m-th root is produced by replacing each such pair with the ones of sizes $s + i$, and $l - i$, for special s, l, and i, see Example 4.2. In a similar manner, new solutions for the matrix equation $M\underline{x}^T = \underline{a}^T$ of Corollary 3.6 can be obtained from a solution $\underline{x}$ having $x_s > 0$, $x_l > 0$ for s and l satisfying the hypothesis of that theorem, see Example 4.3.

In the second part of Section 4, we prove Theorem 1.3 which is about the existence of m-th roots of A^k, for $A = E + F$ where E and F are commuting matrices over various $\mathbf{k}$, see Corollary 4.4 where $\mathbf{k}$ is the complex numbers.

Theorem 1.3. *Suppose that $A = E + F$ is a $d \times d$ matrix over a field $\mathbf{k}$ and $m > 1$ is a fixed integer.*

1. *If $KL = LK$ where K and L are n-th roots of E and F, respectively, then A has an n-th root provided that $\mathrm{char}(\mathbf{k}) = p > 0$ and $n = p^m$ for some m, or $\mathrm{char}(\mathbf{k}) = 0$ and $LK = 0$.*
2. *If $EF = FE = 0$, F is nilpotent of nilpotency t and E^k has an m-th root for $k \geq t$, then A^k has an m-th root for any $k \geq t$. In particular, if the field $\mathbf{k}$ is the complex numbers, the hypothesis that E^k has an m-th root for $k \geq t$ can be replaced by E is diagonalizable or nonsingular.*

2. Preliminaries. This section contains an introduction to the existence problem of m-th roots of matrices for a positive integer m, and the preliminary lemmas about nilpotent matrices. The property of having m-th roots is invariant under the similarity relation of matrices. Hence, we can assume that A is in the Jordan canonical form, J_A, whenever it exists. For nilpotent matrices over any field, the Jordan canonical form exists as the only eigenvalue is 0. All nonsingular complex matrices have m-th roots for any m, and the sizes of the Jordan blocks are the same for the matrix and its m-th roots, see pages 231–232 in [2].

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However, a singular matrix has an m-th root if the elementary divisors of the matrix form an "admissible system," roughly speaking, the splittings are compatible, see page 238 in [2]. The problem of existence of an m-th root of a nonzero singular complex matrix A is reduced to considering a nilpotent matrix, because $J_A = \operatorname{diag}(R, N)$, where R is nonsingular and N is nilpotent. Recall that for a nilpotent matrix X with entries in a field, *the nilpotency* is n provided that $X^n = 0$ and $X^{n-1} \neq 0$; equivalently, n is the size of the largest Jordan block in its Jordan canonical form J_X. That is, if we denote the multiplicity of the upper triangular $k \times k$ nilpotent Jordan block $[j_k]$ by x_k, for $k = 1, \ldots, n$, we write

$$J_X = \operatorname{diag}([j_n]^{(x_n)}, [j_{n-1}]^{(x_{n-1})}, \ldots, [j_2]^{(x_2)}, [j_1]^{(x_1)}) \quad \text{or} \quad J_X = x_1[j_1] \oplus \cdots \oplus x_n[j_n],$$

when taking k-th power of the matrix, to avoid notation like $([j_s]^{(x_s)})^k$ we use the latter one.

A matrix is called *rootless* if it has no m-th root for any $m > 1$. An example of a rootless matrix is $[j_k]$ for any $k > 1$, see Corollary 3.5 (3); however, the diagonal block matrix $\operatorname{diag}([j_k]^{(2)})$ is not rootless as it is similar to $[j_{2k}]^2$, see Proposition 2.3. The fact that $\operatorname{diag}([j_k]^{(2)})$ has a square root similar to $[j_{2k}]$ demonstrates that there is a splitting of the Jordan block sizes after taking powers of nilpotent matrices which is the source of the problem for the existence of m-th roots of nilpotent matrices.

In Theorem 3.2 in [7], Psarrakos proves the following. "A complex matrix A has an m-th root if and only if the ascent sequence $d_1 \geq d_2 \geq \cdots$ of A has no more than one element between mk and $m(k+1)$ for every integer $k \geq 0$, where d_i is the number of Jordan blocks of size at least i. Moreover, if A is singular and $d_2 > 0$, then for every integer $m > d_1$, A has no m-th roots." He also gives a construction for an m-th root of a nilpotent matrix when it exists. Higham and Lin refine the first part of Psarrakos's Theorem for real matrices in Theorem 2.3 in [3]. The sixth section of Otero's article [5] and [8] are devoted to nilpotent matrices. Otero's Theorem 13 in [5] and Schwaiger's Theorem 1 (1) in [8] give a necessary and sufficient condition for a nilpotent matrix to have an m-th root. Otero uses the sequence $e_1 \geq e_2 \geq \cdots$ of the exponents of the elementary divisors of the matrix (i.e., the sizes of the Jordan blocks occurring in the Jordan canonical form of the matrix), Schwaiger uses the a_i's where $a_i (= d_i - d_{i-1})$ is the number of Jordan blocks of size i in the Jordan canonical form of the matrix. We also use a_i's in our Theorems 1.1 and 1.2. Theorem 1.1 can be viewed as a refinement of Schwaiger's Theorem 1 (1) in [8].

We start by a lemma relating m and d, and the nilpotency t of a nilpotent matrix A having an m-th root.

Lemma 2.1. *If A is a nonzero $d \times d$ nilpotent matrix having an m-th root for some $m > 1$, then $m < d$. Moreover, $m \leq d - t + 1$, where t is the nilpotency of A.*

Proof. Let B be an m-th root for A. Since B is nilpotent of size $d \times d$, the characteristic polynomial of B is of degree d and $B^d = 0$. Since $B^m = A \neq 0$, we have $m < d$. The second statement is proved in [11] in Theorem 3.2. □

The following lemmas are essentially from [6]; we state them here with the necessary modifications. Their proofs are obtained from the indicated ones in [6], mainly by replacing p^s, p^r, and p^t with the m, t, and s of this article, respectively.

Lemma 2.2. *Suppose that X is a $d \times d$ nilpotent matrix with nilpotency t and of Jordan type $\underline{x} = (x_1, \ldots, x_t)$. If the nilpotency of X is not known, replace t with d in this lemma. Then*

1. $\operatorname{rank}(X^i) = \displaystyle\sum_{k=1}^{t-i} k x_{i+k}$ *for* $0 \leq i < t$.

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2. $x_i = \operatorname{rank}(X^{i-1}) - 2\operatorname{rank}(X^i) + \operatorname{rank}(X^{i+1})$ *for* $1 \leq i < t$.
3. $\sum_{i=k}^{l} x_i = \operatorname{rank}(X^{k-1}) - \operatorname{rank}(X^k) - \operatorname{rank}(X^l) + \operatorname{rank}(X^{l+1})$ *for* $1 \leq k$, *and* $k+2 \leq l \leq t$; *in particular,* $\sum_{i=1}^{t} x_i = d - \operatorname{rank}(X)$.

Proof. (1), (2), and (3) follow from Main Lemma (1), (2), and (4) in [6], respectively. ∎

The following classical result is used in the proof of Proposition 3.1. It is the main ingredient for determining the Jordan type of the m-th power of a nilpotent matrix in [5], [8], and [9].

