SIXTH AND SEVENTH-GRADE STUDENTS' STRATEGIES IN SOLVING PROPORTIONAL PROBLEMS SUPPORTED WITH INTERACTIVE SIMULATIONS

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submitted by **AHU CANOĞULLARI** in partial fulfillment of the requirements for the degree of **Master of Science** i**n Mathematics Education in Mathematics and Science Education, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Erdinç Çakıroğlu
Head of the Department, **Mathematics and Science Education** ________________

Prof. Dr. Ayhan Kürşat Erbaş
Supervisor, **Mathematics and Science Education, METU** ________________

Examining Committee Members:

Prof. Dr. Erdinç Çakıroğlu
Mathematics and Science Education, METU ________________

Prof. Dr. Ayhan Kürşat Erbaş
Mathematics and Science Education, METU ________________

Assist. Prof. Dr. Mutlu Pişkin Tunç
Mathematics and Science Education, BEUN ________________

Date: 22.12.2021

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name : Ahu Canoğulları

Signature :

ABSTRACT

ALTINCI VE YEDİNCİ SINIF ÖĞRENCİLERİNİN İNTERAKTİF SİMÜLASYONLARLA DESTEKLENMİŞ ORANTISAL PROBLEMLERİ ÇÖZME STRATEJİLERİ

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The general aim of this study was to investigate sixth and seventh-grade students' strategies in solving proportional problems supported with interactive simulations. In this respect, the first aim was to investigate how sixth and seventh-grade students classify proportional problems. The second aim was to examine the strategies that sixth and seventh-grade students employ while solving proportional problems supported with interactive simulations and determine which levels these strategies correspond to on The Ongoing Assessment Project (OGAP) Ratio and Proportion Progression (Petit et al., 2020). The third purpose was to investigate the role of the numerical structure of the problems (i.e., integer ratio, non-integer ratio, letters) on students' selection of strategies in solving problems supported with interactive simulations. The data were collected from seven sixth graders and seven seventh graders during the June-July period of 2021. Interviews were held online via the Zoom platform. Each interview was recorded and transcribed for analysis. Findings revealed that sixth and seventh-grade pupils classify proportional problems based on problems' contextual aspects, problems' nature, and the words that sounded reminiscent of each other. In addition, one seventh-grade student focused on the interrogative word of the question in the problem. Moreover, among all the strategies proposed in The OGAP Ratio and Proportion Framework, only three of them were observed among students' answers, either as a single strategy or multiple strategies. Non-proportional strategies were primarily used in the non-integer ratio numerical structure. Early ratio strategy was employed only on the first question, and lastly, the use of proportional strategies was seen almost in each question. When the multiple strategies of the students were examined, it was found that all the multiple strategies started from a lower level and moved to a higher level. In addition, among all question types, the use of multiple strategies was not observed only in problems involving letters.

Keywords: Student strategies, Proportional Reasoning, Sixth Grade, Seventh Grade

ÖZ

ALTINCI VE YEDİNCİ SINIF ÖĞRENCİLERİNİN İNTERAKTİF SİMÜLASYONLARLA DESTEKLENMİŞ ORANTISAL PROBLEMLERİ ÇÖZME STRATEJİLERİ

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Bu çalışmanın genel amacı altıncı ve yedinci sınıf öğrencilerinin interaktif simulasyonlar ile desteklenmiş orantısal problemlerde kullandıkları stratejileri incelemektir. Bu kapsamda çalışmanın ilk amacı, altıncı ve yedinci sınıf öğrencilerinin orantısal problemleri nasıl sınıflandırdıklarını incelemektir. İkinci amacı, altıncı ve yedinci sınıf öğrencilerinin interaktif simülasyon ile desteklenmiş orantısal problemleri çözerken kullandıkları stratejileri incelemek ve bu stratejilerin The On Going Assessment Project (OGAP) Ratio and Proportion (Petit et al., 2020) çerçevesinde hangi seviyelere karşılık geldiğini belirlemektir. Bu çalışmanın son amacı, problemlerin sayısal yapısının (tamsayılı oran, tamsayılı olmayan oran, harfler) öğrencilerin dinamik simülasyon ile desteklenmiş orantısal problemlere ürettikleri stratejilerinin üzerindeki rolünü incelemektir. Bu çalışmanın verileri Haziran-Temmuz 2021 döneminde yedi tane altıncı sınıf ve yedi tane yedinci sınıf öğrencisinden toplanmıştır. Zoom platformu üzerinden online olarak yapılan görüşmeler analiz için kayıt altına alınmış ve transkript edilmiştir. Bulgular, altıncı ve yedinci sınıf öğrencilerinin orantısal problemleri sınıflandırırken, problemlerin bağlamsal yönlerine, problemlerin doğasına ve birbirini andıran kelimelere dikkat ettiklerini göstermektedir. Buna ek olarak, bir yedinci sınıf öğrencisinin soru köküne odaklandığına rastlanılmıştır. Ayrıca, çerçevedeki tüm stratejiler içerisinden yalnızca üçü ya tek strateji ya da çoklu strateji olarak öğrencilerin çözümleri arasında görülmüştür. Orantısal olmayan stratejiler en çok tam sayı olmayan oran içeren problemlerde kullanılmıştır. Erken oran stratejisinin sadece birinci soruda uygulandığı ve orantısal stratejilerin neredeyse her soruda kullanıldığı gözlenmiştir. Öğrencilerin çoklu stratejileri incelendiğinde, hepsinin düşük seviyeden yüksek seviyeye geçtiği görülmektedir. Ek olarak, tüm soru türleri içerisinde sadece harfler içeren soru türünde çoklu strateji kullanımı gözlenmemiştir.

Anahtar Kelimeler: Öğrenci stratejileri, Orantsısal düşünme, Altıncı sınıf, Yedinci sınıf

To my family

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ABBREVIATIONS

CMP	Curriculum Mathematics Project
CCSSI	Common Core State Standards for School Mathematics
MoNE	Ministry of National Education
NCTM	National Council of Teachers of Mathematics

CHAPTER 1

INTRODUCTION

Mathematics is an essential means for explaining the world around us and a helpful tool in solving daily life problems. Many everyday problems contain mathematical concepts, and it requires developing mathematical thinking skills to solve them effectively. For this reason, students must have certain mathematical skills to solve problems they encounter in daily life. Among these, proportional thinking requiring a multiplicative comparison between quantities while determining one value in terms of another (Mcintosh, 2013) is a significant ability that students must acquire throughout the middle grades (Langrall & Swafford, 2000) and a cornerstone for developing other mathematical and scientific concepts (Carney et al., 2015). Its presence and commonality in daily life require development in this area since there are many situations in which we need to think proportionally (Fernández et al., 2010). For instance, when preparing lemonade, we use proportionality to adjust the sourness of the drink. Similarly, while shopping, we use proportionality to decide which product might be the better choice by calculating the unit price. We also use proportionality to find a vehicle’s speed by establishing a relationship between distance and time. As a central skill that students must acquire, the importance of proportional thinking is included in different standards and curriculums. National Council of Teachers of Mathematics (NCTM, 2000) pointed out that in grades 6-8, students are expected to comprehend and practice ratios and proportions to demonstrate quantitative relationships. Similarly, Common Core State Standards Initiative (CCSSI, 2010) expects that in 6^{th} grade, students should learn the ratio concept to define the relationship between two quantities and apply their reasoning to solve problems involving ratio. In Turkey, the Ministry of National Education (MoNE, 2018) also stresses the importance of comprehending the ratio concept by suggesting that students in 6^{th} grade should use the ratio to compare the quantities and demonstrate the ratio with different forms, and students in 7th grade should decide if the given quantities are proportional by doing investigations within the real life context. Not only middle school students but also primary school students can have proportional thinking skills (Small, 2015). Although proportional reasoning is not officially stated as a subject in the Common Core Math Curriculum until Grade 6, its origins show up even prior (Small, 2015). According to Small (2015), proportional reasoning starts to be employed even in third grade while students are operating in basic multiplication and division, as it requires an understanding of how much one quantity is larger or smaller than the other in a multiplicative manner. Multiplication and division problems presented in the third grade in which the unit rate is given can be considered as unit-rate problems, and students have an inherent familiarity with the unit rate because they encounter incidences in daily life where they need to buy products (Cramer, Bezuk, et al., 1989). Degrande, Van Hoof, Verschaffel, & Van Dooren (2018) also pointed out that children learn to solve multiplicative word problems after third grade, and from the end of the fourth grade, they start to learn missing value word problems requiring multiplicative reasoning. In point of fact, from third and fourth grade, students are confronted with problems requiring proportional reasoning (Van Dooren et al., 2005). Studies conducted with primary grade students indicate that they can shave an understanding of the proportional reasoning concept. For example, Tourniaire's (1986) findings indicated that third, fourth, and fifth-grade students had some knowledge about the notion of proportion. Furthermore, studies by Van Dooren, De Bock, Evers, and Verschaffel (2009) included students from fourth to sixth grade to examine the overuse of proportionality, and Van Dooren et al. (2005) included students from second to eighth grade to examine the misemployment of proportional reasoning regarding students' age and educational level. These studies could be a starting point on which primary school teachers can benefit regarding how their students can be provided with opportunities to tackle simple proportional reasoning problems and reflect on the idea underlying the situations.

Given the place and time devoted to teaching this concept in the curriculum, one can expect students to become experts in proportional reasoning at the end of middle school. However, despite its significance, students have difficulties attaining this concept (Lanius & Williams, 2003; Steinthorsdottir & Sriraman, 2009). The abundant number of studies in this field have shown that students prone to select incorrect strategies while solving problems involving ratio and proportion and there are several factors that influence and account for students' selection of strategies such as their grade level, the nature or the numerical structure of the problem (e.g., Degrande, Van Hoof, Verschaffel, & Van Dooren, 2018; Degrande, Verschaffel, & Van Dooren, 2019; Van Dooren, De Bock, Evers, & Verschaffel, 2009; Van Dooren et al., 2005; Van Dooren, De Bock, Verschaffel, & Janssens, 2003a; Van Dooren, De Bock, & Verschaffel, 2010a; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010b).

Among the difficulties, the common one is that students apply the proportional rules even in the problems that demand additive reasoning, and this tendency increases with age (Van Dooren et al., 2009). In their study, Van Dooren et al. (2009) studied with fourth, fifth, and sixth graders, and students were given eight missing-value word problems. Their results revealed that for non-proportional problems, the number of proportional responses increased from fourth to sixth grade. In another study, working with 12-to-16-year-old students, Van Dooren et al. (2003) found a tendency among students to use proportional methods in non-proportional situations even though they realized that the method was inappropriate for that context. In parallel with the results in the literature, it was found that students' employ additive methods mostly in elementary school and tend to select proportional methods in middle school, and there is a time between these years that students employ additive and multiplicative methods at the same time based on the numerical structure of the problem (Van Dooren, De Bock, & Verschaffel, 2010a). These studies indicated that students employ wrong strategies in problems, and students are more likely to overapply proportional thinking methods as their grade level increases.

These difficulties could be stem from the fact that students could not discriminate between proportional situations from nonproportional ones (Hilton et al., 2012; Toluk-Ucar & Bozkus, 2018; Weiland et al., 2019). Hilton et al. (2012) developed a diagnostic instrument to recognize situations where students can employ proportional thinking and reveal the forms of thinking they utilize. Their pilot instrumentation results revealed that distinguishing proportional situations from nonproportional ones was a significant issue for students since most students select multiplicative strategies to solve non-proportional problems and select additive strategies for situations that entail multiplicative reasoning. Not only students but also teachers had difficulties in distinguishing problems involving proportional reasoning from problems that require additive strategies. In their study, where they investigated elementary school students' and preservice teachers' ability to discriminate proportional and non-proportional situations, Toluk-Ucar and Bozkus (2018) found that preservice teachers tended to use multiplicative methods to solve problems entailing additive reasoning. Similar results were obtained by Weiland et al. (2019), who investigated middle school teachers' skills to recognize situations that require proportional reasoning and found that although teachers were successful in identifying proportional problems, they had difficulty recognizing these problems as suitable for proportional reasoning.

In addition, the literature shows that not only age but also the nature of the problem or the numerical structure play a role in students' choices of strategies and their performances in proportional reasoning tasks, in particular, the integer or non-integer relationship between the quantities affects students' strategies in solving ratio and proportion problems (Artut & Pelen, 2015; Degrande, Verschaffel, & Van Dooren, 2014; Fernández et al., 2010; Van Dooren et al., 2009). Van Dooren et al. (2009) investigated if the numbers in the problems affected students' strategies, and they administered a test containing both proportional and non-proportional problems to fourth, fifth, and sixth graders. Their results revealed that non-integer ratios appeared in the proportional problems negatively affected students' performances, and the existence of non-integer ratios in non-proportional problems led students to use proportional methods less. Similarly, in another study, where third and sixth-grade students' development of qualitative analogical reasoning to missing-value word problems were examined, Degrande et al. (2014) found that the problems involving integer ratios led students to give more proportional answers in comparison with problems involving non-integer ratios. Furthermore, after working with sixth-grade students, Artut and Pelen's (2015) findings were consistent with Degrande et al. (2014). Their results manifested that students gave multiplicative answers when the problems contained integer ratios and selected additive methods when they did not see integer ratios regardless of the problem being proportional or non-proportional. Moreover, Fernández et al. (2010) studied students from diverse levels from fourth to tenth grade and found that students chose additive methods when they were confronted with non-integer ratios, which indicated that ratio type influenced students' strategies. It is evident that the existence of a non-integer ratio has caused students' performance to decrease and their thinking about the problem to change (Cramer et al., 1993). These studies indicate that students' answers vary depending on whether the ratio is an integer or non-integer.

Teachers' teaching methods in schools might be a reason why students apply incorrect strategies and become confused while discriminating proportional situations from the ones that are not. de la Cruz and Garney (2016) argued that students' difficulties could be attributed to teaching strategies, specifically the emphasis on cross multiplication rule in solving missing-value proportional situations. Students are apt to emphasize the solution techniques and are more willing to apply them in missing value situations than recognizing the relations among the ratios and proportions embedded in problems (Lim, 2009). Langrall and Swafford (2000) asserted that the cross-product algorithm is taught too early without letting students build their models. In this regard, they suggested that the instruction should support students to visualize the problem situation. In addition, focusing on the cross-multiplication rule affects students' understanding of proportional thinking in a negative way, so this algorithm should not be introduced to students without having adequate problem-solving experience in proportionality (Cramer, Post, et al., 1989). Likewise, Toluk-Ucar and Bozkus (2018) supported the idea that students' overuse of proportional methods might occur because they learn proportionality by cross multiplication first.

Considering these difficulties, one can argue that traditional techniques might not be sufficient to promote students' advancement in proportional reasoning. Due to limited variation of the activities performed in the classrooms, the learning progress might not develop properly (Behr, Harel, Post, & Lesh, 1992). Neither students nor the middle school teachers have sufficient knowledge about multiplicative situations due to inadequate educational activities (Behr et al., 1992). Moreover, students' performance in mathematical operations does not measure their understanding of the concept underlying their computations (Dole et al., 2012), and providing the correct answers does not certify the presence of proportional reasoning (Lamon, 2007). Therefore, the knowledge of what to teach should be supported and combined with the knowledge for how to teach.

Despite the fact that the traditional technique, commonly known as the cross-product rule, can be used to resolve problems efficiently, it may prevent students from attaining proportional reasoning (Stemn, 2008). On the other hand, a classroom environment in which students are given an opportunity to share and reflect upon each others' ideas was found to be helpful for students to promote their understanding of proportion concepts (Stemn, 2008). Another instructional idea that might be helpful for learners to construct proportional reasoning might be to provide them with a setting in which they could put an effort on distinguishing proportional problems from non-proportional ones as it is necessary to understand the disparities between these situations to fully comprehend the proportional reasoning (Dole & Shield, 2008). Van Dooren, De Bock, Vleugels, and Verschaffel's (2010b) findings where they had sixth-graders classify and solve a set of problems supported this argument. More specifically, they found that students who did the classification first had better scores than those who did the solution task first. They concluded that classification tasks positively affected students' performance in solving problems. Not only classifying proportional situations from non-proportional ones but also judging problems if they could be mathematically related to each other may be a promising means to give a grasp of how necessary knowledge to solve mathematical problems could be processed and recalled (Silver, 1977, 1979). In his work, where he analyzed different mathematical ability students' informational gathering, processing, and retention, Kruketskii (1976) found that mathematically capable students could notice the essential information given in the problems whereas less capable students perceived irrelevant aspects even when they were supported.

Another alternative means might be providing students with technology-integrated activities since technology can enhance students' understanding (Akpan & Beard, 2014). The potential of technology in mathematics education has long been discussed in several studies, and there is an extensive body of literature on that topic (Heid, 2005; Srisawasdi & Panjaburee, 2014). Technology is an essential characteristic of mathematics classrooms (Powers & Blubaugh, 2005) and has a central role in improving students' learning (Srisawasdi & Panjaburee, 2014). Computers can be utilized to support students to originate mathematical notions. Technology integration activities should be adapted in mathematics lessons to permit students to test and create models for clarifying a mathematical concept by collecting data and doing an experiment, adjusting, refusing, or approving hypotheses (Cuoco & Goldenberg, 1996). Moreover, technology can provide effective new methods to accustomed problems and exploration to the various component of mathematics (Corbitt, 1985). However, using technology without purpose does not assure the students' success. Clements (2000) argued that teachers' usage of computers should go beyond the drill and practice of the concept. Instead, technology should allow students to work on engaging problems or projects from which they can learn (Clements, 2000). This argument was supported by Roschelle et al. (2000). Although computer-assisted implementations that stimulate learners to produce arguments and give justifications have the potential to enhance learning, activities focused on drills and practice appeared to reduce achievement (Roschelle et al., 2000). In this regard, it is notable for highlighting some of the essential characteristics of particular technological tools, which may help to provide creative environments in which students can explore and discuss various ideas, such as dynamism. With the help of the dynamism feature found in dynamic environments, students can explore mathematical notions by manipulating objects since these environments render the relationship between the mathematical ideas more concrete (Hoyles & Noss, 2009). Similar idea was also suggested by the Roschelle, Noss, Blikstein, and Jackiw, (2017). As Roschelle et al. (2017) pointed out, the fact that the dynamic feature makes a mathematical situation more concrete and understandable assists students in recognizing the relationships in these situations and building their models.

Therefore, it appears that to facilitate the development of proportional reasoning and to reduce the difficulties stem from the various factors mentioned earlier, teachers should provide their students such an atmosphere where their students can reflect on the problems before solving them, and make explorations and produce diverse strategies through technological tools in the course of problem solving.

1.1 Purposes of the Study

The purposes of this study are threefold. The first purpose of this study is to investigate how sixth and seventh-grade students classify proportional problems. The second purpose of this study is to examine the strategies that sixth and seventh-grade students produce while solving proportional problems supported with interactive simulations and determine which level these strategies correspond to on the Ongoing Assessment Project (OGAP) Framework (Petit et al., 2020). The third purpose of this study is to investigate the role of the numerical structure (i.e., integer ratio, non-integer ratio, letters) of the problem on students' selection of strategies while solving proportional problems supported with interactive simulations.

1.2 Research Questions

1. How do sixth and seventh-grade students classify proportional problems?

2. Which strategies do sixth, and seventh-grade students use while solving proportional problems supported with interactive simulations?

> **2a.** Based on the OGAP Ratio and Proportion Progression (Petit et al., 2020), which strategies do sixth and seventh-grade students produce while solving proportional problems?

3. What is the role of numerical structure on sixth and seventh-grade students' selection of strategies while solving proportional problems?

1.3 Definition of the Important Terms

Ratio, Rate, and Proportion

There are various definitions proposed for rate, ratio, and proportion concepts in the literature. As stated by Lim (2009), these terms refer to different concepts. In this study, the following definitions are adopted.

A ratio can be defined as a multiplicative relationship between two quantities (Lim, 2009; Petit et al., 2020) or between similar quantities (Lamon, 2007). According to Behr, Lesh, Post, and Silver (1983), the ratio can be described as a relationship that expresses the concept of relative quantities.

The rate can be defined as a ratio expressed by distinct quantities (Lamon, 2007; Lim, 2009) or a particular type of ratio wherein one of the amounts being compared is represented as a unit (Petit et al., 2020).

Proportion is a mathematical situation in which two ratios are tantamount to each other (Kilpatrick et al., 2001; Lim, 2009; Petit et al., 2020; Riehl & Steinthorsdottir, 2019; Tourniaire & Pulos, 1985).

Proportional Reasoning

Proportional reasoning entails the usage of multiplicative relations to make a comparison between amounts and figure out the value of one amount in terms of another (Mcintosh, 2013). In other words, proportional reasoning requires the analysis of quantities relatively instead of absolutely (Fielding-Wells et al., 2014; Lamon, 2010). As Lamon (2007) claimed, being able to recognize the multiplicative relationship between two quantities and applying this relationship to other amounts, and at the same time providing justifiable arguments for the underlying causes of these operations, is defined as proportional reasoning. Karplus, Pulos, and Stage (1983) defined proportional reasoning as recognizing the linear relationship of two quantities with a multiplicative invariant between them. Moreover, proportional reasoning as a type of mathematical thinking focuses on making interpretations and estimations about the situations involving covariation (Cramer, Post, et al., 1989).

Types of Proportional Problems

Numerical Comparison Problems

In numerical comparison problems, two ratios are given, and the comparison of these ratios is expected (Cramer & Post, 1993a). For example, when deciding the speeds of two cars, the ratio of distance traveled and time elapsed for both must be calculated and compared. In these situations, two ratios should be formed, and a decision should be made by making a numerical comparison.

Missing Value Problems

In missing value problems, three quantities are known, and one quantity is asked for (Cramer & Post, 1993a). An example of a missing value problem would be as follows. If a person buys 5 kilograms of bananas, for which she paid 45 Turkish Liras for 3 kilograms, how much should she pay? For this specific case, one can calculate the unit price and use it to find the target amount.

Qualitative Comparison and Qualitative Prediction Problems

Qualitative comparison and qualitative prediction problems demand comparisons not dependent on particular numeric values (Cramer & Post, 1993a). The following two problems can be given as examples of qualitative prediction and qualitative comparison, respectively.

> "If Deyan ran fewer laps in more time than she did yesterday, would her running speed be (a) faster, (b) slower, (c) exactly the same, (d) not enough information to tell." (Cramer & Post, 1993a, p.166)

> "Mary ran more laps than Greg. Mary ran for less time than Greg. Who was the faster runner? (a) Mary, (b) Greg, (c) same, (d) not enough information to tell." (Cramer & Post, 1993a, p.166)

As seen, there are no numbers to operate within both problems to predict or compare, which is why they are called qualitative.

Multiplicative and Additive Reasoning

Multiplicative reasoning, or proportional reasoning, involves utilizing ratios to compare quantities. In contrast, additive reasoning requires discerning the additional relationship, such as sums of differences between the numbers (Bright, Joyner, & Wallis, 2003). Van Dooren et al. (2010a) argue that the constant difference between the two quantities is considered additive reasoning. Therefore, an addition operation should be employed to find the unknown quantity. In contrast, the constant ratio among the two quantities is considered in the multiplicative type of reasoning. Therefore, a multiplication operation should be employed to find the unknown quantity.

Measure Space

Vergnaud (as cited in Cramer et al., 1993) introduced the concept of the measure space to help explain the nature of the multiplicative relationship between quantities in situations involving proportional thinking. Accordingly, measure space can be considered as physical magnitudes as length, weight, money, etc. (Cramer et al., 1993). In situations involving proportionality, there is always a multiplicative relationship among the quantities between or across measure spaces (Cramer et al., 1993).

Within vs. Between Relationships

Within relationship is the multiplicative relationship between components in the same ratio, whereas between relationship is the multiplicative relationship between the matching amounts of different ratios (Steinthorsdottir & Sriraman, 2009). Within relationship can also be characterized as a scalar relationship, whereas between relationship is known as a functional relationship (Steinthorsdottir & Sriraman, 2007).

According to (Fernández et al., 2010), there are two kinds of relations between the quantities; within and between relationships. In a within relationship, also called an internal ratio, the relationship exists between the same nature of quantities (Fernández et al., 2010). In between relationship, also called an external ratio, the relationship exists between the different nature of quantities (Fernández et al., 2010).

Unit Rate

The constant factor among the quantities is called the unit rate (Cramer et al., 1993).

Integer vs. Non-integer Ratio

If the multiplicative relationship between proportional quantities is not an integer, this is called a non-integer ratio. In other words, if the unit rate is a non-integer, then there is a non-integer ratio between the amounts compared (Cramer et al., 1993).

If the multiplicative relationship between proportional quantities is an integer, it is called the ratio, which is an integer. In other words, if the unit rate is an integer, then there exists an integer ratio between the amounts that are compared (Cramer et al., 1993).

Dynamic Mathematics Environment

Dynamic mathematics environments often refer to instructional software that supports students' acquisition of mathematical ideas through multiple and dynamic representations, allowing transform and explore mathematical relations and properties (Jobrack et al., 2018).

In this study, GeoGebra software, which is a free program that has properties of dynamic geometry software and computer algebra systems (Hall & Chamblee, 2013), has been used to provide students with dynamic environments where they can explore variation and covariation of quantities in proportional situations and make generalizations in these situations based on the feedback received from the software. Throughout the chapter, I refer to dynamic mathematics software as a dynamic simulation or interactive simulation to emphasize the dynamic nature of the GeoGebra since students in this study were expected to solve problems within GeoGebra by benefiting the dynamism aspect of the GeoGebra via the slider.

1.4 Significance of the Study

Proportional reasoning is an essential ability in the mathematics curriculum (Langrall & Swafford, 2000). Acquisition of this skill is regarded as a significant step in students' mental growth (Cramer & Post, 1993a). Moreover, it is a capability that leads to the achievement of advanced mathematics and science content when it is improved (Carney et al., 2015). There is a demand for research concerning how students promote understanding of this concept (Carney et al., 2015). As argued, it is one of the fundamental mathematical skills as it has a vital role in the students' mathematical improvement (Fernández et al., 2010) and a concept that we come across and widely use in daily life. Numerous daily activities demand proportional reasoning that requires recognizing and evaluating the contexts of comparison of quantities relatively (Dole et al., 2015). For instance, it is necessary to use proportional reasoning in daily life situations in which we calculate the unit price of a product to decide which one would be a better choice to buy, or we make conversions about measurement or money (Im & Jitendra, 2020). As it is a critical concept, students need to gain the ability to reason proportionally. However, despite the value of this concept in students' understanding, students had difficulties in fully understanding proportional reasoning (Çalışıcı, 2018; Stemn, 2008). One of the main difficulties is that students cannot discriminate the proportional situations from non-proportional ones (Arıcan, 2019), although it is essential for proportional reasoning to recognize the difference between proportional and non-proportional situations (Cramer et al., 1993; Fernández et al., 2010; Lim, 2009). This difficulty makes it harder for them to choose appropriate strategies for the problems. For example, Arican (2019) found that almost half of the seventh-grade students in their study could not differentiate nonproportional situations from proportional ones. . Similar results have been obtained by Hilton, Hilton, Dole, and Goos (2013), who assessed middle school students' proportional reasoning. They found that when they were given a problem requiring multiplicative reasoning, only 16.4% of students correctly responded, and 30.9% of the students thought that the problem required additive reasoning. As mentioned in the previous section, it is evident that numbers influenced students' strategies in the problems. More specifically, having integer or non-integer numbers in either within or between ratios affects students' solution strategies as they are solving the problems (Artut & Pelen, 2015; Degrande, Verschaffel, & Van Dooren, 2014; Fernández et al., 2010; Van Dooren et al., 2009). Thus, it is suggested that future studies should consider using letters or symbols instead of numbers in the problems (Degrande et al., 2019; Van Dooren et al., 2009) to see how the replacement of numbers with letters or symbols affect pupils' choices of strategies. Therefore, in this study, students were provided with several problems that have numerical structures ranging from integer ratios to non-integer ratios, along with the problems involving letters instead of numbers.

Nasution and Lukito (2015) argued that students' difficulties might arise from the traditional instruction focusing on practicing algorithms. They suggested that the instruction should be designed so that it helps students acquire the meaning of the proportion concept and helps them make progress in proportional reasoning. Carney et al. (2015) also highlighted that the instruction should be modified to support students to produce their strategies and enhance proportional reasoning instead of having traditional methods where the algorithms are mostly emphasized. In this respect, integrating different activities into the lesson might help foster students' thinking and might allow them to develop their strategies. Bright et al. (2003) suggested that it is critical to provide students an environment where they can employ additive and multiplicative thinking strategies together both erroneously and adequately. This idea was supported by Van Dooren, De Bock, and Verschaffel (2010a) as well. Van Dooren, De Bock, and Verschaffel (2010a) pointed out that without investigating additive and multiplicative problems together, it is doubtful to disclose and identify students' authentic skills of comprehending the multiplicative relations which discriminates additives situations from proportional ones. In parallel with these ideas, Bright et al. (2003) recommended integrating various types of questions into assessments that range from additive to multiplicative structure to make room for students to reveal their understanding. In fact, the findings of the study of Lim and Morera (2010), conducted with pre-service teachers, indicated a decrease in the overapplication of the proportional methods when confronted with non-proportional problems.

So, it can be inferred from the researchers' conclusions that it might be necessary to use alternatives such as classification tasks in the classroom so that students can think about and reflect on the problems before attempting to solve them. Therefore, it might be valuable for students to categorize problems based on their commonalities before answering them, and if they can do so, they will be less prone to use wrong strategies in the problems (Larsson & Pettersson, 2015). The findings of Van Dooren et al. (2010b) have supported this argument. In their study, Van Dooren et al. (2010b) revealed that students who did the classification first had better scores than those who solved the problems first and then made classification. The researchers concluded that classification tasks, where students try to identify the differences between proportional and non-proportional situations, were helpful for students to first understand the problem before solving it and played a role in decreasing the overapplication of proportional reasoning. This study is inspired by the idea of giving students some time to consider the problems and, in turn, providing them an opportunity to understand the mathematical situation behind the problems before jumping to a conclusion. In this regard, students were asked to compare the problems according to their common points before solving them. By doing so, the researcher had a chance to understand which characteristics of the problems students focus on while classifying proportional problems.

So far, the affordances of different instructional strategies in students' learning of proportional reasoning have been mentioned. There is also one component to make learning more robust and engaging: using technology-integrated activities. As argued, it might be critical to integrate technology into the mathematics lessons since technology plays a vital role in improving learning quality (Srisawasdi & Panjaburee, 2014), and computers can be beneficial for helping students to create mathematical ideas (Cuoco & Goldenberg, 1996). Among the benefits of technology integration in education discussed in the literature is the issue of how dynamic and visual aspects of the technology can contribute to students' understanding of the concepts by making ideas more tangible. Roschelle et al. (2000) argued that technologies that involve dynamism are powerful sources in assisting students to better envision and comprehend the ideas behind situations. For instance, Fathom software is considered a powerful vehicle for students in embodying statistical notions since it confers the possibility to observe numerous portrayals of a phenomenon through its dynamism feature (Chance et al., 2007). Also, Computer Algebra Systems (CAS) have the same power as it is possible to simultaneously observe the changes in a representation due to changing the values for a parameter within the CAS environment (Heid et al., 2012). Another example is the dynamic geometry or mathematics environments. They also provide a setting for their users to explore the changes on a diagram and allow making inferences regarding the mathematical situation based on those changes (Laborde et al., 2006).

All in all, examining students' way of thinking while working on the proportionality problems can help teachers decide which activities can support their students' understanding (Lamm & Pugalee, 2009). Evaluation of strategies that will support the development of multiplicative strategies can contribute to the design of educational plans that emphasize the order in which students should perform activities that are at different levels (Tourniaire & Pulos, 1985). Therefore, it is essential to investigate the students' strategies while working with problems supported by dynamic simulations so that educators can make insightful and informed decisions when designing learning activities for particular concepts.

Therefore, this study integrated the situations that have been investigated separately. In this sense, students were expected to classify questions before solving them and use technology. Unlike previous studies, the problems presented to students included numbers and letters. This study differs from other studies in that it includes all these components simultaneously. This study's results might give insight into the students' understanding of proportionality in a technology-supported environment in which they were expected to classify and then solve the problems. To sum up, this study's findings would contribute to our understanding of the strategies students use and the levels of these strategies as they solve problems having different numerical structures within an interactive simulation environment.

CHAPTER 2

LITERATURE REVIEW

This chapter contains seven sections that present a compilation of studies within the framework of the main topics of this study. The first section describes the definition and importance of proportional reasoning, and the abilities demanded to be a proportional thinker. The second section touches upon the difficulties that students experience in solving proportional problems. This section is further divided into two subsections. The first section synthesizes the studies on the erroneous use of multiplicative strategies, and the second compiles studies in the literature addressing the misuse of additive strategies. The third section discusses various strategies that students use while solving proportional thinking problems, and the fourth section provides the factors that may account for the variability in those strategies. The fifth section provides some evidence on the transition of learning proportional reasoning from primary to middle school and discusses several strategies for teaching proportional reasoning. The sixth section of this chapter discusses the potential of using technology in mathematics learning, particularly dynamic mathematics software. Lastly, the seventh section presents a summary of the literature review.

2.1 Proportional Reasoning

Proportion can be described as equating two ratios (Ben-Chaim et al., 1998; Tourniaire & Pulos, 1985), and proportional reasoning was characterized as the capacity to distinguish a multiplicative relationship between two amounts and then apply this relation to the others (Lamon, 2007). Lamon (2007) suggested that proportional reasoning implies explaining a context involving the covariation of the quantities whose ratios remain constant. It was also defined by Karplus, Pulos, and Stage (1983) as reasoning about a situation involving two variables with a fixed ratio and a linear relationship between them. Proportional reasoning, which entails the comparison of ratios, is regarded as a fundamental ability that students should acquire in middle school and considered a factor that determines students' achievement outside of the school (Hilton, Hilton, Dole, & Goos, 2016). One comes across the proportionality concept in mathematics and daily life (Modestou & Gagatsis, 2010). For instance, speed and unit price can be classified as rates we usually encounter daily. In contrast, the lengths of the sides of a photograph or the components of a recipe are examples of proportions that are dimensionless and different from the rates that have dimensions (Karplus et al., 1983), both of which are related to the proportional reasoning concept. Many studies agree on the importance of proportional thinking in the mathematics curriculum (Cramer & Post, 1993a; Dole & Shield, 2008; Modestou & Gagatsis, 2010; Weiland, Orrill, Nagar, Brown, & Burke, 2020). It is an essential and fundamental topic related to other mathematics domains such as fractions, percents, algebra, and probability (Dole & Shield, 2008). NCTM (2000) suggests that the development of proportional reasoning includes understanding the multiplicative relationship between quantities and demonstrating this relationship with the aid of tables, graphs, and equations, beyond finding the missing term by equalizing two ratios. Students who are unable to improve proportional thinking skills can face obstacles not only in the fields of science, engineering, or mathematics that require the use of this skill but also in real-life situations (G. Hilton et al., 2013). Langrall and Swafford (2000) also advocate the idea that proportional thinking is one of the important skills to be acquired in middle school and a key issue in making sense of advanced mathematics topics, and students who fail to acquire this skill are likely to face difficulties in advanced mathematics, particularly algebra. Langrall and Swafford (2000) asserted that there are four foundational skills to be a proportional reasoner. The first one is being able to discern the difference between absolute and relative change, in other words, the difference between the additive and multiplicative relationship. Based on the absolute change, the relationship between the amounts is considered additively, whereas, for the relative change, the multiplicative relationship should be established between the amounts being compared. Langrall and Swafford (2000) emphasized that to develop proportional reasoning, in other words, to realize the multiplicative thinking between the amounts, the teachers should pose students how many questions instead of how many questions. The other important component that should be acquired to be a proportional reasoner, according to Langrall and Swafford (2000), is being able to realize if using a ratio is appropriate or not for a given problem situation. They suggested that students should be able to notice whether the problem requires a comparison of ratios before attempting to solve it. The third essential component, as they argued, in proportional reasoning is understanding that the amounts that constitute a ratio change so that their relationship stays constant. According to this idea, students should perceive that even though the quantities change, the multiplicative relationship between them remains constant. The last, but as important as the other components is that knowing the unitizing approach (Langrall & Swafford, 2000). As claimed by this idea, students should determine a unit that can be built up to form new pair of amounts that have the same ratio with the initial pair of amounts. Langrall and Swafford (2000) argued that these four fundamental skills form the basis for students to become proportional thinkers. Similar to Langrall and Swafford's (2000) suggestions about the requirements of being proportional thinker, Cramer and Post (1993b) also proposed critical abilities that a proportional reasoner should have. They argued that one should first recognize the mathematical features of proportional problems and one also be able to discern whether the problem requires proportional thinking or not. Furthermore, Cramer and Post (1993b) emphasize how it is important to produce numerous methods in solving proportional problems without influenced by the numbers or the context of the problem and solving qualitative problems that also requires proportional thinking to be able to a proportional thinker. Before explaining the factors that students are affected by while solving problems, the difficulties that students frequently encounter with proportional thinking will be mentioned under the next heading, due to the significance of addressing this issue and its relevance with students' choices of solution strategies.