Proposition 2.3. *For positive integers* $m < s$, *with* $s = km + r$ *for some* $k \geq 1$ *and* $0 \leq r < m$, *it holds that*

$$[j_s]^m = [j_{km+r}]^m \sim (m-r)[j_k] \oplus r[j_{k+1}];$$

in particular,

$$[j_{km}]^m \sim m[j_k].$$

In other words, for positive integers s, m, *and* k *with* $km \leq s \leq (k+1)m$, *it holds that*

$$[j_s]^m \sim ((k+1)m - s)[j_k] \oplus (s - km)[j_{k+1}].$$

Proof. For positive integers $m < s$, with $s = km + r$ for some $k \geq 1$ and $0 \leq r < m$, we compute the Jordan type $\underline{x} = (x_1, \ldots, x_s)$ of $X = [j_s]^m$ using Lemma 2.2 (2). Since $[j_n]^l$ is the zero matrix for $l \geq n$ and $\operatorname{rank}([j_n]^l) = n - l$ for $1 \leq l < n$, we have $X^{k+1} = [j_{km+r}]^{km+m} = 0$, and $\operatorname{rank}(X^k) = \operatorname{rank}([j_{km+r}]^{km}) = r$, $\operatorname{rank}(X^{k-1}) = \operatorname{rank}([j_{km+r}]^{km-m}) = m + r$. Therefore, by Lemma 2.2 (2) we obtain

$$x_{k+1} = \operatorname{rank}(X^k) = r \quad \text{and} \quad x_k = \operatorname{rank}(X^{k-1}) - 2\operatorname{rank}(X^k) = m - r.$$

Since X is $s \times s$ and $(k+1)r + k(m-r) = km + r = s$, there is no room for any other Jordan blocks, that is, $x_1 = x_2 = \ldots = x_{k-1} = 0$. Hence, $[j_{km+r}]^m$ is similar to $(m-r)[j_k] \oplus r[j_{k+1}]$ as claimed. The second statement is an alternative form of writing the same result avoiding the computation of the remainder r in the division of s by m. ∎

Lemma 2.4. *Suppose that* A *is a nonzero* $d \times d$ *nilpotent matrix and* B *is an* m*-th root of* A *with* $1 < m < d$, $\underline{a} = (a_1, \ldots, a_d)$, $\underline{b} = (b_1, \ldots, b_d)$ *are the Jordan types, and* t *and* s *are the nilpotencies of* A *and* B, *respectively. If* $d = qm + r$ *for some* $q \geq 1$ *and* $0 \leq r < m$, *then we have the following:*

1. $a_i = mb_{im} + \sum_{j=1}^{m-1} j\left[b_{(i-1)m+j} + b_{(i+1)m-j}\right]$ *for* $1 \leq i \leq q-1$ *and*

$$\begin{aligned} a_q &= b_{(q-1)m+1} + 2b_{(q-1)m+2} + \cdots + (m-1)b_{(q-1)m+m-1} + mb_{qm} \\ &\quad + (m-1)b_{qm+1} + (m-2)b_{qm+2} + \cdots + (m-r)b_{qm+r}, \\ a_{q+1} &= b_{qm+1} + 2b_{qm+2} + \cdots + rb_{qm+r}. \end{aligned}$$

2. $a_t = b_{(t-1)m+1} + 2b_{(t-1)m+2} + \cdots + (m-1)b_{(t-1)m+m-1} + mb_{tm}$, *where* $b_i = 0$ *for* $s < i \leq tm$, *implying* $(t-1)m + 1 \leq s \leq tm$.
3. *In general,* $t \leq q+1$. *If* $s = mk$ *for some* k, *then* $t = k$, *if* $s = mk + l$ *for some* $k \geq 1$, $0 < l < m$, *then* $t = k+1$.

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Proof.

(1) By Lemma 2.2 (2) and (1), we have

$$\begin{aligned} a_i &= \operatorname{rank}((B^m)^{i-1}) - 2\operatorname{rank}((B^m)^i) + \operatorname{rank}((B^m)^{i+1}) \\ &= b_{mi-m+1} + 2b_{mi-m+2} + \cdots + (m-1)b_{mi-1} + mb_{mi} \\ &\quad + (m-1)b_{mi+1} + \cdots + 2b_{mi+m-2} + b_{mi+m-1}. \end{aligned}$$

When $d = qm + r$ for some $q \geq 1$, and for some $0 \leq r < m$, we have

$$\begin{aligned} a_q =& b_{(q-1)m+1} + 2b_{(q-1)m+2} + \cdots + (m-1)b_{(q-1)m+m-1} + mb_{qm} \\ &+ (m-1)b_{qm+1} + (m-2)b_{qm+2} + \cdots + (m-r)b_{qm+r}, \\ a_{q+1} =& b_{qm+1} + 2b_{qm+2} + \cdots + rb_{qm+r}. \end{aligned}$$

(2) Since $a_{t+1} = 0$, by (1) a_t takes the given form. Since $a_t \neq 0$ and b_i's are nonnegative, there is j in the set $\{(t-1)m+1, (t-1)m+2, \ldots, (t-1)m+m-1, tm\}$ such that $b_j \neq 0$. Hence, $(t-1)m \leq s \leq tm$.

(3) By part (1), we have $t \leq q+1$. Since $0 \neq A = B^m$, we have $1 < t$ and $m < s$. Since $B^{tm} = A^t = 0$ and $B^{(t-1)m} = A^{t-1} \neq 0$, we have $(t-1)m+1 \leq s \leq tm$. By part (1), if $s = km$, then $0 \neq b_s = b_{km}$ appears as the last term in a_k, and $a_i = 0$, for $i \geq k+1$, that is, $t = k$, if $s = km+l$, with $0 < l < m$, then $t = k+1$. ∎

See Example 3.2 (1) and (2) for the examples of the cases $tm \leq d$ and $d \leq tm$, respectively.

Lemma 2.5. *Suppose that A is a nonzero $d \times d$ nilpotent matrix, B is an m-th root of A with $1 < m < d$, and $\underline{a} = (a_1, \ldots, a_d)$, $\underline{b} = (b_1, \ldots, b_d)$ are the Jordan types of A and B, respectively. Then*

1. *if m divides $\sum_{j=1}^{m-1} jb_{(i-1)m+j}$, or $a_{i-1} = 0$, then m divides $\sum_{j=i}^{d} a_j$.*
2. *if $a_{i-1} = a_{l+1} = 0$, then $\sum_{j=i}^{l} a_j = m\sum_{j=0}^{lm} b_{im+j}$ for $1 < i \leq l < d$; in particular, if $i = l$, then $a_i = mb_{im}$.*

Proof. Note that (2) follows from (1). For (1), without loss of generality, assume that the nilpotency of A is t and write the a_i's using Lemma 2.4 as follows:

$$\begin{aligned} a_{i-1} &= b_{im-2m+1} + 2b_{im-2m+2} + \ldots + mb_{(i-1)m} \\ &\quad + (m-1)b_{im-m+1} + \ldots + 2b_{im-m+m-2} + b_{im-1}, \\ a_i &= b_{im-m+1} + 2b_{im-m+2} + \ldots + (m-1)b_{im-1} + mb_{im} \\ &\quad + (m-1)b_{im+1} + \ldots + 2b_{im+m-2} + b_{im+m-1}, \\ a_{i+1} &= b_{im+1} + 2b_{im+m-2} + \ldots + (m-1)b_{im+m-1} + mb_{(i+1)m} \\ &\quad + (m-1)b_{im+m+1} + \ldots + 2b_{im+2m-2} + b_{im+2m-1}, \\ &\vdots \\ a_{t-1} &= b_{tm-2m+1} + 2b_{tm-2m+2} + \ldots + (m-1)b_{(t-1)m-1} + mb_{(t-1)m} \\ &\quad + (m-1)b_{tm-m+1} + (m-2)b_{tm-m+2} + \ldots + b_{tm-1}, \\ a_t &= b_{tm-m+1} + 2b_{t-m+2} + \ldots + (m-1)b_{tm-1} + mb_{tm}. \end{aligned}$$

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The pattern for the coefficients of b_j's in a_i's is very useful. Observe that in $a_1, \ldots, a_t$, the numbers $b_1, \ldots, b_{m-1}$ appear only in a_1 with respective coefficients $1, \ldots, m-1$, and b_{im} appears only in a_i with coefficient m, for $i = 1, \ldots, t$. However, the remaining b_j's appear twice, one in a_i and the other in a_{i+1} for some i with coefficients adding up to m. The sum $\sum_{j=1}^{m-1} jb_{(i-1)m+j}$ in the hypothesis of (1) is the sum of the first $m-1$ terms of the equation for a_i above and it is is 0 if $a_{i-1} = 0$. Thus, if m divides $\sum_{j=1}^{m-1} jb_{(i-1)m+j}$ or $a_{i-1} = 0$, then m divides $\sum_{j=i}^{d} a_j$. ∎

The equations listed in the proof of Lemma 2.5 are essentially from the proof of Main Lemma in [6]; we include them here with the necessary modifications, by replacing p^s, p^r, and p^t in [6] with the m, t, and s of this article, respectively.

3. Jordan type of m-th root/power, rootless nilpotent matrices. This section is devoted to our results on nilpotent matrices that have m-th roots or are rootless. Before proving our main theorem, Theorem 1.1, we give a result on nilpotent Jordan blocks which is used in Thereom 1.2. It is a generalization of Lemma 1.3 in [9] (the case $i = 1$ is Lemma 1.3 [9]) and it is of general interest.

PROPOSITION 3.1. *For nonnegative integers $k \geq 0, m > 1$, and $s, l, i \geq 1$ satisfying*

$$0 \leq mk \leq s < s + i \leq m(k+1) \quad \text{and} \quad 0 \leq mk \leq l - i < l \leq m(k+1),$$

it holds that $([j_s] \oplus [j_l])^m \sim ([j_{s+i}] \oplus [j_{l-i}])^m$.

Proof. Suppose the nonnegative integers $k \geq 0, m > 1$, and $s, l, i \geq 1$ satisfy $0 \leq mk \leq s < s + i \leq m(k+1)$ and $0 \leq mk \leq l - i < l \leq m(k+1)$. Note that, for square matrices X and Y we have $(X \oplus Y)^m = X^m \oplus Y^m$. Hence, calculating $[j_s]^m$, $[j_l]^m$, $[j_{s+i}]^m$, and $[j_{l-i}]^m$ by Proposition 2.3 and adding the multiplicities of $[j_k]$ and $[j_{k+1}]$ for each side of the similarity symbol in the statement gives the result. ∎

3.1. Theorem 1.1 and several corollaries.

Proof of Theorem 1.1. Let A be a nilpotent matrix of nilpotency t and $\underline{a} = (a_1, \ldots, a_d)$ be its Jordan type. Suppose A has an m-th root B with nilpotency s and Jordan type $\underline{b} = (b_1, \ldots, b_d)$. The desired equalities and inequalities follow from Lemma 2.4. Conversely, suppose that there exist nonnegative integers $b_1, \ldots, b_d$ satisfying the equations given in (1.1), set $B = \operatorname{diag}([j_d]^{(b_d)}, [j_{d-1}]^{(b_{d-1})}, \ldots, [j_2]^{(b_2)}, [j_1]^{(b_1)})$. Then B has Jordan type $(b_1, \ldots, b_d)$, and by Lemma 2.4 B^m have the Jordan type $(a_1, \ldots, a_d)$. Hence,

$$A \sim \operatorname{diag}([j_d]^{(a_d)}, [j_{d-1}]^{(a_{d-1})}, \ldots, [j_2]^{(a_2)}, [j_1]^{(a_1)}) \sim B^m.$$

Since having an m-th root is a similarity-invariant property, A has an m-th root. ∎

EXAMPLE 3.2 (Immediate applications of Theorem 1.1). Suppose that A is a $d \times d$ nonzero nilpotent matrix of nilpotency t with Jordan type $\underline{a} = (a_1, \ldots, a_t)$, and B is a third root of A, of nilpotency s and Jordan type $\underline{b} = (b_1, \ldots, b_s)$. Assume that $t \leq 4$. By Theorem 1.1 we have $4 \leq s \leq 12$, and

$$\begin{cases} a_1 &= b_1 + 2b_2 + 3b_3 + 2b_4 + b_5, \\ a_2 &= b_4 + 2b_5 + 3b_6 + 2b_7 + b_8, \\ a_3 &= b_7 + 2b_8 + 3b_9 + 2b_{10} + b_{11}, \\ a_4 &= b_{10} + 2b_{11} + 3b_{12}, \end{cases} \tag{3.3}$$

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where $b_i = 0$ for all i with $s < i \leq 12$. The system of linear equations in (3.3) can be written as $M\underline{b}^T = \underline{a}^T$, where

$$M = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}. \tag{3.4}$$

Clearly, $d = a_1 + 2a_2 + 3a_3 + 4a_4$. It is possible that $d \geq tm = 3t$ or $d \leq tm = 3t$ as shown in (1) and (2) below, respectively.

(1) Let $\underline{b} = (b_1, \ldots, b_{12}) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0)$. Then the product $M\underline{b}^T$ gives that $\underline{a} = (0, 0, 4, 2)$, hence, $t = 4$, and $d = 20 \geq tm = 12$.
(2) Let $\underline{b} = (b_1, \ldots, b_{12}) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$. Then the product $M\underline{b}^T$ gives that $\underline{a} = (0, 0, 2, 1)$, hence, $t = 4$, and $d = 10 \leq tm = 12$.
(3) Let $\underline{a} = (a_1, a_2, a_3, a_4) = (0, 0, 1, 1)$. Substituting $a_4 = 1$ in the last equation in (3.3) gives $b_{10} = 1$, and $b_{11} = b_{12} = 0$ as $b_{11}, b_{12} \geq 0$. This forces $1 = a_3 \geq 2b_{10} = 2$ which is a contradiction. Hence, A cannot have any third root.
(4) Let $\underline{b} = (1, 1, 1, 1)$. Hence, $s = 4$. In this case, due to the fact that $b_i = 0$ for $i = 5, 6, \ldots, 12$, the 4×12 matrix M can be reduced to a 4×4 matrix by deleting the columns $5, 6, \ldots, 12$. Then $\underline{a} = (8, 1, 0, 0)$, hence, $t = 2$ and $tm = 6 \leq 10 = d$. Thus, the equations in (3.3) can be be used as long as $t \leq 4$ and $s \leq 12$.

We collect many implications of Theorem 1.1 in the following two corollaries.

Corollary 3.3. *Let m be an integer with $1 < m < d$. Suppose that A is a nonzero $d \times d$ nilpotent matrix over a field having an m-th root B, with Jordan types $\underline{a} = (a_1, \ldots, a_d)$ and $\underline{b} = (b_1, \ldots, b_d)$, respectively. Then the following hold.*

1. *$\sum_{j=1}^{d} a_j \geq m$; more generally, if the sum $\sum_{j=i}^{d} a_j > 0$, then the sum $\sum_{j=i}^{d} a_j \geq m$ for $i = 1, \ldots, d$.*
2. *If $a_{i-1} \cdot a_i \cdot a_{i+1} \neq 0$ for some $i = 2, \ldots, d-1$, then at least one of $a_{i-1} + a_i$ or $a_i + a_{i+1}$ is greater than m.*
3. *If $a_{i-1} = 0 = a_{k+1}$ for some i, k with $2 \leq i \leq k \leq d-1$, then m divides the sum $\sum_{j=i}^{k} a_j$; in particular, if $i = k$, then m divides a_i.*
4. *If $a_i = 0$ for some i with $1 \leq i \leq d-1$, or, alternatively, if m divides the sum $\sum_{j=1}^{m-1} jb_{im-m+j}$, then m divides the sum $\sum_{j=i}^{d} a_j$.*

Proof. Consider the equations for a_i's written in the proof of Lemma 2.5 keeping in mind the non-negativity of all b_j's. Since A is nonzero, its nilpotency $t \geq 2$. Hence, $a_t > 0$ for some $t \geq 2$ implying $a_1 + a_2 + \cdots + a_d = a_1 + \cdots + a_t > 0$.