2.2 Students' Difficulties on Proportional Reasoning

Proportional reasoning, a significant characteristic of formal thinking, is attained throughout middle school (G. Hilton et al., 2013) and requires the capacity to compare quantities in ratio situations (A. Hilton et al., 2016). However, one's ability to provide accurate responses does not assure that they have proportional reasoning since proportional situations can be solved employing a cross-multiplication algorithm and rote procedures (Lamon, 2007). Understanding proportional reasoning requires the ability to distinguish the proportional situations from the non-proportional ones (Cramer et al., 1993; Lim, 2009) since discerning whether or not mathematical situations are proportional is a significant aspect of proportional reasoning (Brown et al., 2019; Cramer & Post, 1993a; Weiland et al., 2020). Nevertheless, literature shows that students have difficulties differentiating the proportional situations from the non-proportional ones (Fernández et al., 2008; A. Hilton et al., 2013). In a study conducted by Degrande et al. (2019), 246 third, fourth, fifth, and sixth-grade students' problem-solving behavior and their solution preferences were investigated through four instruments called a word problem test, a preference test, a computation test, and a discrimination test, each of which involving multiplicative and additive word problems. According to their results, in the word problem test consisting of six additive and six multiplicative word problems, 62.6% of the students answered at least one additive problem in a multiplicative way, and 34.1% of the students answered all additive problems by using multiplicative methods. Similarly, for the multiplicative problems, 39.8% of the students answered at least one problem by using additive methods, whereas 18.7% of the students answered all multiplicative problems additively. Based on this evidence, it can be argued that some students used additive and multiplicative methods interchangeably without understanding the problem situation, which is evidence of not distinguishing proportional situations from the ones that are not.

Based on the literature findings, it is reasonable to infer that difficulties in discerning whether a situation is multiplicative or not may lead students to use additive and multiplicative strategies either erroneously or interchangeably. The studies on students' difficulties in solving proportional problems revealed that the difficulties were generally grouped under two main topics: erroneous use of multiplicative strategies in non-proportional situations or erroneous use of additive strategies in proportional situations. Therefore, in this section, students' difficulties were discussed under these two issues.

2.2.1 Erroneous Use of Multiplicative Strategies

Not all problems involving multiplicative structures are solved with correct methods that explain and resolve the problem situation. In fact, the consolidation of proportionality in many cases and teaching algorithmic methods to solve proportionality problems triggers both children and adults to employ proportional strategies in all situations (Van Dooren et al., 2003a). Many students are inclined to apply proportional rules even if they are not appropriate, which is what Van Dooren et al.(2003b) called the illusion of linearity. For instance, students having this propensity believe that if the sides of a shape are doubled, so does its area (Van Dooren et al., 2003b). Studies in the literature have shown that primary school students tend to employ strategies requiring proportional thinking, even if the method is not suitable when encountering word problems with missing value structure (Van Dooren et al., 2009). In their study, Van Dooren et al. (2010b) compared two groups of sixth-grade students' answers on comparison and solution tasks involving proportional and non-proportional situations, where one group of sixth graders did classification task first and, then solution task; and the other group did the solution task first and then the classification task. Nine experimental word problems, consisting of three proportional, three additive, and three constant problems, were presented to students. In the classification task, students were expected to classify problems by deciding which problems belong to each other. After they grouped the problems, they were asked to put each problem in different envelopes and write down what they had in common. In the solution task, students were expected to solve the nine problems and then make a classification. Accordingly, findings showed that even for the non-proportional problems, out of three students, two of them gave proportional answers, indicating their tendency to use proportional methods although the problems do not demand. In another study conducted by De Bock et al.(2002), 20 seventh grade and 20 tenth grade students were given a problem that did not require proportional thinking and were asked to solve it aloud. The interviews with the students consisted of five stages. At each stage, the students encountered a cognitive conflict, which might have affected students' responses, and asked if they wanted to change their answer with the presented solution. If the students initially answered the question incorrectly, a stronger cognitive conflict was presented at each stage. According to the findings of the study, only two of the 10th-grade students answered the question correctly in the first stage, which again reveals students' tendency to apply proportional methods in situations that are not suitable to use. Similar findings have been obtained from Pelen and Artut's (2016) work. The researchers conducted a study with 331 seventh-grade students, who were given a test involving 24 open-ended problems with three types: direct proportional, inversely proportional, and non-proportional problems. Their findings revealed that students tended to overapply proportional methods even in situations that did not entail proportional reasoning. It is also worth mentioning that students' tendency to employ proportional reasoning strategies within problems requiring additive reasoning increases as they move from elementary to middle school (Degrande et al., 2014). The study of Degrande et al. (2014) revealed that there is an increase in proportional responses and a decrease in additive responses of word problems from 3rd to 6th grade. In another study, Van Dooren et al. (2005) examined the progress of misusage of proportional thinking regarding age and educational level of pupils. One thousand sixty-two students from second to eighth grades were given a paper-pencil test involving numerous forms of proportional and non-proportional problems with a missing value format. According to their results, students revealed a tendency to employ proportional methods in situations where they were not appropriate. Their findings also suggested that as students' competency of proportional reasoning increases from grade 2 to grade 5, their tendency to overapply proportional methods also increases. They concluded that even though after 6^{th}-grade students started to discern if the situation is appropriate for applying proportionality, there were some errors in responses in eight graders, which revealing their overuse of proportionality. In another study, Riehl and Steinthorsdottir (2014) studied with more than 400 middle school students' solution strategies and their thinking regarding proportional reasoning. For that purpose, students were expected to solve a problem called Mr. Tall and Mr. Short. Researchers divided students' correct and incorrect strategies and formed a category for each. Accordingly, students' answers were corresponded to five categories namely illogical, additive, build-up, ambiguous, and multiplicative. Multiplicative category was further divided into three categories which are within, between and ambiguous. Moreover, illogical, and additive answers indicated erroneous answers whereas the remaining categories specified correct answers. Their results revealed a trend on a grade-level basis. While 48% of fifth graders gave answers that corresponds to illogical category, this rate was 23% in sixth graders, 23 % in seventh graders and 22 % in eighth graders. These results display that as students progressed from the fifth to the eighth grade level, they provide less illogical answers. It is interesting that none of the fifth or sixth graders gave multiplicative answers without providing ambiguous responses. In other words, only 2% of the fifth graders and 3% of the sixth graders solved the given task multiplicatively but they were provided ambiguous responses as well. Overall results revealed that seventh and eighth graders performed better than fifth and sixth graders. Furthermore, out of 412 students, 29% of the students solved multiplicative problems correctly and 44% of the students chose strategies that required additive reasoning even though the problem was not suitable to apply additive methods. In another study, A. Hilton et al. (2013) developed and implemented a two-tier instrument to investigate over 2000 middle school students' understanding of proportional reasoning whose grade levels ranged from 5 to 9 over the course of three years. In the first tier, all problems required only true-false answers, and the problems in the second tier were used to recognize participants' solutions involving additive and multiplicative reasoning as well as their capability to discern whether the presented situation is proportional or not and their interpretation skills on the depiction of proportional problems. According to their results, most of the students erroneously applied multiplicative strategies in non-proportional problems, which indicated multiplicative thinking in situations involving additive thinking. These results show students' disposition to multiplicative strategies even in the cases that require additive ones. Literature indicates that this attitude of students may continue at later ages when they become mathematics teachers. Even teachers have difficulties justifying why the amounts being compared vary together, let alone provide different solution methods to the problem (Cohen, 2013). In their study, where they examined mathematics teachers' knowledge in situations requiring proportional thinking, Brown et al. (2019) observed that mathematics teachers thought proportional in situations where three known terms and one unknown term are given, regardless of the context of the problem. In addition, they observed that for a particular problem which entails additive thinking, mathematics teachers were prone to select strategies requiring multiplicative thinking. Another study, conducted with 33 preservice teachers, Cramer et al. (1993) found that out of 33 students, 32 of them solved an additive problem by using multiplicative procedures. Although these students were able to set up an equation and solve the proportion, they were not able to differentiate why the problem presented to them was not appropriate to solve multiplicatively. Lim (2009) also conducted a study with preservice teachers and presented five missing value problem tasks with all having the same context, but requiring either additive or proportional thinking. Also, each question consisted of subquestions namely a and b. As his findings demonstrated, 17 students employed a proportional approach whereas five students chose to employ additive approach for both problems. Among all students only four of them properly applied additive method for task 4a and proportional method for task 5a as the tasks demanded. Based on these results, it is clear that not only students but also middle school teachers do not have the sufficient knowledge of proportionality to differentiate situations if they require proportional reasoning or not and that they tend to rely on multiplicative strategies even in the additive situations.

2.2.2 Erroneous Use of Additive Strategies

Although it is not mentioned as frequently as the widespread use of multiplicative strategies and cases where additive strategies are misused are also included in the literature. Several factors can account for the students' inclination to use additive methods in proportional situations, such as the type of the proportional problem (Singh, 2000a), the numerical structure (Van Dooren et al., 2010a), or the nature of the problems (Van Dooren et al., 2009). In a study conducted by Singh (2000a), 423 ninth-grade students from Malaysia were given 25 problems with different types and contexts to measure their understanding of proportional reasoning. The test consisted of missing value problems, numerical comparison problems, qualitative prediction and comparison problems, and non-routine problems. Students' strategies were scored on a scale ranging from 0 to 3, 0 representing no work and 3 representing correct answers. It has been observed that, in missing value problems, most students employed additive strategies rather than multiplicative ones. In another study, where they examined 132 eighth and ninth-grade students' proportional reasoning through a test consisting of 5 items, four of which were multiple choice types and one of them was an open-ended problem, Bright et al. (2003) reported 14 students' responses and the solution strategies on the five items in the test. Among these 14 students, five responded to a question that requires determining the similarity of given rectangles to a square by thinking additively. Instead of finding the ratio of the sides of the rectangles and deciding if the ratio is close to 1, these students found the difference between the sides of the rectangles while deciding their similarity to a square.

Another study, which examined the students' inclination of additive strategies in proportional situations, was conducted by (Misailidou & Williams, 2003). In their study, pupils whose ages ranged from 10 to 14 were given two types of proportional tasks, one supported by visuals and the other not. According to their findings, numerical structure and kind of the problem were considered to be the factors that played role in students' inclination to use additive methods. Even open problems, which required neither additive nor multiplicative reasoning, it has been observed that students prefered to use additive methods than multiplicative methods (Degrande et al., 2018). It can be deduced from these findings that students' additive responses can be influenced by various factors and that these factors could be regarded as the possible reasons of students' tendencies to emply particular strategies.

2.3 Students' Strategies on Proportional Problems

Cramer and Post (1993a) discussed that proportional problems can be presented within four different structures; missing value, numerical comparison, qualitative prediction, and qualitative comparison. Each of these problems has distinct solution ways and may require various strategies (Johnson, 2010). When the literature on this topic was reviewed, it was seen that strategies could be categorized as informal, formal (Chapin & Anderson, 2003), and progressional (Tourniaire & Pulos, 1985). Building up the quantities, finding the unit rate, and the factor of change are examples of informal strategies, whereas using a standard algorithm, namely cross-multiplication, can be regarded as a formal strategy (Tunç, 2020). Tourniaire and Pulos (1985) argued that there is also a progressional course of strategies that give rise to proportional thinking. In this section, all of these categories will be explicated in detail.

2.3.1. Informal Strategies on Proportional Problems

Informal strategies can be characterized as non-algorithmic procedures that take their foundation from conceptual understanding and that are carried out to solve proportional problems (Christou & Philippou, 2002). One of the well-known strategies is called the build-up strategy, which involves building up the quantities additively by preserving the ratio among them simultaneously (Tourniaire & Pulos, 1985). This strategy is often used before proportional reasoning is formally learned, and it is considered the bridge between additive and multiplicative reasoning (Parker, 1999). Degrande et al. (2019) advocated using build-up strategies to solve missing value proportional problems considering that these strategies may picture a proper reflection of the multiplicative relations within the problem situation.

Cramer and Post (1993b) discussed other common strategies used in solving proportional problems. One of them is the unit rate which involves seeking the per amount in terms of the other (Cramer & Post, 1993b). Students have an innate recognition of unit rate, and even from 3rd grade, they encounter basic problems in which it is possible to use this strategy (Cramer, Bezuk, et al., 1989). This approach is widely used while solving proportional problems in a missing value structure by finding and then multiplying the unit rate by one of the quantities to find out the unknown amount (Breit-Goodwin, 2015). The findings of the studies in the literature support this notion. Fernández et al. (2008) administered a test consisting of 7 questions to 135 students with ages ranging from 12 to 13 and investigated these students' strategies on proportional and non-proportional problems. Their findings revealed that one of the frequently used strategies among students to solve missing value word problems was unit rate. In a similar vein, in another study, in which the participants were 149 sixth, seventh, and eighth-grade students, and which investigated the difficulties that students experienced while solving problems involving ratio, Soyak and Işıksal (2012) found that the students by a majority gave preference to using unit rate strategy in solving problems.

Another common strategy observed in students' responses to proportional problems is the factor of change, which involves finding how many times one amount is as much as the other (Cramer & Post, 1993b). It is such a strategy that may lead to realizing within and between relationships as well as the unit rate (Ercole et al., 2011), and its usage should be supported along with the other informal strategies to understand better the multiplicative structures (Avcu & Avcu, 2010; Breit-Goodwin, 2015; de la Cruz & Garney, 2016). Its importance and connection to the standard proportion algorithm have been emphasized by Boston, Smith, and Hillen (2003). As they argued, adequate time should be devoted to working on the factor of change strategy before students understand why the cross-product works in proportional situations. Studies in the literature also mention the presence and prevalence of this strategy among students. In her study Tunç (2020) investigated 101 sixth and eighth-grade students' solution ways in solving proportional and non-proportional problems and found that the factor of change was the frequently preferred strategy among sixth graders. Likewise, Artut and Pelen (2015) conducted a study with sixth-graders to observe their proportional and non-proportional problems strategies. Their results revealed that the most common strategy that students employed to solve proportional problems was factor of change. It is evident from the literature that even fourth and fifth graders were able to use the factor of change strategy in solving proportional problems (Christou & Philippou, 2002).

The other strategy is called fraction, which involves determining the equivalence of the fractions (Cramer & Post, 1993b). It is one of the strategies that is observed among students' answers to proportional problems. Although its use is not as widespread as other strategies, the literature reveals that this strategy is one of the strategies used by the students when dealing with proportional problems. As an example, in their survey study, wherein they sought for the strategies used by sixth graders, Avcu and Avcu (2010) found that even though the prevalence of its application changed across different items, among six strategies that have been observed in students' responses, the frequency of equivalent fraction strategy in solving proportional problems was obtained as 0.58%. Similar results have been reported in a study by Duatepe-Paksu, Akkuş, and Kayhan (2005). Duatepe-Paksu et al. (2005) carried out a study investigating sixth, seventh, and eighth graders' solution strategies on proportional problems. They asked ten open-ended questions containing five different problem styles to a total of 295 students, most of whom were in the sixth grade, and examined the trends in their strategies. Apart from qualitative comparison and inversely proportional problems, it has been observed that one of the strategies observed in students' answers to a missing value, numerical comparison, and additive problems was equivalent fraction strategy.

2.3.2. Formal Strategies on Proportional Problems

The widely known and formal (Tunç, 2020) strategy to solve proportional problems is called cross product, i.e., rule of three (Fernández et al., 2008, 2010; Silvestre & da Ponte, 2012), which is opted for many students without meaning (Cramer & Post, 1993b). Cross multiplication strategy involves the multiplication of the numerator of the first fraction by the denominator of the second fraction and equalizes the result of this operation to the multiplication of the denominator of the first fraction by the numerator of the second fraction. Several studies revealed the tendency among students to use a cross-multiplication strategy in missing value proportion problems (Avcu, 2010; Avcu & Avcu, 2010; Duatepe-Paksu et al., 2005; Tunç, 2020). Avcu and Avcu (2010) administered a test containing eight open-ended items to 278 sixth-grade students. They observed that the most common strategy used by sixth-graders in solving proportional problems was cross multiplication. Similarly, Duatepe-Paksu et al. (2005) studied sixth, seventh, and eighth-grade pupils and revealed that among all strategies that were used to solve missing value proportional problems, cross-multiplication took place on the top with 49.7% frequency. Nevertheless, regardless of how common this strategy is, it lacks meaning (Lobato et al., 2010) and does not assure that students have a good command of this subject just because they can perform the operations correctly (Sumarto et al., 2014). Because literature shows that students just memorize and blindly use the algorithm without paying attention to the underlying concepts (Sumarto et al., 2014), the cross-multiplication algorithm leads students to overuse proportionality (Toluk-Ucar & Bozkuş, 2018). Put it differently; in the absence of sufficient understanding of why the algorithm helps us to solve proportional problems, students are prone to over-rely on it for every occasion without paying attention to whether the situation actually necessitates proportional reasoning (Lanius & Williams, 2003). Therefore, it is necessary to support intuitive strategies, and their connection to the algorithm to build a robust understanding of the proportion concept and realize why the algorithm is useful to explain proportional situations (Boston et al., 2003). Students' experiences should be taken as a base while introducing them to the standard algorithm to make meaningful learning (Ercole et al., 2011).

2.3.3 Progressional Strategies on Proportional Problems

There is also a progressional route of strategies which mainly show the path followed in the transition from additive thinking to multiplicative thinking (Tourniaire & Pulos, 1985). Numerous frameworks provide a categorization of students' strategies concerning their sophistication (Langrall & Swafford, 2000; Petit et al., 2020; Silvestre & da Ponte, 2012).

Langrall and Swafford (2000) proposed four levels of strategies, namely non-proportional, informal reasoning, quantitative reasoning, and formal reasoning. Non-proportional reasoning encompasses strategies that base their foundation not only on additive reasoning but also on random computations. Strategies at this level do not comprise proportional reasoning and do not reveal evidence of multiplicative thinking. Strategies at the informal reasoning level involve the comparison of quantities qualitatively, and they may be seen accompanied by a pictorial representation. Strategies at the quantitative reasoning level comprise several techniques such as finding a unit rate, establishing equivalent fractions, or building up the amounts being compared. Lastly, the formal proportional reasoning level strategies consist of the cross-multiplication method with a robust understanding of the multiplicative reasoning underpinning the situation.

Following the strategies existing in the literature, Silvestre and da Ponte (2012) also sorted out students' strategies under four categories to proportional problems; pictorial, additive, both additive and multiplicative, and multiplicative strategies. While the pictorial strategy involves drawing and counting, the additive strategy involves making additive comparisons between the quantities. The additive and multiplicative strategies involve the utilization of additive and multiplicative strategies concurrently. Lastly, the multiplicative strategy involves being cognizant of the multiplicative situation and being able to apply the ratio in one situation to other situations.

Another framework called the On Going Assessment Project (OGAP) Ratio and Proportion Progression proposed by Petit et al. (2020) demonstrates development in students' strategies in the proportional reasoning concept. As Petit et al. (2020) contended, the primary objective of this framework is to guide teachers' teaching practices on assisting their students to make flexible and meaningful shifts among the levels. According to the OGAP Ratio and Proportion Progression, there are five primary levels that students' strategies might correspond to non-proportional, early ratio, early transitional, transitional and proportional (Petit et al., 2020). The non-proportional strategy involves attending to the additive difference between the amounts and makes an additive comparison, early ratio strategy involves building up the units by means of addition, the early transitional strategy involves using additive and multiplicative strategies interchangeably, the transitional strategy involves scaling up or down the quantities in a multiplicative manner, and proportional strategy involves coherently using multiplicative reasoning throughout the process (Petit et al., 2020).

2.4 Factors Affecting Students' Strategies on Proportional Problems

There are several studies in the literature reporting that students are prone to select incorrect strategies while solving word problems regarding proportional reasoning concept (Van Dooren, De Bock, Evers, & Verschaffel, 2009; Van Dooren, De Bock, & Verschaffel, 2010a; Van Dooren, De Bock, Vleugels, et al., 2010b). In a typical test, students frequently use signals, such as particular key terms, the characteristic of numbers, the section in which the problem exists, or the situation presented in the problem, to appropriately select which operation is supposed to solve a specific word problem (Van Dooren et al., 2005).

According to Van Dooren et al. (2009), the numerical structure of the problem and students' grade levels play a role in their solution strategies. In their study, where they investigated whether the numbers in the problems made students more inclined to use proportionality, Van Dooren et al. (2009) administered a test including both proportional and nonproportional problems to 508 fourth, fifth and sixth graders. The number in the test comprised of integer and non-integer ratios. Their findings revealed that students gave fewer correct answers when they were confronted with non-integer ratios in proportional problems than they did in integer ratios. On the other hand, students used proportionality less often in non-proportional problems when they were confronted with non-integer ratios. Moreover, they added that fourth graders had more difficulties working with non-integer ratios than sixth graders. These studies show that numbers in the problems and the age of the students are critical factors accounting for students' strategies in problems.

In another study conducted by Steinthorsdottir (2006), 53 eighth-grade students were given 16 problems that differed in terms of the numbers and the context. The problems were divided into four groups: well-chunked problems, part-part-whole problems, associated sets problems, and symbolic problems. Each group of problems involved four questions having different number structures such as integer or non-integer relationships among ratios, and the answers of the problems were either integer or non-integer. Students' strategies were categorized into six hierarchical groups. Results revealed that each problem's contextual structure had elicited a different type of strategy, which was an indication of an influence of the problem context on students' responses. For instance, for part-part-whole problems, some students counted on a strategy in which they found the difference between the numbers in the ratios and applied this difference to the second ratio to obtain the missing value in the second ratio. Steinthorsdottir (2006) reported that 55% of these problems were responded to in a multiplicative manner while 20% of these problems were responded by finding the difference between the numbers in the ratios, which was an indication of additive thinking. Moreover, among the four contextual structures, the most significant frequency difference between the multiplication strategy and the ratio difference strategy was observed for the part-part-whole problems. Overall, for well-chunked problems, associated sets problems, part-part whole problems and symbolic problems the most frequent solution strategy was reported as multiplicative method and the least frequent strategy was reported as qualitative method. Furthermore, along with the problems' contextual structure, the number structures played a role in students' responses. In other words, students' solutions were influenced by the numbers in the ratio being integer or non-integer. For the problems, whose ratios are integer and also their solutions are integer, 82% of these problems were solved through multiplicative methods whereas this frequency dramatically decreased to 25% when the problems involve non-integer answers and non-integer ratios. Another interesting finding was that the ratio difference strategy referring to the additive strategy was mainly used when the problems' answers and the ratios were non-integers. In fact, 16% of these problems, whose answers and ratios were non-integer, were solved by the ratio difference strategy. In contrast, this frequency was 5% for the problems whose answers and ratios were integer. It can be argued that students struggled to solve problems involving non-integer ratios. They tended to apply additive methods when confronted with non-integer ratios or answers.

In another study, Van Dooren et al. (2010a) investigated additive and multiplicative strategies of 325 third, fourth, fifth, and sixth-grade students and how their strategies are influenced by their age and the ratio type embedded in the problems that were in the form of missing value. They presented students with 15 problems, four of which were experimental. Two of the experimental problems required to use additive thinking, and the remaining two entailed multiplicative thinking. The problems presented had different number structures; half of the problems contained integer ratios, whereas half involved non-integer ratios. They analyzed students' answers on these four problems. Their results revealed that some students employ proportional methods while solving additive problems, which shows an inappropriate application of proportional reasoning. According to their findings, numbers played a critical role in students' responses. They argued that problems involving integer ratios produced more proportional answers than the problems whose ratios were non-integer. Van Dooren et al. (2010a) concluded that at the beginning of elementary school, there was a tendency among students to apply proportional methods in any problem and a tendency to apply proportional methods in later years. Among these two periods, several students had a midway period where they concurrently employed additive methods to proportional problems and proportional methods to additive problems, shifting among them based on numbers presented in the problem.

In another study, conducted by Fernández et al. (2011), 551 secondary school students whose levels ranging from first to fourth grade were administered a test involving proportional and non-proportional problems that vary in terms of the number structure and nature of the quantities. Their findings suggested that the students performed better in proportional problems involving integer ratios than those involving non-integer ratios. In additive problems, however, the opposite of this statement was true. Students performed better in additive problems involving non-integer ratios than the additive problems involving non-integer ratios. The findings displayed that the number structure of the problems affected the students' performance and their strategies. Accordingly, in problems involving non-integer ratios, students preferred to use additive strategies more often than as they did for the problems involving integer ratios.

In parallel with these results, in their study, Artut and Pelen (2015) carried out a survey study with 165 sixth-grade students. They investigated their strategies on proportional and non-proportional problems and whether these strategies were affected by the numerical structure of the problem and the kind of the problem. Students were expected to solve a series of questions involving proportional and non-proportional problems that differed in numerical structure. According to their analysis, students' solutions were influenced by the number structure. As a matter of fact, problems involving inter ratios provoked students to apply multiplicative solution ways. In contrast, problems without integer ratios led students to resort to additive methods regardless of the problem type.

A year later, Pelen and Artut (2016) conducted another study with 331 seventh-grade students. This time, they presented students with a test that involved 24 open-ended problems with three different types, namely direct proportional, inversely proportional, and non-proportional problems. The students' answers were scored to obtain success rates on the diverse problems. Their analysis revealed that students performed best in direct proportional problems among the three problem types and the worst in the non-proportional problems. Pelen and Artut (2016) stressed that problem type also determines seventh-grade students' success rates on problems.

Another study, conducted by Singh (2000b), was investigated the development of two sixth grade students' proportional reasoning through proportion tasks involving various contexts. One of the students solved the problem involving integer ratio in a multiplicative manner by finding unit ratio, while the same student chose to solve another question involving non-integer ratio in an additive manner. More specifically, this student correctly answered how many hours a person should work to earn 36 dollars if that person earns 12 dollars after 3 hours of work. However, when another problem was presented in the context of similarity involving proportional thinking, she established an additive relationship between the sides of the given rectangle. She adapted this relationship to the second rectangle to find the missing length. It can be inferred that both the context of the problem and whether the ratio in the problem is an integer or not play a significant role in the student‘s way of thinking.

2.5 Teaching and Learning Proportional Reasoning

Teaching and learning about proportionality concepts in middle school years are acknowledged as highly complicated procedures, so it is of great importance to study this concept thoroughly (Adjiage & Pluvinage, 2007). Although proportional thinking is seen as a middle school topic and has a place in the sixth-grade mathematics curriculum, students have been using proportional thinking indirectly, even in simple multiplication problems, since primary school (Small, 2015). In her book, in which she describes the development of proportional thinking among different mathematics fields and across different grade levels, Small (2015) states that students can begin to develop proportional thinking skills starting from kindergarten as they come to realize counting, grouping, or decomposing quantities. According to Small (2015), by picturing and identifying equal groups or understanding how a particular length can be expressed in terms of another length, students at kindergarten can start to develop proportional reasoning skills. Just as the kindergarteners, first and second graders should also be introduced to how one amount can be represented by means of another amount. They should be encouraged to measure the same lengths by using different units (Small, 2015). After third grade, once students learn how to do multiplication and division, they can solve basic measurement problems involving proportional reasoning, and in fourth grade, conceiving multiplication as a means to compare two numbers multiplicatively promotes them to see amounts multiplicatively instead of additively (Small, 2015). In fourth grade, students have much more opportunities to encounter proportional situations, such as they perform operations with fractions and decimals, as well as they can solve problems involving conversion of measurement units (Small, 2015).

Some studies in the literature have shown that students can solve problems that require proportional thinking even from primary school. In a study, Tourniaire (1986) studied with 60 third, fourth, and fifth-grade students and administered two interviews with each student. Students were expected to solve three proportional problems in the first interview and four proportional problems in the second interview. The four-level scale consisted of incomplete, qualitative, additive, and proportional strategies have been created to evaluate students’ strategies. According to the overall results, it was revealed that 58% and 68% of the students were successful in solving problems in the first interview and the second interview, respectively. Comparison of successful performance across grades revealed a noteworthy increase in the success between fourth and fifth grades, which shows that even in the late years of primary school, students had an understanding of proportions and the ability to solve the problems correctly.

Although there are studies demonstrating that students struggle to understand proportional reasoning and its fundamental principles, some studies suggest that the development of this skill can be encouraged by targeted and planned teaching (A. Hilton et al., 2016). With well-designed and engaging activities, students can strengthen their background and understanding of proportionality and the problems with more open and adaptable ideas (Miller & Fey, 2000). Bearing that in mind, it should be emphasized that in building students‘ proportional reasoning, teachers had an important role to play in their classrooms. Teachers should create an environment such that students‘ opinions and various solution methods should be valued and that discussion on these ideas should take place in an attempt to check whether they are mathematically meaningful (Parker, 1999). Moreover, it should not be the only goal of the lesson to teach cross-multiplication where ratios were equalized, since, in this way, students may fail to discern the reasoning inherent in the problem situation that is connected to proportionality (Banker, 2012). Banker (2012) also points out how the interpretation of the context is essential in making a decision about the problem. When students encounter a problem within a context, they are stimulated to construe with the context and align their strategies with that context (de la Cruz, 2013). The idea of providing the cross-product method ahead of time was also criticized by Markworth (2012). Establishing a proportion then using the cross-multiplication algorithm too early can hamper students' comprehension of covarying nature of the amounts and the multiplicative structure underlying the contextual scenario (Markworth, 2012). For example, when students do not understand why cross multiplication works in proportional situations, they tend to use it even in situations that do not involve proportional thinking (Lanius & Williams, 2003). Even though it is a handy algorithm to apply, students can effectively use it without indicating proportional reasoning (Cohen, 2013). In other words, when students learn the cross-multiplication algorithm, they tend to apply it mechanically and ignore the strategies they have already learned (de la Cruz & Garney, 2016). Therefore, students should thoroughly learn more intuitive techniques, particularly factor of change or unit rate strategies, before cross-multiplication is implemented since these strategies promote them to grasp better the multiplicative relationship between the ratios (de la Cruz & Garney, 2016). Furthermore, solely teaching how to use the cross-multiplication method in proportional situations does not help foster students' proportional reasoning abilities since proportional reasoning is a form of thinking, and it is beyond memorizing an algorithm to solve problems (Thompson & Bush, 2003). Therefore, connecting the understanding of proportional reasoning to the cross-multiplication rule should aid students transfer toward a profound comprehension of proportionality and beyond rote learning (Ercole et al., 2011). Students should know why the cross-multiplication algorithm works in proportional situations. Being able to explain why the operations in the procedure are meaningful enables students to practice reasoning, one of the basic skills of mathematics (Cengiz & Rathouz, 2018).

Moreover, to enhance students’ understanding, it is essential to use familiar contexts and pose questions to students that allow them to interpret the notion of proportion within the problem context while answering the question (Banker, 2012). Many studies emphasize the importance of students encountering problems in different contexts and encouraging them to produce their own solution methods. Martinie and Bay-Williams (2003) argued that if students were inspired to solve various problems that involved multiplicative comparison in different contexts and were promoted to build their models, they would strengthen their competence in proportional reasoning. Creating learning environments that improve students‘ understanding and allow them fully comprehend similar mathematical systems in various contexts will facilitate generalization (Chapin & Anderson, 2003). Moreover, students who learn by experimentation and investigation are more accomplished in successfully solving problems and presenting their solutions with reasonable arguments (Miller & Fey, 2000). Students can construct an understanding that reinforces their capacity to produce persuasive arguments through familiar contexts and corresponding words, units, and depictions (Cengiz & Rathouz, 2018). Language and representations in the contexts are also important, as Cengiz and Rathouz (2018) suggested. Just as the context of the problem is an essential basis for students to develop the appropriate mathematical language, using mathematical phrases that are relevant and a foundational for understanding of the concept of proportions helps building a connection with the context, and to understand the multiplicative relationship between quantities (Cengiz & Rathouz, 2018). More specifically, using the words ‘for each’ and ‘times as much’ to correlate the quantities also plays an essential role in making sense of two different ways of thinking about ratios and relating ratios and fractions (Cengiz & Rathouz, 2018). Therefore, it is necessary for prospective teachers to grasp and clarify what composed units mean and how multiplicative comparison works in proportional situations (Cengiz & Rathouz, 2018). As could be seen from the literature, the context of the problems, the various representation forms used in the question, and the language of the text and expressions are significant considerations that contribute to the growth of the proportional thinking abilities of students. For this reason, it is of great importance that teachers and pre-service teachers have sound knowledge of the different meanings of the ratio, present the questions to their students through various contexts and representations, and be able to use mathematical language appropriately and flexibly. On the assumption that traditional and rote methods prevent or impede learning to think proportionally, how to teach as well as what to teach is an issue that needs to be paid attention.

Numerous curriculum initiatives have investigated alternative methods, such as team works involving tackling real-life problems instead of learning solely rule-based methods to reinforce students' learning of proportional reasoning (Miller & Fey, 2000). The study of Miller and Fey (2000) demonstrated that students who explored concepts and developed their skills with the assistance of problem-based teaching developed proportional thinking abilities more than students who studied with the traditional curriculum and conventional teaching methods.

Contemporary school curriculums have been created for the subjects of middle school mathematics, particularly for the concept of proportional reasoning. Substantial evidence indicated that students had higher achievement in proportional reasoning when they were allowed to create their understanding through actively engaging in problem-solving tasks rather than learning from the traditional methods where they follow teacher‘s instructions (Ben-Chaim et al., 1998). In their study, Ben-Chaim et al. (1998) studied two groups of seventh-grade students learning in different curricula on proportional reasoning. Students in one group of the study solved problems related to proportional thinking through the materials provided by the Curriculum Mathematics Project (CMP), one of the new secondary school curriculum projects inspired by the standards of NCTM documents. The fundamental aim of CMP was to help students build their knowledge and realize the relationships between mathematical ideas and learn the applications of mathematics. The CMP curriculum expects students to develop proportional thinking over contextual problems and share their ideas and strategies in this process with their friends through classroom discussions. Furthermore, standard algorithms are not taught to students in any part of the CMP curriculum. In contrast, the second group of the study was given traditional instruction where the teacher showed the solutions to the problems to the students. Then the students were expected to solve similar problems by adhering to the similar way that their teacher showed to them. In this respect, students in the two groups of this study worked on proportional thinking on different teaching methods. The findings of their study displayed that the students in the CMP group, who received problem-based learning and were encouraged to produce and use their own strategies, could produce more logical and practical solution strategies to the given problems compared to the students who studied with the traditional method.

Another experimental study obtained relevant findings to that of Ben-Chaim et al. (1998). Adjiage and Pluvinage (2007) examined the learning experiences of sixth and seventh-grade students in rational numbers and proportionality in their two-year experimental study. Two classes were formed, and the same materials were used in both classes. However, one of the classes was the quasi-experimental group in which the teacher used his own traditional teaching methods, and the other class was the experimental group using a series of activities and a computer environment created within a framework. At the end of the study, they compared both groups and observed that the experimental group made more progress in making sense of fractions and using them in proportional situations than the quasi-experimental group did. Similar findings have been obtained in another study conducted by Howe et al. (2015). In their study, Howe et al. (2015) shared the results of construction, application, and assessment of a mathematics module regarding fractions, ratios, and proportions prepared within the scope of a project called the Effecting Principled Improvement in STEM Education, epiSTEMe, which sought to promote development in science and mathematics domains. The module, prepared with the teachers, included topics related to rational numbers and proportional thinking following the secondary education curriculum. After being applied to 11 classes whose lessons included sharing ideas and discussions on the problems, the results were compared with 16 control classes taught by the teachers' usual teaching methods. The results revealed that the students studying with the epiSTEMe module both made more progress on the subject than the students in the control group and displayed a more optimistic attitude concerning their learning practices. In addition to new teaching approaches designed for students to enhance their learning, another study revealed the significance of authentic problems in learning proportional reasoning. In their paper, Raymond and Reeder (2018) reported a lesson designed based on a book that focused on proportional reasoning. This book included numerous demographic information that students could use while solving mathematical problems regarding proportional reasoning. For instance, students were asked to find the most crowded continent or the most common language spoken by those people. Students were given a paper to write down their answers, and they were also expected to justify their solutions. Charts were created for each question that enables students to post their notes on the charts that correspond to each question. All in all, students orchestrated a discussion on solving these problems with their peers and produce various strategies in the process. Researchers argued that classroom discussion assisted students in building their own reasoning by starting from producing intuitive strategies, such as additive thinking or making a guess, to develop proportional reasoning (Raymond & Reeder, 2018). Moreover, Raymond and Reeder (2018) stressed the importance of real-life context in the problems since they supported students to be more engaged in the activity and helped them see the application of mathematics in daily life. All in all, these studies demonstrate how it is important for students to construct their strategies for making sense of proportional situations and for solving related problems and how instruction should be designed so that it supports and encourages students to do that with the aid of well-prepared tasks.

2.6 Potential of Using Technology in Mathematics Learning

Instructional technology is more available than it was ten years back and will undoubtedly occupy a progressively significant role in the near future (Cheung & Slavin, 2013). Currently, classroom conditions afford students and teachers settings that will enable them to access technology and instructional resources instantly; thereby, integrating technology into lessons is much more important than before (Doğan, 2012). From a research perspective, technology has a significant role to play in educational studies since it can make specific parts of the activities apparent in an attempt to record, debate, and assess (Loomes et al., 2002). From a teaching perspective, technology is a way to reinforce the learning of concepts provided that it is properly integrated into the lessons (Ranasinghe & Leisher, 2009). Thanks to the environment provided by technology, students have the opportunity to work on computers individually or as a group at their own pace (Fabiola & Ledesma, 2010). Technology also plays an essential role in attracting students' attention to the lesson, and when it is integrated into lessons in a meaningful way, it helps students both have fun and learn (Hicks, 2011). Considering the prevalence of technology in education, it is noteworthy to discuss its role in learning and the advantages brought by the use of technology in mathematics education.