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(1) If only $a_t \neq 0$ in $a_2+\cdots+a_t$, then $a_{t-1}=0$; hence, $a_t = mb_{tm} > 0$. Thus, $a_1+a_2+a_3+\cdots+a_t \geq m$. If $a_1+\cdots+a_t$ has more than one nonzero term, then there are two cases, a_{t-1} is zero or nonzero. If $a_{t-1}\neq 0$, then $a_{t-1}+a_t \geq m$. Hence, $a_1+a_2+a_3+\cdots+a_t \geq a_{t-1}+a_t \geq m$. If $a_{t-1}=0$, then $a_t = mb_{tm} > 0$ which implies $a_1+a_2+a_3+\cdots+a_t \geq a_t = mb_{tm} \geq m$. A similar argument works for the second half of the statement.

(2) If a_{i-1}, a_i, a_{i+1} are nonzero, then either $a_{i-1}+a_i \geq m$, or $a_i+a_{i+1} \geq m$.

The parts (3) and (4) are obtained by Lemma 2.5 (2) and (1), respectively. ☐

Remark 3.4 (Counterexample to an erroneous statement). Theorem 1 (2) in [8] states that "If a nilpotent matrix A has an m-th root, then m divides the total number of Jordan blocks in the Jordan form of A." This is not true in general. As a counterexample, let $A = \mathrm{diag}([j_2],[j_1]^{(6)})$, that is, $\underline{a}=(6,1)$. Consider $B=\mathrm{diag}([j_4],[j_3],[j_1])$. Then $B^3 \sim A$ by Proposition 2.3, that is, A has a third root. However, $m=3$ does not divide the number of Jordan blocks of A which is 7. Corollary 3.3 (4) provides a more general statement and an alternative one to the erroneous statement. The other parts of Corollary 3.3 provide similar statements.

The third inequality in Corollary 3.5 (1) provides a stronger statement than the second statement of Psarrakos's Theorem 3.2 in [7] stated in the Preliminaries by removing the hypothesis $d_2 > 0$.

Corollary 3.5. *Suppose that A is a nonzero $d\times d$ nilpotent matrix over a field, with Jordan type $\underline{a}$, rank r and nilpotency t.*

1. *A cannot have an m-th root whenever $m \geq d$, or $m > d-t+1$, or $m > d-r$.*
2. *If A has an m-th root and $\underline{a}$ contains the sequence $e,1,f$, then $e \geq m-1$ or $f \geq m-1$.*
3. *If $\underline{a}$ contains the sequence $0,1,0$, or $a_{t-1}=0$ and $a_t=1$, then A is rootless.*
   In particular, $A=[j_d]$ is rootless.
4. *If A has no square root and $\underline{a}$ contains the sequence $1,1,1$, then A is rootless.*
5. *If $\underline{a}=(a_1,\ldots,a_t)=(*,\ldots,*,0,p-1,1)$, where p is a prime number, then A may have only p-th roots. In particular, $\mathrm{diag}([j_t],[j_{t-1}]^{(p-1)})$ has only p-th roots, and every p-th root is similar to $[j_{(t-1)p+1}]$.*

Proof.

(1) The first and second inequalities follow from Lemma 2.1. By Lemma 2.2 (3), we have $d-r = a_1+\cdots+a_d$. Therefore, if A has an m-th root, then Corollary 3.3 (1) implies $d-r\geq m$ proving the third inequality.
For (2)–(5), consider the equations for a_i's written in the proof of Lemma 2.5 keeping in mind the nonnegativity of all b_j's.

(2) Set the equations for a_{i-1}, a_i, and a_{i+1} equal to e, 1 ,and f, respectively. Since b_j's are nonnegative, the only way to get $a_i=1$ is either the first term $b_{im-m+1}=1$ and the remaining b_j's are zero, that is, $b_{im-m+2}=b_{im-m+3}=b_{im-m+4}=\cdots=b_{im+m-1}=0$, or, the last term $b_{im+m-1}=1$ and the remaining b_j's are zero, that is, $b_{im-m+1}=b_{im-m+2}=b_{im-m+3}=b_{im-m+4}=\cdots b_{im+m-2}=0$. If $b_{im-m+1}=1$, then $a_{i-1}\geq m-1$. If $b_{im+m-1}=1$, then $a_{i+1}\geq m-1$.

(3) By part (1), we know $(0,1,0)=(a_{i-1},a_i,a_{i+1})$ is not a possibility. Similarly, one can see that $(a_{t-1},a_t)\neq(0,1)$.

(4) The consecutive multiplicities a_{i-1},a_i,a_{i+1} are all 1 only when $m=2$.

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(5) Consider the equations $a_{t-2}=0$, $a_{t-1}=p-1$, and $a_t=1$. The equality $a_t=1$ is true only when $b_{tp-p+1}=1$ and $b_{tp-p+2}=b_{tp-p+3}=\cdots=b_{tp}=0$. Since the coefficient of b_{tp-p+1} in a_{t-1} is $p-1$, we must have $b_{tp-2p+1}$, $b_{tp-2p+2}$, $b_{tp-2p+3},\ldots,b_{tp-1}$ are all zero except for $b_{tp-p+1}=1$. Hence, whether A has a p-th root or not depends on the remaining $a_1,a_2,\ldots,a_{t-3}$. The matrix A has a p-th root if and only if $a_1,a_2,\ldots,a_{t-3}$ satisfy the equations in Theorem 1.1. Assume that A has a p-th root B. On the other hand, we know that if B is an m-th root of A with $m=nk$, then B^n is a k-th root of A. Since p is prime, there is no chance for any root other than the p-th root B. In the particular case that the Jordan type of A is $\underline{a}=(a_1,\ldots,a_t)=(0,\ldots,0,p-1,1)$, the only possible solution $\underline{b}$ for the Jordan type of a p-th root of A is $\underline{b}=(b_1,\ldots,b_{tp-p+1})=(0,\ldots,0,1)$. ∎

3.2. Characterization of existence of an m-th root for a nilpotent matrix by two equivalent matrix equations. The necessary and sufficient conditions of Theorem 1.1 for the existence of an m-th root for a nilpotent matrix can be written as a system of linear equations $M\underline{b}^T=\underline{a}^T$ as in Example 3.2 which can equivalently be written as the matrix equation:

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ b_4 & b_5 & b_6 & b_7 & b_8 \\ b_7 & b_8 & b_9 & b_{10} & b_{11} \\ b_{10} & b_{11} & b_{12} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}^T. \tag{3.5}$$

Note that b_{3i} appears in a_i for $i=1,2,3,4$. The 4×12 matrix M with entries $0,1,2,3$ in Example 3.2 can be reduced to a $4\times s$ matrix by deleting the last $12-s$ columns if the nilpotency s of B is known because of the equalities $b_{s+1}=\cdots=b_{12}=0$. Namely, when $s=10$ as in the parts (1) and (2), the equations for a_3 and a_4 becomes shorter; $a_3=b_7+2b_8+3b_9+2b_{10}$, and $a_4=b_{10}$, see the equations in (3.3). Deleting the rows after the t-th one reduces M to a $t\times s$ matrix.