NCTM (2000) emphasized the role of technology in mathematics education. NCTM (2000) suggested that technology is an essential tool in mathematics education since the presence of technological tools assists students in enriching mathematical concepts with visuals, enabling them to organize and analyze the data, and make calculations, and discover ideas within various fields of mathematics through reasoning and problem-solving. Furthermore, technology can motivate students, visualize mathematical concepts and help students solve complicated problems involving real-life situations (Soucie et al., 2010). In cases where physical objects are limited, without technology, teachers can make limited use of the resources they have in the exploration of mathematical concepts (Drijvers et al., 2016). Fortunately, with the availability and use of the technology, various mathematical concepts and demonstrations are accessible since there exist many virtual tools to emphasize numerous mathematical concepts and their relations (Drijvers et al., 2016). For instance, computers are of significant importance as they offer new possibilities in teaching mathematics. They enrich the learning environment so that students take an active role in their learning process (McCoy, 1996). Different technologies reveal their powers in mathematics by allowing data interpretation and analysis or creation and investigation of simulations (Drier et al., 1999).

Knowing these advantages and designing mathematics lessons in a way that reinforces learning brings a further emphasis on a topic, which is the purpose of using technology. The fact that technology is so widespread raises the question of whether or not to use technology in classrooms, but how to effectively integrate the technological tools to be used in the lessons (Cheung & Slavin, 2013). For the last few years, the advent of innovative technologies has changed how we can motivate learners to discuss mathematical concepts and tackle mathematics problems (Hollenbeck & Fey, 2009). When internet-based systems are introduced to teaching activities, they offer unique ways to change the learning environment of mathematics (Hollenbeck & Fey, 2009). Today, different technological applications can be integrated into the lessons. Teachers can use technology in their classrooms for different purposes. For instance, they can use interactive boards as a tool for projecting the content of the lesson on the board through a PowerPoint presentation, or they can integrate various activities into their lessons utilizing different mathematical applets, virtual manipulatives, or dynamic geometry software, which are free and available on the internet. Drijvers (2012) identified two principal utilities of technologies in mathematics education, namely doing and learning mathematics. Under the learning mathematics perspective, he proposed two further aims for using technology, one of which is practicing skills and the other one is developing concepts. All these branches are considered to be helpful for educators in making pedagogically sound decisions on using technology in their classrooms (Drijvers, 2012). Considering these purposes, selecting the appropriate technologies and designing relevant and compatible activities with the mathematics curriculum is of incontrovertible importance. On this issue, Zbiek, Heid, Blume, and Dick (1992) suggested two different types of activities to be used in the process of technology integration in mathematics lessons, one of which was technical activities such as numerical and algebraic operations, construction of geometric figures or objects and forming tables, diagrams, or graphs, whereas the other one was called conceptual activities such as exploration of patterns, conducting generalization activities, or validating or refusing the arguments emerged from the process of experimentation with the tools. By focusing primarily on these conceptual activities that Zbiek et al. (1992) proposed within the perspective of developing concepts (Drijvers, 2012), the next section will converse about the advantages of dynamic geometry environments in learning mathematics.

2.6.1 Potential of Dynamic Geometry Environments in Learning Mathematics

Dynamic geometry software is a program that allows the generation and manipulation of geometric figures and shapes (Bantchev, 2010). Cabri 3D, GeoGebra, and Geometer's Sketchpad are among the most frequently used dynamic geometry software in mathematics classrooms. Such software can be supportive means for understanding mathematical contents, specifically geometry (Andreasen & Haciomeroglu, 2013), and they permit students to operate with the objects in a dynamic way (Belnap & Parrott, 2020). Dynamic geometry environments have many helpful features. In particular, the dragging option found in the dynamic mathematics environments is the most prominent and exclusive means that enables persons to choose one or several figures and move these figures repeatedly (Sinclair & Robutti, 2013). Thanks to the dragging option, students have an opportunity to explore shapes and create mathematical arguments (Leung, 2011). It also enables the recognition of the invariable properties of a shape whose geometric properties change (English, 2002). For example, by enlarging or shrinking a rectangular picture in GeoGebra with the help of the dragging option, it can be demonstrated that although the size of the rectangle changes, the ratio between its congruent sides remains constant. Hollenbeck and Fey (2009) argued that the significance of proportional reasoning could be increased by showing how proportionality could apply in the resemblance of geometric figures. Visualization of proportions, for instance, using an enlarged picture context, offers an enticing background for a student to discover the concept (Hollenbeck & Fey, 2009). Moreover, since proportional thinking involves the multiplicative relationship of quantities, the key aspects are what changes and what remains constant. Technology offers a good opportunity for students at this point. Thanks to the dynamic feature of technology, it becomes easier for students to observe change (Kaput, 1992).

Another powerful feature of dynamic geometry programs is the measurement tool. Students can build mathematical sense with the measuring tool, generate assumptions, and use these assumptions in the proof-making phase (Sinclair & Robutti, 2013). As in the above example, students can use a measurement tool to observe if the length of the parallel sides are equal to each other after they drag the rectangular shape from its vertices.

There are numerous studies in the literature, especially in geometry, displaying that the use of dynamic geometry software is a robust tool in improving learning and increasing mathematics achievement of both students at different educational levels and teacher candidates. In their meta-analysis, Chan and Leung (2014) examined nine articles that compare the achievements of students who learned the subject by using dynamic geometry software at the K-12 level and those who learned the concepts through traditional teaching methods. The results show that dynamic geometry software significantly affects students' mathematics achievement at all grade levels, mainly elementary school students. They argued that since dynamic geometry software concretizes the concepts and allows students to apply their knowledge, it made it easier to attain geometric information. In a study where 31 thirteen and fourteen-year-old teenage students' understanding of enlargement and similarity concepts were investigated in the context of dynamic geometry software, Denton (2017) observed that the dynamic geometry environment helped learners build connections among factor of scale and ratio concepts.

One of the dynamic geometry software that can be used in mathematics classrooms is the Cabri 3D program. The Cabri 3D program has a two-dimensional interface that permits the building of three-dimensional figures, viewing them from different angles, and examining their opened versions (Kösa & Karakuş, 2010). In his study, Guven (2012) investigated the effect of Cabri on 68 eighth-grade students' knowledge of transformation geometry. A multiple-choice test that consisted of 15 problems, all of which intended to measure the students' achievement in transformation geometry, and an open-ended test consisted of 15 problems, all of which measured students' learning levels in transformation geometry were presented to students as a pretest and a posttest. Thirty-six of the students were assigned to the experimental group, and 32 were assigned to the control group. Students in the experimental group were given worksheets, and they were given time to study themselves on the Cabri. The students in the control group used the same materials as the experimental group did, except the control group did not use the software during the activities; instead, they used paper and pencils while working. The pretest results measuring students' achievement in transformation geometry revealed no significant difference before the experimentation. However, after an 8-hour long implementation, results showed that students in the experimental group outperformed their counterparts in the control group on the achievement test. Guven (2012) pointed out that Cabri positively affected students and improved their achievement in transformation geometry because, in the experimental group, students had an opportunity to identify and fix their errors with the feedback given from the computer. Guven (2012) stressed that this was the main reason for the differences between the students in the experimental and control groups.

Another software that can be used in mathematics lessons is GeoGebra. Geogebra is free software that permits to draw or construct various figures, determine their relationships, and dynamically manipulate them to see the relationships between these figures (Lo & White, 2020) and allows investigating numerous algebraic and geometric notions with the help of figures, graphs or formulas (Garber et al., 2010). In his study, Kutluca (2013) investigated the effect of GeoGebra on 42 eleventh grade students' level on Van Hiele's geometry understanding. Among the 25 problems of Usiskin's (1982)Van Hiele Level of Geometric Understanding Test, the first 15 were selected as pretest and posttest. The last ten problems were not included since these problems were not regarded as appropriate for students' grade level. Students were divided into experimental and control groups. The experimental group received instruction regarding circles through GeoGebra, while the control group received traditional instruction and followed the textbook. Findings demonstrated that although a significant difference between the pretest and posttest has been obtained for the experimental group, there was not a significant difference between the pretest and posttest of the control group. Kutluca (2013) concluded that dynamic geometry software positively impacted students' Van Hiele levels of geometry understanding. Kutluca (2013) attributed students' achievement to the nature of GeoGebra since it allowed students to move and construct the shapes and build their own knowledge.

Studies in the literature show that dynamic geometry software is beneficial not only for students but also for prospective teachers to improve their thinking in terms of geometry. In a study conducted by Christou, Mousoulides, Pittalis, and Pitta-Pantazi (2009), six preservice teachers were given two problems regarding geometry, both of which gave students a chance to produce novel problems, and the aim was to gain insight into how dynamic geometry software, assisted students in solving and creating problems. Preservice teachers used the Geometer's Sketchpad during problem-solving. In the first phase, students were asked to solve the problems. In the second phase, they were expected to both justify their solutions and generate new problems. Their results indicated that dynamic geometry software was helpful for students in terms of both understanding the problems and providing solutions to them. Particularly, the availability of dragging and measurement options enabled students to explore the new information themselves and use that knowledge to solve the problems. Moreover, being able to visualize the problem situation with the help of the dynamic geometric program help them make and test their assumptions. Similar results were obtained in another study. Baki, Kosa, and Guven (2011) examined 96 preservice mathematics teachers' spatial visualization skills through a test that consisted of 36 questions regarding solid geometry. Preservice teachers were assigned into three groups. One of which used dynamic geometry software, namely Cabri 3D, during instruction, one group used physical manipulatives, and one group received traditional instruction. Pre-test and post-test administered, and results revealed that students in Cabri 3D and physical manipulatives outperformed students in the control group who received traditional instruction. They emphasized that students could explore the solid figures and reinforce their spatial visualization skills by dragging and measuring the figures they built on the program.

Kepceoğlu (2018) investigated 30 preservice teachers' drawing abilities through an experimental study. Teachers were assigned experimental and control groups. Students in the experimental group attended a workshop on Cabri 3D that involved activities ranging from drawing to rotating 3D figures and explained how to use Cabri 3D while engaging in these activities. In contrast, students in the control group received traditional instruction on the same topic. All students were given a pretest and a post-test consisting of eight questions asking for drawing 3D figures. Their findings revealed that although there was no significant difference between the pretest scores of the experimental and control group, the post-test results showed statistically significant growth in the success of both groups. However, it was found that this growth was much higher for students in the experimental group than students in the control group, which suggests that the computer-based teaching method was more influential in the participants' success than the conventional learning approach.

2.7 Summary of the Literature Review

Studies conducted on the concept of proportional reasoning point out the students' difficulties and variety of strategies in solving proportional and non-proportional problems within a wide range of contexts and number structures. These difficulties can fall into two major categories, erroneous use of multiplicative strategies in non-proportional situations (De Bock et al., 2002; Degrande et al., 2014; Pelen & Artut, 2016; Van Dooren et al., 2009; Van Dooren, De Bock, Depaepe, et al., 2003; Van Dooren, De Bock, Verschaffel, et al., 2003; Van Dooren, De Bock, Vleugels, et al., 2010), and that of additive strategies in proportional situations (Bright et al., 2003; Degrande et al., 2018; Misailidou & Williams, 2003; Singh, 2000a; Van Dooren et al., 2009; Van Dooren, De Bock, & Verschaffel, 2010).

Literature shows that many factors could account for why students with different age groups opt for certain strategies over others and why particular problems are challenging for them. Foremost among these are the numerical structure (Fernández et al., 2011; Van Dooren et al., 2005, 2009), contextual aspects of the problems (Singh, 2000b; Steinthorsdottir, 2006), and the grade level of students (Van Dooren et al., 2009; Van Dooren, De Bock, & Verschaffel, 2010). Diversification of these factors also leads students to apply numerous informal strategies such as build-up (Tourniaire & Pulos, 1985), unit rate, factor of change, and equivalent fractions (Cramer & Post, 1993b) or formal cross-multiplication strategy (Fernández et al., 2008, 2010; Silvestre & da Ponte, 2012). As reported in the literature, there are also progressional strategies that involve and require different reasoning forms, such as additive and multiplicative (Tourniaire & Pulos, 1985). Several frameworks have been developed to classify students' strategies in proportional problems (Langrall & Swafford, 2000; Petit et al., 2020; Silvestre & da Ponte, 2012). Among them, a relatively new one has been proposed by (Petit et al., 2020). Pondering on progressional strategies and determining where students are in terms of their knowledge on ratio and proportion might be valuable for understanding students' reasoning and supporting their development in this concept. In regard to this, literature offers open-ended problems, engaging activities (Miller & Fey, 2000), and classroom discussion (Parker, 1999) that provide opportunities for students to go beyond memorized rules and bring their informal strategies into the open by means of fruitful discussions. It may not be easy for teachers to keep their students engaged all the time, especially within crowded classrooms where a lot of distracting elements are present. Moreover, instructional strategies that limit teaching and learning as a transfer of rule-based knowledge from teachers to students, in this particular case, memorization of cross-product rule, may prevent students from recognizing the multiplicative nature embedded in proportional problems(Banker, 2012). At this point, technology become involved as a promising agent that can attract students' attention (Hicks, 2011), delineate complex parts of the real-life mathematics problems (Soucie et al., 2010), and how mathematical concepts are related (Drijvers et al., 2016). In particular, dynamic mathematics and geometry environments were found to expand students' understanding of a particular concept (Guven, 2012; Kutluca, 2013). However, little is known about how and in what ways technology can support or hinder middle school students' understanding of proportions while solving problems that are different in terms of numerical structure. Therefore, this study sought to investigate sixth and seventh-grade students' strategies in solving proportional problems supported with interactive simulations.

CHAPTER 3

METHODOLOGY

This chapter contains seven separate sections, each of which gives insight into the methodology of this study. The first section provides the research design and the rationale for opting for it. The second part includes information about the participants, sampling technique, and research setting. The third part touches on the access and permission issues. The fourth section provides a detailed explanation of how the tasks were prepared. More specifically, this section addresses the phases of the task preparation procedure and which modifications were made to the problems within the task to optimize them for this study based on the feedback received from the experts and pilot studies. The fifth section explicates the instrumentation procedure. The sixth section provides an overview of how the trustworthiness of the findings was ensured. The last two parts of this chapter concentrate on the researcher's role and data analysis approach utilized in this study.

3.1 Research Design

Case studies reflect on one or more occurrences of a specific event to provide a comprehensive explanation of the activities, interactions, incidents, or procedures in that specific circumstances (Denscombe, 2010). Creswell and Poth (2018) defined a case study as a qualitative design that aims to provide a comprehensive description of a case or cases via multiple sources used to collect data. Making sense of a case accompanied by its process and activities is one of the primary aims of case studies (Stake, 2006). Yin (2003) proposed four types of case study designs, each of which aims to analyze contextual circumstances pertaining to the case. Accordingly, a case study design consists of either a single case or multiple cases dependent upon the number of cases being investigated, and further, each, in itself, may be holistic or embedded based on how many units are expected to be analyzed (Yin, 2003). Among these, embedded multiple-case design refers to having more than one unit of analysis within more the cases being investigated (Yin, 2003).

In this study, I determined two cases, namely, sixth and seventh graders. Within each case, I wanted to investigate three units of analysis: classification of problems, levels of strategies, and the role of numerical structure. Put differently, I aimed to investigate how sixth and seventh-grade students classify proportional problems, the levels of their strategies to the proportional problems supported with interactive simulations, and the role of numerical structure on their strategies. Therefore, to provide a detailed analysis on these matters, an embedded multiple-case design was seemed to be an appropriate design that aligned with the purposes of this study and that may address my research questions.

3.2 Participants, Sampling, and Research Setting

In this study, the participants were selected from sixth and seventh-grade levels. In total, there were 14 participants, 7 of which were sixth-graders (i.e., S1, S2, S3, S4, S5, S6, S7) and 7 of which were seventh-graders (i.e., S8, S9, S10, S11, S12, S13, S14). The reason for selecting sixth and seventh-grade students was that they were all expected to be familiar with at least ratio and proportion concepts and accustomed to solving word problems involving multiplicative comparison of two quantities since third grade. In other words, middle school students, particularly sixth and seventh-graders, were found to be the well-suited cases that could provide relevant information about the phenomena of interest. Table 3.1 represents the characteristics of the participants of this study.

Table 3.1

Participants of This Study

	Gender		School Type	
Grade Level	Male	Female	Private School	Public School
6^{th}	3	4	1	6
7^{th}	2	5	3	4

The participants of this study were middle school students who dwell in different cities in Turkey. As Table 3.1 demonstrated, among seven sixth graders, three of them were male, and four of them were female students. Only one of the sixth grade students was attending a private school, and the rest were attending a public school. Among seven seventh-graders, two of them were male, and five of them were female students. Also, three of these students were attending a private school, and four of them were attending a public school. All students were characterized by their teachers as mathematically above-average. Moreover, all of them had an internet connection and a computer; therefore, they were able to participate in my study.

Because of the covid pandemic, it was not possible to go to public or private schools to select participants or carry out face-to-face meetings. Therefore, a combination of purposeful and convenience sampling was used to reach the participants. Purposeful sampling is the selection of individuals in terms of their abundance to have information about the related aspect of the study being investigated and that will best answer the research questions with their contributions to the study (Patton, 2015). The rationale for selecting the case or cases and thoroughly explicating the sampling procedure is essential when doing purposeful sampling (Lunenburg & Irby, 2008). This was the reason for selecting participants from sixth and seventh graders since they were assumed to be the ones that would provide relevant and accurate information that could be used to answer the research questions of this study. Moreover, my intention to select mathematically above achievers lied in my assumption that these students could be convenient candidates that could provide mathematically sound and rich answers.

Convenience sampling was the second to be chosen as a sampling method since it rests on the appliance for the researcher to reach the participants readily without wasting too many sources even though it might have a possibility to be incongruous with the meticulousness of the study (Denscombe, 2010). Since this study was conducted on a small scale and did not intend to use a representative group or generalize about a situation, opting for the convenience sampling method was not seen as a major concern. In this regard, participants were selected through their availability and willingness to participate and consisted mainly of the students of teachers with whom the researcher communicates.

The research setting of this study was carried out over the Zoom platform since, at the data collection time, online interviews were the only applicable method for data collection due to the covid-19 pandemic. Moreover, as Merriam (2009) proposed, in deciding the number of participants the researcher needs to interview, one key element is attending to the responses that frequently appeared or repeated within the data set. This research was carried out with 14 participants, and after investigating students' responses, it has been observed that the variety of the answers provided to the problems was not chancing, and nearly the same responses have occurred. Therefore, it seemed sufficient to work with 14 participants to obtain the necessary data sources that will be analyzed.

3.3 Access and Permission

Since the participants of this study were minors, their parents' consent was taken. They were all assured of the confidentiality of the data their child would provide for this study. They were provided a document that signifies the aim of the study and that they were requested to read the information written on the paper carefully and sign the paper if they agreed with the terms. Prior to the interviews, all students were informed that their voice would not be shared by any means and their responses to the problems would only be used for academic purposes.

3.4 Tasks Used to Collect the Data of This Study

Two tasks were used as a means of collecting data in this study. One of them is called the classification task, and the other is called the problem-solving task. The classification task aimed to gain insight into how students classify proportional problems and how they group the problems as similar or different. On the other hand, the problem-solving task proposed obtaining more information about the students' strategies to proportional problems. The problem-solving task intended to provide a setting for a researcher to observe the levels of the strategies given to proportional problems based on the OGAP Framework.

In this study, three problems were prepared to investigate students' strategies and the levels of these strategies on proportional problems involving different numerical structures. Each proportional problem was further divided into three problems in their own right. Among three problems, one problem consisted of integer ratios, one problem consisted of non-integer ratios, and one problem consisted of letters. Option A in each problem set had an integer ratio in either within or between measures. Option B in each problem set had a non-integer ratio in either within or between measures. Option C in each problem set had letters in either within or between measures. Thus, in total, there were nine problems planned to be administered to the students (see Table 3.2).

Table 3.2

Problem Types

Proportional problems (Either numerical comparison problems or missing value problems)	
Integer ratio	1A, 2A, 3A
Non-integer ratio	1B, 2B, 3B
Letters	1C, 2C, 3C

As mentioned earlier in the introductory section, Cramer and Post (1993a) proposed three types of proportional problems: numerical comparison problems, missing value problems, and qualitative prediction or qualitative comparison problems. In this study, numerical comparison and missing value problems were selected as part of the questions because one of the purposes of this study was to investigate how the numerical structure of the problems influences students' responses to the word problems, and qualitative problems did not lend themselves to fulfill this goal. The following section will provide detailed information on the process of preparation of the tasks by focusing on the expert opinion taken by educators to improve the clarity of the questions in the tasks and to see if they were appropriate for sixth and seventh-grade levels, and that pilot studies conducted with pupils to see whether the problems served coherently in line with the objectives of the study.

3.4.1 Preparation of the Tasks

Creswell (2012) recommended revising an instrument prepared beforehand rather than producing the new one. Therefore, instead of creating new problems, the problems in the problem-solving task were revised or modified from Lamon's (2020) book, whose primary focus was on ratio and proportion concepts.

Two tasks were prepared to collect data, one was a classification task, and the other was a problem-solving task. The problem-solving task of this study consists of three main questions, each of which contains three subquestions, such as A, B, and C, in their own right. Options A are problems involving integer ratio, options B are problems involving non-integer ratio, and options C are problems involving letters. Among six questions, only A options were presented to students in the classification task because otherwise, it was highly probable that students would relate the problems mostly because options A, B, and C were the different versions of the same problems in terms of their numerical structure. While determining which problems should be involved in the instrument, mathematics objectives in the Turkish Elementary and Middle School Mathematics Curriculum prepared by the Turkish Ministry of Education (2018) for the sixth and seventh grade were taken into consideration so that their grade level was appropriate to all students who participated in this study. Even though some contexts like similarity or proportion are not taught formally in sixth grade, after pilot studies, it was seen that students were capable of formulating at least an idea or informal ways to produce an answer. Also, I intended to observe the strategies of those who have not been formally taught the concepts mentioned above.

All three problems in the instrument were involved ratio and proportion. The first question in the problem-solving task was a missing value proportional problem related to the enlargement of a rectangular picture. This problem was a modified version of one of the problems in Lamon's (2020) book on fractions and ratios. The second problem was in comparison structure and required determining the intensity of a color, which was also a modified version of a problem in Lamon's (2020) book. The third problem required determining the rectangles' similarity to a square. Again, this problem was a modified version of one of the problems in Lamon's (2020) book on fractions and ratios. This type of comparison problem, asking to determine the similarity of the rectangles to a square, was also seen in the literatüre (e.g., see Bright et al., 2003). The following table presents the proportional problems in the problem-solving instrument. It is seen from the table that one of the proportional problems was in missing value structure, and two of them were in numerical comparison structure. It should also be noted that presenting problems with different structures was not the concern since the primary objective was preparing questions to provoke students to resort to the technology. Problems in the literature were investigated, and the ones assumed to have the highest possibility of encouraging students to use technology were selected. Table 3.3 represents the problems administered to the students.

Table 3.3

Type and Structure of the Proportional Problems Used in the Current Study

Proportional Problems	Problem Type	Numerical Structure
1A: The long side of a rectangular picture is 6 cm, and the short side is 3 cm. The picture is enlarged without distortion. If the short side of this picture is 15 cm, how many cm is the long side?	Missing value	Integer
1B: The long side of a rectangular picture is 12 cm, and the short side is 8 cm. The picture is enlarged without distortion. If the long side of this picture is 26 cm, how many cm is the short side?	Missing value	Non-integer
1C: The long side of a rectangular picture is A cm, and the short side is B cm. The picture is enlarged without distortion. If the long side of this picture is C cm, how many cm is the short side?	Missing value	Letter

Table 3.3 (continued)

Proportional Problems	Problem Type	Numerical Structure
2A: Elif wants to obtain green color by mixing blue and yellow paints. She adds 150 ml of blue paint and 300 ml of yellow paint to the first container. She adds 160 ml of blue paint and 480 ml of yellow paint to the second container. Which container has the darkest color? *Note*. Blue color darkens green, while yellow lightens green.	Numerical Comparison	Integer
2B: Elif wants to obtain green color by mixing blue and yellow paints. She adds 310 ml of blue paint and 150 ml of yellow paint to the first container. She adds 180 ml of blue paint and 120 ml of yellow paint to the second container. Which container has the darkest color? *Note*. Blue color darkens green, while yellow lightens green.	Numerical Comparison	Non-integer
2C: Elif wants to obtain green color by mixing blue and yellow paints. She adds A ml of blue paint and B ml of yellow paint to the first container. She adds C ml of blue paint and D ml of yellow paint to the second container. Which container has the darkest color? *Note.* Blue color darkens green, while yellow lightens green.	Numerical Comparison	Letter

Table 3.3 (continued)

Proportional Problems	Problem Type	Numerical Structure
3A: 10 units – 30 units 20 units – 40 units Which of the rectangles whose side lengths are given is more like a square?	Numerical Comparison	Integer
3B: 16 units – 24 units 27 units – 36 units Which of the rectangles whose side lengths are given is more like a square?	Numerical Comparison	Non-integer
3C: A units – B units C units – D units Which of the rectangles whose side lengths are given is more like a square?	Numerical Comparison	Letter

The second step in the preparation of the task was obtaining expert opinions regarding the suitability of the problems for the purposes of this study. After deciding the problems that would be asked of the students, three experts' opinions were taken regarding the clarity of the problems, their appropriateness to students' grade levels, and their suitableness to the integration of technology. Two of the experts were professors of mathematics education in a public university. Regular meetings were held with one of them, who is the supervisor of this study, after each pilot was conducted. Only one meeting was conducted with the other professor, and discussion was mainly made on the problems' clarity and their appropriateness for selected grade levels. The last expert was a master's student in mathematics education, who is also a mathematics teacher at a private school. Again, the abovementioned issues were discussed throughout this meeting with the last expert. Based on the feedback taken from each of them, a consensus was reached on the problems' clarity, their appropriateness for the sixth and seventh-grade mathematics, and suitableness for adaptation of technology. One of the issues that kept appearing and took considerable time to resolve was related to the way and purpose of technology integration. Since the primary objective of this study was to observe how the presence of the technology would play a role in assisting students while solving problems, all problems were designed in a way that students would feel a need to use technology, albeit partially. Six pilot studies were conducted until the issue was resolved. The following section discusses this process in depth.

3.4.2 Pilot study

The last step of the preparation procedure was conducting pilot studies to see if the problems would assess the issues intended to be investigated. Pilot studies were conducted with two sixth-graders and four seventh graders. The responses and ideas of the students have shaped several problems and led to some modifications in terms of both the numerical structure and the problems themselves.

In the first pilot study, a seventh-grade student was given all nine problems and asked to group them in terms of their commonalities. It was observed that the student solely focused on verbal clues such as 'calculate' or 'compare' while categorizing the problems. Moreover, having provided all nine questions to the student might have been caused him to make grouping among the problems that were already named the same such as 1A, 1B, and 1C. With that in mind, it was considered necessary to give only one option to participants so that they would not group the questions labeled with the same number just because of their wording. The second and the third pilot studies were conducted based on this modification. The second pilot study was conducted with a seventh-grader. In contrast, the third pilot study was conducted with a sixth-grader. Based on the first three pilot studies, it has been observed that the numbers in the problems were so small and close to each other that students could solve them quickly without the assistance of the technology. As observed, students did not feel a need to resort to technology while engaging in problems. Instead, they either preferred to use it to check their answers or preferred not to use it. Therefore, another modification was done for the fourth pilot study regarding the numbers used in the problems. In the fourth pilot study, the numbers within the problems were presented bigger than that of the previous pilot studies so that the answer could not be explicitly seen on the screen, and students could feel a need to utilize technology. Instead, technology could open a way for generalization of the situation by serving as a simulator or representing the given problem. However, changing the numbers could not give the expected result. Students kept using paper and pencil for solving some problems regardless of their numerical structures, especially for the first and the third questions. Therefore, the first and third problems were changed to avoid overuse of the by-hand method and encourage students to use the technology. For the first problem, instead of asking which person on a table eats more cake among two tables, a picture was presented to a student within an enlargement context. Students were asked to find the length of the missing side by encouraging them to use the slider in the program to observe the change of the sides. It is also noteworthy to point out that the type of the problem has been changed by this modification. The cake problem belonged to the numerical comparison category, while the enlargement of a picture problem was in the missing value structure. Since it did not matter for this research to change the problem structure as long as it could be considered proportional, this modification seemed necessary and worth trying. For the third problem, the same situation had happened when students were asked to compare the speed of two cars. They were reluctant to use the technology. For that matter, instead of asking which car's speed was faster than the other, two rectangles were presented to students and asked which of them was more similar to a square. With this change, students felt a need to use the program and observe the changes of the sides of the rectangles to see which one was more similar to a square.

3.5 The Process of Instrumentation

Data collection of this study was held online through the Zoom platform with one-to-one interviews. During the interviews, the participants were expected to complete two tasks, one of which was called classification task and the other was called problem-solving task. In the classification task, students were expected to reflect on the problems and classify the problems they thought to be mathematically similar. Only A options of each question were presented to each student in the classification task. In other words, only the problems involving the integer ratio were presented on a word document. I opened the document on my computer, shared my screen, and asked students to classify the problems and share their thoughts with me after they finished categorization. This process approximately lasted five to ten minutes. The classification task purposed to encourage students to reflect on the problems prior to solving them to prevent forejudging and draw quick conclusions without sufficient consideration.

The second task of this study consisted of three problems, each of which had three subquestions having different number structures. The average time devoted for each interview was approximately two hours. In this process, while solving the problems, students were requested to use the slider option for the A and B options. The underlying reason for this demand was to observe how they solve problems within the GeoGebra environment. Also, they were expected to write down their responses on paper. All nine problems in the problem-solving task were presented within the GeoGebra environment to students. The researcher sent a link to all participants, and they could work on their computers during the process. Since participants were only asked to use the slider, pre-instruction on using the program was not administered before data collection. For the first question, namely 1A, students were given the problem displayed in Figure 3.1. They were asked to use the slider to change the side lengths and make an observation while solving the problem. Figure 3.1, Figure 3.2, and Figure 3.3 represent the A options of the problems in the problem solving task (see Appendix B for the rest of the problems).

Figure 3.1

Problem 1A in the problem-solving task

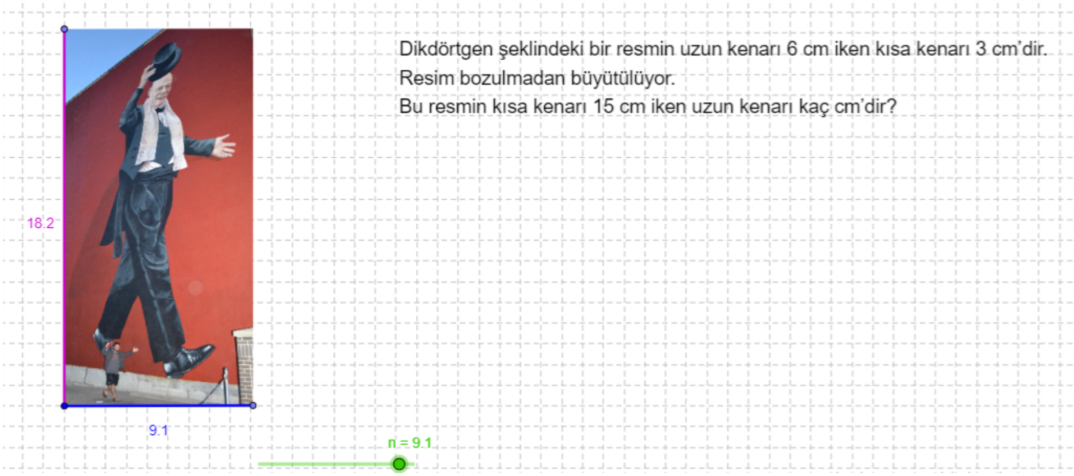


Figure 3.2

Problem 2A in the problem-solving task

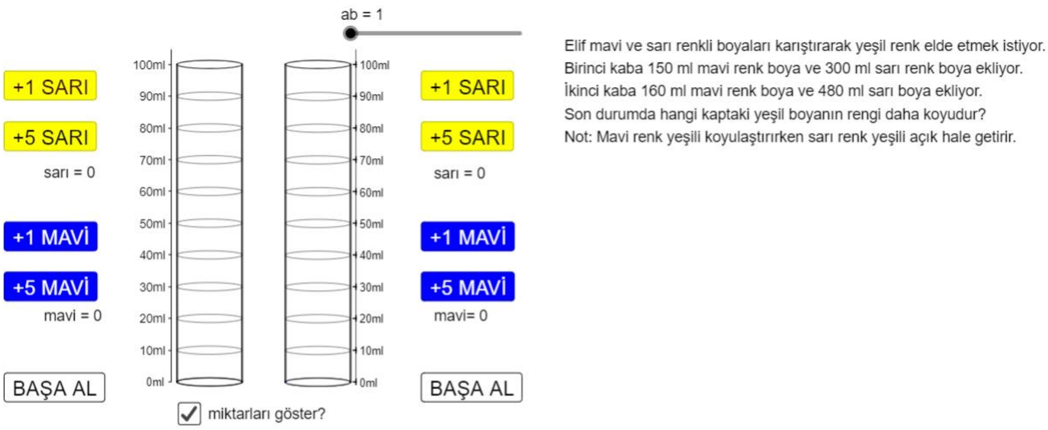


Figure 3.3

Problem 3A in the problem-solving task

Students were given ample time to think and explicate their ideas and were explicitly requested to provide rationals for their responses as much as possible for the sake of clarity and facilitate the interpretation of their statements by the researcher. At the end of each question, students were expected to write their answers in detail on the paper so that the researcher would understand and make an in-depth analysis of their solutions. While they were solving the problems, their ideas and solutions were noted down as a field note. After completing the interviews, participants were asked to send their notes as well. All interviews were videotaped, and all discussions were transcribed for analysis.

3.6 Trustworthiness

It is indisputable that validating findings in any research is a major concern of the study to demonstrate its trustworthiness (Creswell, 2012). In a straightforward sense, credible research offers confidence that gathering and analyzing the data are fulfilled appropriately (Robert, 2016). The credibility of research is predominantly dependent on the extent to which the researcher puts effort into assurance (Golafshani, 2003). It is possible to use various techniques under different paradigms to ensure validity

(Creswell & Miller, 2000). Even though validity was not seemed to be appropriate to use in qualitative research, researchers agree with the idea that there is a necessity to assess the quality of the study in some way (Golafshani, 2003; Winter, 2000). Since qualitative studies involve naturalistic characteristics (Fraenkel & Wallen, 2009), it is appropriate to build trustworthiness through naturalistic paradigms regarding the research's social aspects (Guba & Lincoln, 1982). In qualitative studies, the issues of reliability and validity are not addressed independently; instead, the terms trustworthiness or credibility are generally used to imply both (Golafshani, 2003). In an attempt to reduce the ambiguity, I opt for using the term 'trustworthiness' to comprise them all. For establishing trustworthiness in a qualitative study, Lincoln and Guba (1985) recommended four benchmarks; credibility, transferability, dependability, and confirmability. The following four sections revolve around those issues.

3.6.1 Credibility

Credibility is the degree to which the research findings convey the truth (Lincoln & Guba, 1985). Several techniques can be applied to assure credibility, such as prolonged engagement, persistent observation, triangulation, member checking, negative case analysis, referential adequacy, and peer debriefing (Lincoln & Guba, 1985). In this study, persistent observation and triangulation techniques were used to enhance credibility.

Persistent observation refers to determining factors pertinent and non-relevant to the subject under investigation to provide an elaborate explanation (Guba, 1981; Lincoln & Guba, 1985). It also includes the formation and revision of codes and categories due to spending adequate time and with the support of careful scrutiny, which would, in turn, possibly facilitate the comprehension of data (Korstjens & Moser, 2018). In this study, I started investigating the findings by transcribing interviews verbatim. Then, I scrutinized each answer given to the problems and tried to locate students' responses on the OGAP Framework. I investigated the responses that fall under each strategy. Also, I paid attention to their consistency by reading the responses over and over again.

Triangulation is another technique that was used to enhance the credibility of findings. It is a technique that allows reducing personal bias that could be involved in the interpretation and analysis of findings due to a lack of diverse methods or points of view (Denzin, 1978). Denzin (1978) suggested four types of triangulation techniques: using multiple data sources, investigators, theories, or methodologies to increase credibility. Among them, investigator triangulation was used, and a second coder was involved in the analysis process. Denzin (1978) recommended that observers, coders, or interviewers be selected based on their ability to analyze. In this regard, I asked a Master's student in the mathematics education program to be the second coder and analyze the two students' responses. The second coder had the relevant knowledge on the field being studied as she and I were enrolled in a graduate project course and studied the concept of proportional reasoning throughout a semester. Moreover, we conducted a small project in a graduate course and gained experience analyzing qualitative data. In order to decide the number of data that needed to be analyzed, Creswell and Poth (2018) proposed to obtain an intercoder agreement, with at least two different coders, by asking the second coder to analyze 10% to 25% of the data. Since there were 14 participants and 14 separate transcripts belonging to each, two of them were selected randomly, which was about 14% of the data. I asked the second coder to analyze these two transcripts by providing her the necessary knowledge about the study, its purposes, and also the framework used. After completing the analysis, 88% agreement was reached. The second coder and I discussed the parts with which we disagreed until a complete agreement was obtained.

3.6.2 Transferability

Transferability is the counterpart of the external validity of quantitative research (Lincoln & Guba, 1985). It deals with providing other researchers with an idea on whether the study could be repeated under different circumstances with different participants (Lincoln & Guba, 1985). To do so, an adequate amount of knowledge, what Lincoln and Guba (1985) called thick description, should be shared with audiences about the methodological aspects of the study (Korstjens & Moser, 2018). Thick description allows audiences to see whether the findings in one context can be applied to another context without aiming to generalize the findings (Guba, 1981). In this regard, I gave as detailed information as possible concerning the methodological issues of this study. I explicated every step that I have taken in detail to give a clear insight into the process that I followed through.

3.6.3 Dependability and Conformability

Dependability is used in qualitative studies in substitution for reliability in quantitative studies and deals with the consistency of the findings obtained (Lincoln & Guba, 1985). An important factor here is considering how consistent the findings are (Lincoln & Guba, 1985). Since the instrument is considered to be the researcher in qualitative studies, it is essential for the researchers who conduct a study under a naturalistic paradigm to give much more thought to find the ways to ensure dependability by considering the inconsistencies stem from the research design or the subject being studied (Lincoln & Guba, 1985). In fact, they are considered as valuable instead of a hindrance for qualitative studies provided that they are articulated explicitly (Guba & Lincoln, 1989). However, it should also be noted that since reliability is a prerequisite for validity, checking the credibility may suffice for guaranteeing reliability (Lincoln & Guba, 1985). Lincoln and Guba (1985) suggested using some techniques such as overlap methods, stepwise replication, and inquiry audit to ensure dependability. They criticize the stepwise replication method for becoming skeptical, partly because independent researchers could follow different paths and obtain inconsistent results, making it even harder to ensure dependability. Therefore, I preferred to use the inquiry audit technique as a means of enhancing dependability. The inquiry method expects an auditor to examine the research process and make comments on whether an appropriate procedure was followed in accordance with the selected design (Lincoln & Guba, 1985). Since this dissertation study will be reviewed by the supervisor and the examining committee members, the dependability issue will be handled. Likewise, conformability can also be confirmed through auditing (Creswell & Poth, 2018), and both dependability auditing and conformability auditing should be performed together (Guba & Lincoln, 1989).

3.7 Researcher's Role

The data of this study were collected through semi-structured one-on-one interviews. Creswell (2012) argues that one-on-one interviews are good ways to enable reluctant individuals to speak comfortably. During the interviews, my role was to take notes of students' answers and listen to them carefully to understand what they meant while engaging in solving the problems. Occasionally, I used probes to either elucidate some points that I did not grasp, what Creswell (2012) called clarifying probes, or expand on the issue that we made a discussion on, which is what Creswell (2012) referred to as elaborating probes. It caught my attention that some of the participants struggled to give reasonable explanations to their answers. It might have happened because their problem-solving experience might have been more like "solve the question and say the answer" rather than "solve the question out loud and explain the process in detail." Therefore, occasionally it was necessary to make probing such as "What do you mean by that?" or "Can you explain what did you do?" to clarify what they meant. Moreover, when they contradicted themselves during problem-solving, I confronted them to their responses and asked them exlain to me why this was the case. My objective for doing so was discern whether they mistakenly followed a wrong path or deliberately tried alternative ways to see which one would work.

3.8 Data Analysis

Data analysis refers to the procedures taken to interpret what the data reveal and answer the study's research questions by making necessary organizations on the data (Merriam, 2009). One of the techniques that can be used to make sense of the qualitative data is to carry out a content analysis (Merriam, 2009), which allows researchers to interpret any data to arise from communication, including songs, pictures, texts, etc. (Denscombe, 2010; Fraenkel et al., 2012). Data of this study consisted of students' written and verbal answers and the researcher's field notes taken during the interviews.

The first purpose of this study was to investigate how sixth and seventh-grade students classify proportional problems. In an attempt to answer the first research question, students' statements in the classification task were examined, and their responses were grouped under several categories. These were focusing on the contextual aspects of the problem, focusing on the terms that may sound reminiscent of each other, focusing on the nature of the problem, and focusing on the interrogative word in the problem. For instance, if a student attended to the contextual aspect of problems such as mathematical shapes involved in problems while relating them, their criterion was labeled as 'focusing on the contextual aspects of the problems.' In a similar vein, if they associated problems based on the presence of words such as paint and picture with which somehow they found reasonable to associate, their strategy was named as "focusing on the terms that may sound reminiscent of each other." Another criterion that some students resorted to was named "focusing on the nature of the problem." If the students realized the mathematical situation embedded in the problem, such as increase or decrease of amounts or shrinking or enlarging a picture, then their response was incorporated into this category. Lastly, if the students paid attention to the interrogative word of the problems, the response was assigned to the "focusing on the interrogative word in the problem" category.

The second purpose of this study was to investigate students' solution strategies on proportional problems and determine which level their strategies corresponded to on the OGAP Ratio and Progression. In order to answer the second research question, students' responses to the problems were carefully examined, and their strategies were assigned to the levels on the OGAP Ratio and Proportion Progression. As proposed by (Petit et al., 2020), these strategy levels are proportional, transitional, early transitional, early ratio, non-proportional. The proportional strategy level shows evidence of efficient use of multiplicative strategies, including unit rate and scale factor comparison of fractions or use of the cross-product rule. If the students correctly manipulated any of these strategies and provided evidence of multiplicative reasoning, their response was assigned to the proportional strategy level. At the transitional strategy level, these strategies do not necessarily be efficient, according to Petit et al. (2020). Transitional level strategies may encompass visual aids such as ratio tables. If the students tried to apply multiplicative strategy but could not use it efficiently and somehow strategy did not help them come up with a correct solution partly due to a misinterpretation of multiplicative relationship, their strategy was assigned to a transitional level. At the early transitional level, additive strategies may also appear together with multiplicative ones. If the students employed both additive and multiplicative thinking within a single strategy, their response was characterized as being at the early transitional level. Early ratio level strategies, on the other hand, involve solely additive reasoning. At this strategy, students form a composed unit and build upon it either mentally or using a ratio table until the target amount is obtained. If the students used a building-up strategy and constantly iterated the composed unit, their strategy was regarded as being at an early ratio level. Lastly, non-proportional level strategies comprise taking additive differences between the amounts, guessing the answer without providing a meaningful explanation, using inaccurate ratio referent, misconceiving wording or conceptual aspect of the problem, and comparing numbers rather than ratios. If the students used any of the abovementioned strategies, their response was assigned to the non-proportional strategy level. Table 3.4, adapted from the book of Petit et al.'s (2020), which was written on ratio and proportion concepts, shows distinguishing evidence for each strategy level on OGAP Ratio and Proportion Progression Framework.

Table 3.4

OGAP Ratio and Proportion Progression Framework

Strategy Level	Distinguishing Evidence
Proportional	✓ A student is able to recognize the multiplicative relationship and employ it efficiently throughout problem-solving. ✓ Application of a unit rate, construction of an equation that could express a proportional situation, comparison of ratios or fractions, application of the cross-product algorithm, and recognition and application of factor of change are examples of proportional strategy.

Table 3.4 (continued)

Strategy Level	Distinguishing Evidence
Transitional	✓ A student can notice the multiplicative relationship between different quantities yet may not be able to generalize this relationship to find the missing amount. Instead, a student can form a multiplicatively related pair of quantities with the aid of a ratio table to find out the target quantity.
Early Transitional	✓ A student can recognize the multiplicative relationship within the same quantity. Yet, s/he may not be able to notice the multiplicative relationship between different amounts. ✓ A student can discern the scale factor within the same amount and multiplicatively build up quantities within the ratio table to find the target amount. ✓ A student may also interchangeably use additive and multiplicative strategies within a single solution.

Table 3.4 (continued)

Strategy Level	Distinguishing Evidence
Early Ratio	✓ A student forms a composed unit and iterates this unit until s/he can find the target quantity. S/he may not necessarily use a model.
Non-proportional	✓ A student compares quantities additively instead of multiplicatively, may compare numbers rather than ratios, may use an incorrect ratio referent, may guess the answer or perform random operations, may misinterpret the problem or the concept, may use proportional strategy in a non-proportional problem.

Note. Adapted from *A Focus On Ratıos And Proportıons: Bringing Mathematics Education Research To The Classroom* (p. 82) by M. M. Petit et al., 2020, Routledge. Copyright 2020 by M.M. Petit. et. al.

The last purpose of this study was to investigate the role of numerical structure on students' strategies. Again students' answers to each problem were examined, and how their strategies were altered across different numerical structures were addressed. There was not any categorization or coding for answering the third research question. Instead, I have presented the frequency of a particular strategy used for each problem with different numerical structures on the table at the end of the section, which reported the findings of that particular strategy.

CHAPTER 4

FINDINGS

This chapter contains three main sections, each of which proves a detailed explanation for each research question of this study. In the first section, the characteristics of the problems sixth and seventh-grade students focused on while classifying problems are presented as an answer to the first research question. Subsequently, in the second section, strategies that sixth and seventh-grade students used while solving proportional problems with and without the use of interactive simulations are explicated in accordance with the progression of ratio and proportion proposed by (Petit et al., 2020). As Petit et al. (2020) argued, each strategy can be at a different level at the OGAP Ratio and Proportion Progression. Bearing that in mind, the second section aims to provide a comprehensive analysis of the categorization of the responses into those levels along with the changes in students' responses with the presence of the interactive simulation. The last chapter of this section, which purposes to answer the third research question of this study, provides an analysis of the role of numerical structure on students' strategies in solving proportional problems. This section aims to provide a clear explanation regarding the possible changes in the students' responses or the levels of the evidence in their responses with respect to the numerical structure of the problem.

4.1 Students' Classification of Proportional Problems

To answer the first research question, I investigated how sixth and seventh-grade students classify proportional problems. It has been observed that regardless of the grade level, all students pretty much determined the same aspects while classifying the problems. The following table represents the aspects of the problems that students from each grade level provided in the process of classification. The numbers next to

‘S’ from 1 to 7 indicate the sixth-graders, while 8 to 14 represent the seventh-grade pupils.

Table 4.1

Students’ Classification of Proportional Problems

Sixth-Graders	Seventh-Graders
• Focusing on the contextual aspects of the problems (S3, S4, S5) • Focusing on the terms that may sound reminiscent of each other (S6) • Focusing on the nature of the problems (S1, S7)	• Focusing on the contextual aspects of the problems (S12) • Focusing on the terms that may sound reminiscent of each other (S14) • Focusing on the nature of the problems (S10, S13) • Focusing on the interrogative word in the problem such as “which one” (S9)

Students' responses in the classification process were categorized under four main headings, the contextual aspects, the nature of the problems, the reminiscent terms, and the interrogative word in the problem. The responses that were assigned to the contextual aspects category involved focusing on the mathematical shapes presented in the problems, such as rectangles, and the mathematical concepts that the figures were associated with, such as area. The responses considered under the nature of the problems were related to the mathematical situation underlying the problem, such as enlargement or minimization. The third main heading involved the responses that consisted of the association of reminiscent terms, either having a mathematical meaning such as milliliter and unit or a semantic connection such as paint and picture. The last major category into which students’ responses were categorized was attending to the interrogative words such as ‘which one.’

It is noticeable from the table that the common characteristics that sixth-graders focused on while classifying problems were the contextual aspects of the problems. Out of seven sixth-grade students, three of them used this aspect in their categorization process. The below responses demonstrate these three students' responses while classifying proportional problems.

> S3: 1 and 3 look similar because they involve geometric shapes and can be related to area problems.
>
> S4: 1 and 3 are similar because they both involve rectangles.
>
> S5: 1 and 2, their question type, is related to the area.

These three students focused mainly on the figures or the concepts these figures might relate to, such as area.

Another characteristic that was used by a sixth-grader was the terms that may sound reminiscent of each other. Only S6 from sixth-graders associated some terms in categorizing problems. Her response was as follows:

> S6: Teacher, do the second and the third questions similar to each other? Millilitre, unit…

The last characteristic used by sixth-graders in the classification of problems was the nature of the problems. Two of the sixth-graders used this characteristic in the classification process.

> S1: 1 and 3 are similar to each other. They either enlarged or shrunk.
>
> S7: Teacher, in the first question, it is enlarged with pressure. In the second question, it is about the same color, but it is either dark or light.

The explanation of S7 may not sound as clear as that of S1's. By saying pressure, S7 referred to the imbalance of the colors. In the first problem, by saying pressure, he referred to the force that should be applied to the figure to enlarge it, whereas in the second, by saying pressure, he meant that one of the colors should outweigh or press the other to be darker than the other.

As sixth graders did, one of the seventh-graders used contextual aspects of the problem in classification.

> S12: In terms of similarity, the first and the third questions, side….that is… in the first question the side length of a rectangle is given, in the first and the third question, the sides...

Apart from that, one criterion that was used by one of the seventh-graders was the terms that may be reminiscent of each other. For instance, S14 saw a relation between the painting involved in the second problem with the picture presented in the first problem:

> S14: In the first, it says picture, and in the second, there is paint. Two of them seemed reminiscent.

Attending to the nature of the problems in classification was also among the characteristics that seventh-graders used in their categorizations. Two seventh-graders who focused on the nature of the problems gave the following responses:

> S10: Teacher, in third, it does not ask us to find something. It is a comparison question. Therefore, since none of them have a comparison, I do not associate this question with others. In second, here the colors are increasing. The darkness of the colors is given in terms of their milliliters. Actually, teacher, here, there is also a comparison.
>
> S13: 3 and 2 are similar to each other. They both have a comparison.

Only one of the seventh-graders attended to the interrogative word in the problem and associated the which one questions.

> S9: The second question is a little similar to the third question, I think. Both of them have the same thing. I think it asks which one is more.

All in all, it is noteworthy to state that the sixth and seventh-grade students focused on a diversity of points, ranging from contextual or verbal aspects of the problems to the nature of the problems while classifying them.

4.2 Students' Strategies on Proportional Problems Supported with Interactive Simulations

In this section, strategies employed by students to problems requiring proportional reasoning were examined under two main headings; single strategy and multiple strategies. Under the title of a single strategy, the problems solved with only one strategy and their corresponding levels are mentioned. Since there were three proportional problems administered to the students, discussing each problem separately under each strategy level seemed appropriate to provide a clear picture of the diversity of responses among sixth and seventh-graders for each of these problems. In the following section, labeled as multiple strategies, the problems in which students employed more than one strategy are discussed, and the progression of these strategies from the lowest to the highest and the highest to the lowest are explained with possible underlying causes. Moreover, students' strategy shifts over different numerical structures, and the potential influence of using technology on these changes are presented in this section.

4.2.1 Single strategy

The findings of this study revealed that even though students shifted their strategies in the course of problem-solving, it has been observed that they preferred to employ a single strategy in some of the proportional problems. As findings demonstrated, students applied non-proportional, early ratio, and proportional strategies as a single strategy in solving proportional problems. This section presents only the problems that led students to employ a single strategy with their corresponding levels on the OGAP Ratio and Progression Framework.

4.2.1.1 Non-proportional Strategy

Students who use non-proportional strategies in proportional problems may provide evidence of additive reasoning, use the ratio referent inaccurately, apply proportional strategies, get confused about the verbal aspects of the problems, make operations randomly, or simply try to guess the answer (Petit et al., 2020). As the findings demonstrated, non-proportional strategies were frequently used by the majority of the students regardless of the numerical type of the problem. Table 4.2 displays the problems solved only with non-proportional strategies, and that reveals which students employed these strategies on the given problems. It is evident that out of forty-nine responses given to the problems presented in the table, twenty-two of them belong to problem 3, which asked students to determine which of the rectangles was more similar to a square; nineteen of them belong to problem 2, which asked students to decide the darkness of the green color; and eight of them belong to problem 1, which asked students to find the length of the missing side of the enlarged version of the rectangular picture.

Table 4.2

Problems Solved by Non-proportional Strategy

Problem name	**Sixth grade**	**Seventh grade**
1A		S8
1B		S8, S14
1C	S3, S6	S9, S12, S14
2A	S1, S4, S6	S8, S12, S13
2B	S1, S3, S4, S5, S6, S7	S8, S10, S11, S12, S13, S14

Table 4.2 (continued)

Problem name	Sixth grade	Seventh grade
2C	S1, S4, S5	S8, S12, S13
3A	S5, S6	S8, S12, S13, S14
3B	S1, S5, S6, S7	S8, S9, S10, S12, S13, S14
3C	S1, S3, S5, S7	S8, S12

Among the problems that provoked students to apply non-proportional strategies, one of them was the first problem involving enlargement context and missing value structure. As mentioned earlier, this problem required students to find the missing length of the enlarged rectangular picture. Two sixth-graders and four seventh-graders provided evidence of non-proportional strategy in their responses to problem 1. In problem 1A, only one seventh-grader, namely S8, used a non-proportional strategy. She followed an interesting way to find the missing side length. Her solution was as follows:

> S8: Since this is the long side and this is the short side, we will multiply… It becomes 36 divided by 9. [I do not know why she said division because she did an addition operation]. It becomes 48. It says that the short side of the picture is 15cm. I mean, there will be 15 instead of 3. First, we do.. 15 plus 15… It becomes 45. Then, we…. What is the difference between them in the earlier? We subtract 9 from 36. It becomes 27, I think. I can try… 17 from 25. It becomes 41. Then our long side will be 41.

She squared the long side and short side and then added them. She said that in this way, she could find the area of the rectangle. After a long process of random calculations, she did not reach a conclusion. Above speech was just a part of our interview with her revealing her random and erroneous operations in her attempt to find the area of the rectangles. As seen, there was not an indicator of multiplicative reasoning or meaningful explanation that would support her arguments and calculations. Therefore, this strategy was considered non-proportional. In another problem, namely problem 1B, only two seventh-graders employed a non-proportional strategy. These students were S8 and S14. S8 tried to find the areas of the rectangles by doing random operations, and S14 focused on the additive differences between the sides. The following speeches were part of our separate conversations with S8 and S14, respectively.

> S8: I think this is the same as earlier. But this time, it is a multiple of it. This time it asks us (to find) the short side. I will do the thing I did earlier. I will use the same tactic and do not confuse my mind. 12 times 12 becomes 144. 8 times 8 becomes 64. When we add them up, it becomes its area. I will try to do it based on average. The difference between them 4. I mean, there are 4 numbers between them. I can try this way. Actually, it is short and practical. It says 26. Its four more will be 30. But I do not think that the answers will be the same.
> R: Do the short side 30 and the long side 26?
> S8: No. It cannot be. Then it will be meaningless. It becomes 22. Since it asks for the short side. From proportion. Does it a proportion? Since there is 4 difference between them like a proportion, then I said their difference will also be 4 and found 22.

> S14: I found the answer as 22…Again, I made the same arrangement. I made [the slider] 8. The long side becomes 12. Then I found the difference between them and subtracted fit from 26.

As seen from the above responses, S8 used random calculations to find the area of the rectangles, but her method did not lead to a conclusion. She actually resorted to the strategy that she used in the previous problem. Then, she attempted to use an additive strategy to find the missing side. Therefore, her strategy was considered non-proportional. Similar to what S8 did, S14 focused on the additive differences between the side lengths and found the result the same as with S8. S14's strategy was also considered a non-proportional one since it involved additive thinking.

Problem 1C was another problem that caused students to apply non-proportional strategies. In this problem, two sixth graders and three seventh graders used a non-proportional strategy. The following speeches were part of our separate conversations with sixth graders, namely, S3 and S6.

> S3: It becomes AB…or it becomes only A. If it becomes C, it is enlarged. It cannot become A. C is bigger than A and B. They should not be equal as well. Then it should become A, I say A.

> S6: We can add C and A, or subtract A from C [or C from A, she did not specify]. Since it asks for the short side, we can subtract A from C [or C from A, she did not specify].

It is clear that S3 made random calculations without grounding his reasoning on meaningful mathematical arguments. S6, on the other hand, used additive reasoning in her solution. Both solutions provided by these students were considered a non-proportional strategy since they involve either a random operation that lacks meaning or involve additive comparison that was not applicable to the multiplicative situation.

As sixth graders, seventh graders also employed a non-proportional strategy in problem 1C. These students were S9, S12, and S14. The following speech demonstrates S9's reasoning on problem 1C.

> S9: It becomes 1 letter away then. If the long side is C, then the short side becomes D.
>
> R: What if I said that there is no relation between the order of the letters in the alphabet and this problem. What would we do? How would we solve this problem in a different way?

S9: It becomes an algebraic expression then. A+1, since it is 1 more than.. the long side [she meant the short side] is A, it becomes A+1. If it is C, then it becomes C+1.

As seen from the above speech, S9 first focused on the alphabetic rule and thought that there could be some sort of relationship between the alphabetic rule and the solution of the problem. When I asked her what she would do differently if there was not the case, she applied additive strategy. However, her solution could not lead her to find the correct answer since she thought additively in a proportional situation. Like S9, S12 also tried to make an algebraic connection between the problem and its solution when he saw the letters. Nevertheless, he was not able to express the multiplicative relationship between the sides. Instead, he assigned values to the unknowns randomly and constructed an additive relationship between them.

S12: It looks like algebra. As far as I know, we give value. Let the long side be 3, and the short side is 2. If we think it increases by 1, we can say C is 4, and the short side is 3. I mean, we can give any value.

Among the seventh-graders, another strategy that was given to problem 1C and that was categorized as non-proportional was that of S14's. The following transcript was part of our discussion with S14.

S14: I found the answer 6.

R: How did you do?

S14: I counted the squares here.

R: What is the number of squares? Where are they?

S14: The long side 16, the short side 10. When I subtracted, I found 6.

[Silence]

R: Anything that comes to your mind?

S14: Will I construct an equation?

> R: If you want, you can.
>
> S14: I thought of ABC as a right angle. Then I divided 90 by 3. It became 30. I consider the answer as 30.
>
> R: Well, why do you think so? Why do you draw a right angle?
>
> S14: ABC looks like a right angle. Since there are 3 letters, I divided the result by 3.

As seen above, S14's response was ambiguous, irrelevant to the problem contexts, and lack of meaning. Since she randomly operated with numbers and could not provide a meaningful explanation for her reasoning, her response was considered to be non-proportional.

In addition to problem 1, some of the sixth and seventh-graders also provided non-proportional strategies on problem 2. Unlike problem 1, problem 2 was a comparison type of proportional problem involving mixture context. In this problem, five sixth-graders and six seventh-graders used a non-proportional strategy. Among the sixth graders, S1, S4, and S6 used a non-proportional strategy in their solution to problem 2A. The following speeches demonstrate their solutions.

> S1: In the first container, there is a 150 ml blue color, and in the second container, there are a 160 ml blue container. There is a 10 ml difference. The first one should be 10 darker than the second one. However, yellow makes the color lighter. Therefore…. In the first container, there is a 300 ml yellow color, and in the second container, there are 480 ml yellow color. The second one is 180 ml lighter than the first one.
>
> S4: Teacher, because in this container, the amount of blue color is less [than the first container] and the amount of yellow color is more [than the first container]. However, in the first container, the amounts of blue and yellow colors are more than in the second container. Then, to see the reason…I will

find the difference between the corresponding yellow amounts and corresponding blue amounts. Teacher, the difference between the yellows [amount of yellow colors] is 12, and the difference between the blues [amount of blue colors] is 24.

S6: Teacher, at first, we tried to reduce the numbers and make them fit in this container. Then we looked at which one was darker. It was darker in the first container because there was less yellow. That's why we chose the first container.

R: But blue is also less. Is not it interesting? Blue is less than the second, yellow is also less than the second. But it seems darker.

S6: Teacher, since yellow makes the color lighter and it is lighter than blue, it makes the color lighter. So, the yellow should be less so that we have darker green.

As seen above, S1 compared the colors additively to decide which container had a darker color. Therefore, her strategy was considered non-proportional. Similarly, S4 applied a non-proportional strategy. She first simplified the amounts to fit them into the containers. However, after simplification, to decide their darkness, she used additive reasoning. Therefore, her response was categorized as a non-proportional strategy. Likewise, S6 applied a non-proportional strategy. Even though S6 simplified the amounts correctly, she could not provide an explanation that was multiplicative in nature. She tried to compare the simplified amounts additively. Therefore, her strategy was regarded as non-proportional.

In problem 2A, among seventh-graders, S8, S12, and S13 used non-proportional strategies. The following speeches were part of our discussions with S8 and S12 on problem 2A. Also, Figure 4.1 and Figure 4.2 illustrate S12’s and S13’s solutions on problem 2A, respectively.

S8: As I understand from the question, the more I add blue, the more I should add yellow so that the green color becomes… sorry. The more I add yellow, the more I should add blue so that the color becomes darker because the question asks us (to find) the darker color. In the first container, to find the less amount, I subtract 150 from 300. Again it will be 150. In the second container, I subtract 480 from 160. It becomes 320. It is more than our container. Then how can I do it? Since its limit is 100 .. Then should I take the multiples of it? For instance, should I divide it by 5? What does that have to do with anything?

The above speech demonstrates that S8 started her solution by comparing the amounts additively to find which container had darker green color. However, the amounts she obtained were beyond the limits of the containers' volume. Therefore, she could not fit them into the containers. She tried to simplify the colors, but at that point, her explanations became ambiguous and lacked meaning. She started to perform random operations, but it did not lead her to any conclusion. Therefore, her strategy was considered non-proportional.

S12: When I subtracted 150 from 300, it becomes 150 since 150 is half of 300. If we subtract 160 from 480, I found 320. I was afraid of obtaining an answer which is half of 480. But it did not.

R: Why does the answer make you afraid if it becomes half of 480?

S12: Because in this case, I would not know which operation to do.

Figure 4.1

S12's strategy to the problem 2A

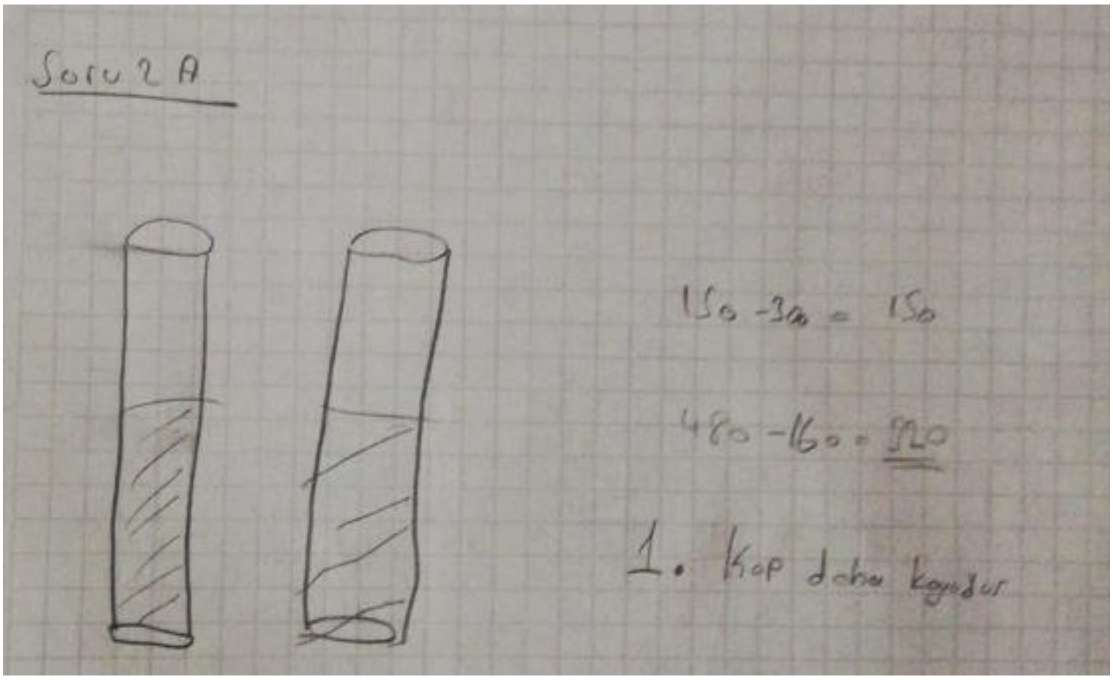
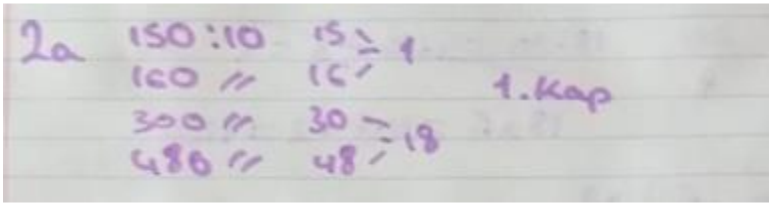


Figure 4.2

S13's strategy to the problem 2A



As seen from both responses, S12 and S13 resorted to additive reasoning in solving proportional problems. Although S12 took the halves of the numbers, he also made a subtraction operation to determine the difference between two amounts, which revealed that his focus was on additive differences rather than the multiplicative relationships between the amounts of two colors. S13 started her solution by division and simplified the amounts of colors to make them fit into the containers. However, to decide the density of the colors, this student paid attention to the additive differences between the amounts and made a subtraction operation. Since both students focused on the additive differences between the quantities they compared, their strategies were regarded as non-proportional.

Like problem 2A, problem 2B also provoked students to use non-proportional strategies. In fact, the frequency of non-proportional strategies was the highest in this problem. Six sixth-graders and sixth seventh-graders applied a non-proportional strategy in problem 2B. The following six speeches were from our separate interviews with sixth graders, namely S1, S3, S4, S5, S6, and S7, respectively.

> S1: We divide 310 by 10, it becomes 31, and we divide 150 by 10, it becomes 15. In the second container, we divide 180 by 10, it becomes 18, and we divide 120 by 10, it becomes 12. The first container is darker than the second. Both 18 and 12 are multiples of 6. We can divide both by 6. For the former case, we obtain 3, and for the latter case, we obtain 2.

> S3: In the second container, there is a 60 ml difference. In the first container, there is a 160 ml difference. It means that since the first container has more amount of blue, its color might be darker.
> R: Is it more than the yellow [of the first container] or the blue color of the second container?
> S3: The difference between 310 and 150 is much more than that of 180 and 120.

> S4: This time, I will divide both of them by 10. In the first container, the amount of blue color is 31 ml, and the amount of yellow color is 15 ml. The second container has 18 ml blue color and 12 ml yellow color. The difference between the amounts of blue and yellow in the second container is 6. In the first container, the difference is 16. Therefore, the first container is darker than the other.

S5: We first deleted the zeros [S5 simplified the amounts by 10] in the first and the second container. Since the amount of blue color in the first container preceded that of the yellow colors, it gives a darker color to it. In the second container, since the difference between the amounts of blue and yellow colors is not too much, this [the second container] becomes lighter. That is, the second container has lighter color.

S6: This is considered exactly the same. But what does it want this time? Again, it is asking for us to find the darker one. Again I reduce the amount of blue to 31. I make the yellow 15. For the second container, the blue is 18, and the yellow is 12, teacher. Teacher, the first one again looks darker.
R: Well, I wonder what is the reason?
S6: Teacher, this time in the first container, the amount of blue is more. With respect to the blue, the yellow is less. In the second container, the [amount of] blue is less [than that of the first], but it is more in comparison with yellow [in the second container], but it is less [than the amount of blue] in comparison with the first. The yellow is less, but as I said, since the blue is lesser, it becomes lighter.

S7: Teacher, if we divide 180 by 3, it becomes 60. If we divide 120 also by 3, it becomes 40. Then we divide 40 by 2, and it becomes 20. I make it 20 because here, there is a 40 ml difference between 180 and 120. That means we fill 60 ml blue and 20 ml yellow into the container. Teacher, to make the color the same, the difference between 60 and 40 and 180 and 120 should be the same.

From the above speeches taken from different interviews, it is seen that S1 simplified the amounts to fit them into the containers. However, during our interview, she could not proceed after that and could not provide a sufficient explanation as to why she made a simplification. Her response was regarded as a non-proportional strategy due to a lack of explanation that could be used to explain the multiplicative situation embedded within the problem. S3 compared the amounts of colors additively by taking the differences between the amounts of colors within each container. S4 first tried to simplify the quantities by 10 to fit them into the containers, but then she used an additive strategy while comparing the darkness of the colors. In a similar manner, S5 simplified the quantities by deleting zeros from the quantities and then compared the amounts additively. Likewise, S6 simplified the amounts by 10, then made an additive comparison between them to decide which container had a darker green color. Lastly, S7 simplified the amounts of yellow and blue colors until he obtained the same difference between them for both containers. He thought that in order to obtain the same darkness for both containers, he should make the differences between the amounts of yellow and blue colors for both containers the same. All these four sixth graders' reasoning was quite similar to each other. All these students tried to simplify the amounts so that the quantities could fit into the containers. However, to decide the darkness of the green color, instead of providing multiplicative reasoning, they focused on the additive differences between the yellow and blue colors, and thereby, their strategies were considered as a non-proportional strategy. Similar to sixth graders, the seventh graders also used a non-proportional strategy in solving problem 2B. These students were S8, S10, S11, S12, S13, and S14. The following speech was part of our interview with S8.

> S8: The amounts change and but the question does not. Therefore, we can use it the same way, but I could not find it earlier. This time I will add [meant subtraction] those amounts. [310 and 180, 150 and 120]. I will subtract 180 from 310. The blue becomes 130. To find the yellows, I will subtract 120 from 150. It becomes 30. The yellows become 30. Then, I will divide 130 by 2.
>
> R: If this 130 is exactly blue, how do we find the blue inside it by dividing it two?

S8: Since there are two containers, I will divide 150 by 2. Then I write the result here [pointing to the top of the first container]. And then, since there are two colors, I will divide them by 2 again. When I divide 130 by 2, 65, but I cannot divide 65 by 2.

S8 first subtracted the amounts to find the difference between the same colors, and then she divided the amounts by 2. She reasoned that the amounts should be divided by two because there were two containers to be filled. She tried to apply the same method that she used in the previous problem. However, since additive thinking was not applicable in the context, her reasoning did not lead her to a conclusion. Likewise S8, S10 also used a non-proportional strategy in his solution. The following speech was part of our interview with S10.

S10: Teacher, because.. between them.. the difference between their ml is equal. Both of them 60.
R: Are they the same?
S10: In the second container, it is 60. In the third container [he meant the first], it is 160. Then, the first container will be darker because blue makes the color darker. Since the difference between them is 160, the yellow color makes it lighter, but it does not influence much. It will be less because the blue color is twice as much as yellow. In the second container, the difference between them is only 60. Therefore, the lighter green colors complete themselves with a bit of difference. Since in the first container, the dark color is more, and the lighter color is less, yellow color has a minor influence. But the difference between the colors in the second container is more minor. Even though the blue makes the color darker, the yellow color makes the color lighter, even though not as much as blue.
R: You said that in the first container, the amount of blue is twice that of the yellow. What about the second container?
S10: It does not one times more, just 60 ml more.

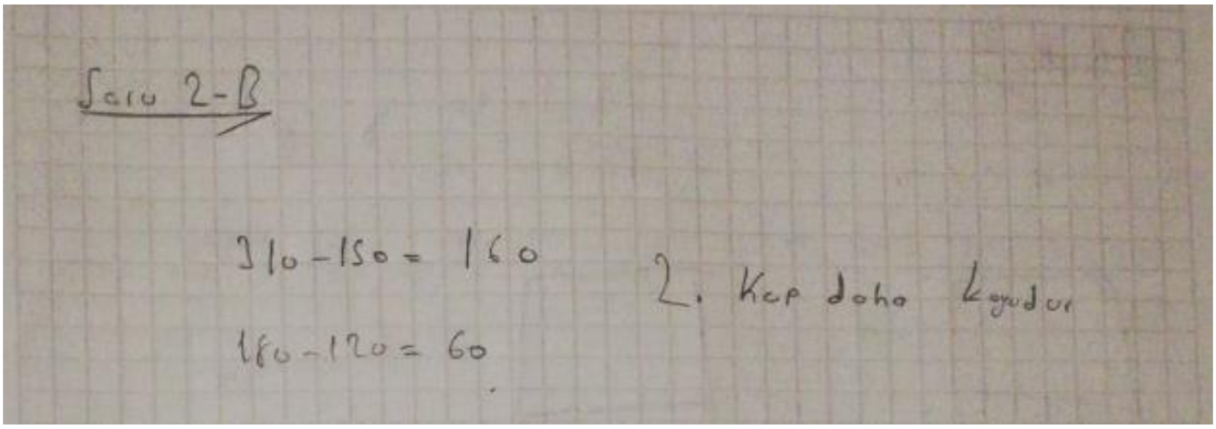
To decide the intensity, S10 first looked at the additive differences between the amounts and primarily focused on how the colors would balance themselves. In the process, he realized that the blue amount is twice as much as the yellow amount for the second container, whereas the case is not the same for the first container. He could not provide multiplicative reasoning as there was not an integer multiplier between the amounts he compared. When I asked him to explain the case in the first container, he responded that the color was not one times more than the other. He said that it was just 60 ml more. All in all, not being able to recognize the non-integer multiplier led him to compare the amounts of colors additively. Similarly, S11 applied a non-proportional strategy in problem 2B. The following speech demonstrates her reasoning.