In general, if $d=mq+r$, for some r with $0\le r<m$, then $t\le q+1$ by the equations (1.1) and (1.2). The numbers $a_1,\ldots,a_q$, and a_{q+1} can be zero depending on the nilpotencies t and s. It may be of interest to find an m-th root B of specific nilpotency s provided that $(t-1)m+1\le s\le tm$ holds; see Example 3.10.

Corollary 3.6 giving the general forms of the matrices of the equations (3.4) and (3.5) is as follows.

Corollary 3.6. *Suppose that B is a $d\times d$ matrix with the Jordan type $\underline{b}=(b_1,\ldots,b_d)$. For $1<m<d$, where $d=qm+r$ for some $q\ge 1$, $0\le r<m$, and $A=B^m$ with the Jordan type $\underline{a}=(a_1,\ldots,a_d)$, there is a $(q+1)\times d$ matrix M with nonnegative entries $0,1,2,\ldots,m$ satisfying $M\underline{b}^T=\underline{a}^T$, where $M=(R_1,\ldots,R_d)^T$, for $i=1,2,\ldots,q-1$,*

$$R_i=(\underbrace{0,\ldots,0}_{(i-1)m\ times},1,2,\ldots,m-2,m-1,m,m-1,m-2,\ldots,2,1,0,0,\ldots),$$

$$R_q=(\underbrace{0,\ldots,0}_{(q-1)m\ times},1,2,\ldots,m-2,m-1,m,m-1,m-2,\ldots,m-(r-1),m-r),$$

$$R_{q+1}=(\underbrace{0,\ldots,0}_{qm\ times},1,2,\ldots,r-1,r).$$

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That is, M is an echelon matrix as shown, where the entries not written are all zeroes:

$$M = \begin{array}{l}
\begin{array}{ccc}
\overbrace{\hspace{7em}}^{\text{1st block of } m \text{ columns}} & \overbrace{\hspace{9em}}^{\text{2nd block of } m \text{ columns}} & \overbrace{\hspace{9em}}^{\text{3rd block of } m \text{ columns}}
\end{array}\\
\left[\begin{array}{ccccccccccccc}
1 & 2 & \cdots & m & m-1 & \cdots & 1 & 0 & & & & & \\
 & & & & 1 & \cdots & m-1 & m & m-1 & \cdots & 1 & 0 & \cdots \\
 & & & & & & & & 1 & \cdots & m-1 & m & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \cdots
\end{array}\right.
\end{array}$$

$$\begin{array}{l}
\begin{array}{ccc}
\overbrace{\hspace{9em}}^{(q-1)\text{-th block of } m \text{ columns}} & \overbrace{\hspace{9em}}^{q\text{-th block of } m \text{ columns}} & \overbrace{\hspace{12em}}^{r \text{ columns}}
\end{array}\\
\left.\begin{array}{cccccccccccc}
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
m-1 & \cdots & 1 & 0 & & & & & & & & \\
1 & \cdots & m-1 & m & m-1 & \cdots & 1 & 0 & & & & \\
 & & & & 1 & \cdots & m-1 & m & m-1 & \cdots & m-(r-1) & m-r \\
 & & & & & & & & 1 & \cdots & r-1 & r
\end{array}\right].
\end{array}$$

The matrix M can be reduced to a $(q+1) \times s$ matrix by deleting the last $d-s$ columns, where s the nilpotency of B. The matrix M can be reduced to a $t \times d$ matrix by deleting the last $q+1-t$ rows, where t is the nilpotency of A and satisfies $(t-1)m+1 \leq d$.

The matrix M can be reduced to a $t \times s$ matrix with $(t-1)m+1 \leq s \leq tm$, where t and s are the nilpotencies of A and B, respectively. In this case, the matrix equation $M\underline{b}^T = \underline{a}^T$ is equivalent to the matrix equation:

$$\mathcal{B}'(1, 2, \ldots, m-1, m, m-1, \ldots 2, 1)^T = \underline{a}^T,$$

where $\mathcal{B}'$ is $t \times (2m-1)$ matrix with nonnegative entries in the set $\{0, b_1, b_2, \ldots, b_d\}$ as follows:

$$\mathcal{B}' = \left[\begin{array}{c|c|c}
b_1\ b_2\ \cdots\ b_{m-1} & b_m & \\
\cline{1-1}
 & b_{2m} & \mathcal{B} \\
\mathcal{B} & \vdots & \\
\cline{3-3}
 & b_{tm} & 0\ \cdots\ 0
\end{array}\right], \quad \mathcal{B} = \begin{bmatrix}
b_{m+1} & b_{m+2} & b_{m+3} & \cdots & b_{2m-1} \\
b_{2m+1} & b_{2m+2} & b_{2m+3} & \cdots & b_{3m-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
b_{(t-2)m+1} & b_{(t-2)m+2} & b_{(t-2)m+3} & \cdots & b_{(t-1)m-1} \\
b_{(t-1)m+1} & b_{(t-1)m+2} & b_{(t-1)m+3} & \cdots & b_{tm-1}
\end{bmatrix}$$

where $\mathcal{B}$ is a $(t-1) \times (m-1)$ matrix, with $b_{d+1}, b_{d+2}, \ldots, b_{tm}$ defined as 0 if $tm > d$.

Proof. If $A = B^m$ and t is the nilpotency of A, then by Theorem 1.1, the equation $a_t = b_{(t-1)m+1} + 2b_{(t-1)m+2} + \cdots + (m-1)b_{(t-1)m+m-1} + mb_{tm} \neq 0$ implies that at least one of its terms is nonzero. The greatest index s for which $b_s \neq 0$ is the nilpotency of B, hence, $(t-1)m+1 \leq s \leq tm$. In addition, $A^k = (B^m)^k = (B^k)^m = 0$ implies that $t \leq s \leq tm$. Since $s \leq tm$, $b_i = 0$ for $s \leq i \leq d$, there is no need to include them in the matrices. $\square$

Remark 3.7. Note that when $\underline{a}$ is given and we look for a solution $\underline{b}$ with nonnegative entries to $M\underline{b}^T = \underline{a}^T$, there is no guarantee that there will be such a solution, regardless of the fact that the system is consistent and has many free variables. See Example 3.2 (3) and Example 3.10 where no third root exists

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for a nilpotent A. See Corollary 3.11 where an m-th root of smallest nilpotency which is an m-th root of A is obtained provided that $\underline{a}$ satisfies some conditions.

3.3. Computing Jordan type of m-th power/root of a nilpotent matrix by $M\underline{b}^T = \underline{a}^T$. Consider the matrix equation $M\underline{b}^T = \underline{a}^T$ given in Corollary 3.6, where $\underline{b}$ and $\underline{a}$ are the Jordan types of B and $B^m = A$, respectively. The size of M reduces naturally to $t \times d$, $d \times s$, and $t \times s$, where t and s are the nilpotencies of A and B, respectively. Computing the Jordan form of the m-th power is reduced to multiplying M by $\underline{b}$ without any reference to eigenvectors. Thus, $M\underline{b}^T = \underline{a}^T$ provides an alternative easy algorithm for computer computations of the Jordan canonical forms of the m-th power of a nilpotent matrix when the eigenvectors are not needed. Conversely, the existence of an m-th root for a matrix with Jordan type $\underline{a}$ is reduced to the existence of a nonnegative integer solution $\underline{b}$ to the equation, and this can be implemented by computers as well. We should note here that when there is a solution with a pair of Jordan blocks of special sizes, Theorem 1.2 gives a method of producing other solutions with a different Jordan type which can also be implemented by computers.