> S11: Let us think that as 2.5 times. Now we will do 180 divided by 120. It becomes 1.5. In addition, I will do it that way for one second. If we subtract 150 from 310, it becomes 160. If we subtract 120 from 180, it becomes 40. [she made a mistake]. It is something like. If we consider the difference between them is 4k, it says 310 for the first and 150 for the second. I do the first container first. Actually, I made the operations, but I do not understand how to make a thing. Because the difference between 310 and 150 is 60, and the difference between 180 and 120 is 40. But if we divide, for the first, it becomes 2.06 repeating decimal, and for the other, it becomes 1.5.

S11 could not decide if she should use an additive or multiplicative strategy. She stated that she struggled when she encountered decimals. Therefore, after obtaining a decimal, she became prone to think additively. However, towards the end, she said she was not sure how to solve it. Another student who used a non-proportional strategy in problem 2B was S12. Figure 4.3 illustrates S12's strategy for problem 2B.

Figure 4.3

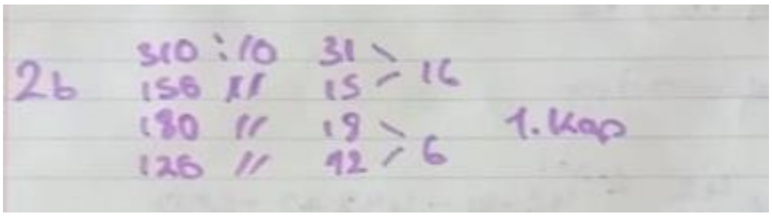
S12's strategy to the problem 2B



As demonstrated, S12 compared the quantities additively rather than multiplicatively and thereby made a decision based on a non-proportional strategy. Similar to the strategy of S4, S13 also employed a non-proportional strategy. She first divided the amounts of colors by 10 to simplify them, and then she compared them additively to decide which container had a darker color. Figure 4.4 demonstrates her strategy on problem 2B.

Figure 4.4

S13's strategy to the problem 2B



S14 was another seventh-grade student who utilized a non-proportional strategy in solving problem 2B.

S14: I divided 310 by 5. It becomes 62. I divided 10 by 5. It becomes 30. I divided 180 by 3. It becomes 60. I divided 120 by 3. It becomes 40. The first one [is darker]. Based on the result, I would determine the darkness, for instance, by adding 30 and 60 in the first container and 40 and 50 in the second container. The first container would be 92, and the second container would be 90. Then I would understand that the second container is darker.

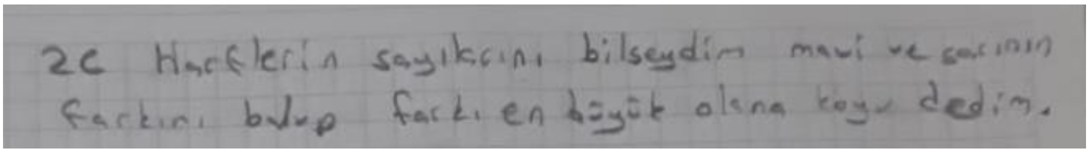
At first, S14 simplified the amounts of colors to fit them into the containers. However, when I asked her what she would do to decide the darkness of the colors if she did not use the technology, she provided a non-proportional strategy. More clearly, she combined the different amounts of colors and thought that the more the total amount, the darker the color. It is clear that S14, like other students who followed a similar path, did not have trouble fitting the colors into the containers. However, in her decision related to the darkness of the colors, she used additive thinking.

Problem 2C was another problem that students used a non-proportional strategy. In this problem, three of the sixth graders and three of the seventh graders used non-proportional strategies. Among the sixth graders, S1, S4, and S5 used a non-proportional strategy in this problem. The following speech demonstrates S1's solution, and Figure 4.5 illustrates the solution of S4 to problem 2C.

S1: If A is equal to 1, then B should be 2. The same thing should be applied to C and D as well. For instance, if C is equal to 3, then D should be equal to 4.

Figure 4.5

S4's strategy to the problem 2C



As seen from S1's response, she assigned values to the unknowns and thought that the difference between the amounts should be the same. Therefore, her response was categorized as a non-proportional strategy. Likewise, S4 employed a non-proportional strategy. Because she stated that if she knew the numbers, she would be able to calculate the difference between the amounts of blue and yellow colors and would decide that the greater the difference between the colors, the darker the color. Her solution demonstrates a non-proportional strategy. Similarly, S5 used a non-proportional strategy in his answer. The following speech was part of our interview with him.

> S5: Since the numbers are not known, we will determine them. For instance, we say 20 for the blue and 15 for the yellow [For the first container]. In the second container, we say 15 for the blue and 20 for the yellow, for example. Since the numbers are not known, we determine a number that continues [could be added up by] with 5 or 1. And based on that, we change the place. In the end, one needs to be darker. Therefore, one needs to be darker, and one needs to be lighter.

The above speech shows that S5 tried to fill the containers so that each could have the same amount of colors. His response was regarded as non-proportional because it was based on a random calculation. He thought that the amounts could only be increased by 1 or 5 because, in simulation, the buttons were filling the container either by 1 ml or 5 ml. However, not being able to recognize the intermediate values between these numbers was not the main cause why his strategy was regarded as non-proportional. It was non-proportional because his response lacked meaning as he could not provide sufficient explanation for his operations and could not support his arguments.

In problem 2C, there were also seventh graders who applied non-proportional strategies in their solutions. These students were S8, S12, and S13. The following speeches were part of our interviews with S8, S12, and S13.

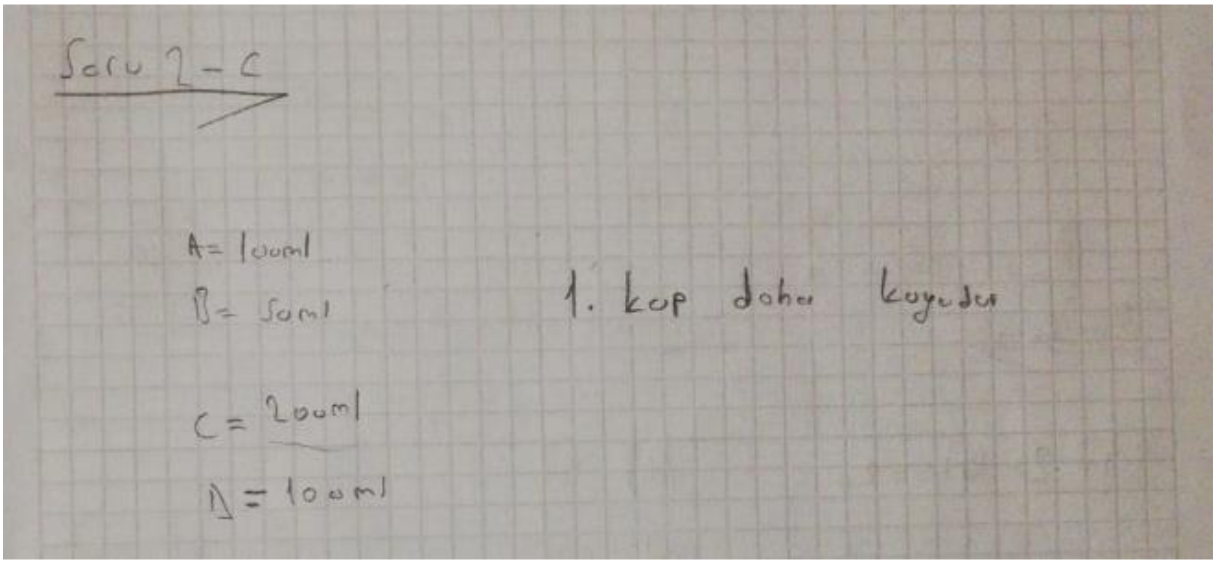
> S8: For instance, this time, let A be 10, B is 5, then I can add 12 to C and again 7 to D. When I do that when I add that… But again, different answers will obtain it depends on the number.

As seen from S8's response, she randomly assigned numeric values to the unknowns. However, she did not provide any explanation for why she performed these operations as she did not exactly know what to do when encountered with letters. Therefore, her response was considered a non-proportional strategy. S12 also used a non-proportional strategy in his solution. The following speech and Figure 4.6 demonstrate his reasoning.

> S12: Let us say that they add 100 ml for A and 50 ml for B and 200 ml for C, and 100 ml for D. It asks again the darker one. Can we go back to the previous question? I guess the second container may not be darker. According to the numbers that I give.. the first container is darker since the difference between them is 50. Any number can be given, I think if there are no options.

Figure 4.6

S12's strategy to the problem 2C



Lastly, S13 was another seventh-grader who used a non-proportional strategy in problem 2C. The following speech and Figure 4.7 illustrate her solution.

S13: Since I did not know the values, I will make a guess.

R: Will you make a guess.

S13: On second thought, I do not think so.

R: Can you do operations with values?

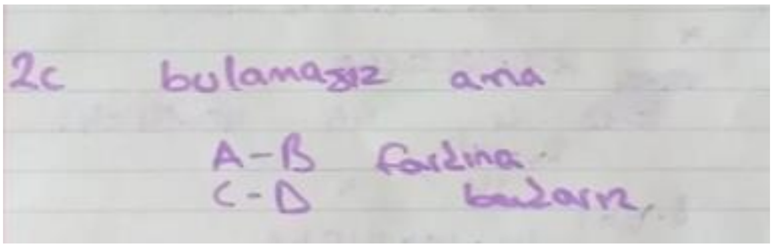
S13: No, there is nothing that we know.

R: What if we know?

S13: I look for the difference between them.

Figure 4.7

S13's strategy to the problem 2C



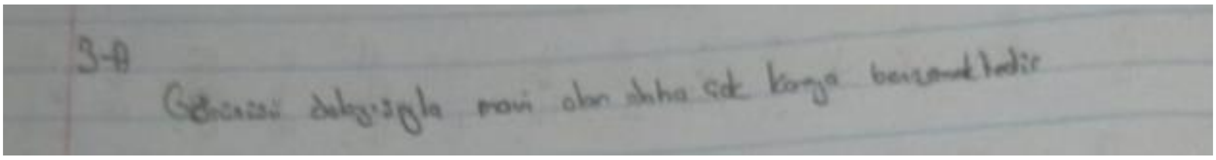
When we reviewed these three solutions, we noticed that all students (i.e., S8, S12, and S13) stated that the solution would change depending on the numbers we would place in letters. All of them tried to give random numbers and made their calculations based on these numbers. Therefore, their methods were regarded as a non-proportional strategy.

As in problem 1 and problem 2, students used non-proportional strategies in problem 3 as well. Five of the sixth graders and five of the seventh graders used non-proportional strategies in problem 3. In problem 3A, S5 and S6 used non-proportional strategies. The following speech shows S5's response to problem 3A, and Figure 4.8 illustrates S6's strategy.

> S5: The blue one is more like a square. Because there is not much difference between them [pointing to the first rectangle]. There is not also much difference between 1 and 3. However, here [pointing to the first], when we add 2 to 2, we can get 4, but when we add 1 to 1, it becomes 2. It does not become 3. I think the one that is most likely to be a square is the blue one.

Figure 4.8

S6's strategy to the problem 3A



As seen from students' responses, S5 focused on the additive differences between the side lengths, and S6 focused on the appearance of the shape to determine if it was similar to a square. Like these two sixth graders, four seventh-graders also applied non-proportional strategies in this problem. The following two speeches were taken from our interviews with S8, and S12, respectively.

> S8: It is like..there can be differences in appearance. For instance, this one more [its sides] longer. More looks like a rectangle. But this, at first glance, seems like a rectangle, but without numbers, it both looks like a rectangle and a square. This and this [adjacent sides] are similar to each other.
> R: If you would like, we can enlarge the figures.
> S8: But this time, the difference between them becomes smaller. The blue is more looks like a square.

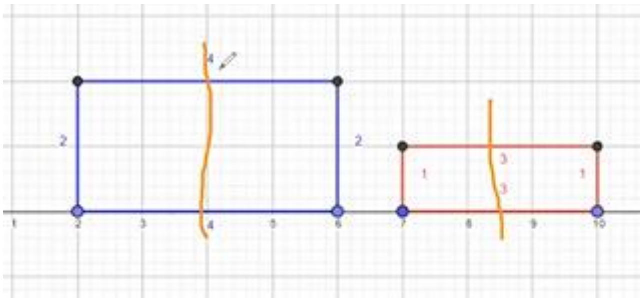
> S12: 20 units and 40 units. There are 20 units difference between them. (he wanted to enlarge the rectangle so that it became 20 by 40 and wanted to do the same thing for the other rectangle). As far as I know, the sides of the square are equal. As for me, they are both similar to a rectangle.

As seen, S8 attended to the appearances of the rectangles to determine if they were similar to a square, and S12 compared the side lengths additively while deciding which one was more similar to a square. Like, these two students, S13 and S14, employed a non-proportional strategy in solving problem 3A. The following speech was part of our interview with S13, and Figure 4.9 displays her solution to problem 3A.

> S13: When I divide this by 2, it will be exactly square, but the second does not. Therefore I say the first.

Figure 4.9

S13's solution to the problem 3A



The above method demonstrates that S13 divided the rectangles right into the middle. Doing so led her to realize that the first rectangle involves two squares. Yet the second rectangle does not. Therefore, she decided that the first rectangle was more similar to a square. Since her strategy did not manifest multiplicative reasoning, and it was based on a random technique that was not applicable and useful to solve the problem, her response was considered a non-proportional strategy. Likewise S13, S14 also used an interesting way to solve this problem. The following speech was taken from our discussion with S14, and Figure 4.10 displays her solution to the same problem.

> S14: I made 20 units 2 and 40 units 4. [She shrunk the rectangle]. Then, I did the same thing for the other rectangle. Then I associated the red one with the square.

R: Why do you think so? What is your reasoning? Do not hesitate to respond. It is not important whether you give a correct or incorrect answer. I give value to your reasoning.

S14: The square blocks are equally based on my result.

R: Do you mean it is the case for the red?

S14: Yes.

R: Does not it the case for the blue?

S14: No.

R: What is the case in the blue?

S14: In blue, it is a rectangle.

R: Well, can you show me which one is the red rectangle?

[She pointed to the red one]

R: Okay, I get it. Now, since the blue and red merged, the red one does not appear. Actually, the red one is this. [She shrunk the blue rectangle to separate the rectangles from each other and made her see that her vision is not correct].

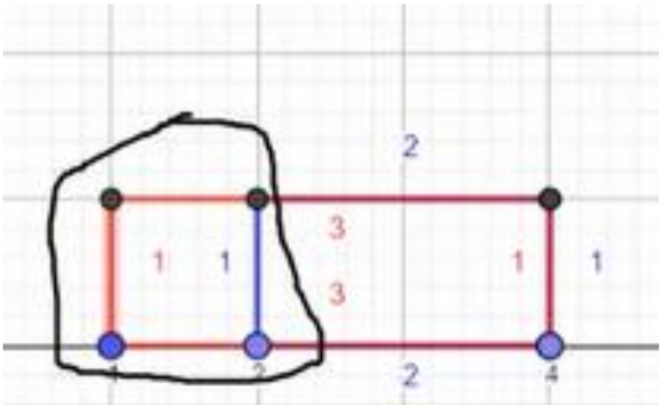
S14: I again said red.

R: Can you say your reasoning again?

S14: When I do the same operations, the red one is more similar to the square.

Figure 4.10

S14's strategy to the problem 3A



As displayed in Figure 4.10, as a result of the overlapping of two rectangles and having two quadrilaterals within a rectangle, S14 thought that she formed two figures, one of which was a square and the other was a rectangle. She stated that the left one, displayed in a closed curve in Figure 4.10, was more similar to a square. However, it did not represent any of the rectangles given in the problem. I realized that, and I asked her to move the slider so that she could separate the rectangles and see more clearly which one represented the blue and which one represented the red. Even after separation, she insisted on her response and was quite hesitant to share how she thought.

Like, problem 3A, in problem 3B, some of the sixth and seventh graders used non-proportional strategies. Among the sixth graders, S1, S5, S6, and S7 used a non-proportional strategy in their solutions. The following speeches were from our interviews with these sixth graders.

> S1: There is no multiplicative relationship between them. However, for instance, if we compare the units, in the first one there are 8 units, and in the second one, there are 9 units. Actually, the first one more looks like a square because the more close the sides are to each other, the more it looks like a square. Therefore, I would say the first one [blue one].

> S5: I guess the one that is most likely to be a square is the red one. The same thing [happens] again. I added the length of the short sides. But this time, since the total length is more than the long side, I also added the long side. For instance, when I combined the length of the short sides is become 12, and when the long sides are added up, it becomes 16. The difference is 4. [He meant that the difference between 16 and 12 is 4]. Here [pointing to the first rectangle, blue one], I added the [length of the] short sides, it becomes 8. Then I added 6 to 6. It becomes 12. I guess I made it wrong. [He thought that he made something wrong since the difference he obtained was again 4]. If

we add the short sides, it becomes 12, the difference between 12 and 8 is 4 [He added the short side lengths and then subtracted from the long side to find the difference in the second rectangle], and the difference between 6 and 8 is 2 [in the first rectangle]. Then, the blue one more looks like a square.

S6: This time, based on its appearance, the red one more looks like a square.

S7: Teacher, again it is blue. Because the difference between them is 8. And the closer the sides, the more they look like a square.

As seen from the above responses, students used a variety of techniques, all of which corresponded to non-proportional reasoning. For instance, S1 could not find a multiplicative relationship between the side lengths and thought that additive reasoning could be applied if there was no integer multiplier between the quantities being compared. S5 followed an interesting way. He tried to add the lengths of the short sides and compared the total length with that of the long side. He also added the shorts sides to each other and did the same thing for the long sides. Then, he compared the differences between the numbers he obtained for each addition. All in all, he used additive reasoning throughout the process. S6, on the other hand, made a judgment based on the appearances of the rectangles. Lastly, S7 focused on the differences between the side lengths.

Like sixth graders, seventh graders also applied non-proportional strategies in problem 3B. Among the seventh graders, S8, S9, S10, S12, S13, and S14 applied a non-proportional strategy in this problem. The following speech was part of our interview with S8.

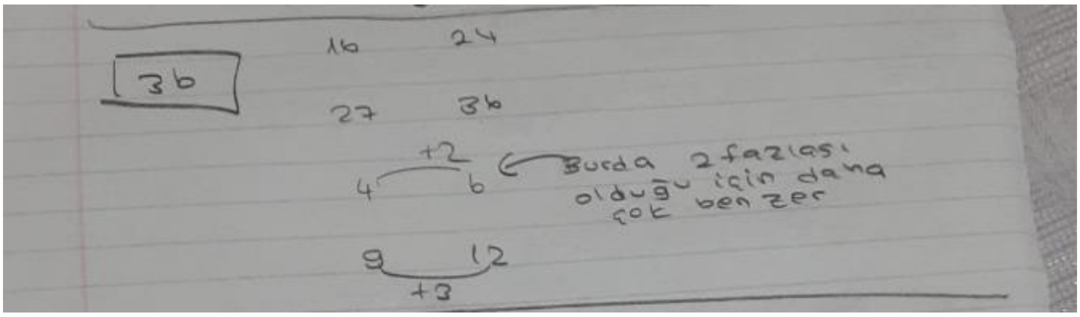
S8: I subtract 16 from 24 to find the ratio/unit/how many difference between them. There is an 8 difference between 16 and 24. But this time for red, this difference is 9. This time again, I will say the blue one, but it becomes unreasonable. The multiples [she meant the sides] of blue is a multiple of 8. The answer becomes 8 by 12. It seems possible for now. 27, 36.. hmm… They are multiple of 9. The difference between them is 4. For red, it is 3 [she was supposed to find the difference between the simplified versions as 1 after simplifying 27 and 36 by 9], which is less than this time. But again, if we look at from appearance, as to me the red one is exactly a square. It's not obvious that it's a rectangle. For the blue, since the difference between them is more, it more looks like a square. But the difference between them for red is less. Therefore, I say the red one is similar to the square.

The above speech shows that S8 was familiar with the term ratio. However, she used it interchangeably with the term difference. She simplified the rectangles and tried to compare the adjacent side lengths additively. Put differently, she simplified the first rectangle's sides by 2 and obtained 8 and 12 cm for its short and long sides. However, while simplifying the side lengths of the second rectangle by 9, she made a mistake and obtained 1 and 4 instead of 3 and 4. Therefore, she found the difference between the side lengths of the first rectangle as 4 and the difference between the side lengths of the second rectangle as 3. Her simplification led her to find the correct solution by chance. Because the blue one actually was more like a square. All in all, throughout the process, she actually focused on the additive differences between the side lengths. Therefore, her strategy was regarded as non-proportional. Similar to S8, S9 also provided a non-proportional strategy to problem 3B. The following response demonstrates his reasoning on this problem.

S9: [by referring to 16 and 24] Since they are both multiples of 4, I simplify [them] by 4. They become 4 and 6. [by referring to 27 and 36]. The other two are multiples of 3. I simplify [them] by 3. They become 9 and 13. Now, when it is a square when one is added to 0, then the other should appear. 4 and 6. When we add 2 to 4, we obtain 6. For the other, when we add 3 to 9, we obtain 12. Since it is closer to 0, the first rectangle [blue] is closer to square.

Figure 4.11

S9's strategy to the problem 3B



As seen above response and Figure 4.11, S9 simplified amounts to make numbers smaller and manageable. She used an additive comparison to decide which one was more similar to a square. Therefore, her response was considered a non-proportional strategy. Likewise, S10 based his reasoning on additive differences to decide whether a rectangle was similar to a square. The following speech reveals his thinking.

> S10: Teacher, the red.
>
> R: Why?
>
> S10: Teacher, because there is less difference between their sides.
>
> R: What is the difference between them?
>
> S10: Teacher, when we subtract 4.5 from 6, we are left with 1.5. I mean, the difference between the short and long sides is 1.5 in the red rectangle. The difference between the sides is 2 for the blue one. In a square, when we subtract [the sides], the ones that are zero, the ones that are closest to zero, is more similar to the square.
>
> R: Well, what do we see when we look at the differences between 16 and 24, 27 and 36?
>
> S10: One of the differences is 8, and one of the differences is 9. The one with having the difference of 8 is the smallest. Is it the blue one [pointing to the rectangles whose dimensions were 16 and 24] teacher?

R: Yes.

S10: Then the teacher, this time, is blue.

As the above response demonstrated, S10 started to work on the shrunken versions of the rectangles displayed on the GeoGebra screen. Then, I remarked that the given rectangles were shrunken versions of the rectangles whose dimensions were given in the problem. I asked him to consider the original rectangles that I presented in the problem. Regardless of having been shrunken or enlarged, he focused on the additive difference between the adjacent side lengths of each rectangle and stated that the closer the difference to the zero, the more the rectangle would be similar to a square. His response manifested that he used a non-proportional strategy as he attended to the additive differences. In a similar manner, S12 also used a non-proportional strategy in this problem. The following speech displays his argument for problem 3B.

> S12: Again, in this problem, the blue is more similar to the square. The reason is the same, the difference between them is small. [He meant the difference between 16 and 24 is smaller than that of 27 and 36].

As seen, S12 focused on the additive differences between the long side and the short sides of each rectangle and made a comparison among these differences in his decision. S13 used a very similar process to that of S8. The following speech was part of our discussion with her.

> S13: I will try to do it the same way. They both can be divided by 8. Therefore, I will divide it by 8. So, they become 2 and 3. This [27] can be divided by 8. If there is a common divisor, I will find it. Ha, it can be divided by 9. Then, this will be 3, this will be 4. Therefore, in the same way, I will make it 3 by 4. The short side will be 3. Now I will use my previous method to find which one is more similar to a square. I cannot divide the first square

[she meant rectangle], but there is a short and long side in the second square. But in the other [blue], it more looks like a square. Therefore I say the first one [blue].

As demonstrated above, after simplifying the side lengths, S13 made an additive comparison in her decision. However, since the difference between the side lengths for both rectangles was the same, she focused on the appearances and came to the wrong conclusion. Lastly, and in a similar manner, S14 focused on the appearances of the rectangles in her decision. The below response illustrates her thinking on problem 3B.

S14: The red one looks more square.
R: Why?
S14: When we put the numbers into place, the red one is smaller and looks like a square. The blue one is more similar to a rectangle.
R: So you decided based on their appearances?
S14: Hıhı

As seen, even though S14's answer was correct, she obtained her answer based on a guess by focusing on the appearances of the rectangles. Therefore, her response was categorized as non-proportional.

Lastly, in problem 3C, non-proportional strategies appeared again. Four of the sixth graders and two of the seventh graders used a non-proportional strategy in their solutions. Among sixth-graders, S1, S3, S5, and S7 used a non-proportional strategy in their solutions. The following speeches were part of our interviews with S1, S3, S5, and S7.

S1: Any value could be given. If we made A 1 and B is 2, then the difference between them would be 1. If we made C 3 and D 4, then the difference between them would be 1. They look the same. As a result, when they are divided into 2 parts, they become square. Therefore, I cannot say anything.

S3: Again, we can determine the lengths. Let A be 5, B be 8, C be 4, and D be 10. I think the first one looks more like a square. As I said earlier, since the numbers are more close to each other…

S5: Here, the values are not given. Now, one looks like a square, and the other looks like a rectangle. We assume A and B are 4 and 6. We assume C and D are 6 and 8. Again, if we add them, the short sides, if the addition is close to the long side, then it looks like a square. I would say so.

S7: Teacher, for instance, A is 8 and B is 24, C is 12, and D is 32. The rectangle with the smaller difference between its side would be less similar to the square. The rectangle with the larger difference between its side would be the most similar to the square.

It is clear that all students assigned numeric values to the unknowns before deciding which of the rectangles was more look like a square. Then, all of them additively compared the adjacent sides of each rectangle. Therefore, their strategies were regarded as non-proportional.

Like, sixth graders, seventh graders also employed a non-proportional strategy in this problem. The following speeches were part of our interviews with S8 and S12, respectively.

S8: For instance, here [bigger rectangle] it looks like a square but here [smaller rectang], it does not.

S12: I really do not know how to solve this type of problem. As we did earlier, we say a random number. Then we will find the differences. The fewer the

> differences. I mean, the square has equal sides. If all of them are the same, then the difference will be zero. The closest to the zero the similar to the square.

As seen, S8 looked directly at the appearances of the rectangle while deciding which one was more similar to a square. Her response was based on a guess. Therefore, it was categorized as a non-proportional strategy. S12, at first, was confused about how to solve the problem since there were not any numbers. Then, he decided that he would give a random number and additively compare the side lengths. All in all, these two students employed non-proportional strategies in their solutions.

With regard to the role of numerical structure, there was diversity among the number of non-proportional strategies applied to each problem context. Table 4.3 demonstrates the number of non-proportional responses provided to each context and number structure.

Table 4.3

The Number of Non-Proportional Strategies Provided to Each Problem

		Number of non-proportional strategies			
Problems	Context	Total	Option A	Option B	Option C
1	Enlargement	8	1	2	5
2	Mixture	24	6	12	6
3	Similarity	22	6	10	6
		54	13	24	17

As Table 4.3 clearly demonstrated, option B had the highest frequency of non-proportional strategies (24 responses out of 54 responses), especially for the second (12 responses out of 54 responses) and the third (10 responses out of 54 responses) proportional problems. On the other hand, the ratio of responses given to the B option of the first problem to the total number of responses was 2 out of 54. Furthermore, among the problem's numerical structures, option A had the lowest frequency (12 responses out of 54 responses) of non-proportional strategies. For the first problem, this ratio was 1 out of 54. For the second problem, this ratio was 6 out of 54, and for the third problem, this ratio was 6 out of 54. Moreover, as Table 4.3 demonstrated, mixture and similarity context elicited more non-proportional strategies than enlargement context.

4.2.1.2 Early Ratio Strategy

Early ratio strategies involve an iteration of paired quantities or building up the corresponding measures additively and occasionally resorting to the help of models or tables in the solution process (Petit et al., 2020). Table 4.4 demonstrates the students who applied the early ratio strategy in their solutions. Findings revealed that three of the sixth graders and one of the seventh graders resorted to this strategy only in the first problem.

Table 4.4

Problems Solved by Early Ratio Strategy

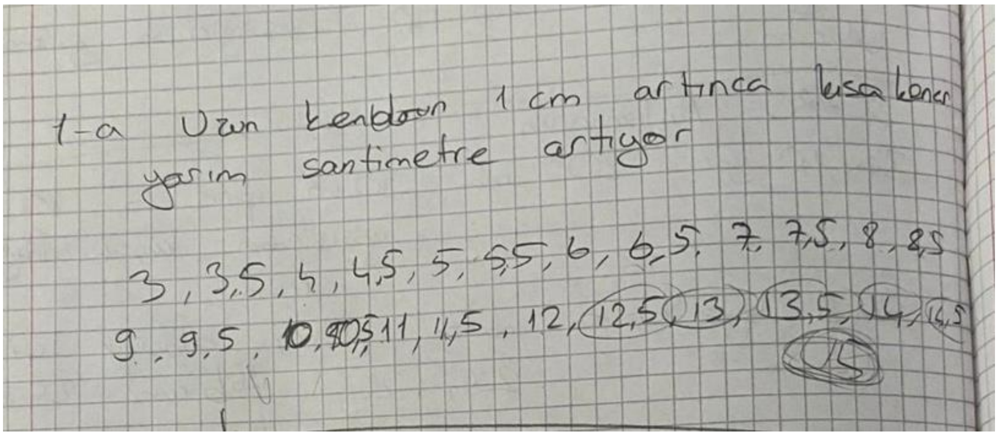
Problem name	Sixth grade	Seventh grade
1A	S2	
1B	S2, S4, S6	S12
1C	S2	
2A		
2B		
2C		
3A		
3B		
3C		

As seen from Table 4.4, only S2 provided evidence of a build-up strategy in her solution to problem 1A. The following speech and Figure 4.11 demonstrate S2's solution to problem 1A.

> S2: When the long side is increased by 1 cm, the short side is increased by half cm. In my mind, I added half cm to the 3. I continued to the 15. I found the long side as 23 cm. I found the long side as 25 cm and the short side as 15 cm. I increased 3 by half up to 15 and wrote down these numbers on paper. I found 25. From 3 to 15, there are 25 halves. This will be the long side's length.

Figure 4.12

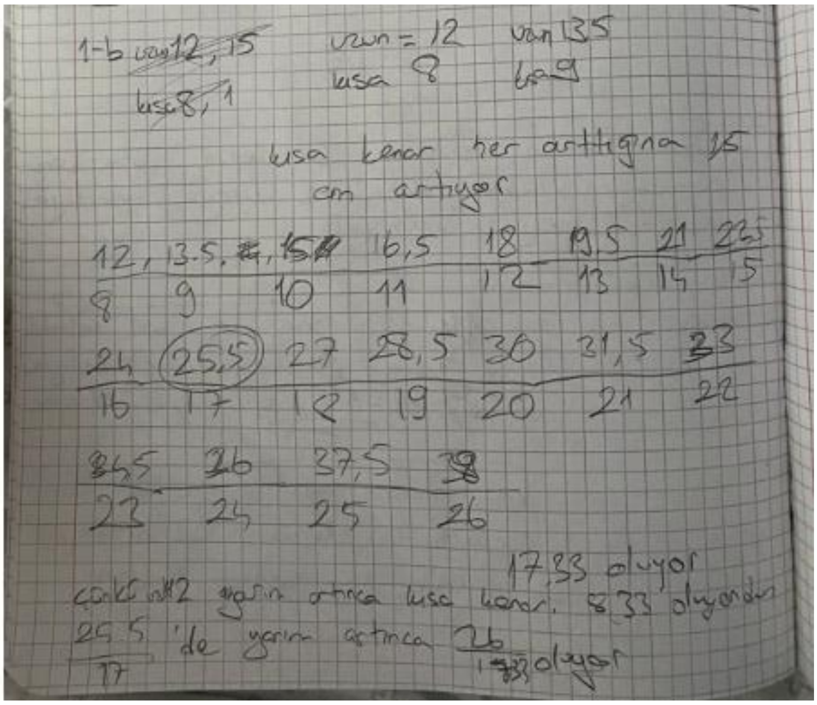
S2's strategy to the problem 1A



As seen above, S2 built up the quantities additively, which is evidence of an early ratio strategy. On the other hand, problem 1B evoked more students to use the early ratio strategies. Three sixth-graders and one seventh-grader applied an early ratio strategy in this problem. Figure 4.13 illustrates S2's solution strategy on problem 1B.

Figure 4.13

S2’s strategy to the problem 1B



S4’s strategy was also considered to be an early ratio. The following speech was part of our interview with S4 and demonstrates her reasoning.

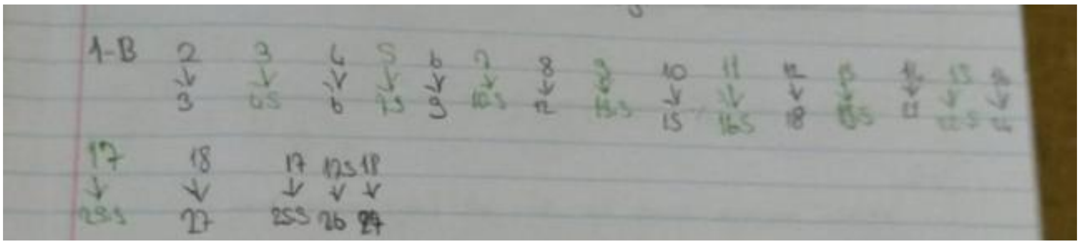
> S4: Teacher, when we make 10, it [the long side] becomes 5 more [than the short side]. When we make it 9, it becomes 4.5 more. When we make it 8, it becomes 4 more. Teacher, for each time, we lose half of it…[after some time and a discussion]… Teacher, I would like to add 5.5 to 11. It becomes 16.5. It is a long side. Then I want to add 6 to 12. Now I will add 6.5 to 13, which becomes 19.5. Then I will add 7 to 14, which becomes 21. 15 and 7.5, 22.5. 16 and 8, 24. 17 and 8.5, 25.5. 18 and 9, 27. Teacher, then, the number corresponding to 26 is 17.5.

As seen above, this student spent much time finding the differences between the sides as she increased them spontaneously. Soon, she realized that each time the difference would be a half more than the previous difference. She built up the composite unit after she obtained the missing short side length that corresponded to the long side of the enlarged rectangle. Therefore, her solution was considered an early ratio strategy.

Another sixth-grader who applied an early ratio strategy in problem 1B was S6. Figure 4.14 displays her solution to this problem.

Figure 4.14

S6's strategy to the problem 1B

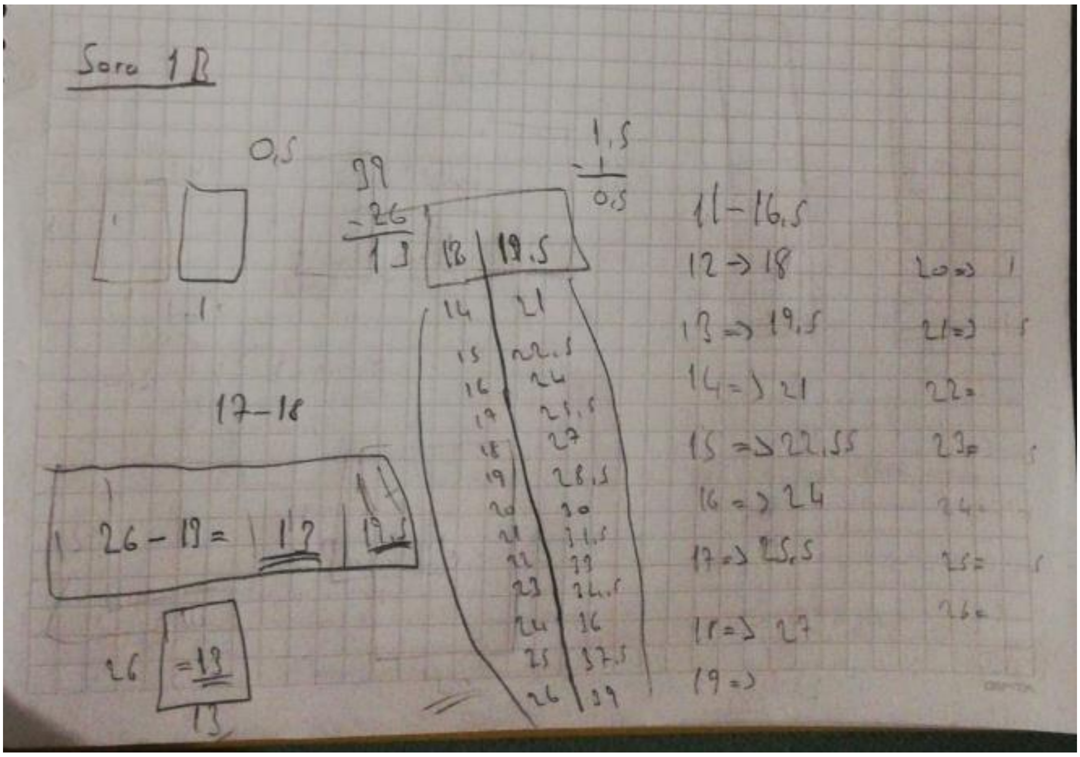


The above response clearly manifests that the student formed a table and built on the quantities until she found out the length of the missing short side. Therefore, her response was regarded as an early ratio strategy.

Another student who provided an early ratio strategy was S12. Figure 4.15 represents S12's solution strategy on the same problem.

Figure 4.15

S12's strategy to the problem 1B



As seen from the above response, it is clear that S12 iterated composite units on a table to find the missing side length of the enlarged rectangle.

All in all, the early ratio strategy was observed only in the first problem having an enlargement context with a missing value structure. Moreover, it is noticeable that students mostly applied early ratio with the presence of non-integer ratio. Table 4.5 illustrates the number of early ratio strategies provided to each problem.

Table 4.5

The Number of Early Ratio Strategies Provided to Each Problem

			Number of early ratio strategies		
Problems	Context	Total	Option A	Option B	Option C
1	Enlargement	6	1	4	1
2	Mixture	0	0	0	0
3	Similarity	0	0	0	0
		6	1	4	1

Clearly, there were only six strategies that corresponded to an early ratio strategy in total. Four of which were observed in option B. Moreover, as Table 4.5 demonstrates, the early ratio strategy was not employed by students in problem 2 and problem 3 as a single strategy.

4.2.1.3 Proportional Strategy

According to Petit et al. (2020), proportional strategies involve a wide range of techniques to solve proportional problems. These strategies encompass applying cross-product algorithm (cross multiplication strategy), comparing ratios to decide their equivalence (equivalent fraction strategy), focusing on between or within relationships (factor of change strategy), determining the unit rate (unit rate strategy), drawing a linear graph on the coordinate passing through the origin, and accurately form a ratio referent to make decisions (Petit et al., 2020). Table 4.6 represents the problems in which students only used the proportional strategy in this study.

Table 4.6

Problems Solved by Proportional Strategy

Problem name	Sixth grade	Seventh grade
1A	S1, S3, S4, S5, S7	S9, S10, S11, S12
1B	S3	S10, S11, S13
1C	S1, S4, S5, S7	S10, S11, S13
2A	S3, S5, S7	S9, S10, S11, S14
2B		S9
2C	S3	S9, S10
3A	S2, S3, S7	S9, S10
3B	S2, S3, S4	S11
3C	S2, S4	S9

As seen from the table, in problem 1, five sixth-graders and five seventh-graders applied a proportional strategy. In problem 1A, five sixth-graders, namely S1, S3, S4, S5, and S7, applied a proportional strategy. The responses of the sixth graders were as the following.

> S1: The short side is half of the long side. Therefore, the long side is twice the short side. If one of the sides is equal to 15, we can find the other side by multiplying that length by 2. So, we multiply 15 by 2, and we get 30 cm. *(Proportional factor of change strategy)*

Figure 4.16

S3's strategy to problem 1A (proportional factor of change strategy)

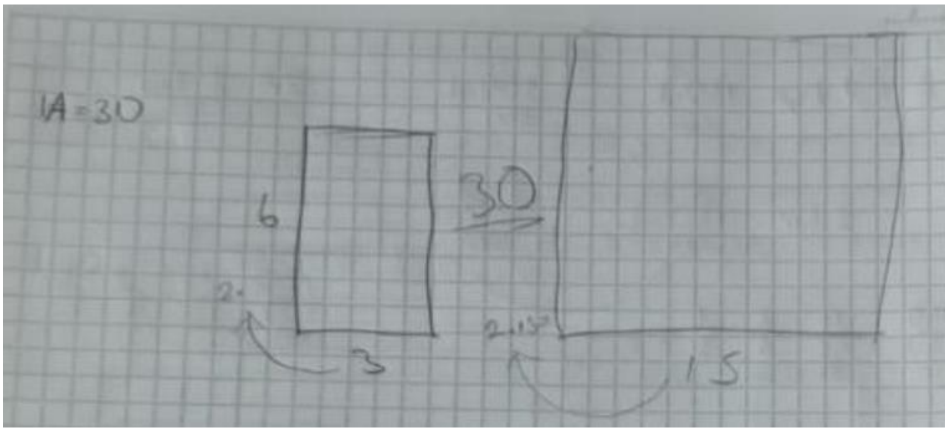
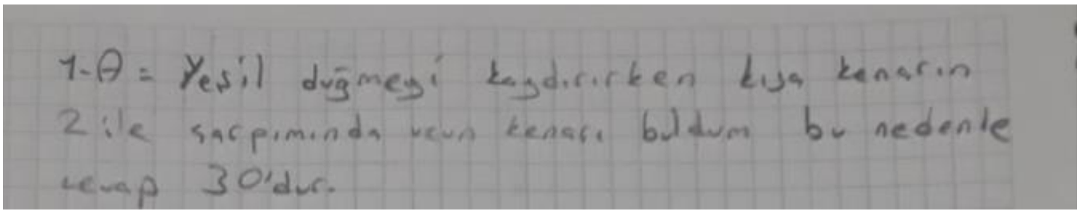


Figure 4.17

S4's strategy to problem 1A (proportional factor of change strategy)



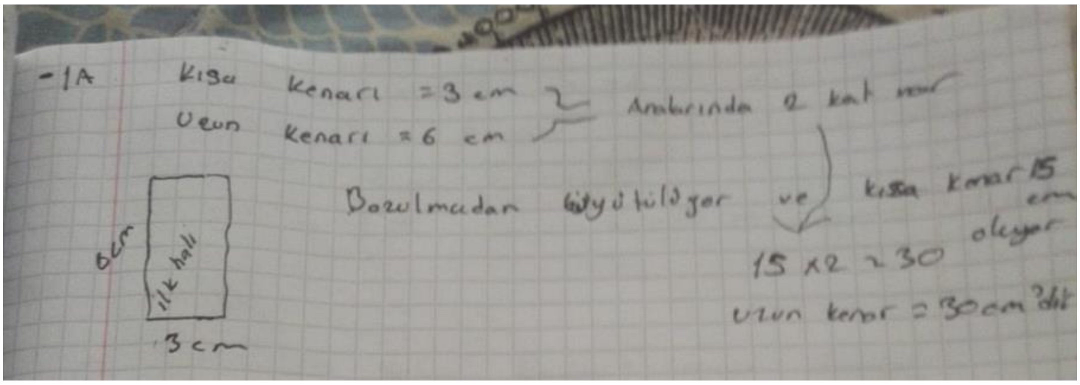
S5: If the short side is equal to 15 cm, then there should be an increase in the long side based on that amount. The short side is half of the long side, I think. *(Proportional factor of change method)*

R: What will be the long side if the short side is equal to 15 cm?

S5: 30

Figure 4.18

S7's strategy to problem 1A (proportional factor of change strategy)



As seen from the sixth graders' responses, all of them used a factor of change method, which is a proportional strategy. They found the ratio between the side lengths and applied that ratio to the other ratio to find the missing side length.

Likewise, sixth graders, seventh graders also used a proportional strategy in problem 1A. These seventh graders were S9, S10, S11, and S12. The following responses illustrate their thinking.

> S9: The long side is half of the short side. *(Proportional factor of change strategy)*
>
> R: Well, what will be the length of the short side when the long side is 15 cm?
>
> S9: It becomes 30.

As seen above, S9's strategy was a proportional factor of change strategy since the focus was on the multiplicative relationship between the long and the short side lengths. Likewise S9, S10 used a proportional strategy as well. The following transcript demonstrates S10's solution to problem 1A.

> S10: Teacher, we will do direct proportion. If the short side is 3 and becomes 15, the increment is 5 times. Then with the same ratio, the long side should

be increased. That is, if we multiply 6 by 5, it becomes 30. *(Proportional factor of change strategy)*

S10 realized a direct proportional situation and multiplicatively related to the length of the short sides of each rectangle. He found the ratio between the short side lengths, applied that ratio to the long sides of each rectangle to find out the missing long side. His strategy was also called a factor of change since he attended to the multiplicative relationship between the length of the sides. S11, as well, used a proportional strategy. However, her strategy was based on the cross-product rule.

S11: We will establish a ratio, actually. When it is 3, it will be 6, then it is 15. What should it be? Something like that. It says when the long side is 6. Let us decrease the long side to 6. We said that when it is 15, it will be x. We will establish a direct ratio. If we multiply by 3.. let us look at… *(Proportional cross-product strategy)*
R: What did you find the answer to that?
S11: 45, teacher.
R: How did you do?
S11 When it is 3, it will be 6. When it is 15, it will be x. Then I establish a proportion. Since both of them increased, I did in this way. This increased by 3. Therefore I multiplied 15 by 3.
R: Well, does 3 increase 3 times? [to be 6]
S11: Aa teacher 2 times. I am very careless in these things. It becomes 30, teacher.

At first, S11 articulated that she would establish a ratio to find the missing side length. Then, in the process, she made a mistake. However, her strategy corresponded to proportional since she used a cross-multiplication algorithm in her solution. S12 also used a proportional strategy in solving problem 1A. His strategy is given below:

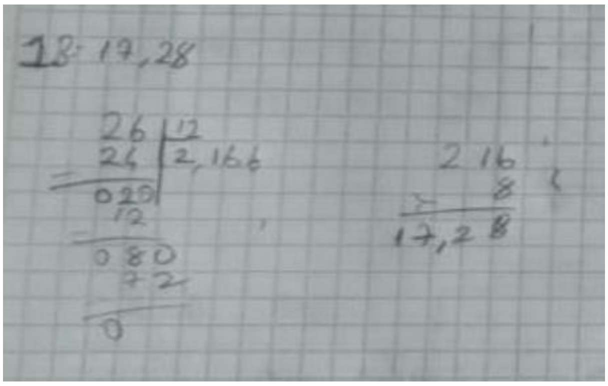
S12: If the long side is twice as long as the short side, to find the long side, we should multiply the short side by 2. *(Proportional factor of change strategy)*

As some of the students, S12 also used the factor of change method since he focused on the multiplicative relationship between the long and the short side and then applied the scalar factor to the other ratio to find the missing side length, which was also considered as a proportional strategy.

In problem 1B, one-sixth grader and three seventh graders applied a proportional strategy. The following response displays S3's solution to problem 1B.

Figure 4.19

S3's strategy to problem 1B (proportional factor of change strategy)



The above strategy demonstrates that S3 established a multiplicative relationship between the long sides of the rectangles, found their ratio, and used that ratio to find the missing short side length of the enlarged rectangle. As S3, seventh-graders applied a proportional strategy in problem 1B as well. These students were S10, S11, and S13. The following response demonstrates S10's reasoning on problem 1B.

S10: For each 1 unit increase in here [short side], the other increase 1.5 [units].

R: Well, what will be the length of the short side when the short side is 26?

S10: Teacher, 12 is increased by 14 to 26. We will divide 14 by 1.5 and add 8 to the result.

R: What do you find when you divide 14 by 1.5?

S10: Teacher, 9.5

[After that point, he added 9.5 to 8 and found 17.5]

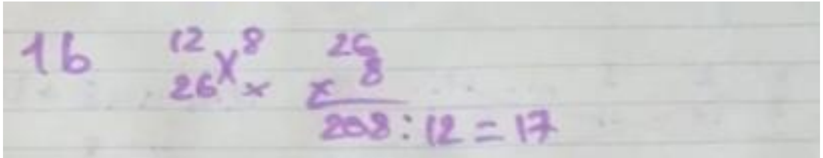
At first, S10 found the unit rate. In other words, he found the unit amount of change on one side with respect to the other side. Then, he reasoned that if each one-unit increase on the short side corresponds to a 1.5 units increase on the long side, then 14 units on the long side should correspond to the 14/1.5 increase on the other side. His strategy was based on the application of the cross-multiplication method and was regarded as proportional. Similarly, S11, as well, used a proportional strategy in her solution. The following response illustrates her reasoning on problem 1B.

S11: Again, we will apply the same logic. When the length is 12, the short is 8. We said that when the long is 26, the short is *x*. This is again direct proportion. It does not become 16, which is twice as long as 8. Then we will do cross multiplication. *(Proportional cross-product strategy)*

S11's solution indicates that she applied the cross-product rule to find the missing side. Therefore, her strategy was considered proportional. Similarly, S13 used the cross-product method in her solution to find the missing side length. Figure 4.20 displays her strategy.

Figure 4.20

S11's strategy to problem 1B (proportional cross-product strategy)



Like problems 1A and 1B, the proportional strategy was used by students in problem 1C as well. In problem 1C, four sixth-graders and three seventh graders applied a proportional strategy. The following answers demonstrate sixth graders' solutions to problem 1C.

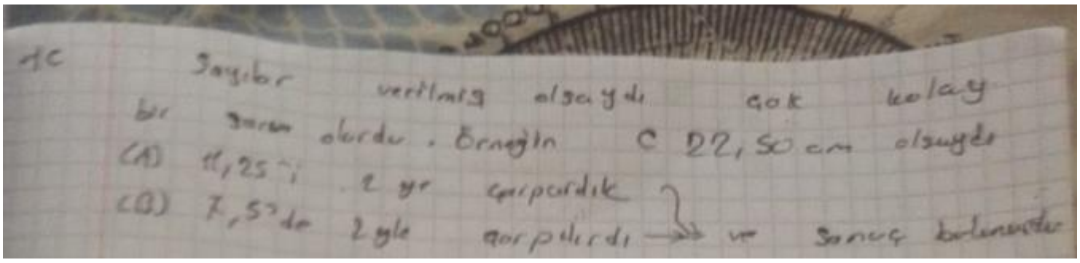
> S1: If A is equal to 2, then B should be 1. If C is equal to 3, then the below side should be half of the C. It should be 2,5. *(Proportional factor of change strategy)*

> S4: If we say A is 2, then when it is enlarged, it becomes 4. If we say B is 1, then it becomes 2. Because it [gets larger] at the same time with A. *(Proportional factor of change strategy)*

> S5: Let A be 4 and B is 2. When we enlarged it, the long side would be B cm, and we would not know the length of the short side. Since we did not know to what extent we enlarged it, we enlarged a particular amount. I multiplied 4 by 5, I completed it to 20. When I multiplied 4 by 5, I multiplied the short side 2 by 5, it became 10. That is, A becomes C, the unknown amount becomes 10 cm. *(Proportional factor of change strategy)*

Figure 4.21

S7's strategy to problem 1C (proportional factor of change strategy)



As seen above, even though S1 made an operational mistake, she actually compared the amounts multiplicatively. In other words, her response demonstrated a proportional factor of change strategy. S4 similarly used multiplicative reasoning and established a multiplicative relationship between the amounts she compared. S5 also used a proportional strategy since he kept the ratio the same while enlarging the dimensions of the rectangle that he determined. Lastly, S7 used a proportional strategy as he recognized the multiplicative relationship between the long sides of the rectangles and applied this ratio to find the missing side length of the enlarged rectangle. Despite the fact that all these students could not provide an algebraic expression that could manifest a multiplicative relationship between the numbers that letters could represent, they were able to acknowledge the multiplicative nature between the amounts they compared. Therefore, their strategy was considered a proportional strategy.

Similar to sixth graders, seventh-grade students also employed a proportional strategy in problem 1C. Below, the responses that were given by seventh graders were provided.

> S10: I give them their numerical value in the alphabet. The place of the A in the alphabet is the first. For the C, it is third, and for the B, it is second. Since C is three times greater than A, and B is 2, it [the value of D] becomes 6. *(Proportional factor of change strategy)*

S11: We say for A, it is B, and for C, it is x. Again since there is no number, we cannot….

R: Well, what would you do if you had numbers?

S11: If there was a number, again, I would do the same operation. I would construct a ratio and proportion. I would say for A, it is B, and for C, it is x. *(Proportional cross-product strategy)*

S13: We cannot find it since we do not know their values. I will read the question again. I can use the previous method. For A, it is B. For C, what will it be? C times B divided by A. *(Proportional cross-product strategy)*

The above responses manifested that some of the seventh graders (i.e., S11 and S13) were able to construct an algebraic expression that reflected the multiplicative nature of the situation. Also, one of the students concentrated on the alphabetical rule while assigning numerical values to the unknown. Yet, he was able to construct a multiplicative relationship between the values, and his response was categorized as a proportional strategy.

As problem 1, problem 2 invoked sixth and seventh-grade students to apply a proportional strategy. In this problem, four sixth-graders and four seventh-graders employed a proportional strategy. Among the sixth graders, S3, S5, and S7 used a proportional strategy in their solutions. The following speeches demonstrate their responses to problem 2A.

S3: There are 150 ml blue color and 300 ml yellow color [in the first container]. There is 2 as much…She [Elif, the girl in the question] puts 2 as much as blue. In the second container, there are 160 ml blue colors and 480 yellow colors. That is 2 times 160 means 320. It means that the second container has lighter color since there are 3 times….Because there is a note there saying that yellow makes the color lighter. Therefore, the first container has a darker color. *(Proportional equivalent fraction strategy)*

As seen above, S3's response reflects an equivalent fraction strategy as he compared the two ratios that he obtained for each container. He decided that the larger the multiplicative difference between the yellow and the blue amounts, the lighter the color. In a similar manner, S5 used a proportional equivalent fraction strategy by taking half as a reference point. The following speech reveals his reasoning on problem 2A.

S5: The second container requires a more yellow color. I perceive the color in the second container as lighter since the amount of yellow in the second container is more than that of the first container.
R: But, the amount of blue in the second container is also more than the first container. Does not it balance the amount of yellow?
S5: It does not because it is half of it.
R: Who is half of who?
S5: In the first container, the amount of blue is half of the amount of yellow. In the second container, the amount of blue is half of the amount of yellow.
R: Are you sure that the second is the half?
S5: Ha, it does not. In the first, the blue is half of the yellow, but in the second, the yellow is more [He realized that the amount of yellow in the second container is more than twice the amount of the blue]. But in any way, the second container has more. *(Proportional equivalent fraction strategy)*

Clearly, S5 realized that in the first container, the amount of yellow color is twice as much as that of blue color. However, in the second container, this ratio is not the same. At first, he made an operational mistake and stated that both of the ratios were equal. Then, I asked him if he was sure of his answer. He realized that the amount of yellow is more than twice the amount of blue in the second container. Therefore, his solution was regarded as a proportional equivalent fraction strategy as he compared the two ratios. In addition to S3 and S5, S7 also used a proportional method in solving problem 2A. The below speech demonstrates his reasoning.

> S7: Teacher, there is 2 times difference between them. [He meant that one amount is twice as much as the other amount]. To fit them into the container, we simplify 150 [blue] ml to 25 ml. Then for 300 [yellow], we simplify and make it 100 first, and again, to fit it into the container, we simplify it to make 50. Teacher, blue becomes 25 ml, and yellow becomes 50 ml. The quantities are the same teacher. [He referred to the ratio being remained for both cases]. I mean, teacher 300 is twice as much as 150. For the second container teacher, we simplify 160 [blue] by 2. It becomes 80. Again we simplify 80 and make it 40. Teacher, we simply 480 [yellow] by 2 and make it 240. We can again simplify 240 by 2 and make it 120. Or we can directly simplify 480 by 4 and make it 120. Then we divide 120 by 2 and make it 60 and divide 40 by 2 and make it 20.
>
> R: Well, I wonder, did you choose the numbers that you used to make simplification randomly?
>
> S7: No teacher, I chose them if they could be divided by [the amount of color] without a remainder. I stopped [when I obtained] 25 and 50 because 300 is twice as much as 150.
>
> R: I get it. Do you want to preserve the two times relationship?
>
> S7: Yes, teacher.
>
> R: Would it be okay we do not preserve this relationship?
>
> S7: Teacher, [in that case] the darkness would change.
>
> R: Well, what is your final decision? Which container is darker?
>
> S7: Teacher, the left container looks darker.

At first, I thought he randomly simplified amounts to fit them into the containers as the containers could hold 100 ml amount of color at maximum. Therefore, I asked him if he selected numbers purposefully or not. Then, he explained that he tried to keep the ratio of 300/150 the same so that the darkness of the green color would not change. To elaborate on his answer more, I asked him that what if we would not preserve the relationship that he established. As a response, he stated that the darkness would change when we changed the [multiplicative] relationship. Then, I realized that even though this student did not state the ratio term explicitly, he was aware that the amounts should be blended based on a multiplicative relationship so that the darkness of the color would not change.

Seventh-graders also used proportional strategies in solving problem 2A. The following four transcripts demonstrate our separate discussions with four seventh-graders on problem 2A.

> S9: I give both of them 5. 5 blue in the first container, 5 blue in the second container. In the first container, there will be 10 yellow. In the other container, there will be 15 yellow. If the blue makes [the color] darker, since the first container has less yellow, the first container becomes darker. In the second container, there is more yellow. It becomes lighter. *(Proportional equivalent fraction strategy)*

It is evident from the above response that S9 simplified the amounts and equalized the amount of blue color in both containers so that she could compare the amount of yellow color. This case is similar to finding the common denominator of the two fractions and comparing the numerators. Therefore, it was considered to be a proportional strategy. The following three strategies demonstrate S10's, S11's, and S14's solutions to problem 2A.

> S10: 15 and 30. If we look at the note, it says that blue makes the color darker. They are multiple of each other. If blue is 150 and yellow is 300, since yellow makes the color lighter, the first container has lighter green. In the second container, 3 times of each other, I mean, there is three times difference

between them. In the first, there is two times difference. Again, it is lighter green. But there are three times difference between them, the green will be darker in comparison to the other. *(Proportional factor of change strategy)*

S11: At first, let us find the thing between them. For 150, it is 300. For 160, it is 480. One second I need to do an operation here. Now, 150, 300… There are two multiplicative relationships between them. Let us say 2k. But.. it is not the case for 480. One second I am going to do its operation as well. There are three multiplicative [relationships] between them. The second container will be much lighter. *(Proportional factor of change strategy)*

S14: I directly found this way. When I divide 160 by 480… It becomes 3. In the first container, when I divide 300 by 150, it becomes 2. Therefore I think that the second is darker. *(Proportional equivalent fraction strategy)*

As seen above, the strategies of S10 and S11 to problem 2A were very similar. Both students compared the amounts multiplicatively by focusing on the between relationships. S14 started her solution by simplifying the amounts of colors. After spending time on GeoGebra, she realized that she should have compared the amounts multiplicatively. In other words, she realized that the ratios of the amounts should have been compared and that the greater the ratio, the darker the color would be.

In problem 2B, only S9 applied a proportional strategy. The following speech was taken from our interview with S9.

S9: We can equalize them at 3. If there are 3 ml yellow colors in the first container, there will be 6.2 [ml] blue colors. In the second container, if there are 3 ml yellow color, then there will be 4.5 ml blue color. Since there are more blue amounts in the first container, it will be darker. *(Proportional equivalent fraction strategy)*

The above response represents the equivalent fraction strategy, which again corresponds to the proportional strategy. In an attempt to determine the darkness of the colors, S9 wanted to equalize the amounts of the yellow color in both containers so that she could observe and compare the amounts of blue color in the containers.

In problem 2C, one sixth-grader, S3, and two seventh-graders, namely S9 and S10, applied a proportional strategy. The following responses demonstrate these students' strategies on problem 2C.

> S3: I give values [to the letters]. Let A be 100, B be 50, C is 180, and D be 60. Again, I simplify. Since we consider A as 100 and B as 50, I simplify them by 50. It becomes 2/1. It means that 2 ml blue and 1 ml yellow. Since I determined C as 180, and D as 60, 180/60, I simplified by 60, which becomes 3/1. 3 ml blue and 1 ml yellow. When we look at this, we see that the second container has a darker color.
>
> S9: Again, I wanted to make the blue (amounts) equal to find the others. Then the way that makes them equal is to multiply them by each other.
> R: By whom [which letter] you multiplied A?
> S9: I multiply A by C. When C is also multiplied by A, they become equal. They are both equal to A times C.
> R: What about B and D?
> S9: Since there is a ratio again, I need to multiply them again by the same number. I multiplied B by C and D by A.
> R: Well, how do we interpret? Can we make a relationship between BC and DA? How can we decide? By the way, they represent yellow, right?
> S9: Hıhı, the lesser amount will be darker.

S10: Let us say that A is bigger than B. Then… y times…. A is y times bigger than B, or B is bigger than A. If we say A is y times bigger than B, therefore C is y times bigger than D… here they become equal.

R: I get it. What about one of them becomes bigger? For instance, let us say that one of them is y and the other is x. How do we interpret?

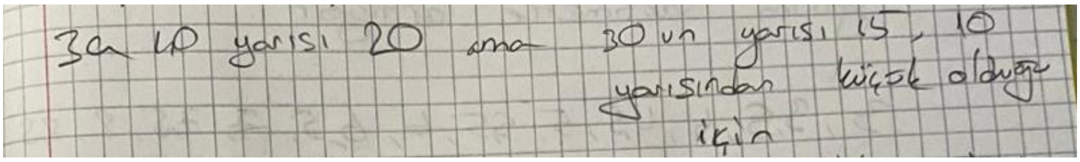
S10: Teacher, it depends on whether A is light or dark. But since A is blue, there is a bigger difference between them and makes the color darker. The first container will be darker

In problem 3A, three sixth graders (i.e., S2, S3, S7) and two seventh graders (i.e., S9, S10) used a proportional strategy. The following responses demonstrate all these students' solutions to problem 3A.

S2: 20 is half of the 40, but 10 is less than half of the 30. Therefore, the first one more looks like a square.

Figure 4.22

S2's strategy to problem 3A (proportional equivalent fraction strategy)



As seen from Figure 4.22 and her explanation, S2 realized that the ratio of 40 to 20 is smaller than that of 30 to 10, which makes the rectangle more similar to a square. Her strategy was a proportional equivalent fraction method. Similarly, S3 also used a proportional strategy. His response was given as follows:

S3: If we make the short side twice as long, then we obtain the long side. For the second, if we make the short side three times longer, then we obtain the long side. Therefore, the first side is much longer, I think.

R: Does having one side is a multiple of the other while we are comparing rectangles make the shape more rectangular or square?

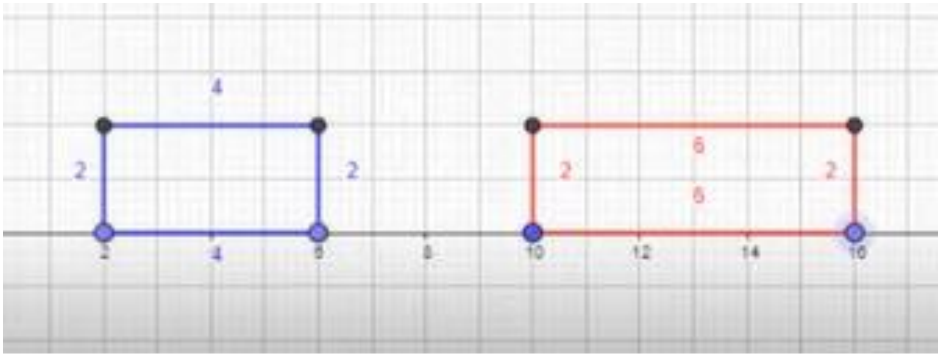
S3: For instance, 12 and 2. 6 and 2. 6 and 2 is more like a square, 12 and 2 is more like a rectangle.

R: Is there a reason for making the short sides the same?

S3: I wanted [to make] one of the multiples [sides] the same.

Figure 4.23

S3's strategy to problem 3A (proportional equivalent fraction strategy)



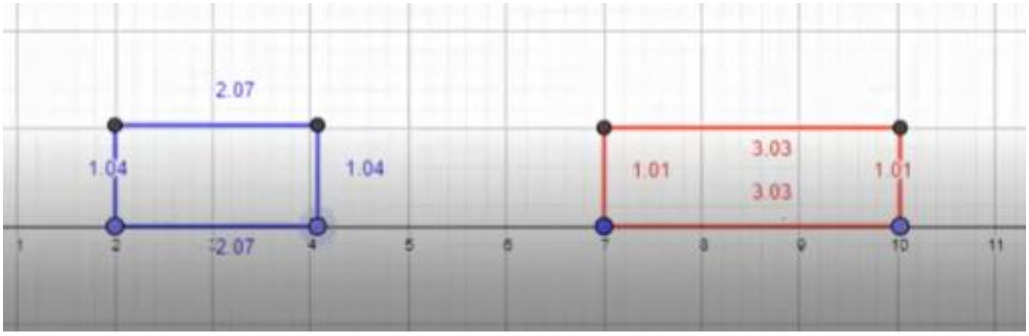
As seen from the above speech and Figure 4.23, it is clear that S3 tried to equalize the short sides of the rectangles to compare their long sides and decide which one of them was more similar to a square. This was considered as a proportional equivalent fraction strategy because the student was aware that the sides should be equalized in order to make a comparison between the other sides. The fractions he formed represented the relationship between the long and short side lengths of each rectangle. After he found these ratios, he was able to decide the rectangle that was more similar to a square by comparing the ratios he found. In a very similar manner, S7 applied a proportional strategy by constructing and comparing two ratios. The following speech and Figure 4.23 illustrate his reasoning on problem 3A.

> S7: If it were 15 by 30, it would be exactly half of it. 20 and 40 are more similar to square than 10 and 30, teacher. Because it is half of it. If we subtract 10 units from 40 and add it 20, it becomes a square. But, teacher, 10 and 30,

to make it square, we need to move all the edges. We make their lengths equal Teacher when they are in half, it looks similar to a square. If 10 and 30 became 15 and 30, here, I would say they are the same. [He meant both of the rectangles equally look like a square]. Teacher, the rectangle whose sides are 20 and 40 is more similar to square because the sides are in half of each other.

Figure 4.24

S7's strategy to problem 3A (proportional equivalent fraction strategy)



As the above speech and Figure 4.24 display, the student tried to equalize the short side lengths of each rectangle. His explanation supported his strategy as well. He noticed that in the first rectangle, the ratio of the long side to the short side was 2, whereas, in the second rectangle, this ratio was 3. Clearly, he used a proportional equivalent fraction strategy. S9 also applied a proportional strategy in deciding which of the rectangles was more similar to a square. The following speech was part of our interview with S9, and Figure 4.25 demonstrates her solution to problem 3A.

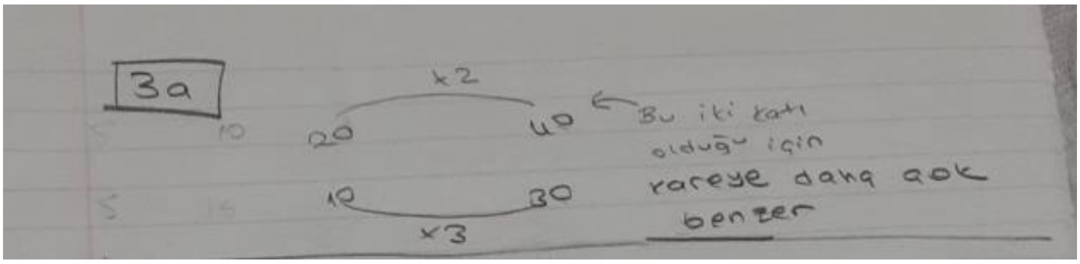
S9: I think this rectangle [she pointed to the blue one] more looks like a square.

R: Why do you think so?

S9: Because now, it is 20 by 40. I thought that it was twice as long. 30 is three times as long as 10. Since 2 times is lesser in comparison with three times, it [the blue one] is closer to the square.

Figure 4.25

S9's strategy to problem 3A (proportional equivalent fraction strategy)



As the above solutions demonstrate, S9 used a proportional strategy. She first identified the multiplicative difference between the short and the long side of each rectangle and compared these ratios. Therefore, her strategy was categorized as a proportional equivalent fraction strategy. Lastly, S10 used a proportional strategy in his solution to problem 3A. He confused the colors of the rectangles, but the solution he provided manifested proportional reasoning. The below speech illustrates his opinion on this problem.

> S10: Teacher, I think the red [He actually referred to the blue one. He confused the colors]. When we want to equalize them, the numbers we need to multiply are 2 and 4. For the above, it is 1 and 3. There is a 3 times difference between 1 and 3. I mean, the blue [He confused the colors, he actually referred to the red one] is 3 times. There is a 2 times difference between 2 and 4. I mean, the ones with the least multiple between them will be closest to the square. This is red. Cannot we say red [He confused the colors, he meant blue rectangle]?

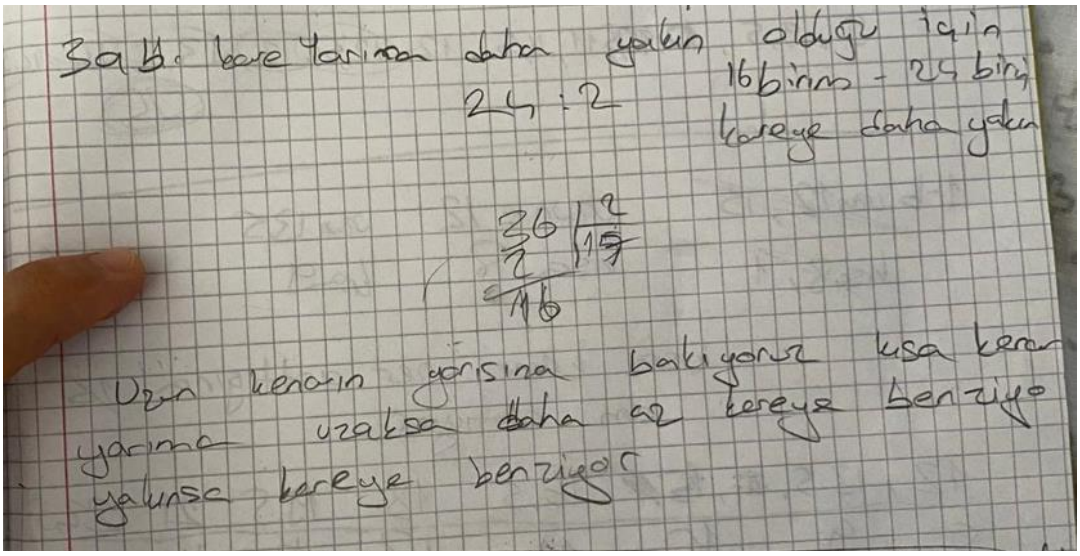
As seen above, S10 compared the short and the long side lengths of each rectangle multiplicatively. Like the other students who used a proportional strategy, he formed two ratios and compared them to decide which rectangle was more similar to a square.

In problem 3B, three sixth graders (i.e., S2, S3, S4) and a seventh-grader (i.e., S11) used a proportional strategy. The following speech demonstrates S2's solution to problem 3B.

> S2: Again, I am looking for the halves of the sides. The half of the 24 is 12, but this side is 16 units. So, it is closer to being a square. The half of the 36 is 17 [she made a mistake]. This is also close. One second… 16 is close to 12, but 27 is far from 17. Therefore, the red one, I think.

Figure 4.26

S2's strategy to problem 3B (proportional equivalent fraction strategy)

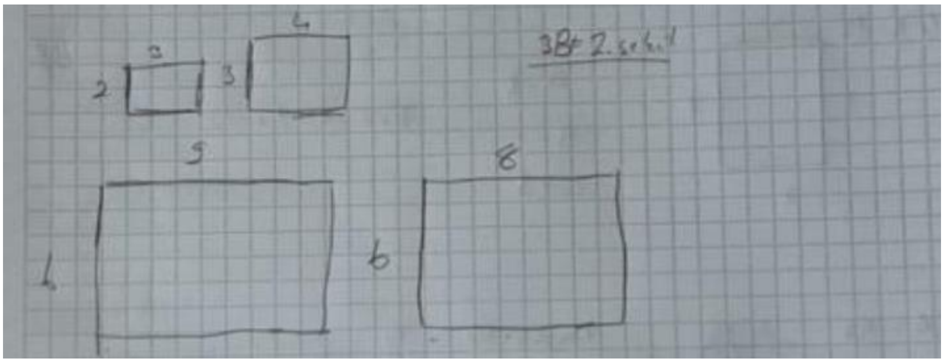


As seen from Figure 4.26 and the above speech, S2 determined the ratio between the sides of the rectangles and used that ratio to compare it with the other ratio formed by the dimensions of the second rectangle. She realized that the first ratio was greater than a half. Yet, the second ratio is greater than that of the first ratio. Since her solution involved forming two ratios and making a comparison between them, her strategy was categorized as a proportional equivalent fraction strategy. S3 followed a different way in his solution to problem 3B. However, he also applied a proportional strategy. The following response and Figure 4.27 demonstrate his strategy to problem 3B.

> S3: Let us simplify the sides. I mean, shrink them.
>
> R: What do you obtain?
>
> S3: If we divide both of them by 4, we obtain 4/6. If we divide by 2, it becomes 3/2. 'We divide the sides of the second rectangle by three and obtain 9/12, then we divide again by 3 and obtain 3/4.
>
> R: Well, which one do you think is more like a square?
>
> S3: Let use the previous strategy and equalize them. Which one? We can equalize 3 with 6. So, one of them becomes 6 by 8, and one of them becomes 6 and 9. By looking from here…6 and 8 again closest to each other. So, it looks more like a square [referring to the second rectangle].

Figure 4.27

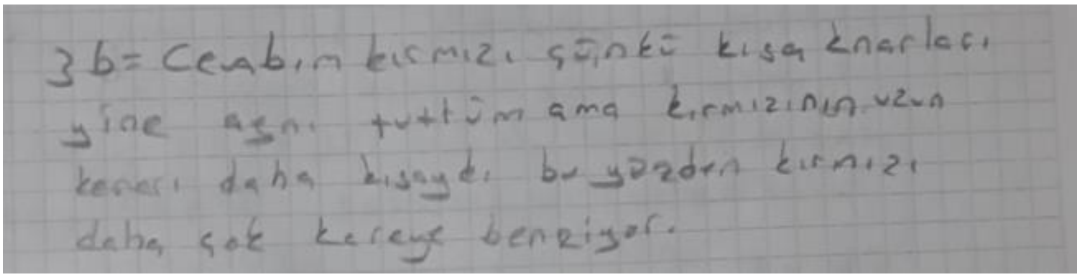
S3's strategy to problem 3B (proportional equivalent fraction strategy)



Clearly, S3 tried to equalize the short side lengths of both rectangles in order to compare the long side lengths. He divided the adjacent side lengths of each rectangle to obtain a ratio and compared these ratios. Therefore, his strategy was also considered as a proportional equivalent fraction strategy. S4 used the same strategy as well. Figure 4.28 manifests her strategy to problem 3B.