Example 3.8 (Computing the Jordan type of the third power of a matrix). If B is a 16×16 nilpotent matrix with Jordan type $\underline{b} = (1, 1, 1, 0, 2)$, then B is of nilpotency 5 and $d = 16$. The Jordan type of B^3 is obtained by the matrix multiplication suggested by Corollary 3.6. Namely, consider the 4×5 submatrix of M in the equation (3.4) by taking the first five columns and multiplying it with $\underline{b} = (1, 1, 1, 0, 2)$ gives the Jordan type of B^3 as $(8, 4, 0, 0)$.

Example 3.9 (Matrices with no third roots). In Example 3.2 (3), there is a Jordan type $\underline{a}$ for which no nonnegative solution $\underline{b}$ exists to $M\underline{b}^T = \underline{a}^T$. For another example, let A be of Jordan type $\underline{a} = (1, 1, 2, 1)$. There is no $\underline{b}$ satisfying the equation (3.3). Hence, A cannot have a third root.

It may be desirable to specify the nilpotency s of an m-th root of a nilpotent matrix of nilpotency t provided that $(t-1)m + 1 \leq s \leq tm$, see Theorem 1.1.

Example 3.10 (Matrix having a third root of nilpotency 11 but not 10). Suppose that A is a nilpotent matrix with nilpotency $t = 4$ and has a third root B of nilpotency s. Then, $10 \leq s \leq 12$. Assume that $s = 10$. We can write the Jordan type of A as $\underline{a} = (a_1, \ldots, a_4)$ and the Jordan type of B as $\underline{b} = (b_1, \ldots, b_{10})$. Note that $d \geq 10$ and $a_4 \geq 1$. By Theorem 1.1, $a_3 = b_7 + 2b_8 + 3b_9 + 2b_{10}$ and $a_4 = b_{10}$. Hence, $a_3 \geq 2a_4$ implying a contradiction for $\underline{a} = (0, 0, 1, 2)$ because $a_3 = 1 \ngeq 2a_4 = 4$. Therefore, A has no third root of nilpotency 10. However, A has a third root B of nilpotency 11 which has the Jordan type $\underline{b} = (b_1, \ldots, b_{11}) = (0, \ldots, 0, 1)$ as it satisfies the equations in (3.3).

Corollary 3.11. *Suppose that A is a nilpotent matrix of nilpotency t and of Jordan type $\underline{a}$. For $j = 0, 1, \ldots, t-1$, define $b_i = 0$ for $i \neq jm + 1$ and*

$$b_{jm+1} = \sum_{k=1}^{t-j} (-1)^{k-1}(m-1)^{k-1} a_{k+j}. \tag{3.6}$$

If $b_{jm+1} \geq 0$ for $j = 0, 1, \ldots, t-1$, then A has an m-th root B of smallest possible nilpotency $(t-1)m + 1$ whose Jordan type is $\underline{b} = (b_1, \ldots, b_{(t-1)m+1})$.

Proof. If B is an m-th root of nilpotency s of A, then the smallest s is $(t-1)m+1$ by Theorem 1.1 and there is a $t \times s$ matrix M of rank t satisfying $M\underline{b}^T = \underline{a}^T$ by Corollary 3.6. The leading entry of M in the $j+1$-th row is at the column $jm+1$ for $j = 0, 1, \ldots, t-1$. Reducing the augmented matrix $[M \mid \underline{a}^T]$ so that the only nonzero term in the columns of leading entries is 1 and letting the free variables $b_i = 0$ for $i \neq jm+1$,

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$j = 0, 1, \dots, t-1$ gives the equations in (3.6), in particular $b_{(t-1)m+1} = a_t \geq 1$. Whenever $\underline{a} = (a_1, \dots, a_t)$ makes $b_{jm+1} \geq 0$ for $j = 0, 1, \dots, t-1$, any matrix B with Jordan type $\underline{b} = (b_1, \dots, b_{(t-1)m+1})$ is an m-th root of A. ∎

3.4. There are rootless and not rootless nilpotent matrices of nilpotency t for $2 < t < d$. By Lemma 2.2 (3), the rank r of a $d \times d$ nilpotent matrix of nilpotency t with Jordan type $\underline{a} = (a_1, \dots, a_t)$ is $r = d - (a_1 + a_2 + \cdots + a_t)$. Corollary 3.12 is an analog of Theorem 2 in [8] replacing rank with nilpotency.

Corollary 3.12. *Let $0 \neq A$ be a $d \times d$ nilpotent matrix of nilpotency t.*

1. *For $2 < t \leq d$, there is A which is rootless.*
2. *For $2 \leq t < d$, there is A which is not rootless; in particular, for $t = 2 < d$, there is no rootless $d \times d$ nilpotent matrix of nilpotency t.*

Proof. Since t is the degree of the minimal polynomial of A, $t \leq d$. Since A is nonzero $t \geq 2$, hence, $2 \leq t \leq d$.

(1) Consider the matrix $A = [j_d]$. When $t = d$, A is rootless by Proposition 2.3. Assume that $2 < t < d$. Consider the matrix $A = \operatorname{diag}([j_t], [j_1]^{(d-t)})$ (of nilpotency t and rank $r = t-1$). The result follows by Corollary 3.5 (3) because $a_t = 1$ and $a_{t-1} = 0$ and $t \geq 3$.

(2) Assume that $2 < t < d$. Consider the matrix $A = \operatorname{diag}([j_t]^{(2)}, [j_1]^{(d-2t)})$ (which is of rank $r = 2t-2$). By Proposition 2.3, A is similar to the square of the matrix $\operatorname{diag}([j_{2t}], [j_1]^{(d-2t)})$. Hence, A is not rootless of nilpotency t. In particular, for $t = 2$, the Jordan type of A is $\underline{a} = (a_1, a_2)$, the rank r of A is $r = a_2 \geq 1$, and $d = a_1 + 2a_2$. By Proposition 2.3,

$$A \sim \operatorname{diag}([j_2]^{(r)}, [j_1]^{(d-2r)}) \sim (\operatorname{diag}([j_{2r}], [j_1]^{(d-2r^2)}))^r.$$

Therefore, there is no rootless matrix of nilpotency $t = 2 < d$. ∎

The rootless matrices in the proof of Corollary 3.12 (1) have rank less than nilpotency. This need not be the case as the following corollary shows. To produce rootless matrices, we can use Corollary 3.6. We arrange a nonnegative $\underline{a}$ such that $M\underline{b}^T = \underline{a}^T$ has no nonnegative integer solution $\underline{b}$.

Corollary 3.13. *Let A be a nonzero $d \times d$ nilpotent matrix over a field, of nilpotency t, of rank r.*

1. *For $2 < k \leq d \leq 2k$, the matrix $A = \operatorname{diag}([j_{d-k}], [j_k])$ is rootless of nilpotency $t = d-k$ with $r = d-2$.*
2. *For $3 < k \leq d-2$, the matrix $\operatorname{diag}([j_{d-k}], [j_{k-1}], [j_1])$ is rootless of nilpotency $t = d-k$ with $r = d-3$.*
3. *For $0 \leq k \leq d-2$, the matrix $\operatorname{diag}([j_{d-k}], [j_1]^{(k)})$ is rootless of nilpotency $t = d-k$ with $r = d-k-1$.*
4. *For $3 < k \leq d-2$, the matrix $\operatorname{diag}([j_{d-k}], [j_2], [j_1]^{(k-2)})$ is rootless of nilpotency $t = d-k$ with $t = r$.*

Proof. Write $\underline{a} = (a_1, \dots, a_t)$ of A and argue as in the proof of Corollary 3.5. ∎

In Corollary 3.14, in addition to the ones in the proof of Corollary 3.12 (2), we give more examples of Jordan types of A and B, where B is an m-th root of A.

Corollary 3.14. *Let $1 < m < d$ be integers and A be a $d \times d$ nilpotent matrix of nilpotency t having an m-th root B with respective Jordan types $\underline{a}$ and $\underline{b}$.*

1. *For $t = 2$, if $\underline{a} = (a_1, a_2) = (q + r(m-1), r)$ (A is of rank r), where r is a nonnegative integer, then $\underline{b} = (b_1, \dots, b_{m+1}) = (q, 0, \dots, 0, r)$ (B is of rank mr).*

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2. *For $t = 2$, if A has $\underline{a} = (a_1, a_2) = (q, mk)$ (A is of rank mk), then $\underline{b} = (b_1, \dots, b_{2m-1}, b_{2m}) = (q, 0, \dots, 0, k)$ (B is of rank $(2m-1)k$).*
3. *For $t = 3$, if $\underline{a} = (a_1, a_2, a_3) = (2q, p(m-1), p)$ (A is of rank $k(m+1)$), where k is a nonnegative integer, then $\underline{b} = (b_1, \dots, b_{2m+1}) = (2q, 0, \dots, 0, k)$ (of rank $2km$), or $\underline{b} = (b_1, \dots, b_{2m+1}) = (0, q, 0, \dots, 0, k)$ (B is of rank $2km + q$).*
4. *For $4 \le t \le d$, if $t = 2n$ for some $n \ge 1$, and $\underline{a} = (a_1, \dots, a_t) = ((m-1)k_1, k_1, \dots, (m-1)k_n, k_n)$ (A is of rank $\sum_{i=1}^{n}(2(i-1)m+1)k_i$), where $k_1, \dots, k_n$ are nonnegative, then $\underline{b} = (b_1, \dots, b_{(t-1)m+1}) = (\underbrace{0,..,0}_{m\ times}, k_1, \underbrace{0,..,0}_{2m-1\ times}, k_2, \underbrace{0,..,0}_{2m-1\ times}, k_3, \dots, \underbrace{0,..,0}_{2m-1\ times}, k_{n-1}, \underbrace{0,..,0}_{2m-1\ times}, k_n)$ (B is of rank $\sum_{i=1}^{n}(2i-1)mk_i$).*
5. *For $4 \le t \le d$, if $t = 2n+1$ for some $n \ge 1$, and $\underline{a} = (a_1, \dots, a_t) = (0, (m-1)k_1, k_1, \dots, (m-1)k_n, k_n)$ (A is of rank $\sum_{i=1}^{n}(2i-1)m+1)k_i$), where $k_1, \dots, k_n$ are nonnegative, then $\underline{b} = (b_1, \dots, b_{(t-1)m+1}) = (\underbrace{0, \cdots, 0}_{2m\ times}, k_1, \underbrace{0,..,0}_{2m-1\ times}, k_2, \underbrace{0,..,0}_{2m-1\ times}, k_3, \dots, \underbrace{0,..,0}_{2m-1\ times}, k_{n-1}, \underbrace{0,..,0}_{2m-1\ times}, k_n)$ (B is of rank $\sum_{i=1}^{n} 2imk_i$).*

Proof. In (1)–(5), $\underline{a}$ and $\underline{b}$ satisfy the equations given in Theorem 1.1. $\square$

4. Results for not necessarily nilpotent matrices.

4.1. New m-th roots with different Jordan type from a special m-th root of a singular matrix. The key observation used in the proof of Theorem 1.2 is Proposition 3.1. Since it is on nilpotent matrices, it is given in Section 3.

Proof of Theorem 1.2. Let $m > 1$, $k \ge 0$, $u, s, l, i \ge 1$ be integers satisfying

$$0 \le mk \le s < s + i \le m(k+1) \quad \text{and} \quad 0 \le mk \le l - i < l \le m(k+1),$$

E be any square matrix. Let $\underline{b}$ and $\underline{c}$ be the Jordan types of the nilpotent parts of B, C, respectively, where

$$B = \mathrm{diag}(E, [j_s]^{(u)}, [j_l]^{(u)}) \text{ and } C = \mathrm{diag}(E, [j_{s+i}]^{(u)}, [j_{l-i}]^{(u)}).$$

Permuting the blocks of a block diagonal matrix produces a similar matrix, that is, $X^{(u)} \oplus Y^{(u)} \sim (X \oplus Y)^{(u)}$. Hence,

$$B \sim \mathrm{diag}(E, ([j_s] \oplus [j_l])^{(u)}) \quad \text{and} \quad C \sim \mathrm{diag}(E, ([j_{s+i}] \oplus [j_{l-i}])^{(u)}),$$

implying that

$$B^m \sim \mathrm{diag}(E^m, (([j_s] \oplus [j_l])^m)^{(u)}) \quad \text{and} \quad C^m \sim \mathrm{diag}(E^m, (([j_{s+i}] \oplus [j_{l-i}])^m)^{(u)}).$$

By Proposition 3.1, we obtain

$$[j_s]^m \oplus [j_l]^m = ([j_s] \oplus [j_l])^m \sim ([j_{s+i}] \oplus [j_{l-i}])^m = [j_{s+i}]^m \oplus [j_{l-i}]^m.$$

Hence, for any positive integer u, we have

$$(([j_s] \oplus [j_l])^m)^{(u)} \sim (([j_{s+i}] \oplus [j_{l-i}])^m)^{(u)}.$$

By the transitivity of the similarity relation we obtain $B^m \sim C^m$.

Under the given hypotheses, the Jordan type $\underline{b}$ of the nilpotent part of B is $(b_1, \dots, b_n)$ for some n with $b_s \ge u$ and $b_l \ge u$, and the Jordan type $\underline{c}$ of the nilpotent part of C is $(c_1, \dots, c_k)$ for some k with $c_{s+i} \ge u$ and $c_{l-i} \ge u$. The Jordan type $\underline{c}$ has the same entries as $\underline{b}$ except for the following:
$c_s = b_s - u, \quad c_l = b_l - u, \quad c_{s+i} = b_{s+i} + u, \quad \text{and } c_{l-i} = b_{l-i} + u.$ $\square$

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Remark 4.1. Theorem 1.2 gives a method to produce other m-th roots of a singular matrix from a given m-th root having a pair special size Jordan blocks. Let s, l, k, i satisfying the hypothesis of Theorem 1.2 and $B = \mathrm{diag}(E, [j_s]^{(u)}, [j_l]^{(u)})$ be an m-th root of A. Let $\underline{b} = (b_1, \ldots, b_n)$ be the Jordan type of the nilpotent part of B and $u := \min\{b_s, b_l\}$. Then $u \geq 1$, and $A = B^m \sim C^m$ by Theorem 1.2 where $C = \mathrm{diag}(E, [j_{s+i}]^{(u)}, [j_{l-i}]^{(u)})$. Hence, A has an m-th root similar to C as well. Another interpretation of Theorem 1.2 is for solutions of $M\underline{x}^T = \underline{a}^T$, where M is the matrix given in Corollary 3.6.