Figure 4.28

S4's strategy to problem 3B (proportional equivalent fraction strategy)



As seen above, S4 tried to equalize the short side lengths of each rectangle to be able to compare the long side lengths of them. This was also evidence of proportional equivalent fraction strategy. Lastly, a seventh-grader, namely S11, used a proportional strategy in her solution. Her solution was demonstrated in the below speech.

> S11: If I divide 24 by 16, it becomes 1.5. I mean, if we think it is 1.5 times [of the other side], if I divide 27 by 36, sorry, I will divide 36 by 27, one second… 1.3 again, 3 goes like this. 1.3.
> [She made some manipulations on the GeoGebra screen, equalized the short side lengths to 1 cm, and compared the long side lengths].
> R: Now, which one do you think is more similar to a square?
> S11: The second is more similar [to a square].

It is clear from the above response that S4 formed to ratios. One of which belonged to the first rectangle, and the other one belonged to the second. She divided the adjacent side lengths of each rectangle two obtain the ratios of the short and the long side lengths. Thereafter, she decided that the smaller multiplicative difference indicates the greater similarity to a square. Her strategy, thereby, was categorized as a proportional equivalent fraction strategy.

Lastly, in problem 3C, two sixth graders (i.e., S2, S4) and one of the seventh graders (i.e., S10) applied a proportional strategy. The following response demonstrates S2's strategy in problem 3C.

> S2: We take half of the [long side of the] rectangle. If [the ratio] is less than half, it is more like a square.

As seen from the above response, S2 applied the same reasoning that she employed in previous problems. She considered half as a referent ratio since she observed in the previous problem (i.e., problem 3A) that the short side of one of the rectangles was half of its long side. Even though to be a square, the ratio of the adjacent sides should be 1, and she did not explicitly state this fact, she manifested a multiplicative strategy in her solution. It is also noteworthy to state that it is unknown, however, which strategy she would apply if she were given the C option first. S4 was another student who applied a proportional strategy in problem 3C. The following speech reveals her thought on the problem.

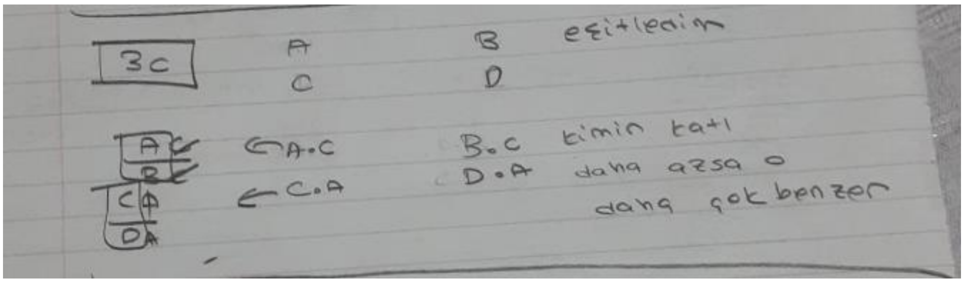
> S4: Teacher, if I knew the numbers, I would use the same reasoning. I would keep the short sides the same, depending on how much I shrunk the short side with respect to the long side. I would shrink the short side depending on and would compare the short and long sides of both. The shortest long side would more likely be a square.

Clearly, S4 suggested equalizing the short sides of each rectangle the same so that she could compare the lengths of the long side for each rectangle. Therefore, her strategy was categorized as a proportional equivalent fraction strategy. S9 followed the similar way that S4 did. The following speech and Figure 4.29 demonstrate S9's reasoning.

S9: Let me equalize the short sides. Again, they will be equal if I multiply them by each other. A times D which becomes AD. Since I multiplied A by D, I also multiply B times D. I multiply C by A. Now, when they are equal, it becomes like this, for instance, if there is a [multiplicative relationship] multiple between them, if there is a lesser multiple, it will be more similar to a square.

Figure 4.29

S9's strategy to problem 3C (proportional equivalent fraction strategy)



As seen above, S9 was able to manipulate letters and could express the situation with algebraic notation. As her response revealed, she equalized the short side lengths so that she could compare the long side lengths. She explained this situation by stating that whoever rectangle had a less multiplier [between the adjacent sides] is the one that would be more similar [to a square].

All in all, a proportional strategy was observed in all problems. Moreover, it is noticeable that students mostly applied proportional strategy with the presence of an integer ratio. Table 4.7 elucidates the number of proportional strategies provided to each problem.

Table 4.7

The Number of Proportional Strategies Provided to Each Problem

		Number of proportional strategies			
Problems	Context	Total	Option A	Option B	Option C
1	Enlargement	20	9	4	7
2	Mixture	11	7	1	3
3	Similarity	12	5	4	3
		42	20	9	13

As seen from Table 4.7, students mostly applied proportional strategies in A options, where numbers constituted an integer ratio (20 out of 42 responses). This ratio was the lowest for option B, where numbers presented in the problem constituted a non-integer ratio. Moreover, as seen from Table 4.7, enlargement contexts elicited more proportional strategies than mixture and similarity contexts.

4.2.2 Multiple Strategies

This study revealed that although students tended to use a single strategy during the problem-solving process in some proportional problems, it has been observed that they shifted their strategy in some of the problems. It is possible that during the course of problem-solving within an interactive environment, they might have developed new ideas as a result of their experience with the dynamism of GeoGebra. In this section, the problems that sixth and seventh-grade students employed multiple strategies along are touched upon.

As Petit et al. (2020) argued, students' development on OGAP Ratio and Progression is not linear, and their strategies may vary depending on the problem. This study revealed that more than half of the students shifted their strategies for some of the problems during the problem-solving process. Both sixth and seventh graders altered their strategies and progressed from a lower level to a higher one, mostly in problem

1. Table 4.8 shows the problems that the students changed their strategies from a lower level to a higher level.

Table 4.8

Multiple Strategies

Problem name	Sixth grade	Seventh grade
1A	S6	S13, S14
1B	S1, S5, S7	S9
1C		
2A		
2B		
2C		
3A	S1, S4	S11
3B		
3C		

As seen from Table 4.8, for problem 1A, one of the sixth graders and two of the seventh graders employed multiple strategies starting from a lower level and shifting from a higher level. Among sixth-graders, S6 provided a non-proportional strategy to problem 1A and then shifted her strategy to a proportional one. The following discussion was part of our interview with S6.

> S6: Teacher, when we make it 5, the long side becomes 10. When we have 15, as in the case of here, when we make the difference 5, our result…Normally, for 10, we obtain 20. For 15, we should obtain 25. *(Non-proportional strategy)*
>
> R: Did you look at the difference between the sides?
>
> S6: Yes, teacher.

R: Well, what about 5 and 10? What would be the difference? How can we be sure of that? [If the difference changes]

S6: I have never thought it.

R: Could you try other numbers?

S6: Okay. Teacher, let us try 7. Teacher, for 7, the long side becomes 14. Teacher, it is like, we do multiplication operation. For instance, when we multiply 7 by 2, it becomes 14. It is similar to that. *(Proportional strategy)*

R: Does this rule apply to the other cases?

S6: It applies for 6. I also apply for 5, teacher. It applies to 4 as well. It applies to 3.

R: What if for 15?

S6: It cannot be. Because when we multiply 15 by 2, it becomes 30.

R: Why not?

S6: I think it could be if we continue to use the same logic. For each time, we obtain [the long side] twice as much [as the short side]. I mean, when the short side is 15, the long side should be 30.

At first, she made observations of the side lengths and investigated their relationship through the slider. Even after some observation, she could not notice the multiplicative relationship and compared the sides additively. Then, I asked her to check her answer by using the slider and to see if it was the case. After observing the changes in the side lengths for a few cases, she realized that for each case, the long side would be twice as long as the short side. Then, she altered her response and used a proportional strategy.

A similar case happened when I asked S13 and S14 to use the slider. These students focused on additive differences between the sides at first, although they used a slider and changed the dimensions of the rectangle. Then I asked them to check their responses to see if this would always be the case. After spending some time observing a few cases, they changed their strategy to a proportional one since they realized that the sides should be multiplicatively related, not additively. The following discussion was part of our interview with S13.

> S13: If this is 10 by 20, then it will be 15 by 25. *(Non-proportional strategy)*
> R: How did you find your answer?
> S13: If 10 [corresponds to] 20, then I thought 15 [corresponds to] 25.
> R: Did you do any operation?
> S13: I found it mentally. I did not know which operation to perform.
> R: I get it. Well, what about other numbers?
> S13: Ha, when it is 4, it becomes 8. It is a multiplicative issue. *(Proportional strategy)*
> R: What about for 5?
> S13: When it [the short side] is 5, it [the long side] becomes 10. It is always twice as much.

The above speech clearly demonstrates that, at first, students did not know how to approach the problem. She could not provide an explanation for her first solution that was considered non-proportional. However, when I asked her to use a slider and observe the relationship for other numbers, she became aware of the multiplicative difference that existed between the side lengths. Likewise S13, S14 also changed her strategy. The following speech demonstrates her thinking on problem 1A.

> S14: I think it is 18.
> R: How did you find it?
> S14: I found the difference between them and then added those differences. *(Non-proportional strategy)*
> R: Well, does it always the case?
> S14: It may not be.
> R: For instance?
> S14: Can I change my opinion?

R: Of course.

S14: I will say 30.

R: What changed your opinion?

S14: I multiplied 15 by 2. Because there is 7.5 [She meant the short side of the rectangle]. When I multiplied 7.5 by 2, it became 15. When I multiplied 15 by 2, it became 30. *(Proportional strategy)*

R: Would it be always true? You can use the slider.

S14: Yes, it would be.

R: Are there other cases that you noticed?

S14: For instance, 8.5 times 2, it becomes 17.

The above discussion demonstrates that S14 could not notice the multiplicative relationship at first, even with the aid of a slider. However, when I asked her to observe different cases and check her response after she proposed one, she realized that the long side was always twice as much as the short side.

For problem 1B, four students, namely S1, S5, S7, and S9, used multiple strategies starting from a lower level and transitioning to a higher level. The following dialog demonstrates our interview with S1 on problem 1B.

S1: We can subtract 8 from 12 to find the difference between them. We can also subtract 4 from 26 because there is 4 cm between the sides. If we subtract 4 from 26, we get 22. So, the length of the short side will be 22 cm. *(Non-proportional strategy)*

R: Well, let us check. What about for 6?

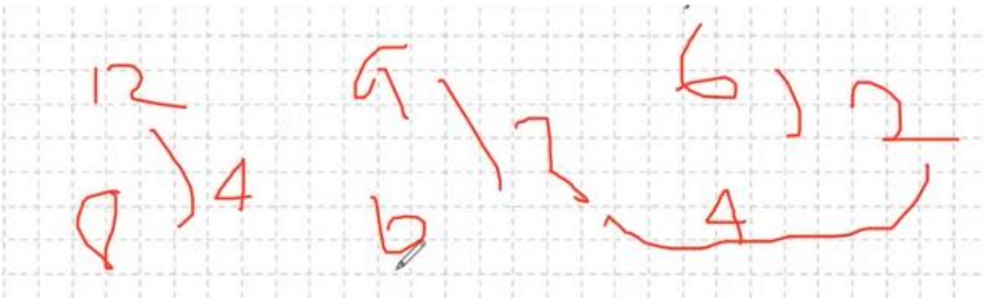
S1: Aaa. What if I use the area? But it is irrelevant. Haa. Okay. One second. The short side is two times 4, and the long side is 3 times 4. Is 26 a multiple of 4? No. Then, how should I do? 12 is three times something. 8 is two times something.

[After that point, she made an observation and realized that the rectangle's dimensions could be 6 by 4 as well. Then, she noticed that 6 by 4 rectangle had dimensions that were half of the dimensions of the original rectangle. Afterward, she made the rectangle's dimensions 6 by 9 and emphasized the additive difference between these two numbers. Then, she made a table for three dimensions that she observed with the aid of a slider and noticed a pattern. Figure 4.30 represents the table she constructed].

S1: It goes by 2 for the below and goes by 3 for the above, teacher.

Figure 4.30

S1's strategy to problem 1B (transition to early ratio strategy)



As seen from Figure 4.30, S1 made a transition from non-proportional to early ratio strategy as she engaged with interactive simulation and observed the dynamically changed sides. Like S1, S5 also shifted his strategy from a non-proportional to an early ratio in problem 1B. The following speech demonstrates his reasoning on this problem.

S5: I could not solve this problem.

R: What was your approach?

S5: At first, I tried to find depending on what it increases. For instance, when we make [the slider] 7, it becomes 10.5, 6 corresponded to 9. I thought that it increased by 2. Then, I wanted to see if it increased by 3. *(Non-proportional strategy).* It did not. When we make [the slider] 3, it becomes 4.5. Then I looked if 12 was completed to 26, it did not. Then I also looked if 8 was completed to 26. It did not also.

[After that point, I asked him to observe some cases]

S5: If that (side) is equal to 7, then the corresponded side should be increased by 3.5 based on the order.

R: What should it be if the difference between the sides is 3.5?

S5: 10.5

R: Try to observe if this really is the case by using the slider.

S5: Indeed.

R: What will be the length of the short side when the long side is 26 cm?

S5: When the short side is 10 cm, then the long side will be 15 cm. If we made the short side 11 cm, then it would be increased by 5.5 and would become 16.5. If we made the short side 12 cm, then it would be increased by 6, and it would become 18 cm. If we made it 13 cm, 6.5, if we made it 14, 7, if we made it 15, it would be increased by 7.5. *(Early ratio strategy)*

R: What if we made 16?

S5: It would be increased by 8.

R: What if we made 17?

S5: 8.5. If we made it 18, it would be increased by 9.

R: When the long side was between 25.5 and 27, then what would be the range of the short side?

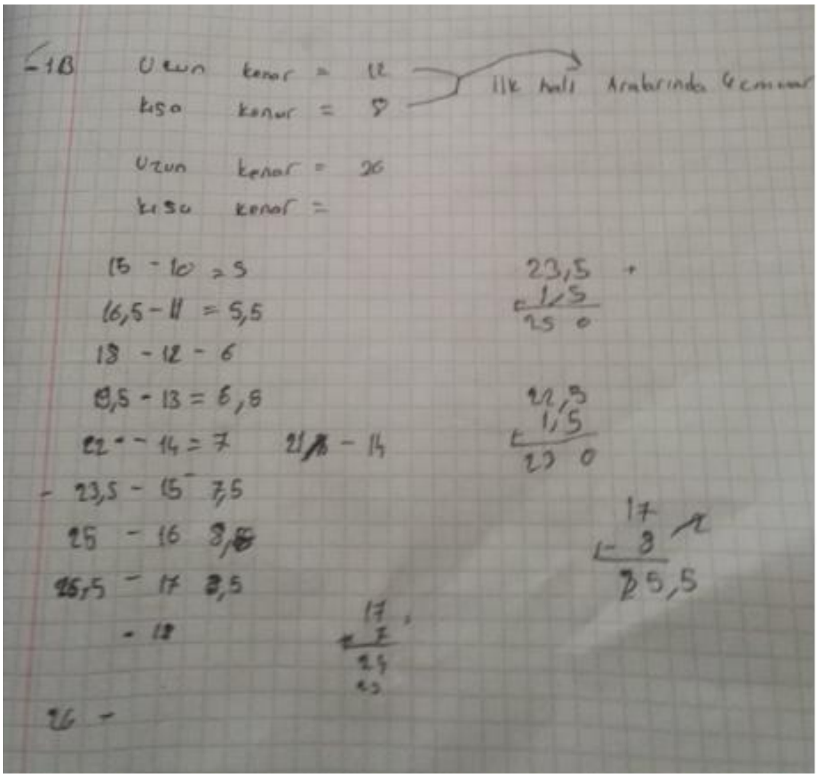
S5: It should be between 17 and 18.

As seen from above, at his first attempt, S5 could not find a solution to problem 1B. He explained that he focused on the additive difference between the side lengths. Then, I asked him to check a few cases. He made an observation and could realize a pattern. By building up the quantities and my probing questions that I stated above, he estimated that the solution would be within the range of 17 and 18. Clearly, he switched from a non-proportional to a proportional strategy. Like S5, S7 followed a similar path. The following speech and Figure 4.31 illustrate S7's reasoning on problem 1B.

> S7: Teacher, there is a pattern between numbers. Teacher, the long side is 12, and the short side is 8 cm. There is a 4 cm difference between them. *(Non-proportional strategy)*
>
> S7: The long side is always increased by 1.5, and the short side is always increased by 1. The difference between the sides is increased by 0.5 in comparison with the previous difference. *(Early ratio strategy)*

Figure 4.31

S7's strategy to problem 1B (transition to early ratio strategy)



As seen above, S7 first attended to the additive difference between the side lengths. Having observed the covariation of the sides via GeoGebra helped him to notice a pattern. He built up the quantities until he found the missing side length. Clearly, he changed his strategy from a non-proportional to a proportional one. Lastly, S9 changed her strategy in the course of solving problem 1B. The following response demonstrates her reasoning.

> S9: Here, the short side is five less than the long side. Let me check if it is the case here…[She tried to add 7.4 to 5 to check if she would obtain 11.1. but she could not.] It is not the case everywhere. *(Non-proportional strategy)*
> R: So, what should we do?
> S9: For instance, here it is 2 more than [the short side is 4 and the long side is 6]. Here I found 2.05 more [the long side is 6.15 and the short side is 4.10]. Here it is 2.10 more [the long side is 6.3, the short side is 4.2]. All of them are different.
> [She made a few trials and obtained different numbers for each case she tried]
> S9: Then, if it is 4 when it is 6, it will be x when it is 26. *(Proportional cross-product strategy)*
> R: Which number do you divide by which number?
> S9: Well, I do a direct proportion. 144 should be equal to 6x. I wrote 104. It becomes 24.
> R: Do you divide 104 by 6?
> S9: It is not 24, one second. It is a repeating decimal.
> R: What did you obtain?
> S9: 17.3 repeating

The above response reveals that S9 started to reason problem 1B by focusing on the additive differences. She used the slider to make observations, and even if I did not ask for it, she checked her result. However, she was not sure about her response. Then, instantly she changed her strategy and applied a cross-product rule.

The last problem that some of the students used multiple strategies was 3A. S1, S4, and S11 used multiple strategies in solving this problem.

S1: To be a square, the side lengths should be equal. So, I look for the lengths. To see whether they look like a square, I need to compare the differences. In the first figure, the difference is 20 units. In the other figure, the difference is 20 units, the same. *(Non-proportional strategy)*

[After that point, she struggled to decide which one of them was more similar to a square because she obtained the differences the same. Therefore, she attempted to the technology.]

S1: In the first figure, the short side is half of the long side. The long side is twice the short side. In the second figure, there is a threefold relationship [The length of the long side is three times greater than that of the short side]. So, I give my answer as the first one. The first figure looks more like a square because there is less.. something.. twofold relationship. The other one has a threefold relationship. So, it is more than the second one. *(Proportional strategy)*

As seen above speech, S1 first compared the adjacent side lengths of each rectangle additively. Having obtained the difference, the same confused her mind. Therefore, she attempted to use the slider and observe how the rectangle's dimensions were changing as she moved the slider. Afterward, she quickly realized that there is a multiplicative relationship between the long and the short sides. She reasoned that the lessor the multiplicative difference, the more the rectangle would be similar to a square. Clearly, S1 changed her strategy from a non-proportional to a proportional. S4 was another student who changed her strategy during problem solving. The following response and Figure 4.32 demonstrate her reasoning on problem 3A.

S4: Teacher, when they are shrunk, the difference between them is 2.21, for instance, the blue one. I subtracted 2.16 from 4.37, which gives 2.21. Normally, the difference between them is 20 units. However, here the difference is 2.21. *(Non-proportional strategy)*

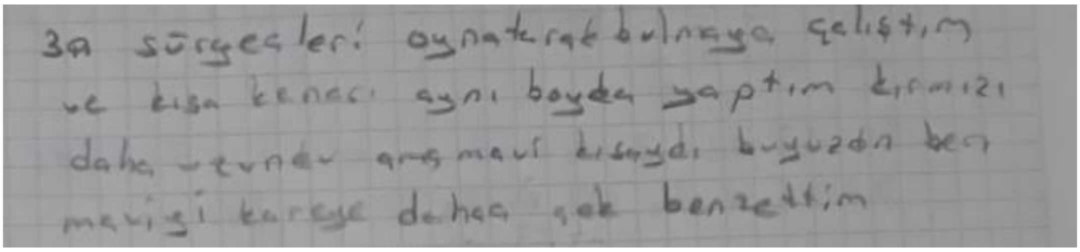
S4: The blue one is more like a square. Teacher, if we subtract 10 from 30, we obtain 20, and if we subtract 20 from 40, we obtain 20 again. One second, teacher. The red one is longer.

R: What does it tell us?

S4: The red one is longer. It more looks like a rectangle. However, the blue one looks more like a square in comparison with the red one.

Figure 4.32

S4's strategy to the problem 3A (transition to proportional strategy)



At first, S4 attended to the additive differences between the adjacent side lengths of the original rectangle and their shrunken versions. Then, she realized that the difference could change. Having seen this led her to change strategy. She thought of equalizing the short side lengths and comparing the long side lengths of each rectangle. She began with a non-proportional strategy. Yet, after engaging in an interactive simulation, she switched her method and used a proportional equivalent fraction strategy. Lastly, S11 switched her strategy from a non-proportional to a proportional one. The following responses demonstrate her solution to problem 3A.

S11: Actually, the first one seems similar [to a square]. The difference between them 2, for each of them. They are equal. If we think mathematically, they are equal, but when I look from here, the first one [blue] more looks like a square because it is bigger. It looks a little wider. *(Non-proportional strategy)*

> S11: There is a difference between the first and the second. Then the first one is more similar to a square. I tried to make the sides of the red rectangle 2 by 4 like the sides of the blue rectangle. But it was not. Then.. the first one is more similar. I am sure. Because for 1 by 3, it will be 2 by 6. But here, it will be 2 by 4. [She enlarges the red rectangle to make both of the short sides 2 cm. Then make a comparison on the basis of long sides]. The first one is more similar [to a square] then. *(Proportional strategy)*

It is clear from the above transcripts that S11 started to solve the problem with a non-proportional strategy as she compared the side lengths additively. However, having observed the changes in the sides by moving the dynamic points of the rectangles led her to switch her strategy from non-proportional to proportional one. S11 used an equivalent fraction strategy by fixing one of the sides and looking for the difference between the other sides. This strategy was considered to be proportional since it involved a multiplicative comparison of quantities in an efficient way. All in all, S11 also changed her strategy after engaging with GeoGebra. Table 4.9 may provide a clear picture of the transitions that students made during the problem solving process.

Table 4.9

Examples of Multiple Strategies

Problem 1A		Problem 1B		Problem 3A	
S6	NP to P	S1	NP to ER	S1	NP to P
S13	NP to P	S5	NP to ER	S4	NP to P
S14	NP to P	S7	NP to ER	S11	NP to P
		S9	NP to P		

*NP: Non-proportional

*ER: Early ratio

*P: Proportional

As seen above table, in total, there were 10 responses categorized as multiple strategies, three of them were given to problem 1A, four of them were given to problem 1B, and three of them were given to problem 3A. In terms of the numerical structure, it is clear that only the A and B options, in which using the slider was applicable, elicited multiple strategies. Moreover, the transition from a non-proportional strategy to an early ratio was only seen in problem 1B, involving a non-integer ratio.

CHAPTER 5

DISCUSSION AND CONCLUSION

This study aimed to offer an insight into three main issues. The first issue was to provide an overview of how sixth and seventh-graders classify proportional problems. The second issue was to examine the level of the students' strategies to the proportional problems based on the OGAP Ratio and Proportion Progression Framework. The last matter was observing the role of numerical structure on students' strategies. This section is meant for providing an overarching discussion on the aforementioned issues on the grounds of the findings of this study.

5.1 Students' Classification of Proportional Problems

According to Van Dooren et al. (2010), students blindly apply proportional methods without paying attention to the underlying mathematical structure, and the operations they have chosen are mainly based on their familiarity with a particular kind of problem. They suggested that, in order to prevent this inclination and reduce the senseless selection of operations, students could be given some time to understand problems (Van Dooren et al., 2010). They also found that students who classified problems before solving them performed better than those who solved problems first. Starting from this point, the first aim of this study was to examine how sixth and seventh-grade students classify proportional problems. Students' answers indicated that the sixth and seventh-grade students focused almost on similar characteristics of the problems in their classifications. Sixth-grade pupils focused on three characteristics while categorizing problems as similar or different. These were the nature and the context of the problems and the terms that many reminiscent of each other. Likewise, seventh-grade students focused on the same characteristics to categorize the problems. Additionally, one seventh-grader used one more criterion, which was the interrogative word of the problem. Silver (1979) obtained similar findings to this study. He asked eighth-graders to classify word problems and decide if they had a mathematical connection or similarity. He found that students mainly used four criteria in their classifications, each of which reflected a particular characteristic of the problem. These were the mathematical structure, contextual aspects, problem's form, and pseudo structure. All of which were also observed in this study. Mathematical structure, which I referred to as the nature of the problem, involved the attention to the mathematical explanation underlying the problem that best expressed the situation, such as increasing, decreasing, or enlargement of some quantity. Contextual aspect criterion, in both Silver's (1977) and my study, involved the interpretation of the problem in its contexts. By saying context, I referred to the attention to the shapes involved in problems (e.g., rectangle-rectangle) or the mathematical notion that the shape was being associated (e.g., rectangle-area). Silver (1977) regarded the attention of the quantities as both contextual details and elements reflecting the structure of the problem. Therefore, he preferred to name this category as pseudo structure. In my study, this criterion corresponded to the terms that may sound reminiscent of each other (e.g., paint-color, milliliter-unit). The last criterion reported in Silver's (1977) study was the problem's form, which I referred to as the focus on the interrogative word within the problem in my work. In both Silver's (1977) and my study, this criterion involved the attention to the question being asked (e.g., what, how, which). I should note, however, that while I was invetigating how students' classify proportional problems, I used open coding. I was not aware of Silver's (1979) results. Therefore, it was surprising to observe almost completely matching results. In another study where they asked high school and college students to categorize 76 algebra problems based on their types, Hinsley et al. (1977) similarly found that contextual aspects, such as area, mixture, triangle, were among the criteria that students used to classify problems. Similar to Hinsley's work, in his thesis, Chartoff (1976) presented students from highschool to college level with algebra word problems and asked them to judge based on their similarity. His results revealed four dimensions that students used to sort problems. Dimension one involved the attention to the ways in which the given problems could be solved, which was very similar to what Silver (1977) called the structure of the problem. Dimension two involved the attention to the contextual aspects of the problems and dimension three included making generalizations about a problem with another problem that was judged to be similar to the original problem. The last dimension encapsulated the classification on the basis of the question being asked in the problem, which was very similar to what Silver (1997) called question form, and what I called interrogative word criterion in my study.

Moreover, based on the results of this study, sixth-grade students' thoughts on the nature of the problem remained more superficial, while seventh-graders more clearly defined the mathematical situation in the problem. For instance, two sixth-graders realized that some problems involved enlarging the figures but could not mention the covariation of the side lengths. On the other hand, two seventh-graders could recognize that two of the problems involve comparing quantities, which was the property that defined these problems as the comparison type of proportional problems.

As to the focus on the contextual aspects of the problems, this study revealed that sixth-grade students who focused on the context of the problems during classification outnumbered those in the seventh grade. While there were three sixth-grade students who attended to the contextual aspects such as involving a rectangle or being related to the area, only one seventh-grader associated some problems due to their context.

Associating problems because they involve the terms reminiscent of each other was another criterion used by two grade levels. One sixth-grader and one seventh-grader used this criterion in their classification. The sixth-grade student's connection had a mathematical foundation since milliliter, and unit terms were associated with this student. On the other hand, the seventh-grade student associated two terms, paint and picture, not because they had a mathematical meaning, but they seemed reminiscent.

The last characteristic that I found a student used in classifying problems, which was only used by a seventh-grader, was the interrogative word of the problem, such as "which one." It was probably because problems 2 and 3 asked students to make a comparison. Therefore, they involved which one questions. Even though this student did not explicitly state that the problems were of comparison type, she noticed that the problems actually demanded the same thing.

What these mutually complementary results reflect might be that before attempting to solve problems, while reading them, students' attention could be attracted by almost every aspect of the problem. This aspect might be the numbers presented in the question or wording of the mathematical situation. My intention to examine how students classify problems was to see what characteristics of the problems they would attend to before solving them. Even though my aim was not to establish a relationship between students' performance on the classification and problem solving tasks, the features of the problems on which students focus may provide an insight into what they understand from the problems and further how they approach them.

5.2 Students' Strategies on Proportional Problems Supported with Interactive Simulations

Under this section, students' single and multiple strategies to proportional problems are discussed with the related literature.

5.2.1 Single Strategy

The second purpose of this study was to investigate sixth and seventh-grade students' strategies on OGAP Ratio and Proportion Progression while solving proportional problems. Based on this study's findings, it could be argued that there is a variance among responses across different grade levels based on the context of the problem. When non-proportional strategies that the sixth and seventh-grade students used in solving proportional problems were examined, it was observed for both grade levels that this type of strategy emerged mainly in the second and the third problems that had a numerical comparison structure and having a mixture and similarity contexts respectively (see Table 4.3). Put differently, in these problem contexts, students tended to apply mostly additive strategies. Especially in the third problem, within a similarity context, students could not give a sufficient explanation on how to measure a squareness of a rectangle or how to determine if a rectangle is similar to a square. Arıcan et al. (2018) obtained parallel results to that of this study. They administered a geometric similarity test to 32 preservice teachers in an attempt to investigate their strategies on similarity problems. As their results illustrated, preservice teachers could not provide a diverse type of strategies on similarity problems and counted primarily on the cross-multiplication method. Lobato et al. (2010) interpreted this situation by proposing that students' difficulties might stem from the fact that similarity, enlargement, and scaling problems do not permit partitioning or iterating composite units. My study supported these ideas for the following reasons. First, like the preservice teachers in Arıcan et al.'s (2018) study, participants of my study provide insufficient and ineffective arguments in explaining the similarity of the rectangle to a square. Second, they could not form a composite unit between the two perpendicular side lengths, as Lobato et al. (2010) suggested, and thereby, could not interpret the similarity situation in the problem. The distinctive aspect of my findings from Arıcan et al.'s (2018) is that in my study, when students struggled to construct a similarity between rectangles, they attempted to apply additive strategies, whereas preservice teachers in Arıcan et al.'s (2018) study showed a tendency to apply cross-multiplication rule. To account for this difference, it is worth mentioning that both sixth and seventh graders were not formally taught similarity concepts in school since congruence and similarity concepts are included in the eighth-grade mathematics curriculum (CCSSI, 2010; MoNE, 2018). In fact, the cross-product algorithm is not a subject to be introduced until seventh grade in the Turkish curriculum, possibly explaining the prevalence of sixth-graders non-proportional strategies on this type of problem. Therefore, it was not unusual to observe that they struggled to interpret the similarity and opted for the strategy that they presumed would work.

Nevertheless, the emergence of the proportional strategy would not be a surprise since even six to eight-year-olds would understand similarity and congruence and how they were related to ratio and proportion concepts (Van Den Brink & Streefland, 1979). The findings of my study supported this notion as well. Because, despite the trend in additive reasoning in similarity context, there were also sixth and seventh-graders who could reason multiplicatively in the same situation, which revealed that even before a formal instruction on the topic, students could produce their own intuitive strategies such as early ratio (see Table 4.4) or proportional factor of change strategy (see Table 4.6) that led them to find an answer This idea was supported by Cramer and Post (1993b) as well. Cramer and Post (1993b) found that seventh-grade students who were not familiar with the cross-product algorithm produced informal strategies such as the unit rate or factor of change, and they were appeared to be ahead of the others, which in fact signified how intuitive strategies could be helpful in proportional situations. One important suggestion from these results might be that even though students are not formally taught some mathematical concepts at a certain grade level, they are capable of producing informal strategies that may support their formal thinking on a mathematical concept.

Another problem that invoked the majority of students to use non-proportional strategies was the first problem which was a missing value proportional problem in an enlargement context. Even though it did not generate non-proportional strategies as much as the second and the third problem did, it was observed that four of the seventh-graders and the two of the sixth-graders used a non-proportional strategy in this problem. In a strict sense, however, it is not clear why more seventh-graders produced non-proportional strategies in this context than the sixth-graders. Lastly, problem 2, a numerical comparison type of proportional problem in a mixture context, was another problem in which non-proportional strategies were observed.

Six sixth-graders and six seventh-graders applied a non-proportional strategy in this problem. In fact, it was the problem that had the highest frequency of non-proportional strategies. Like problem 2, problem 3 also had a numerical comparison structure. It had a similar context and expected students to decide which of the rectangles was more similar to a square. The prevalence of the non-proportional strategy among all students in the solutions of problems 2 and 3 might be explained by the fact that problems 2 and 3 were comparison types of proportional problems, whereas problem 1 had a missing value structure. In other words, problems requiring the comparison of ratios and not asking to find out the missing value might have challenged students because it is possible that they might not know how to approach comparison type of problems, or they might have had little experience with this kind of problem. One thing should be noted here, however. In problem 2, one standard method applied by most students was to simplify the amounts of colors to fit them into the containers and then additively compare the amounts to determine which container had the darkest color. If the solution of this problem were separated into two phases, one would be making operations to pour the amounts into the containers, and the other would be deciding the darkness of the colors. Based on this reasoning, in this study, it has been seen that even though students preferred to simplify the amounts in a multiplicative manner as it should be, they reasoned additively in the second phase, which led their methods to be regarded as a non-proportional strategy. A similar situation happened in problem 3. Problem 3 asked students to find which of the rectangles was more similar to a square. Again, based on the students' answers, the solution of this problem could be divided into two parts, one of which was enlarging or shrinking the rectangles until they seemed somehow suitable to make the comparison. The other phase was deciding their similarity to a square by concentrating on the additive differences between the adjacent side lengths of each rectangle. Both in problem 2 and problem 3, students simplified the given amounts in the first phase and then applied an additive strategy in the second phase. It is possible that their fraction knowledge might have led them to simplify amounts so that they could obtain manageable numbers and easily perform operations. However, simplification of amounts could not alone be regarded as proportional or transitional. In other words, performing multiplication or division operations without connecting meaning to the situation could not be treated as a multiplicative strategy since students made their decisions based on additive reasoning.

Among the non-proportional strategies identified by Petit et al. (2020), the most frequently used non-proportional strategies in this study were attending to the additive difference between amounts and carrying out random operations that lacked meaning. The literature argues that the non-proportional additive strategy is common for students who have not formally covered proportional thinking or have little familiarity with the proportion concept. This study has observed that even students who have formally taught the proportion concept applied the non-proportional strategy in proportional problems.

When it comes to early ratio strategies, which consisted of simultaneously building up the given quantities, it has been observed that four sixth-graders and one seventh-grader applied this strategy. It is also evident from the literature that students may use informal strategies such as the building-up method in solving proportional problems (Ercole et al., 2011). Even though the early ratio strategy was considered to be at the elementary level due to not representing scalar and functional relationships (Christou & Philippou, 2002), this study revealed that middle schoolers also applied it in their solutions to proportional problems. Some researchers argued that the usage of this strategy is common among students who have not formally taught the proportion concept (Parker, 1999). Therefore, for sixth-graders, usage of this strategy seems reasonable and expected since they have not formally taught the proportion concept although they were familiar with the concepts rate and ratio. For seventh-graders, the usage of this strategy could be accounted for by the problems' numerical characteristics, which will be discussed in detail in the following chapters. Tourniaire and Pulos (1985) distinguished multiplicative strategies from building-up strategies by defending this notion even though they considered building-up strategies as a correct method for solving proportional problems. On the other hand, some researchers accepted this method as sufficient to handle missing value problems and defended that it has a multiplicative foundation (Degrande et al., 2018, 2019). When all problems were examined, it was seen that only problem 1 elicited an early ratio strategy, which could be explained albeit partially by the type of the problem. In other words, among the proportional problems, problem 1 was the only one that had the missing value structure. Having three knowns and one unknown in the question might have led students to use this strategy with and/or without consciousness. On the other hand, since problems 2 and 3 did not include a missing value and, instead, required comparison of two ratios, students might not have felt a need to employ this strategy. It was not unexpected to observe a build-up strategy in missing value problems. Like this study, Steinthorsdottir and Sriraman (2009) also observed that some of the fifth-grade students in their study used a build-up strategy in solving a missing value problem given to them. In another study, conducted with middle school students from fifth to eighth grade, Riehl and Steinthorsdottir (2014) observed evidence of a build-up strategy in students' answers to a well-known missing value problem called Mr. Tall and Mr. Short. Collectively, all these results may point that missing number structure permits students to use early-ratio strategy as the structure of the problem allows them to form and iterate composite units. This may account for the trend of early-ratio strategies in the first problem, having enlargement context. At this point, I should say that Lobato et al. (2010) regarded enlargement context difficult as well, as the context does not well-suited for construction and iteration of composite units. Even though the findings of this study is consistent with their suggestion that the similarity, enlargement, and scaling contexts are not well-suited for the formation and iteration of composite units when problem has a comparison structure, this study also manifested that when numbers presented in a missing value structure within enlargement context, this challenge has been partly eliminated.