Example 4.2 (New m-th root from a special one). Let B be a third root of some nilpotent matrix A. Assume that the Jordan type of B is $\underline{b} = (0,0,0,1,3,0,0,0,0,1)$, that is, $B \sim \mathrm{diag}([j_4], [j_5], E)$, where $E = [j_{10}] \oplus 2[j_5]$. By multiplying $\underline{b}$ with the submatrix obtained by deleting the last two columns of the matrix M given in the equation (3.4), we obtain the Jordan type of A as $\underline{a} = (5,7,2,1)$. By Theorem 1.2, $(\mathrm{diag}([j_4], [j_5], E))^3 \sim (\mathrm{diag}([j_6], [j_3], E))^3$, where $u = 1$, $s = 4$, $l = 5$, $i = 2$, $k = 1$. Hence, $s+i = 6$, $l-i = 3$, and C is another third root with $\underline{c} = (0,0,1,0,2,1,0,0,0,1)$, that is, $c_3 = b_3 + 1 = 1$, $c_4 = b_4 - 1 = 0$, and $c_5 = b_5 - 1 = 2$. The Jordan type of C^3 is $(5,7,2,1)$ which can be obtained by multiplying $\underline{c}$ with the submatrix obtained by deleting the last two columns of the matrix M given in the equation (3.4) as well.

Example 4.3 (New solution for $M\underline{x}^T = \underline{a}^T$ from a special one). Let M be the matrix given in the equation (3.4), and $\underline{a} = (5,7,2,1)$. Then $\underline{x} = (0,0,0,1,3,0,0,0,0,1,0,0)$, and $\underline{x} = (0,0,1,0,2,1,0,0,0,1,0,0)$ are solutions to $M\underline{x}^T = \underline{a}^T$ by Example 4.2, where the second one is obtained from the first one using Theorem 1.2.

4.2. Sum of commuting matrices over various fields. Our Theorem 1.3 below is independent from the rest and does not require much preliminary work. It is based on some observations on the sum of two commuting matrices using the binomial theorem. Namely, if E and F are commuting matrices over a field $\mathbf{k}$. If $\mathrm{char}(\mathbf{k}) = p > 0$, then $(E+F)^{p^k} = E^{p^k} + F^{p^k}$. If $\mathrm{char}(\mathbf{k}) = 0$, and $EF = FE = 0$, then $(E+F)^n = E^n + F^n$ for any integer $n \geq 1$. Several nice consequences of these facts in relation to m-th roots are given in Theorem 1.3.

Proof of Theorem 1.3. Suppose that $A = E + F$ is a $d \times d$ matrix over a field $\mathbf{k}$ and $m > 1$ is a fixed integer. (1) Assume that K and L are commuting m-th roots of E and F, respectively. If $\mathrm{char}(\mathbf{k}) = p > 0$, the hypothesis $KL = LK$ implies that $A = K^{p^m} + L^{p^m} = (K+L)^{p^m}$ by the modulo p binomial theorem. If $\mathrm{char}(\mathbf{k}) = 0$, the hypothesis $KL = LK = 0$ implies that $(K+L)^m = K^m + L^m = E + F = A$ by the binomial theorem. Hence, $K + L$ is an n-th root for A where $n = p^m$ if $\mathrm{char}(\mathbf{k}) = p > 0$.

(2) Assume that $EF = FE = 0$, F is nilpotent of nilpotency t and E^k has an m-th root for $k \geq t$. Since $EF = FE = 0$, and $F^k = 0$ for $k \geq t$, by the binomial theorem we obtain $A^k = (E+F)^k = E^k$. Hence, the result follows. If E is diagonalizable (or nonsingular), then all powers of E are diagonalizable (or nonsingular). Thus, $A^k = E^k$ is diagonalizable (or nonsingular) for any $k \geq t$. Clearly, every diagonalizable complex matrix has an m-th root. When $\mathbf{k} = \mathbb{C}$ is the complex numbers the nonsingularity of E implies that of E^k; hence, E^k has an m-th root for any $k \geq t$ by the equation (89) in Chapter VIII of [2]. $\square$

Recall that the Jordan canonical form J_A of a matrix A having its eigenvalues in the field is unique up to the order of diagonal blocks. Let D be the diagonal matrix with same diagonal elements as J_A, and $N = J_A - D$. Then $A \sim D + N = J_A$, where $DN = ND$. Hence, $A = PJ_AP^{-1} = PDP^{-1} + PNP^{-1}$ can be written uniquely up to the order of diagonal blocks as a sum of two commuting matrices, one of which is diagonalizable matrix and the other is nilpotent. If A is nonsingular, then D is nonsingular; hence, $D + N = D(I + D^{-1}N)$ and $A = PDP^{-1}P(I + D^{-1}N)P^{-1}$. We obtain the following corollary of Thorem 1.3.

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COROLLARY 4.4. *Suppose that $A = D + N$ is a $d \times d$ complex matrix, where D is diagonalizable and N is nilpotent of nilpotency t.*

1. *If $ND = DN = 0$, then A^k has an m-th root for any $k \geq t$ and any $m > 1$.*
2. *If A is nonsingular, B and C are commuting m-th roots of D and $(I + D^{-1}N)$, respectively, then BC is an m-th root of A.*

Proof.

(1) Since $ND = DN = 0$, and $N^k = 0$ for $k \geq t$, $A^k = (D + N)^k = D^k$. Diagonalizable matrices over complex numbers have m-roots for any $m > 1$. Hence, for any $k \geq t$, $A^k = D^k$ have an m-th root for any $m > 1$.

(2) Since $BC = CB$, we have $(BC)^m = B^m C^m = D(I + D^{-1}N) = A$. □

EXAMPLE 4.5. Let $A = D + N$ where $D = \mathrm{diag}(0, e, f, 0)$ and $N = [j_4]^3$. Since $N^2 = 0$, $A^2 = D^2$. Then $\mathrm{diag}(0, e', f', 0)$ is an m-th root of $A^2 = D^2$ where e' and f' are some m-th roots of e and f, respectively, for any $m > 1$.

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REFERENCES

[1] J. Gallier. Logarithms and square roots of real matrices: Existence, uniqueness, and applications in medical imaging. "technical report." www.cis.upenn.edu/$\sim$cis610/matlog.pdf.

[2] F.R. Gantmacher. *The Theory of Matrices. Vol. 1.* Translated from the Russian by K. A. Hirsch. Reprint of the 1959 translation. AMS Chelsea Publishing, Providence, RI, 1998. x+374, pp. ISBN: 0-8218-1376-5.

[3] N.J. Higham and L. Lin. On p-th roots of stochastic matrices. *Linear Algebra Appl.*, 435:448–463, 2011.

[4] D.B. Janse van Rensburg, M. van Straaten, F. Theron, and C. Trunk. Square roots of H-nonnegative matrices. *Linear Algebra Appl.*, 621:29–49, 2021.

[5] D.E. Otero. Extraction of the m-th roots in matrix rings over fields. *Linear Algebra Appl.*, 128:1–26, 1990.

[6] S.Ö. Kaptanoğlu. Restricted modules and conjectures for modules of constant Jordan type. *Algebr. Represent. Theory*, 17:1437–1455, 2014.

[7] P.J. Psarrakos. On the mth roots of a complex matrix. *Electron. J. Linear Algebra*, 9:32–41, 2002.

[8] J. Schwaiger. More on rootless matrices. *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.*, 141:3–8, 2005.

[9] C. de Seguins Pazzis. The space of all p-th roots of a nilpotent complex matrix is path-connected. *Linear Algebra Appl.*, 596:106–116, 2020.

[10] B. Yood. Rootless matrices. *Math. Mag.*, 75:219–223, 2002.

[11] B. Yuttanan and C. Nilrat. Roots of matrices. *Songklanakarin J. Sci. Technol.*, 27(3):659–665, 2005.