Transitional strategies may involve the inefficient application of multiplicative relationships to the problem situation or may benefit from multiple representations that display the mathematics within the problem. In contrast, early transitional strategies mainly depend on using additive and multiplicative reasoning interchangeably (Ebby & Petit, 2017). In this study, neither early transitional nor transitional strategies have been observed.

Proportional strategies may involve the application of the cross-product algorithm, comparing ratios, focusing on the between or within relationships or identifying the unit rate, and representing the situation with a linear graph (Petit et al., 2020). In this study, concerning proportional strategy, all of the abovementioned methods were used except drawing a graph. The findings are not unusual since the coordinate system and linearity concepts are introduced in the eighth-grade mathematics curriculum (MoNE, 2018). It has been observed that six of the sixth-graders and six of the seventh-graders used a proportional strategy as a single strategy in their solutions. This study shows that for both grade levels, the problem that the students used the proportional strategy the most was the first problem, which was a missing value proportional problem in an enlargement context. On the other hand, in the second and third problems, the common strategy observed among the answers was a non-proportional strategy, which is primarily based on additive thinking (Petit et al., 2020). These findings are consistent with Singh (2000a), who revealed ninth-grade students performed better in missing value type problems than numerical comparison and qualitative comparison type proportional problems. It is essential to mention that for this study, using a proportional strategy was not considered to be better than the others but is considered to be more sophisticated since it involved understanding the multiplicative nature of the situation. Therefore, Singh's (2000a) findings were similar to that of this study.

Another thing worth mentioning regarding proportional strategies was that in comparison with the seventh-graders, sixth-graders applied informal solution techniques more, even though they used proportional strategies like seventh-graders. Put it differently, it was observed that factor of change strategy, namely within ratio method, was the common answer among sixth-graders in solving problem 1. On the other hand, cross-product strategy was observed more frequently in seventh-graders than sixth-graders in the same problem. This was an expected situation since sixth-graders were not formally taught cross-product algorithm. Nevertheless, they were able to realize the multiplicative relationship within or between the side lengths and could provide the correct answers to problem 1. It is evident that even though they did not receive formal instruction on proportionality, they could recognize the multiplicative nature of the problem, which is consistent with the findings reported in the literature. Tourniaire's (1986) findings demonstrated that this ability could be observed even in the primary grade. In her study, where she worked with third, fourth, and fifth-grade pupils, Tourniaire (1986) observed that primary students could solve basic proportional problems successfully despite not being provided particular training on the proportion subject.

Regarding proportional strategies, it has also been observed that problems 2 and 3 led students to utilize proportional equivalent fraction strategy more than problem 1. In these problems, regardless of their grade levels, students who used proportional strategies mainly attended to the multiplicative relationship between unlike quantities that formed a ratio. It was mainly because these problems were in comparison structure, and it was appropriate to form ratios to decide which of them was greater or smaller. It has also been observed that problem 1 elicited a more proportional factor of change strategy than the second and the third problems.

All in all, the diversity of students' responses might have been occurred because of their experience with proportional problems, irrespective of their grade levels. This study manifested that the fact that the seventh-graders are more familiar with proportional thinking problems than the sixth-graders does not guarantee that they will solve the proportional problems correctly. In addition to that context (e.g., enlargement, mixture, similarity), numerical structure (integer-ratio, non-integer ratio, letters), and problem type (e.g., missing value, comparison) might have affected students' strategies. Studies in the literature are consistent with the findings of this study. As Karplus et al. (1983) argued, the context was among the factors that might account for the diversity of students' strategies to proportional problems. In parallel with this study's findings, Bell et al. (1984) and Tourniaire (1986) found that problem context could impact students' answers.

5.2.2 Multiple Strategies

When multiple strategy usage was examined, it became apparent that there had been a shift from a lower level to a higher level in some of the strategies that students provided. With that being said, these strategies have only been observed in A and B options. It should be noted that students were expected to use technology only in these options where numbers either constituted integer ratio (i.e., option A) or non-integer ratio (i.e., option B). Findings have revealed that using technology may have caused students to alter their strategies as, in these versions, students had a chance to observe covariation and change their strategy if their observation led them to recognize something that they had not noticed earlier.

Problem 1, a missing value type of proportional problem within an enlargement content, elicited more strategies that started with a lower level progressed to a higher level than any other problem in this study. Four of the sixth-graders and three of the seventh-graders started solving this problem with a lower level, and as they engaged in dynamic simulation, their thinking was shaped, which ultimately resulted in an application of a more sophisticated method to solve the problem. Among ten responses categorized under the multiple strategies, seven of them reflected a transition from a non-proportional to a proportional strategy. Three of these seven responses were given to problem 1A, and three of them were given to problem 3A. Only one of them was given to problem 1B. Moreover, among the ten responses, three of them belong to problem 1B and indicated a change from a non-proportional strategy to a proportional one. This could partly be explained by the problem's type and numerical structure. Problem 1B had an enlargement context, missing value structure, and involved an integer ratio. Having a missing value structure and examination of the covariation of the side lengths through interactive simulation might have led students to notice how one quantity was changed in terms of the other. Recognition of this situation further might have invoked them to perceive the two adjacent side lengths as a compose unit, and provide them an opportunity to build up the quantities additively until they obtained the missing side length corresponded to the other side given in the problem. As I stated above, among the four strategies provided to problem 1B that were categorized as multiple strategies, three of them (i.e., S1, S5, S7) manifested a transition from a non-proportional strategy to an early ratio, whereas only one of them (i.e., S9) reflected a shift from a non-proportional strategy to a cross-product. It should also be noted that only S9 was a seventh-grader among these students. This fact could account for her cross-product strategy since sixth graders have not yet taught this algorithm. It is clear that even though technology might have influenced their responses and led them to progress to a higher level of strategies, sixth-graders preferred informal methods, whereas a seventh-grader chose the shortcut and directly applied cross-product algorithm.

5.3 The Role of Numerical Structure on Students' Strategies

The last purpose of this study was to investigate what role would numerical structure play in students' interpretations of the problems and in their strategies. This study manifested the idea that using dynamic software to investigate covarying quantities might have influenced students' way of thinking and enabled them to produce different strategies than they would be able to do without using technology since multiple strategy usage appeared during the process of working with interactive simulations. Therefore, it can be suggested that observing a simultaneous change in the quantities within a dynamic environment, especially in geometrical contexts, may enable students to realize the functional relationship between two covarying quantities. In other words, it may help students come to understand the change in one quantity with respect to another. From this point of view, under this section, I seek to discuss the students' strategy shift and how the aforementioned factors possibly promote or hinder their thinking in each problem context.

5.3.1 Enlargement Context

It is a well-known fact that several factors could be used to account for the diversity of strategies that students choose to apply in solving mathematics problems (Lawton, 1993). In the context of proportionality, these factors could be the numerical structure (Van Dooren et al., 2009), semantic type of the problem (Lamon, 1993), the type of the proportional problem like being an inverse or direct proportion (Fisher, 1988), or contextual aspects of the problem (Karplus et al., 1983; Tourniaire & Pulos, 1985). One of the problems of this study was in an enlargement context and required students to find the missing value of a rectangle that was being enlarged. The literature alleged that among multiplicative problems, the enlargement context was claimed to be more difficult than the others (Bell et al., 1984; Singh, 2000b). This idea was also supported by Lamon (1993), who asserted that enlargement problems were the most challenging kind of problem for novice sixth-graders. On the contrary, the current study revealed that enlargement was not the most challenging type for neither grade level. Notwithstanding, the numerical structure, especially the presence of a non-integer ratio, was amongst the main factors that might have added complexity to the problems, making it challenging for students to notice multiplicative relationships. This was an expected situation in line with the findings obtained from numerous studies conducted with students of different grade levels (Christou & Philippou, 2002; Van Dooren et al., 2009; Van Dooren, De Bock, & Verschaffel, 2010).

Another thing that is also worth mentioning is that the current study demonstrated that non-proportional strategies given to the enlargement context were primarily seen in option C, whereas they were observed least in option A where numbers constituted an integer ratio (see Table 4.3). This was an expected situation because, in option C, there were no numbers presented to students. Not having numbers might account for the lack of realization of the multiplicative nature within the situation since, in this case, some students seemed confused and were not sure how to proceed without numbers to operate with. Early ratio strategies given to the enlargement context were seen mostly in option B and equally observed in option A and C. In other words, the early ratio strategy was mostly observed in the problems where numbers formed non-integer ratios. The presence of technology and the shift in the numerical structure might account for this situation. Using the slider and observing the covariation might have caused students to use the early ratio strategy since it requires building up the quantities several times until the intended outcome was obtained. Another possible explanation that could explain the application of the early ratio strategy in option B was that problems involving non-integer ratios might have pushed students towards using the early ratio strategy due to the lack of an integer multiplier. Lastly, proportional strategies utilized in enlargement context were observed mostly in option A, and they were seen least in option B. One could expect that proportional strategies should have been seen least in option C since this option had the highest frequency of non-proportional answers. However, proportional strategies have been least observed in option B, where the numbers formed a non-integer ratio. This might have partly been explained by the numerical structure and the lack of an integer multiplier, which made it relatively difficult for students to notice the multiplicative relationship and increased their inclination to opt for an additive strategy. Also, not having proportional strategies in option C as much as in option B could be explained by the implementation procedure. Put differently, students have presented problems in the order of options A, B, and C. In some cases, I realized that when they struggled to understand the situation in option C, they tended to utilize the strategies that they previously adopted in other options, and these were mostly the strategies that they used in option A since in these versions, they were able to come up with an answer smoothly. The frequency of proportional responses obtained in option C, thereby, might be explained by the fact that students' disposition to quickly resolve the situation, their avoidance of putting so much effort to understand the problem situation might have led them to adopt a strategy that has been already validated. Furthermore, it is possible that the absence of numbers might have revealed their actual reasoning because there were no numbers to operate with and that there was nothing to steer them to choose a particular operation over the other.

For instance, S4, who was a sixth-grader, used a proportional strategy in problem 1A (see table 4.6). However, she was inclined to use an early ratio strategy in problem 1B, where the numbers constituted a non-integer ratio. Strategy shift across different numerical structures of the first problem could be seen in S9's solutions as well. S9, who was a seventh-grader, used a proportional strategy in option A, preferred using multiple strategies in option B, and opted for a non-proportional strategy in option C. Similarly, Christou and Philippou (2002) observed in their study that fourth and fifth-graders were inclined to switch strategy when they could not develop a method to determine the unit rate easily, especially in the presence of a non-integer ratio. In addition to that, no matter which strategy S4 employed at the beginning, having observed the dynamic aspect of the photograph through the slider and investigating the side lengths with respect to each other caused her strategy to progress to an early ratio. Moreover, her transition to proportional strategy in problem 1C might be accounted for by the fact that she assigned values to the unknowns in a way that constituted an integer ratio. Evidently, both numerical structure and technology influenced S4's solutions. In a very similar manner, Van Dooren et al. (2009) observed in their study that the presence of a non-integer ratio was the main reason that accounted for fourth, fifth, and sixth-grade students' additive methods in proportional problems. Similarly, in this study, not being able to obtain an integer multiplier brought about some of the students either to apply additive methods or to resort to using the technology and made an investigation on the side lengths.

All in all, as mentioned earlier, the variety of responses across different numerical structures provided to the first problem having an enlargement context might also have happened as a result of the experience with the technology. Because interactive simulation provided within the GeoGebra environment allowed students to observe the simultaneous changes in the sides and how they were related to each other.

5.3.2 Mixture Context

Mixture context is different from the enlargement context in a way that it represents an intensive quantity, consisting of the ratio of two extensive quantities (Karplus et al., 1983). Unlike extensive quantities, such as length, area, or volume, intensive quantities could not be gauged directly (Simon & Placa, 2012). As Lamon (2007) argued, intensive quantities such as speed, slope, or density should be considered keystones in developing proportional reasoning. On the other hand, the literature argues that it is not surprising for younger students to apply additive methods in proportional mixture problems, consisting of intensive quantities, and even for older students when they encounter more challenging problems (Van Dooren et al., 2009; Van Dooren, De Bock, & Verschaffel, 2010). The study of Singh (2000a) was an example of this situation that demonstrated students' difficulty in solving a mixture problem involving a non-integer ratio. As concluded by Singh (2000a), many students applied the additive method when they were asked to determine the taste of the two lemonade mixtures by calculating the difference between the amount of lemon juice and that of sugar. In this study, on the other hand, the mixture consisted of blue and yellow colors, and students were asked to determine which of the green color in the containers was darker. Similar to that of Singh (2000a), the results of this study revealed that students especially struggled to recognize the multiplicative nature of the situation and applied non-proportional strategies mostly in option B, where numbers formed non-integer ratio, and that they applied proportional strategies mostly in option A, where numbers formed an integer ratio. Having a non-integer ratio in option B might have caused some students to struggle to recognize the multiplicative relationship between the amounts, and instead, might have led them to use additive strategies.

Apparently, as in the case of enlargement context, the numerical structure of the problem might have led the students to change their strategies in solving mixture problems. Especially in option C, having letters instead of numbers created a challenge for students to interpret what was happening in the problem situation. S3, one of the sixth graders, might be a good example supporting this argument. In problems 2A and 2C, he used proportional strategies, whereas, in problem 2B, he attended to the additive differences. This case was quite similar to that of S10, who was a seventh-grade student. S10 could realize the multiplicative nature in problems 2A and 2C. However, he could not notice the multiplicative relationship within the quantities presented in problem 2B in the presence of a non-integer ratio. Parallel with these results, in his study, where pupils were given a numerical comparison task involving 23 items, each of which had a numerical complexity, Noelting (1980) found that number pairs without a lack of integer multiplier elicited the lowest number of correct answers to the juice concentration problem, where pupils were asked to compare the sourness of two glass of juices. His study has demonstrated that students' performance in solving proportional problems would change under different numerical structures. Similarly, in my study, I observed that some of the students' strategies become sophisticated in favor of an integer multiplier between the quantities being compared. In contrast, they were not successful enough in noticing a multiplicative relationship between the quantities in the absence of an integer multiplier.

Apart from the numerical structure, one more challenge may lie behind students' choice to apply the additive strategy. Mixture problems necessitate an understanding of what one would obtain due to mixing two quantities (Tourniaire & Pulos, 1985). In mixture problems, since the mixture of the quantities creates another entity, students might have been struggled to come to understand the intensive quantity embedded within the mixture problems. Lobato et al. (2010) explained this situation over a vinegar and oil mixture problem. They contended that vinegar and oil within the mixture apparently lose their entities as they were mixed, and the resulting component raises the problem of not recognizing vinegar and oil as separate entities as they were in the first place. Quite similarly, in this study, the green color represented an intensive quantity comprised of blue and yellow colors, and the abovementioned reason might account for why some students who provided non-proportional strategies were not able to interpret in which ratio the two colors have been mixed.

One final comment that is worth discussion was that nearly all sixth and seventh graders applied a non-proportional strategy in a mixture context. However, fewer sixth-grade students (i.e., S3, S5, S7) drew upon proportional strategy as compared to seventh graders (i.e., S9, S10, S11, S14). Ben-Chaim et al. (2012) attributed the sources of these difficulties to cognitive aspects and explained that when the ratio represents an intensive quantity, students not only need to acquire the knowledge of proportions to understand the situation but also figure out how rules work pertaining to the subject in the problem. They further contended that even though pupils had an intuitive understanding of the rules being applied, they should learn how to interpret results qualitatively and should communicate them effectively in a way that portrays the ratio in the situation (Ben-Chaim et al., 2012). In this respect, along with the numerical structure, the cognitive challenges, as Ben-Chaim et al. (2012) claimed, might have added complexity to the problems involving intensive quantities and might have generated an impediment for students. Though, in this study, it is an open question whether grade level could be used to explain the difficulties that emerged from cognitive aspects.

It should also be pointed out that not being able to provide a correct answer to mixture problems is not an indication of inability to think proportionally as the difficulty might have happened peculiar to the specific problem and disappeared in other problems (Lamon, 2020). Moreover, familiarity with the context being presented seemed to be much more important than the context itself (Tourniaire, 1986). These arguments manifested that contemplating the characteristics of the problems that could explain students' difficulties in solving proportional problems is a matter of great importance to providing an accurate assessment regarding learners' understanding. Especially without numbers to operate with, students may reveal their very understanding of the concept, as they would not be influenced by the numerical structure and would be encouraged more on figuring out what the problem demands them to find. Therefore, the findings of this study have demonstrated that the numerical structure and context of the problem, when considered together, play a pivotal role in contributing to our understanding of how students think of proportional situations.

5.3.3 Similarity Context

Deciding the squareness of a rectangle requires forming a ratio that could be characterized as an intensive quantity (Lamon, 2007). As reported in the literature, children struggle to comprehend covariation or the relationship between the quantities when they encounter intensive quantities (Nunes et al., 2003). In this study, for instance, there were students who provided a non-proportional strategy by just focusing on the appearances of the rectangles to decide if they were similar to a square. Bright et al. (2003) obtained a very similar conclusion that some students judged a rectangle similar to a square just because it is smaller, which Bright et al. (2003) argued reflected an absence of proportional reasoning.

This study showed that technology had the potential to allow observations through the slider and elicited multiplicative thinking to a great extent due to enlarging or shrinking the rectangles. For instance, S1, who was a sixth-grader, started to solve the problem by comparing the side lengths additively. Yet, somehow, interaction with interactive simulation helped her to recognize that the relationship between the side lengths should be multiplicative rather than additive. Similarly, S4 benefited from the technology to decide the similarity of rectangles to a square. Even though she started with a non-proportional strategy and attended to the additive difference between the adjacent side lengths, having observed the covariation between the sides as rectangles were being enlarged or shrunken helped her to conjecture that if she equalizes the short side lengths, she could compare the long side lengths, which actually corresponded to proportional equivalent fraction strategy. These findings supported Denton's (2017) assumptions that within a dynamic geometry environment, students had the opportunity to test and revise their conjectures through various representations provided by the technological environment, particularly via the dragging option.

Moreover, the findings of this study seem to indicate that students' strategies to the similarity problem were also changed across different numerical structures. For instance, S3, who was a sixth-grader, could recognize the multiplicative nature within the similarity situation in options A and B. Yet, he was not able to apply this reasoning to option C of the same problem when numbers were replaced with letters. Instead, he used a non-proportional strategy in option C. S9, who was a seventh-grader, could be another example demonstrating a strategy shift across the different numerical structures. Even though she could express the situation multiplicatively in options A and B, she could not provide multiplicative reasoning in option B of the same problem. Similar findings that of this study were obtained in the work of Bright et al. (2003). Bright et al. (2003) implemented a five-item test for 132 eighth and ninth-grade students. Among the reported answers of 14 eighth-graders, it was seen that five of them compared the adjacent sides additively in deciding the squareness of rectangles.

5.4 Implications for Future Studies

This study aimed to gain insight into sixth and seventh-grade students' strategies in solving proportional problems supported with interactive simulations. This section intends to address the possible implications for future studies. Based on the findings of this study, it can be argued that the availability of technology, the numerical structure of the problem (e.g., non-integer ratio, integer ratio, and letters), the context of the problem (e.g., enlargement, mixture, similarity) and the type of the problem (e.g., missing value and numerical comparison) can be considered potential factors that can influence students' thinking and strategies. The diversity of strategies across different numerical structures of the same problem, the changes in strategies observed with the use of technology within a single problem, preference of some strategies over others within a particular problem context, and problem types supported this idea. However, since establishing a cause-and-effect relationship was not among the purposes of this study, I avoid concluding that there is some sort of a relationship between all of these factors. Investigating how and to what extent these factors could relate to each other and how and to what extend one of them could relate to students' understanding of ratio and proportion concepts while the other factors are controlled might give valuable insights into students' understanding and that how it is shaped by particular factors. In my study, I designed problems so that they could be solved within the GeoGebra environment, and I used three different contexts that are applicable and useful to display in a dynamic environment. These were enlargement, mixture, and similarity contexts. Since my aim was not to focus on a particular context or a particular problem type, the problems that I administered to students varied greatly in terms of these factors, and this variation made it difficult to analyze, interpret and report the findings.

Further research might select one problem context and investigate the usage of the dynamic environment on students' understanding to provide a more in-depth analysis of how students reason within a particular context. For instance, designing geometrical similarity problems with different numerical structures and providing students an environment in which they can solve these problems by observing the covariation of the quantities through dynamism and by allowing them to discuss their ideas on problems with their classmates might give further insight into how students are thinking in a geometric similarity context while engaging GeoGebra activities and how peer interaction supports or hinders their way of reasoning. Similarly, in my study, I used both missing value structure and numerical comparison structure to design the problems. Even though I tried to give as detailed analysis as possible for each strategy level applied to these two problem types, one implication for future research might be focusing only on one type, designing problems with different numerical structures, and having students solve these problems through GeoGebra. By doing so, research might provide a more in-depth analysis of students' thinking of ratio and proportion concepts within a particular problem type while they are engaging in GeoGebra activities.

Moreover, especially within the context of dynamic geometry environments, it might be valuable to examine these relationships on a larger group of students with the support of statistical tests to see whether or not some sort of relationship between these factors can be established. Investigating whether or not a relationship between these factors exists or, if it does, how it influences several other factors such as problem-solving skills or achievement on a particular test measuring content knowledge of pupils regarding proportional reasoning might be of value to better understand how tasks could be shaped, designed, or enhanced to fully help students learn ratio and proportion concepts.

Another implication was that in the current study, the three different numerical structure was presented to the students in the order of integer ratio, non-integer ratio, and letters. This alignment might have affected students' thinking and possibly their strategies. In some cases, it was observed that students tried to repeat the same reasoning they used in previous problems when they struggled to solve the current problem. Especially in the last version, in which there were no numbers, the majority of students adopted the methods that seemed easy and applicable for them in the previous problems without understanding what the current problem asked for. In this regard, it is suggested that further studies might change this order and may give priority to the version presented without letters in order to prevent the recurrence of the previously adopted strategies and to explicate better why students prefer some strategies over others within different numerical structures. Especially by prioritizing the situations that numbers are absent, students may be provoked to understand the nature of the relationship between the quantities presented in the problem.

5.5 Limitations of The Current Study

There were also limitations of this study that are worth mentioning related to the methodological aspects, such as participant selection and data collection. Because of the Covid-19 pandemic, the selection of participants was made in accordance with their convenience. Therefore, there was no background information about the students' ability to reason proportionally. In this regard, future studies may obtain an idea of students' understanding of proportional reasoning through a diagnostic test designed to determine the level of students on this subject and make their selection more purposeful. For instance, they may categorize students' levels on some sort of scale and may conduct further examination on students' difficulties or strategy preference to see whether the responses vary across different levels of understanding.

Moreover, again because of the pandemic, the interviews were held online through the Zoom platform, which made it relatively harder to monitor student's written work while they were solving problems, and in turn, made the progress of the discussion difficult because I put a lot of effort into understanding the students. It is suggested that further studies may be conducted face to face or even in the classroom environment with different groups of students to answer how classroom discussion and exchange of ideas resulting from peer interaction may promote or hinder students' understanding.

This study was also limited to sixth and seventh-grade levels and conducted on a small scale with few participants. Future studies may replicate the current study with students from different age groups, particularly with eighth-graders, to observe whether receiving formal instruction on similarity concepts may produce alternative strategies to the problems entailing enlargement and shrinking. Considering the fact that both CCSSI (2010) and MoNE (2018) give emphasis on teaching similarity concept in eighth grade, studying with this grade level, observing how they conceptualize similarity and that comparing how eight graders' strategies, thinking, or difficulties may differ from those who have not been taught similarity concept may provide valuable insight into how different grade levels construct their understanding of similarity and how ratio and proportion concepts reinforce them for doing so.

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APPENDICES

A. CLASSIFICATION TASK

SINIFLANDIRMA GÖREVİ

Bu tablo üç problem içermektedir. Onları çözmeniz beklenmemektedir. Aksine, hangi problemlerin matematiksel olarak veya çözüm yolu olarak birbirine benzer olduğunu bulmanız beklenmektedir.

Soru 1: Dikdörtgen şeklindeki bir resmin uzun kenarı 6 cm iken kısa kenarı 3 cm'dir. Resim bozulmadan büyütülüyor. Bu resmin kısa kenarı 15 cm iken uzun kenarı kaç cm'dir?

Soru 2: Elif mavi ve sarı renkli boyaları karıştırarak yeşil renk elde etmek istiyor. Birinci kaba 150 ml mavi renk boya ve 300 ml sarı renk boya ekliyor. İkinci kaba 160 ml mavi renk boya ve 480 ml sarı boya ekliyor. Son durumda hangi kaptaki yeşil boyanın rengi daha koyudur?

Not: Mavi renk yeşili koyulaştırırken sarı renk yeşili açık hale getirir.

Soru 3:

10 birim – 30 birim

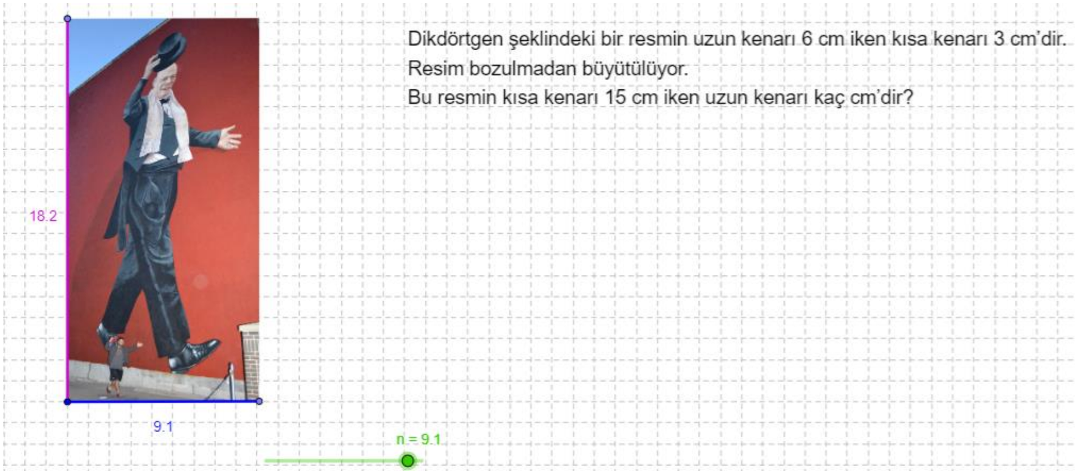
20 birim – 40 birim

- Kenar uzunlukları verilen dikdörtgenlerden hangisi daha çok kareye benzemektedir?

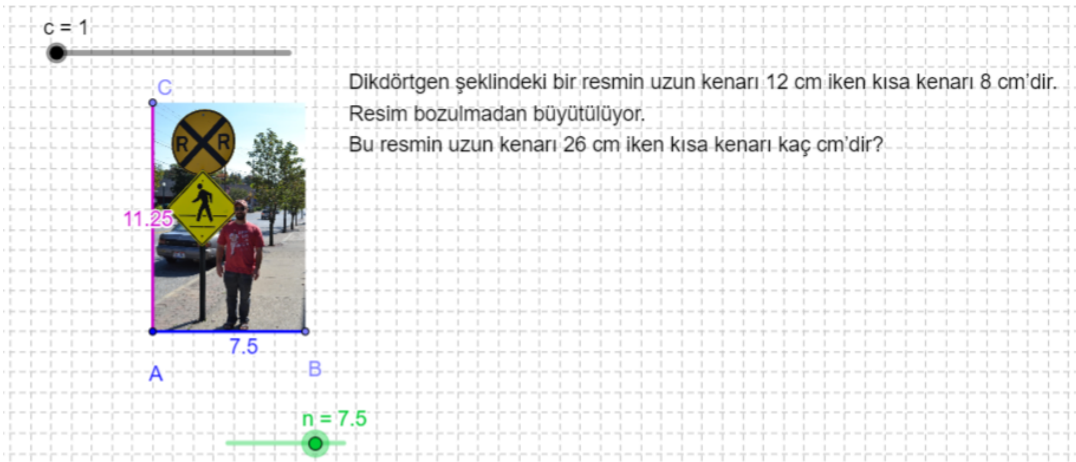
B. PROBLEM SOLVING TASK

PROBLEM ÇÖZME GÖREVİ

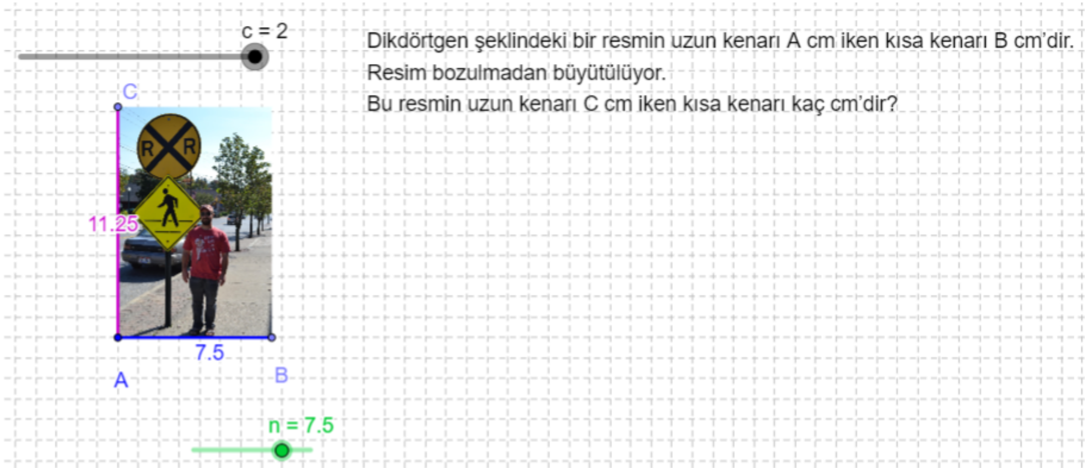
SORU 1A



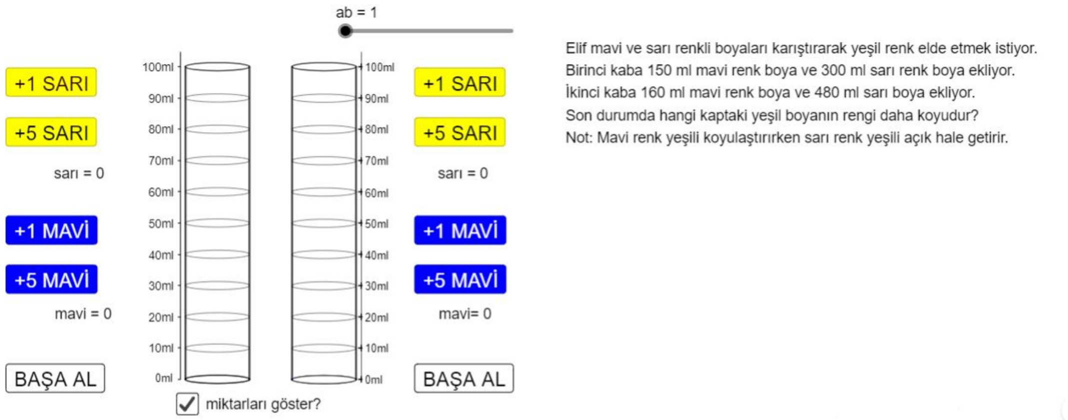
SORU 1B



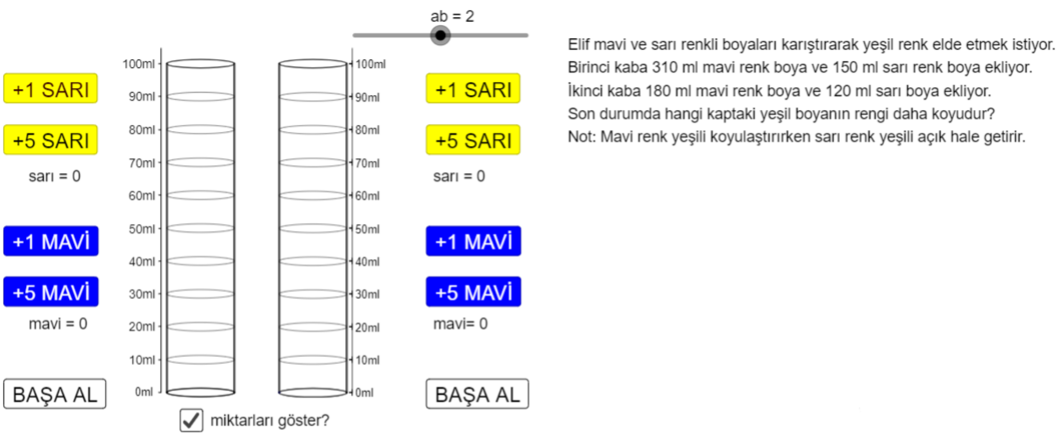
SORU 1C



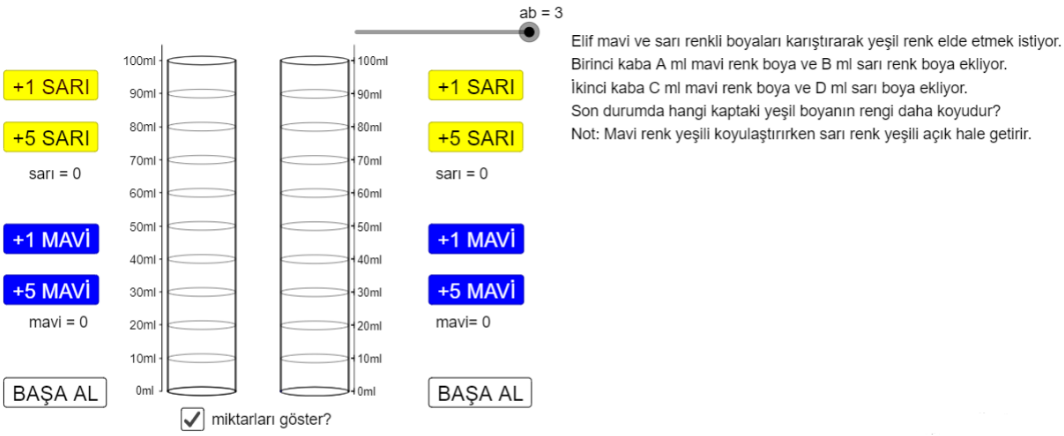
SORU 2A



SORU 2B



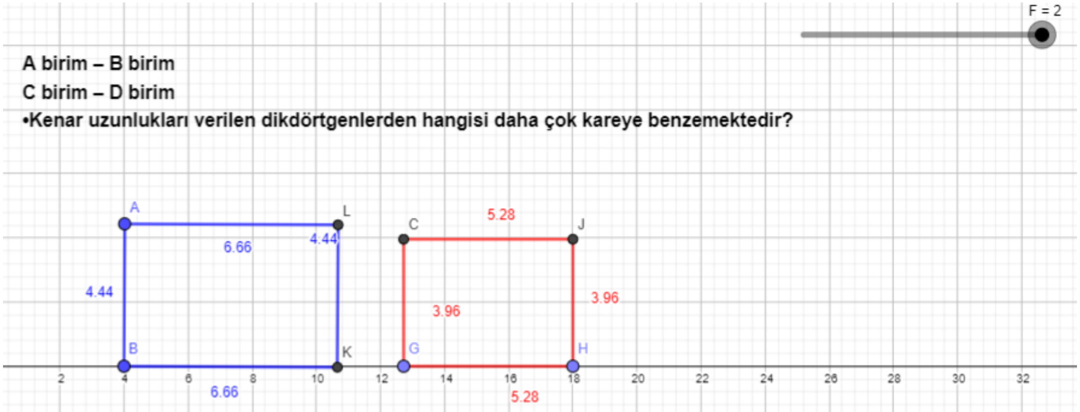
SORU 2C



SORU 3A

SORU 3B

SORU 3C


F = 2
A birim – B birim
C birim – D birim
•Kenar uzunlukları verilen dikdörtgenlerden hangisi daha çok kareye benzemektedir?
A
L
6.66
4.44
4.44
B
K
6.66
C
5.28
J
3.96
3.96
G
H
5.28
2
4
6
8
10
12
14
16
18
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22
24
26
28
30
32

C. APPROVAL OF THE ETHICAL COMMITTEE OF METU

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

DUMLUPINAR BULVARI 06800
ÇANKAYA ANKARA/TURKEY
T: +90 312 210 22 91
F: +90 312 210 79 59
ueam@metu.edu.tr
www.ueam.metu.edu.tr

Sayı: 28620816 / 252

23 Haziran 2021

Konu : Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi : İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof. Dr. Ayhan Kürşat ERBAŞ

Danışmanlığını yaptığınız Ahu Canoğulları'nın "Seventh grade students' understanding and use of proportional reasoning in the context of word problems supported by dynamic mathematics software" başlıklı araştırmanız İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **252-ODTU-2021** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

Dr.Öğretim Üyesi Şerife SEVİNÇ
İAEK Başkan Vekili